Construction of two-bubble solutions for some energy critical wave equations

Jacek Jendrej
École Polytechnique, CMLS

Séminaire Laurent Schwartz
Tuesday, May 3, 2016

Overview of the talk

1. Energy-critical wave equations
2. Soliton resolution
3. Examples of solutions with two bubbles
4. Formal computation
5. Elements of the proof

Critical NLW

Focusing energy-critical power nonlinearity in dimension $1 + N$ (with $N \geq 3$):

$$\begin{cases} \partial_{tt} u(t,x) = \Delta_x u(t,x) + |u(t,x)|^{\frac{4}{N-2}} u(t,x), \\ (u(t_0,x), \partial_t u(t_0,x)) = (u_0(x), \dot{u}_0(x)). \end{cases} \tag{NLW}$$

Notation: $\boldsymbol{u}_0 := (u_0, \dot{u}_0)$, $\mathcal{E} := \dot{H}^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$.

There is a natural energy functional defined on $\mathcal{E}$:

$$E(\boldsymbol{u}_0) = \int \frac{1}{2}|\dot{u}_0|^2 + \frac{1}{2}|\nabla u_0|^2 - \frac{N-2}{2N}|u_0|^{\frac{2N}{N-2}} \,\mathrm{d}x.$$

We consider solutions with radial symmetry: $\boldsymbol{u}(t,x) = \boldsymbol{u}(t,|x|)$.

Comments

- (Local well-posedness in $\mathcal{E}$) [Ginibre-Soffer-Velo, Shatah-Struwe '90]

$$\forall \boldsymbol{u}_0 \in \mathcal{E},\ \exists ! \boldsymbol{u} \in C((T_-, T_+); \mathcal{E}), \qquad T_- < t_0 < T_+.$$

- The energy is conserved; the flow is reversible.
- (Scaling) Let $\lambda > 0$. For $\boldsymbol{v} = (v, \dot{v}) \in \mathcal{E}$ we denote

$$\boldsymbol{v}_\lambda(x) := \big(\lambda^{-\frac{N-2}{2}} v(\frac{x}{\lambda}), \lambda^{-\frac{N}{2}} \dot{v}(\frac{x}{\lambda})\big).$$

We have $\|\boldsymbol{v}_\lambda\|_{\mathcal{E}} = \|\boldsymbol{v}\|_{\mathcal{E}}$ and $E(\boldsymbol{v}_\lambda) = E(\boldsymbol{v})$. Moreover, if $\boldsymbol{u}(t)$ is a solution of (NLW) on the time interval $[0, T_+)$, then $\boldsymbol{w}(t) := \boldsymbol{u}(\frac{t}{\lambda})_\lambda$ is a solution on $[0, \lambda T_+)$.

Ground states

- Explicit radially symmetric solution of $\Delta W(x) + |W(x)|^{\frac{4}{N-2}} W(x) = 0$:

$$W(x) := \Big(1 + \frac{|x|^2}{N(N-2)}\Big)^{-\frac{N-2}{2}}, \qquad \boldsymbol{W} := (W, 0) \in \mathcal{E}.$$

- All the radially symmetric stationary states are obtained by rescaling: $\mathcal{S} := \{\boldsymbol{W}_\lambda\}$,
- Threshold energy for nonlinear behavior (Kenig-Merle '08),
- $\mathcal{S}$ is a non-compact subset of $\mathcal{E}$,
- All the elements of $\mathcal{S}$ have the same energy,
- "Building blocks" of every solution bounded in $\mathcal{E}$?

For $N = 3$ in the radial case every solution bounded in $\mathcal{E}$ decomposes into a finite number of energy bubbles:

Theorem (Duyckaerts, Kenig, Merle '12)

Let $\boldsymbol{u}(t) : [0, T_+) \to \mathcal{E}$ be a radial solution of (NLW), bounded in $\mathcal{E}$.

- **Type II blow-up:** $T_+ < \infty$ and there exist $\boldsymbol{v}_0 \in \mathcal{E}$, $\iota_j \in \{\pm 1\}$, $\lambda_j(t)$ with $\lambda_1(t) \ll \lambda_2(t) \ll \ldots \ll \lambda_n(t) \ll T_+ - t$ as $t \to T_+$ such that

$$\lim_{t \to T_+} \Big\| \boldsymbol{u}(t) - \Big(\boldsymbol{v}_0 + \sum_{j=1}^{n} \iota_j \boldsymbol{W}_{\lambda_j(t)}\Big) \Big\|_{\mathcal{E}} = 0.$$

- **Global solution:** $T_+ = +\infty$ and there exist a solution $\boldsymbol{v}_{\mathrm{L}}$ of the linear wave equation, $\iota_j \in \{\pm 1\}$, $\lambda_j(t)$ with $\lambda_1(t) \ll \lambda_2(t) \ll \ldots \ll \lambda_n(t) \ll t$ as $t \to +\infty$ such that

$$\lim_{t \to +\infty} \Big\| \boldsymbol{u}(t) - \Big(\boldsymbol{v}_{\mathrm{L}}(t) + \sum_{j=1}^{n} \iota_j \boldsymbol{W}_{\lambda_j(t)}\Big) \Big\|_{\mathcal{E}} = 0.$$

Comments

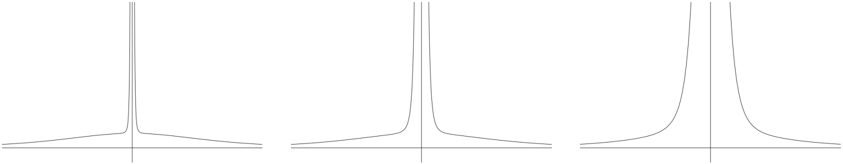
- Such a decomposition *for a sequence of times* holds in the nonradial case for $N \in \{3, 4, 5\}$ (Duyckaerts, Jia, Kenig, Merle '16),
- Similar results for critical wave maps with values in S^2 in the equivariant case (Côte '15),
- Classical problem in the theory of the heat flow from S^2 to S^2 (Struwe, Qing, Topping),
- Do there exist solutions decomposing into more than one bubble?

Unknown for $N \in \{3,4,5\}$. They exist for $N = 6$.

Theorem (J. '16)

Let $N = 6$. There exists a solution $\boldsymbol{u} : (-\infty, T_0] \to \mathcal{E}$ of (NLW) such that

$$\lim_{t \to -\infty} \|\boldsymbol{u}(t) - (\boldsymbol{W} + \boldsymbol{W}_{\frac{1}{\kappa}\mathrm{e}^{-\kappa|t|}})\|_{\mathcal{E}} = 0, \qquad \text{with } \kappa := \sqrt{\frac{5}{4}}.$$



Comments

- A similar result holds in the case of equivariant wave maps with equivariance degree $k \geq 3$, but this time the second bubble concentrates like $|t|^{-\frac{2}{k-2}}$.
- In any dimension $N > 6$ we can expect an analogous result, with concentration rate $|t|^{-\frac{4}{N-6}}$.
- Strong interaction of bubbles: the second bubble could not concentrate without it.
- The two bubbles must have the same sign in the case of (NLW), and opposite orientations in the case of wave maps.
- The possibility of concentration of *one bubble* was proved by Krieger, Schlag and Tataru '08.
- Other constructions of towers of bubbles: heat flow with a well chosen target manifold (Topping '99), Yamabe flow (del Pino '12).

Asymptotic expansion, 1

We search a solution of the form $\boldsymbol{u}(t) \simeq \boldsymbol{W} + \boldsymbol{W}_{\lambda(t)}$.

$$\boldsymbol{u}(t) = \boldsymbol{W} + \boldsymbol{U}^{(0)}_{\lambda(t)} + b(t)\cdot\boldsymbol{U}^{(1)}_{\lambda(t)} + b(t)^2\cdot\boldsymbol{U}^{(2)}_{\lambda(t)} + \ldots,$$

with $\boldsymbol{U}^{(0)} = \boldsymbol{W}$, $b(t) > 0$ and $\lambda(t), b(t) \to 0$ as $t \to -\infty$.

For $v(x) : \mathbb{R}^6 \to \mathbb{R}$ denote:

$$v_\lambda(x) := \frac{1}{\lambda^2} v\big(\frac{x}{\lambda}\big), \quad \Lambda v := -\frac{\partial}{\partial\lambda} v_\lambda|_{\lambda=1} = (2+x\cdot\nabla)v, \quad \Lambda_0 v := (3+x\cdot\nabla)v.$$

Since $u(t) \simeq W + W_{\lambda(t)}$, we have

$$\dot u(t) = \partial_t u(t) \simeq -\frac{\lambda'(t)}{\lambda(t)}(\Lambda W)_{\lambda(t)} \quad \Rightarrow \quad b(t) = \lambda'(t), \boldsymbol{U}^{(1)} = (0, -\Lambda W).$$

Can we find $\boldsymbol{U}^{(2)} = (U^{(2)}, \dot U^{(2)})$?

Asymptotic expansion, 2

Neglecting irrelevant terms and replacing $\lambda'(t)$ by $b(t)$, we compute

$$\partial_t^2 u(t) = -\frac{b'(t)}{\lambda(t)}(\Lambda W)_{\lambda(t)} + \frac{b(t)^2}{\lambda(t)^2}(\Lambda_0 \Lambda W)_{\lambda(t)} + \ldots.$$

Denote $L = -\Delta - f'(W)$ the linearization of $-\Delta u - f(u)$ near $u = W$. A simple computation yields

$$\Delta u(t) + f(u(t)) = -\frac{b(t)^2}{\lambda(t)^2}(LU^{(2)})_{\lambda(t)} + f'(W)_{\lambda(t)} + \ldots,$$

We discover that

$$LU^{(2)} = -\Lambda_0 \Lambda W + \frac{\lambda(t)}{b(t)^2}\big(b'(t)\cdot \Lambda W + \lambda(t)\cdot f'(W)\big).$$

Fredholm condition

$$LU^{(2)} = -\Lambda_0 \Lambda W + \frac{\lambda(t)}{b(t)^2}\big(b'(t)\cdot \Lambda W + \lambda(t)\cdot f'(W)\big). \qquad (1)$$

Due to scaling invariance, $\Lambda W \in \ker L$.

$$\int_{\mathbb{R}^6} \Lambda W \cdot \big(b'(t)\cdot \Lambda W + \lambda(t)\cdot f'(W)\big)\,\mathrm{d}x = 0$$
$$\Leftrightarrow \quad b'(t) = \frac{5}{4}\lambda(t) = \kappa^2 \lambda(t).$$

If this condition is satisfied, we can solve (1) and find $U^{(2)}$. We take $\boldsymbol{U}^{(2)} = (U^{(2)}, 0)$.

$$\begin{cases} \lambda'(t) = b(t) \\ b'(t) = \kappa^2\lambda(t) \end{cases} \quad \text{leads to} \quad \begin{cases} \lambda_{\mathrm{app}}(t) = \dfrac{1}{\kappa}\mathrm{e}^{-\kappa|t|} \\ b_{\mathrm{app}}(t) = \mathrm{e}^{-\kappa|t|}. \end{cases}$$

Main scheme

- Approximate solution:

$$\varphi(\mu, \lambda, b) := \boldsymbol{W}_\mu + \boldsymbol{U}_\lambda^{(0)} + b \cdot \boldsymbol{U}_\lambda^{(1)} + b^2 \cdot \boldsymbol{U}_\lambda^{(2)}$$

- Key idea: construct a sequence of solutions $\boldsymbol{u}_n(t)$ such that we have *uniform* bounds

$$\|\boldsymbol{u}_n(t) - \varphi(1, \lambda_{\mathrm{app}}(t), b_{\mathrm{app}}(t))\|_{\mathcal{E}} \leq C\mathrm{e}^{-\frac{3}{2}\kappa|t|}$$
$$\text{for } t \in [T_n, T_0] \text{ with } T_n \to -\infty.$$

- To do this, we consider the initial data $\boldsymbol{u}_n(T_n) = \varphi(1, \lambda_{\mathrm{app}}(T_n), b_{\mathrm{app}}(T_n))$ (with a correction due to the linear unstable direction).
- Pass to a weak limit.

Note that time reversibility of the flow is crucial.

Control of the error term by the energy method

- Decompose $\boldsymbol{u}_n(t) = \boldsymbol{\varphi}(\mu(t), \lambda(t), b(t)) + \boldsymbol{g}(t)$, with $\mu(t)$ and $\lambda(t)$ defined by natural orthogonality conditions.
- We construct a functional $H(t)$ such that $H(t) \gtrsim \|\boldsymbol{g}(t)\|_{\mathcal{E}}^2$ and

$$\|\boldsymbol{g}(t)\|_{\mathcal{E}} \leq C_0 \mathrm{e}^{-\frac{3}{2}\kappa|t|} \text{ for } t \in [T_n, T] \quad \Rightarrow \quad H'(t) \leq c \cdot C_0^2 \cdot \mathrm{e}^{-3\kappa|t|},$$

  with c arbitrarily small.
- A continuity argument yields the required bounds on $\|\boldsymbol{g}(t)\|_{\mathcal{E}}$.

The functional $H(t)$ is a correction of the energy functional by a localized virial term.

Merci de votre attention.