Strongly interacting kink-antikink pairs for scalar fields in dimension 1+1

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Séminaire Laurent Schwartz
CMLS, École polytechnique, 10/03/2020

Scalar field equation

- We consider nonlinear scalar fields $\mathbb{R}^{1+1} \to \mathbb{R}$ which are critical points of the Lagrangian

$$\mathscr{L}(\phi, \partial_t \phi) := \iint \Big(\frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}(\partial_x \phi)^2 - U(\phi)\Big)\,\mathrm{d}x\,\mathrm{d}t,$$

  where $U : \mathbb{R} \to \mathbb{R}$ is a positive function.

- The Euler-Lagrange equation reads

$$\begin{gathered}\partial_t^2 \phi(t,x) - \partial_x^2 \phi(t,x) + U'(\phi(t,x)) = 0, \\ (t,x) \in \mathbb{R} \times \mathbb{R}, \quad \phi(t,x) \in \mathbb{R}.\end{gathered} \tag{CSF}$$

- "The simplest" nonlinear wave equation.
- This equation and its quantisation are used as toy models in Quantum Field Theory.

Scalar field equation

- Energy:

$$E(\phi, \partial_t\phi) = E_k(\partial_t\phi) + E_p(\phi)$$
$$:= \int_{\mathbb{R}} \Big[\frac{1}{2}(\partial_t\phi)^2 + \Big(\frac{1}{2}(\partial_x\phi)^2 + U(\phi)\Big)\Big]\,\mathrm{d}x.$$

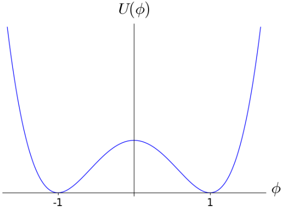
- Momentum:

$$P(\phi, \partial_t\phi) := \int_{\mathbb{R}} -\partial_t\phi\partial_x\phi\,\mathrm{d}x.$$

- $E := E_k + E_p$ and P are conserved quantities.
- If ω is a non-degenerate minimum of U (a "vacuum"), $U(\omega) = 0$ and $U''(\omega) > 0$, then $\phi(t, x) \equiv \omega$ is a trivial stable solution of (CSF). The flow linearised around this solution is a linear Klein-Gordon equation with mass $\sqrt{U''(\omega)}$.

Two vacua and transitions between them

Consider U having many vacua. For the sake of simplicity, we consider a double-well potential, say $U(\phi) := \frac{1}{4}(1-\phi^2)^2$, corresponding to the "$\phi^4$ model".



- The space of finite energy states $(\phi_0, \dot{\phi}_0)$ is the union of four affine spaces (topological classes) characterised by:
  1. $\lim_{x\to-\infty} \phi_0(x) = -1$ and $\lim_{x\to\infty} \phi_0(x) = -1$,
  2. $\lim_{x\to-\infty} \phi_0(x) = 1$ and $\lim_{x\to\infty} \phi_0(x) = 1$,
  3. $\lim_{x\to-\infty} \phi_0(x) = -1$ and $\lim_{x\to\infty} \phi_0(x) = 1$,
  4. $\lim_{x\to-\infty} \phi_0(x) = 1$ and $\lim_{x\to\infty} \phi_0(x) = -1$.
- The global minimisers of energy in classes 1. and 2. are the constant functions. The global minimisers in classes 3. and 4. are (respectively) *kinks* and *antikinks*.
- One says that states in classes 1. and 2. are topologically trivial, because they are homotopic to trivial states.

Two vacua and transitions between them

Let $y_1 < y_2$ and $\phi_1 < \phi_2$. The minimal potential energy of a state connecting $\phi_0(y_1) = \phi_1$ with $\phi_0(y_2) = \phi_2$ is computed using the "Bogomolny trick":

$$\begin{aligned}&\int_{y_1}^{y_2} \Big(\frac{1}{2}(\partial_x\phi_0)^2 + U(\phi_0)\Big)\,\mathrm{d}x = \\ &\quad= \int_{y_1}^{y_2} \sqrt{2U(\phi_0)}\partial_x\phi_0\,\mathrm{d}x + \frac{1}{2}\int_{y_1}^{y_2}(\partial_x\phi_0 - \sqrt{2U(\phi_0)})^2\,\mathrm{d}x \\ &\quad\geq \int_{\phi_1}^{\phi_2} \sqrt{2U(\psi)}\,\mathrm{d}\psi,\end{aligned}$$

with equality if and only if $\partial_x\phi_0 = \sqrt{2U(\phi_0)}$. There is exactly one solution such that $H(0) = 0$ (the kink). All the other solutions are obtained by space translation. We obtain $\lim_{x\to-\infty} H(x) = -1$ and $\lim_{x\to\infty} H(x) = 1$ (with exponential convergence).
The antikinks are $-H$ and its space translates.

Kink-antikink pairs

- Small perturbations of the constant solution $\phi \equiv 1$ have oscillatory behavior (proved by Delort '01 for smooth, compactly supported data).
- Intuitively, one expects the solution to somehow oscillate around $\omega_+ = 1$, as long as the attraction force of ω_+ is not counterbalanced by the other vacuum $\omega_- = -1$ in some space region.
- We will study solutions in class 2. which "dynamically reach -1" and have minimal possible energy.

Definition

We say that a solution ϕ of (CSF) is a (strongly interacting) kink-antikink pair if

- it belongs to class 2, that is $\lim_{x\to-\infty}\phi(t,x) = \lim_{x\to\infty}\phi(t,x) = 1$,
- there exists $x_0 : \mathbb{R} \to \mathbb{R}$ such that $\lim_{t\to\infty}\phi(t, x_0(t)) = -1$,
- $E(\phi, \partial_t\phi) \le 2E_p(H)$.

Kink-antikink pairs

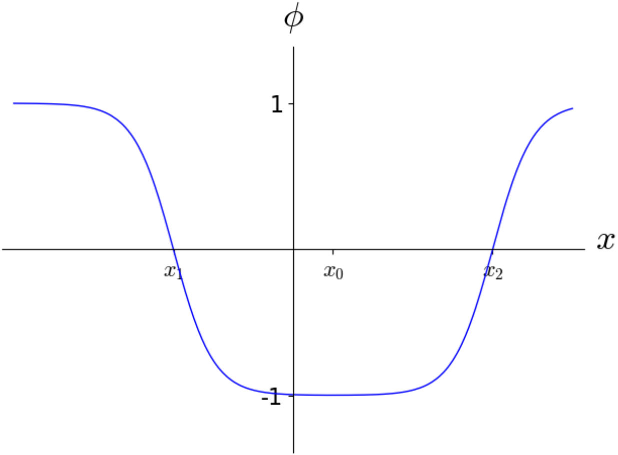


Figure: The shape of a kink-antikink pair. Because of the energy constraint, the shape of the transitions has to be close to optimal.

Kink-antikink pairs

Proposition

A solution ϕ of (CSF) is a kink-antikink pair if and only if there exist $x_1, x_2 : \mathbb{R} \to \mathbb{R}$ such that $\lim_{t\to\infty} \big(x_2(t) - x_1(t)\big) = \infty$ and

$$\lim_{t\to\infty} \big(\|\partial_t\phi(t)\|_{L^2} + \\ + \big\|\phi(t) - \big(1 - H(\cdot - x_1(t)) + H(\cdot - x_2(t))\big)\big\|_{H^1}\big) = 0.$$

Indeed, if $E(\phi, \partial_t\phi) \leq 2E_p(H)$, then both transitions are close to optimal. We are thus in a perturbative setting, except for the "trajectories" (x_1, x_2).

Main result

Theorem (Existence and uniqueness of kink-antikink pairs)

There exist a solution $\phi_{(2)}$ of (CSF) and a function $x : \mathbb{R} \to \mathbb{R}$ such that

- $|x(t) - \log(At)| = O(t^{-2+\epsilon})$ and $|x'(t) - t^{-1}| = O(t^{-3+\epsilon})$,
- $\|\phi_{(2)}(t) - (1 - H(\cdot + x(t)) + H(\cdot - x(t))\|_{H^1} = O(t^{-2+\epsilon})$,
- $\|\partial_t\phi_{(2)}(t) - x'(t)(\partial_x H(\cdot + x(t)) - \partial_x H(\cdot - x(t)))\|_{L^2} = O(t^{-2+\epsilon})$.

Moreover, if ϕ is any kink-antikink pair, then there exist $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}$ such that $\phi(t, x) = \phi_{(2)}(t - t_0, x - x_0)$.

Here, A is an explicit constant depending on U:

$$A := \Big(\int_0^1 \sqrt{2U(\phi)}\,\mathrm{d}\phi\Big)^{-1/2} \exp\Big(\int_0^1 \Big(\frac{1}{\sqrt{2U(\phi)}} - \frac{1}{1-\phi}\Big)\,\mathrm{d}\phi\Big).$$

Comments

- The kink-antikink pair plays the role analogous to a mountain pass of the energy functional (lowest energy non-oscillatory wave).
- We see this result as an analog of results of Merle ('93), and Raphaël and Szeftel ('11) on uniqueness of minimal mass blow-up solutions for critical nonlinear Schrödinger equations.
- We call our kink-antikink pairs "strongly interacting" because the specific logarithmic distance between the kink and the antikink clearly reflects the interaction between them.
- Other works on strongly interacting multi-solitons include Martel-Raphaël ('15) on critical Schrödinger, Jendrej ('16) on critical waves, Nguyen ('17) on Schrödinger and gKdV, Jendrej and Lawrie ('17) on critical wave maps, Jendrej ('18) on gKdV, ... Uniqueness is not (completely) proved in these works.
- The result is also inspired by the theory of the Allen-Cahn equation in two dimensions (which would be obtained by changing t to it), especially the works of del Pino, Kowalczyk, Pacard and Wei.

Modulation analysis – first change of variables

Let ϕ be a solution of threshold energy $E(\phi, \partial_t \phi) = 2E_p(H)$, close to a kink-antikink pair for all $t \gg 1$. Denote $H_j(t, x) := H(x - x_j(t))$. We decompose

$$\phi(t) = 1 - H_1(t) + H_2(t) + g(t).$$

Using the Implicit Function Theorem, one shows that there exists a unique choice of continuous functions $x_1(t)$ and $x_2(t)$ such that $\lim_{t\to\infty}(x_2(t) - x_1(t)) = \infty$ and the following *orthogonality conditions* hold:

$$\langle \partial_x H_1(t), g(t)\rangle = \langle \partial_x H_2(t), g(t)\rangle = 0 \qquad \text{(orth)}$$

(the brackets always denote the L^2 inner product).
The mapping $(\phi, \partial_t \phi) \mapsto (x_1, x_2, g, x_1', x_2', \partial_t g)$ is a change of variables (or unknowns), note however that the space of admissible states $(x_1, x_2, g, x_1', x_2', \partial_t g)$ is not a linear space, but rather a codimension 4 submanifold of $\mathbb{R}^2 \times H^1 \times \mathbb{R}^2 \times L^2$, defined by (orth).

Formal computation – restriction method

- A prediction of the evolution of x_1 and x_2 (at least at the main order) can be obtained by "pretending" that $(g, \partial_t g) = 0$, without changing the Lagrangian.
- The reduced Lagrangian is given by

$$\begin{aligned}\widetilde{\mathscr{L}}(x_1, x_2, x_1', x_2') &:= \mathscr{L}(1 - H_1 + H_2, x_1'\partial_x H_1 + x_2'\partial_x H_2) \\ &\simeq 2E_p(H) + \frac{1}{2}((x_1')^2 + (x_2')^2)\|\partial_x H\|_{L^2}^2 \\ &+ A^2\|\partial_x H\|_{L^2}^2 \mathrm{e}^{-(x_2 - x_1)}.\end{aligned}$$

- The Euler-Lagrange equations are $\frac{\mathrm{d}}{\mathrm{d}t}\big(\partial_{x_j'}\widetilde{\mathscr{L}}\big) = \partial_{x_j}\widetilde{\mathscr{L}}$, yielding

$$x_1'' = A^2\mathrm{e}^{-(x_2 - x_1)}, \qquad x_2'' = -A^2\mathrm{e}^{-(x_2 - x_1)}.$$

- Note the attractive sign of the interaction.
- The solutions of this system are $(x_1, x_2) = (-\log(At), \log(At))$ and its space-time translates.

True computation – another change of variables

- We would like to show that, if $(x_1, x_2, g, x_1', x_2', \partial_t g)$ is any kink-antikink pair, then $x_2(t) - x_1(t) \simeq 2\log(At)$.
- Differentiating in time (orth) yields a system of differential equations for (x_1, x_2, x_1', x_2'), which is the formally obtained system perturbed by the "error term" $(g, \partial_t g)$. We need to show that this perturbation is negligible.
- From the second-order expansion of the energy one obtains $\|(g, \partial_t g)\|^2_{H^1 \times L^2} \lesssim \mathrm{e}^{-(x_2 - x_1)}$, which allows to write differential inequalities in (x_1, x_2, x_1', x_2') only, not involving $(g, \partial_t g)$.
- We introduce auxiliary functions, the *localised momenta*:

$$p_1(t) := \|\partial_x H\|_{L^2}^{-2} \langle \partial_x (H_1(t) - \chi_1(t) g(t)), \partial_t \phi(t) \rangle,$$
$$p_2(t) := \|\partial_x H\|_{L^2}^{-2} \langle -\partial_x (H_2(t) + \chi_2(t) g(t)), \partial_t \phi(t) \rangle.$$

True computation – another change of variables

- Using the bound on $\|(g, \partial_t g)\|_{H^1\times L^2}$, we obtain

$$\left|x_j'(t) - p_j(t)\right| \lesssim \mathrm{e}^{-(x_2(t)-x_1(t))},$$
$$\left|p_j'(t) + (-1)^j A^2 \mathrm{e}^{-(x_2(t)-x_1(t))}\right| \lesssim (x_2(t) - x_1(t))^{-1}\mathrm{e}^{-(x_2(t)-x_1(t))}.$$

- This system can be solved at main order, yielding the following result.

Proposition (Modulation analysis)

Let $(x_1, x_2, g, x_1', x_2', \partial_t g)$ be a kink-antikink pair. Then

$$x_2(t) - x_1(t) - 2\log(At) = O((\log t)^{-1}),$$
$$x_2'(t) - x_1'(t) - 2t^{-1} = O((t\log t)^{-1}),$$
$$\|g(t)\|_{H^1} + \|\partial_t g(t)\|_{L^2} = O(t^{-1}(\log t)^{-\frac{1}{2}}) = o(t^{-1}).$$

- Introducing p_1, p_2 is perhaps analogous to the method of normal forms, but we are not in the setting of Birkhoff normal forms and the formulas were found by trial and error.

Existence and uniqueness – Lyapunov-Schmidt reduction

- Equation (CSF) re-written in terms of $(x_1, x_2, g, x_1', x_2', \partial_t g)$ reads

$$\partial_t^2 g + x_1''\partial_x H_1 - (x_1')^2\partial_x^2 H_1 - x_2''\partial_x H_2 + (x_2')^2\partial_x^2 H_2 \\ - \partial_x^2 g + U'(1 - H_1 + H_2 + g) + U'(H_1) - U'(H_2) = 0.$$

  We solve it in two steps.

- First, for given trajectories (x_1, x_2) we solve the *auxiliary equation*

$$\begin{aligned} &\partial_t^2 g + x_1''\partial_x H_1 - (x_1')^2\partial_x^2 H_1 - x_2''\partial_x H_2 + (x_2')^2\partial_x^2 H_2 \\ &\quad - \partial_x^2 g + U'(1 - H_1 + H_2 + g) + U'(H_1) - U'(H_2) \\ &\quad = \lambda_1\partial_x H_1 + \lambda_2\partial_x H_2. \end{aligned} \tag{A}$$

  We obtain the unique solution
  $(\lambda_1, \lambda_2, g) = (\lambda_1(x_1, x_2), \lambda_2(x_1, x_2), g(x_1, x_2))$.

- We find all the trajectories (x_1, x_2) solving the *bifurcation equations*

$$\lambda_1(x_1, x_2) = \lambda_2(x_1, x_2) = 0. \tag{B}$$

Existence and uniqueness – Lyapunov-Schmidt reduction

The scheme is an instance of a general approach known as the *Lyapunov-Schmidt method* or the *alternative method*.

- The energy is coercive in the direction $(g, \partial_t g)$, thanks to (orth):

$$\langle \mathrm{D}^2 E(1 - H_1 + H_2, x_1' \partial_x H_1 - x_2' \partial_x H_2)(g, \partial_t g), (g, \partial_t g)\rangle \gtrsim \|(g, \partial_t g)\|^2_{H^1 \times L^2}.$$

- The component $(g, \partial_t g)$ is thus non-degenerate from the point of view of energy estimates.
- The component (x_1, x_2, x_1', x_2') is degenerate, due to translation invariance of the energy.
- Lyapunov-Schmidt approach:
    - first solve for the non-degenerate component, for any (reasonable) degenerate component,
    - then find the degenerate component.

Existence and uniqueness – Lyapunov-Schmidt reduction

Proposition

For any pair of trajectories (x_1, x_2) satisfying the estimates obtained in the Modulation analysis, equation (A) has a unique solution such that $\|g(t)\|_{H^1} + \|\partial_t g(t)\|_{L^2} = o(t^{-1})$. This solution satisfies $\|g(t)\|_{H^1} + \|\partial_t g(t)\|_{L^2} = o(t^{-\gamma})$ for any $\gamma < 2$. Moreover, for $\nu > 1$ we have (almost... one should use slightly different norms)

$$\sup_{t \geq T_0} t^{\nu+2} \Big| \lambda_j(x_1^\sharp, x_2^\sharp) - \lambda_j(x_1, x_2) + (-1)^j ((x_j^\sharp)'' - x_j'') + A^2 \mathrm{e}^{-(x_2^\sharp - x_1^\sharp)} - A^2 \mathrm{e}^{-(x_2 - x_1)} \Big| \ll \sup_{t \geq T_0} t^\nu (|x_1^\sharp - x_1| - |x_2^\sharp - x_2|). \quad (*)$$

- The proof is done using the Contraction Principle in the energy space with a time weight. This relies on energy estimates for the linearised equation (with time-dependent potentials), which are achieved using modified energy functionals.
- Bound (*) allows to solve the bifurcation equations.

Modified energy functionals – main idea

Consider a linear wave equation with a (slowly) moving potential:

$$\partial_t^2 h(t) - \partial_x^2 h(t) + V(\cdot - x(t))h(t) = f(t).$$

If one uses the energy

$$I(t) := \int_{\mathbb{R}} \Big(\frac{1}{2}(\partial_t h(t))^2 + \frac{1}{2}(\partial_x h(t))^2 + \frac{1}{2}V(\cdot - x(t))h(t)^2\Big)\,\mathrm{d}x,$$

differentiating gives $|I'(t)| \lesssim \|\partial_t h(t)\|_{L^2}\|f(t)\|_{L^2} + |x'(t)|\|h(t)\|_{L^2}^2$, which is not sufficient, since the error decays only polynomially in time and $|x'(t)| \simeq t^{-1}$.

Instead, one uses a mixed energy-momentum functional (cf. Martel, Merle and Tsai '01, ...):

$$I(t) := \int_{\mathbb{R}} \Big(\frac{1}{2}(\partial_t h(t))^2 + \frac{1}{2}(\partial_x h(t))^2 + \frac{1}{2}V(\cdot - x(t))h(t)^2 - x'(t)\partial_t h(t)\partial_x h(t)\Big)\,\mathrm{d}x.$$

Final remarks and open problems

- We do not know what the behaviour of a kink-antikink pair is for $t \to -\infty$. One expects generically an inelastic collision and modified scattering.
- Kink-antikink pairs with asymptotically distinct speeds are easy to construct, but uniqueness is an open problem, as is describing the collision.
- One can construct kink K-clusters, solutions decomposing asymptotically into K alternating kinks and antikinks, for any $K \in \mathbb{N}$. Uniqueness is an open problem.

Thank you for your attention.