Construction of concentrating bubbles for the energy-critical wave equation

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Critical NLW

Focusing energy-critical power nonlinearity in dimension $1 + N$ (with $N \geq 3$):

$$\begin{cases} \partial_t^2 u(t,x) = \Delta_x u(t,x) + |u(t,x)|^{\frac{4}{N-2}} u(t,x), \\ (u(t_0,x), \partial_t u(t_0,x)) = (u_0(x), \dot{u}_0(x)). \end{cases} \tag{NLW}$$

Notation: $\vec{u}_0 := (u_0, \dot{u}_0)$, $\mathcal{H} := \dot{H}^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$.

There is a natural energy functional defined on $\mathcal{H}$:

$$E(\vec{u}_0) = \int \frac{1}{2}|\dot{u}_0|^2 + \frac{1}{2}|\nabla u_0|^2 - \frac{N-2}{2N}|u_0|^{\frac{2N}{N-2}} \,\mathrm{d}x.$$

We consider solutions with radial symmetry: $\vec{u}(t,x) = \vec{u}(t,|x|)$.
This model shares some features with critical equivariant wave maps:

$$\partial_t^2 u(t,r) = \partial_r^2 u(t,r) + \frac{1}{r} u(t,r) - \frac{k^2}{2r^2} \sin(2u(t,r)). \tag{WM}$$

Comments

- Local well-posedness in $\mathcal{H}$ (conditional)
  - Ginibre, Soffer, Velo (1992)
  - Shatah, Struwe (1994)

$$\forall \vec{u}_0 \in \mathcal{H},\ \exists! \vec{u} \in C((T_-, T_+); \mathcal{H}), \qquad T_- < t_0 < T_+.$$

- The energy is conserved; the flow is reversible.
- Let $\lambda > 0$. For $\vec{v} = (v, \dot{v}) \in \mathcal{H}$ we denote

$$\vec{v}_\lambda(x) := \Big(\lambda^{-\frac{N-2}{2}} v\big(\frac{x}{\lambda}\big), \lambda^{-\frac{N}{2}} \dot{v}\big(\frac{x}{\lambda}\big)\Big).$$

We have $\|\vec{v}_\lambda\|_{\mathcal{H}} = \|\vec{v}\|_{\mathcal{H}}$ and $E(\vec{v}_\lambda) = E(\vec{v})$. Moreover, if $\vec{u}(t)$ is a solution of (NLW) on the time interval $[0, T_+)$, then $\vec{w}(t) := \vec{u}(\frac{t}{\lambda})_\lambda$ is a solution on $[0, \lambda T_+)$.

Ground states

- Explicit radially symmetric solution of $\Delta W(x) + |W(x)|^{\frac{4}{N-2}} W(x) = 0$:

$$W(x) := \Big(1 + \frac{|x|^2}{N(N-2)}\Big)^{-\frac{N-2}{2}}, \qquad \vec W := (W, 0) \in \mathcal{H}.$$

- All radial stationary states are obtained by rescaling: $\mathcal{S} := \{\vec W_\lambda\}$.
- Threshold elements for nonlinear behavior – Kenig, Merle (2008).
- "Building blocks" of every solution bounded in $\mathcal{H}$?

For $N = 3$ in the radial case the answer is "yes":

Theorem – Duyckaerts, Kenig, Merle (2012)

Let $\vec{u}(t) : [0, T_+) \to \mathcal{H}$ be a radial solution of (NLW) in dimension $N = 3$.

- **Type II blow-up:** If $T_+ < \infty$ and $\|\vec{u}(t)\|_{\mathcal{H}}$ is bounded, then there exist $\vec{u}_0^* \in \mathcal{H}$, $\iota_j \in \{\pm 1\}$, $\lambda_j(t)$ with $\lambda_1(t) \ll \lambda_2(t) \ll \ldots \ll \lambda_n(t) \ll T_+ - t$ as $t \to T_+$ such that

$$\lim_{t \to T_+} \Big\| \vec{u}(t) - \Big(\vec{u}_0^* + \sum_{j=1}^{n} \iota_j \vec{W}_{\lambda_j(t)} \Big) \Big\|_{\mathcal{H}} = 0.$$

- **Global solution:** If $T_+ = +\infty$, then there exist a solution $\vec{u}_{\mathrm{L}}^*$ of the linear wave equation, $\iota_j \in \{\pm 1\}$, $\lambda_j(t)$ with $\lambda_1(t) \ll \lambda_2(t) \ll \ldots \ll \lambda_n(t) \ll t$ as $t \to +\infty$ such that

$$\lim_{t \to +\infty} \Big\| \vec{u}(t) - \Big(\vec{u}_{\mathrm{L}}^*(t) + \sum_{j=1}^{n} \iota_j \vec{W}_{\lambda_j(t)} \Big) \Big\|_{\mathcal{H}} = 0.$$

Comments

- Such a decomposition *for a sequence of times* holds in the nonradial case for $N \in \{3, 4, 5\}$
  - Duyckaerts, Jia, Kenig, Merle (2016)
- Similar results for critical wave maps with values in S^2 in the equivariant case
  - Côte (2015)
- The "Soliton Resolution Conjecture" originates in the theory of integrable systems
  - Eckhaus, Schuur (1983)
- Classical problem in the theory of the heat flow from S^2 to S^2
  - Struwe; Qing; Topping

Some natural questions

- Study the dynamics of solutions which remain close to $\{\vec{W}_\lambda\}$ (for example in the energy space: $\|\vec{u}(t) - \vec{W}_{\lambda(t)}\|_{\mathcal{H}} \le \eta \ll 1$ for $t_0 \le t < T_+$).
  How does $\lambda(t)$ behave as $t \to T_+$ in this case?
  - Krieger, Schlag, Tataru (2008, 2009)
  - Rodnianski, Raphaël (2012)
  - Hillairet, Raphaël (2012)
  - Donninger, Krieger (2013)
  - Krieger, Schlag (2014)
  - Donninger, Huang, Krieger, Schlag (2014)
  - Ortoleva, Perelman (2013)
  - Perelman (2014)
  - Krieger, Nakanishi, Schlag (2015)
  - Collot (2014)
- Do there exist solutions decomposing into more than one bubble?

Results

I considered two special cases of the (continuous time) soliton resolution in various dimensions:

- One bubble in the finite-time blow-up case:

$$\vec{u}(t) = \vec{u}^*(t) + \vec{W}_{\lambda(t)} + \vec{h}(t), \quad \lim_{t \to T_+} \lambda(t) = 0, \quad \vec{u}^*(T_+) = \vec{u}_0^*,$$

- Two bubbles without remainder:

$$\vec{u}(t) = \vec{W}_{\mu(t)} \pm \vec{W}_{\lambda(t)} + \vec{h}(t), \quad \lambda(t) \ll \mu(t) \text{ as } t \to T_+,$$

where $\vec{h}(t)$ is an error term which satisfies $\lim_{t \to T_+} \|\vec{h}(t)\|_{\mathcal{H}} = 0$.

Main idea – "modulation theory"

Try to understand the dynamics by "forgetting" $\vec{h}(t)$, hence reducing the equation to an ODE. Then, control $\vec{h}(t)$ using (modified) energy functionals.

Examples of solutions with two bubbles

Theorem – J. (2016)

Let $N = 6$. There exists a solution $\vec{u} : (-\infty, T_0] \to \mathcal{H}$ of (NLW) such that

$$\lim_{t \to -\infty} \left\| \vec{u}(t) - \left(\vec{W} + \vec{W}_{\frac{1}{\kappa} \mathrm{e}^{-\kappa|t|}} \right) \right\|_{\mathcal{H}} = 0, \qquad \text{with } \kappa := \sqrt{\frac{5}{4}}.$$

We have a similar result for wave maps:

Theorem – J. (2016)

Let $k \geq 3$. There exists a solution $\vec{u} : (-\infty, T_0] \to \mathcal{H}$ of (WM) such that

$$\lim_{t \to -\infty} \left\| \vec{u}(t) - \left(-\vec{W} + \vec{W}_{\frac{k-2}{2\kappa}(\kappa|t|)^{-\frac{2}{k-2}}} \right) \right\|_{\mathcal{H}} = 0, \qquad \kappa \text{ constant } > 0.$$

Comments

- A similar result holds for the critical radial Yang-Mills equation (exponential concentration rate); the same would be the case for (WM) with $k = 2$.
- In any dimension $N > 6$ we can expect an analogous result, with concentration rate $\lambda(t) \sim |t|^{-\frac{4}{N-6}}$.
- Strong interaction of bubbles: the second bubble could not concentrate without being "pushed" by the first one,
    - Martel, Raphaël (2015)
    - Gérard, Lenzmann, Pocovnicu, Raphaël (2016).

Asymptotic expansion, 1

We search a solution of the form $\vec{u}(t) \simeq \vec{W} + \vec{W}_{\lambda(t)}$.

$$\vec{u}(t) = \vec{W} + \vec{U}^{(0)}_{\lambda(t)} + b(t) \cdot \vec{U}^{(1)}_{\lambda(t)} + b(t)^2 \cdot \vec{U}^{(2)}_{\lambda(t)} + \ldots,$$

with $\vec{U}^{(0)} = \vec{W}$, $b(t) > 0$ and $\lambda(t), b(t) \to 0$ as $t \to -\infty$.

For $v(x) : \mathbb{R}^6 \to \mathbb{R}$ denote:

$$v_\lambda(x) := \frac{1}{\lambda^2} v\Big(\frac{x}{\lambda}\Big), \quad \Lambda v := -\frac{\partial}{\partial \lambda} v_\lambda|_{\lambda=1} = (2+x\cdot\nabla)v, \quad \Lambda_0 v := (3+x\cdot\nabla)v.$$

Since $u(t) \simeq W + W_{\lambda(t)}$, we have

$$\dot{u}(t) = \partial_t u(t) \simeq -\frac{\lambda'(t)}{\lambda(t)}(\Lambda W)_{\lambda(t)} \quad \Rightarrow \quad b(t) = \lambda'(t), \vec{U}^{(1)} = (0, -\Lambda W).$$

Can we find $\vec{U}^{(2)} = (U^{(2)}, \dot{U}^{(2)})$?

Asymptotic expansion, 2

Neglecting irrelevant terms and replacing $\lambda'(t)$ by $b(t)$, we compute

$$\partial_t^2 u(t) = -\frac{b'(t)}{\lambda(t)}(\Lambda W)_{\lambda(t)} + \frac{b(t)^2}{\lambda(t)^2}(\Lambda_0 \Lambda W)_{\lambda(t)} + \ldots.$$

Denote $f(u) := |u|u$ and $L = -\Delta - f'(W)$ the linearization of $-\Delta u - f(u)$ near $u = W$. A simple computation yields

$$\Delta u(t) + f(u(t)) = -\frac{b(t)^2}{\lambda(t)^2}(LU^{(2)})_{\lambda(t)} + f'(W)_{\lambda(t)} + \ldots,$$

We discover that

$$LU^{(2)} = -\Lambda_0 \Lambda W + \frac{\lambda(t)}{b(t)^2}\big(b'(t)\cdot \Lambda W + \lambda(t)\cdot f'(W)\big).$$

Fredholm condition

$$LU^{(2)} = -\Lambda_0 \Lambda W + \frac{\lambda(t)}{b(t)^2}\big(b'(t)\cdot \Lambda W + \lambda(t)\cdot f'(W)\big). \qquad (1)$$

Due to scaling invariance, $\Lambda W \in \ker L$.

$$\int_{\mathbb{R}^6} \Lambda W \cdot \big(b'(t)\cdot \Lambda W + \lambda(t)\cdot f'(W)\big)\,\mathrm{d}x = 0$$
$$\Leftrightarrow \quad b'(t) = \frac{5}{4}\lambda(t) = \kappa^2 \lambda(t).$$

If this condition is satisfied, we can solve (1) and find $U^{(2)}$. We take $\vec{U}^{(2)} = (U^{(2)}, 0)$.

$$\begin{cases} \lambda'(t) = b(t) \\ b'(t) = \kappa^2 \lambda(t) \end{cases} \quad \text{leads to} \quad \begin{cases} \lambda_{\mathrm{app}}(t) = \dfrac{1}{\kappa}\mathrm{e}^{-\kappa|t|} \\ b_{\mathrm{app}}(t) = \mathrm{e}^{-\kappa|t|}. \end{cases}$$

Main scheme

- Approximate solution:

$$\vec{\varphi}(\mu, \lambda, b) := \vec{W}_\mu + \vec{U}_\lambda^{(0)} + b \cdot \vec{U}_\lambda^{(1)} + b^2 \cdot \vec{U}_\lambda^{(2)}$$

- Key idea: construct a sequence of solutions $\vec{u}_n(t)$ such that we have *uniform* bounds

$$\|\vec{u}_n(t) - \vec{\varphi}(1, \lambda_{\mathrm{app}}(t), b_{\mathrm{app}}(t))\|_{\mathcal{H}} \leq C\mathrm{e}^{-\frac{1}{2}\kappa|t|}$$
$$\text{for } t \in [T_n, T_0] \text{ with } T_n \to -\infty.$$

- To do this, we consider the initial data $\vec{u}_n(T_n) = \vec{\varphi}(1, \lambda_{\mathrm{app}}(T_n), b_{\mathrm{app}}(T_n))$ (with a correction due to the linear unstable direction).
- Pass to a weak limit.
    - Merle (1990); Martel (2005)

Note that time reversibility of the flow is crucial.

Control of the error term by the energy method

- Decompose $\vec{u}_n(t) = \vec{\varphi}(\mu(t), \lambda(t), b(t)) + \vec{g}(t)$, with $\mu(t)$, $\lambda(t)$ defined by natural orthogonality conditions and $b(t) := b_{\mathrm{app}}(T_n) + \int_{T_n}^{t} \kappa^2 \lambda(\tau)\,\mathrm{d}\tau$.
- If we assume that $\|\vec{g}(t)\|_{\mathcal{H}} \leq C\mathrm{e}^{-\frac{3}{2}\kappa|t|}$, we can solve differential inequalities and find $\lambda(t) \simeq \lambda_{\mathrm{app}}(t)$, $\mu(t) \simeq 1$ and $b(t) \simeq b_{\mathrm{app}}(t)$.
- We construct a functional $H(t)$ such that $H(t) \gtrsim \|\vec{g}(t)\|_{\mathcal{H}}^2$ and

$$\|\vec{g}(t)\|_{\mathcal{H}} \leq C_0\mathrm{e}^{-\frac{3}{2}\kappa|t|} \text{ for } t \in [T_n, T] \quad \Rightarrow \quad H'(t) \leq c \cdot C_0^2 \cdot \mathrm{e}^{-3\kappa|t|},$$

  with c arbitrarily small. $H(t)$ is the energy functional, corrected using a localized virial term.
- A continuity argument yields the required bounds on $\|\vec{g}(t)\|_{\mathcal{H}}$. This finishes the proof.
  - Raphaël, Szeftel (2011)

Take $\vec{u}_0^* \in \mathcal{H}$ and let $\vec{u}^*(t)$ be the solution of (NLW) with $\vec{u}^*(0) = \vec{u}_0^*$. We wish to construct $\vec{u}(t) \simeq \vec{u}^*(t) + \vec{W}_{\lambda(t)}$ with $\lambda(t) \to 0$ as $t \to 0$.

Theorem – J. (2015)

Let $N = 5$ and let $\vec{u}_0^* \in H^5 \times H^4$ with $u_0^*(0) > 0$. There exists a solution $\vec{u}(t) : (0, T_0) \to \mathcal{H}$ of (NLW) such that

$$\lim_{t\to 0^+} \left\| \vec{u}(t) - (\vec{u}_0^* + \vec{W}_{\lambda_{\mathrm{app}}(t)}) \right\|_{\mathcal{H}} = 0, \quad \lambda_{\mathrm{app}}(t) := \left(\frac{32}{315\pi}\right)^2 (u_0^*(0))^2 t^4.$$

Theorem – J. (2015)

Let $\nu > 8$. There exists a solution $\vec{u}(t) : (0, T_0) \to \mathcal{H}$ of (NLW) such that

$$\lim_{t\to 0^+} \left\| \vec{u}(t) - (\vec{u}_0^* + \vec{W}_{\lambda_{\mathrm{app}}(t)}) \right\|_{\mathcal{H}} = 0, \quad \lambda_{\mathrm{app}}(t) := t^{\nu+1},$$

where $\vec{u}_0^* = \left(\frac{315\nu(\nu+1)\pi}{32(\nu-1)(\nu+3)}|x|^{\frac{\nu-3}{2}}, 0\right)$ near $x = 0$.

Bounds on the speed of type II blow-up

It seems that there is a relationship between $\vec u_0^*$ and the asymptotics of $\lambda(t)$. One can prove an upper bound:

Theorem – J. (2015)

Let $N \in \{3, 4, 5\}$ and $\vec u_0^* \in H^3 \times H^2$ be a radial function. Suppose that $\vec u(t)$ is a radial solution of (NLW) such that

$$\lim_{t \to T_+} \|\vec u(t) - (\vec u_0^* + \vec W_{\lambda(t)})\|_{\mathcal{H}} = 0, \qquad \lim_{t \to T_+} \lambda(t) = 0, \qquad T_+ < +\infty.$$

Then there exists a constant $C > 0$ depending on $\vec u_0^*$ such that:

$$\lambda(t) \leq C(T_+ - t)^{\frac{4}{6-N}}.$$

- Main idea: $E(\vec u^* + \vec W_\lambda + \vec h) - E(\vec u^*) - E(\vec W) \geq -C\lambda^{\frac{N-2}{2}} + c\|\vec h\|_{\mathcal{H}}^2$.
- As a by-product, if $u_0^*(0) < 0$, then such a solution cannot exist.

Non-existence of two-bubbles with opposite signs

Theorem – J. (2015)

Let $N \geq 3$. There exist no radial solutions $\vec{u} : [t_0, T_+) \to \mathcal{H}$ of (NLW) such that

$$\lim_{t \to T_+} \|\vec{u}(t) - (\vec{W}_{\mu(t)} - \vec{W}_{\lambda(t)})\|_{\mathcal{H}} = 0$$

and

- in the case $T_+ < +\infty$, $\lambda(t) \ll \mu(t) \ll T_+ - t$ as $t \to T_+$,
- in the case $T_+ = +\infty$, $\lambda(t) \ll \mu(t) \ll t$ as $t \to +\infty$.

- Main idea: $E(\vec{W}_\mu - \vec{W}_\lambda + \vec{h}) - 2E(\vec{W}) \geq c\lambda^{\frac{N-2}{2}} + c\|\vec{h}\|_{\mathcal{H}}^2$.
- Compare with an easy variational proof in the case of (WM).
- Negative eigenvalues related to (un)stable directions of the wave flow.
- We can recover coercivity by modulating around the stable variety.
  - Krieger, Nakanishi, Schlag (2015)

Some open problems

- Prove that $\vec{u}_0^*$ determines the asymptotics of $\lambda(t)$.
- Classify the solutions at the energy level $2E(\vec{W})$ and topological degree 0 for (WM).
  - Côte, Kenig, Lawrie, Schlag (2015) for $E(\vec{u}) < 2E(\vec{W})$,
  - Expect a unique non-dispersive solution.

Thank you!