

KNOT INVARIANTS FROM HOMOTOPY THEORY

Embedding calculus and grope cobordism of knots

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Based on [\[Kos20\]](#).

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Introduction

Spaces of embeddings

Goal. Study the homotopy type of the space

$$\text{Emb}_{\partial}(X, M)$$

of smooth neat embeddings between compact manifolds X and M , satisfying a fixed boundary condition $\partial X \hookrightarrow \partial M$.

Remarks.

- Neat means embedding is transverse to ∂M .
- We use Whitney C^{∞} -topology.
- The case of closed manifolds can be reduced to this.

Tools. Goodwillie-Weiss [GW99] embedding calculus

\Rightarrow homotopy limits, Whitehead / Samelson products, configuration spaces, operads, graph complexes...

In this talk...

...we give a geometric interpretation of embedding calculus for

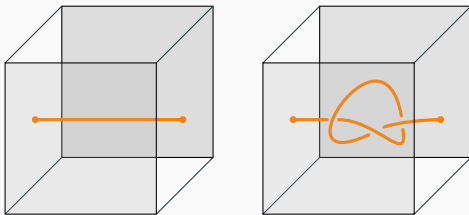
$$\text{Emb}_{\partial}(I, M) \quad \text{with} \quad I = [0, 1] \quad \text{and} \quad \dim(M) = 3$$

and relate it to Vassiliev theory of finite type knot invariants [Vas90].

In particular, for $M = I^3$ one has

$$\pi_0 \text{Emb}(\mathbb{S}^1, \mathbb{S}^3) \cong \pi_0 \text{Emb}_{\partial}(I, I^3) = \{\text{knots}\} / \text{isotopy}$$

which is a commutative monoid:



Geometric approach to embedding calculus

Embedding calculus: the Taylor tower

The outcome of this theory is the tower:

Theorem (Goodwillie-Klein [GK15])

If $(\dim X, \dim M) \neq (1, 3)$ then the map ev_n is k -connected for

$$k := (1 - \dim X + n(\dim M - \dim X - 2)).$$

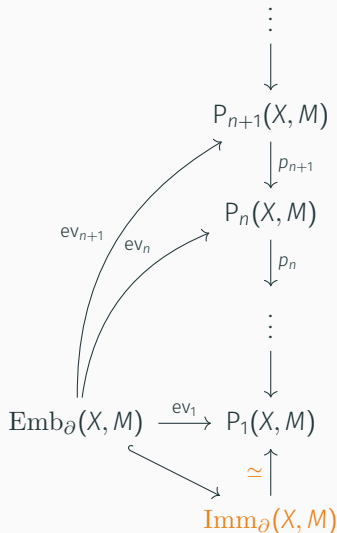
Recall: a map is k -connected if it is an iso on $\pi_{* < k}$ and onto π_k .

Corollary

If $\dim M - \dim X > 2$, then

$$\lim ev_n: \text{Emb}_\partial(X, M) \rightarrow \lim P_n(X, M)$$

is a weak equivalence.



The Taylor tower for knotted arcs

Note. One can show that $\lim ev_n$ for $M = I^3$ is not a weak equivalence. However the connectivity formula predicts:

If $\dim M = 3$, then $ev_n: \text{Emb}_\partial(I, M) \rightarrow P_n(I, M)$ is 0-connected.

This follows from **Main Theorem B** (see p. 9), which gives a geometric interpretation of points in $P_n(I, M)$. We use the following model.

Fix an increasing sequence of disjoint closed subintervals $J_i \subseteq I$.



Definition (Goodwillie's punctured knots model)

$$P_n(I, M) := \text{holim}_{\emptyset \neq S \subseteq \{0, 1, \dots, n\}} \text{Emb}_\partial(I \setminus \bigsqcup_{i \in S} J_i, M)$$

There are natural fibrations p_n and evaluation maps ev_n .

Connection to Vassiliev's theory

Theorem (Budney-Conant-Koytcheff-Sinha [BCKS17])

The set $\pi_0 P_n(I, I^3)$ has a structure of an abelian group, and $\pi_0 \text{ev}_n: \pi_0 \text{Emb}_\partial(I, I^3) \rightarrow \pi_0 P_n(I, I^3)$ is an additive Vassiliev invariant of type $< n$, that is, a map of monoids which factors as

$$\begin{array}{ccc} \pi_0 \text{Emb}_\partial(I, I^3) & \xrightarrow{\pi_0 \text{ev}_n} & \pi_0 P_n(I, I^3) \\ \downarrow & \nearrow \bar{\text{ev}}_n & \\ \pi_0 \text{Emb}_\partial(I, I^3) / \sim_n & & \end{array}$$

Conjecture (BC-Scannell-S [BCSS05])

$\pi_0 \text{ev}_n$ is a **universal** additive Vassiliev invariant of type $< n$, that is, the induced homomorphism $\bar{\text{ev}}_n$ is **an isomorphism of groups**.

Corollary (of Theorem A)

The homomorphism $\bar{\text{ev}}_n$ is surjective.

Connection to Vassiliev's theory

Moreover, we can combine our main theorem and results of Boavida de Brito and Horel to obtain the following.

Corollary (of Theorem B and [BH20])

- $\overline{ev}_n \otimes \mathbb{Q}$ is an isomorphism for all $n \geq 1$.
- $\overline{ev}_n \otimes \mathbb{Z}_p$ is an isomorphism for $n \leq p + 2$.

- Remarks.*
- Thus, the embedding calculus invariants are at least as good as Kontsevich integral (or Bott–Taubes configuration space integrals).
 - Those invariants indeed use integration, so cannot offer answers over \mathbb{Z} (or in characteristic p).
 - As a consequence of Theorem B, we also have that they factor through the Taylor tower, cf. [Vol06].

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Two disguises of trees

There is a geometric approach to Vassiliev's theory using gropes.

A **grope cobordism** of degree n is a certain 2-complex built out of surfaces which are embedded in M . It has an underlying **$\pi_1 M$ -decorated tree** $\Gamma^{g_n} \in \text{Tree}_{\pi_1 M}(n)$.

Here Γ^{g_n} consists of a rooted planar binary tree Γ with n leaves which are enumerated and also decorated by elements

$$g_i \in \pi_1(M), \quad 1 \leq i \leq n.$$

For example,

$$\Gamma^{g_3} := \begin{array}{c} \begin{array}{ccc} 1 & 3 & 2 \\ \swarrow & \nearrow & \swarrow \\ g_1 & g_3 & g_2 \\ \searrow & \swarrow & \searrow \\ & \square & \end{array} \end{array} \in \text{Tree}_{\pi_1 M}(3).$$

Two disguises of trees

Remarkably, the first non-vanishing homotopy group of the layer

$$\mathcal{F}_{n+1}(M) := \text{fib}\left(p_{n+1}: P_{n+1}(I, M) \rightarrow P_n(I, M)\right)$$

in the Taylor tower is **also related** to the set $\text{Tree}_{\pi_1 M}(n)$.

Namely, for any $\dim(M) = d \geq 3$ we show that

$$\pi_{n(d-3)}\mathcal{F}_{n+1}(M) \cong \text{Lie}_{\pi_1 M}(n) := \mathbb{Z}[\text{Tree}_{\pi_1 M}(n)]_{/AS, IHX} \cong \text{Lie}(n) \otimes \mathbb{Z}[(\pi_1 M)^n]$$

where

$$AS: \begin{array}{c} \Gamma_2 \quad \Gamma_1 \\ \diagdown \quad \diagup \\ \text{---} \\ \vdots \end{array} + \begin{array}{c} \Gamma_1 \quad \Gamma_2 \\ \diagdown \quad \diagup \\ \text{---} \\ \vdots \end{array} = 0,$$

$$IHX: \begin{array}{c} \Gamma_3 \quad \Gamma_2 \quad \Gamma_1 \\ \diagdown \quad \diagup \\ \text{---} \\ \vdots \end{array} - \begin{array}{c} \Gamma_3 \quad \Gamma_2 \quad \Gamma_1 \\ \diagdown \quad \diagup \\ \text{---} \\ \vdots \end{array} + \begin{array}{c} \Gamma_1 \quad \Gamma_3 \quad \Gamma_2 \\ \diagdown \quad \diagup \\ \text{---} \\ \vdots \end{array} = 0.$$

Two disguises of trees are compatible

Main Theorem B [Kos20]

Given a grope cobordism \mathcal{G} of degree n there is a point

$$e_{n+1}\psi(\mathcal{G}) \in \mathcal{F}_{n+1}(M)$$

and its path component is precisely given by the class modulo AS, IHX of the underlying tree

$$t(\mathcal{G}) \in \text{Tree}_{\pi_1 M}(n).$$

Remark

Linear combinations of trees are realised by “grope forests” (higher genus gropes) and the appropriate extension of the theorem holds.

See p. 16 for maps e_{n+1} and ψ .

More details

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Finite type knot invariants

Vassiliev '90 studied a stratification of $\text{Map}(l, l^3) \setminus \text{Emb}_\partial(l, l^3)$.

Definition

A knot invariant $v: \pi_0 \text{Emb}_\partial(l, l^3) \rightarrow A$ is of type $< n$ if its natural extension to knots with $< n$ double points vanishes.

Kontsevich '91 defined a universal (additive) invariant of type $< n$

$$Z_{<n}^{\text{Kont}}: \pi_0 \text{Emb}_\partial(l, l^3) \rightarrow \prod_{k < n} \mathcal{A}_k^t \otimes \mathbb{Q}$$

where $\mathcal{A}_k^t := \text{Lie}^{(k)} / \text{STU}^2$ is the group of **Jacobi trees**. Namely, any type $< n$ invariant over \mathbb{Q} factors through this one.

Example. All quantum invariants are of finite type, and can be written as $\omega \circ Z_{<n}^{\text{Kont}}$ using weight systems $\omega_k: \mathcal{A}_k^t \rightarrow \mathbb{C}$.

Question

What is a geometric meaning of Jacobi trees?

Geometric approach to finite type theory

Theorem (Gusarov [Gus00] Habiro [Hab00] Conant-Teichner [CT04])

Two knots $K_0, K_1: I \hookrightarrow I^3$ have the same invariants of type $< n$ if and only if there is a sequence of (capped) grope cobordisms of degree n from K_0 to K_1 . We write $K_0 \sim_n K_1$.

Think of a grope cobordism as an ambient cobordism from a subset of K_0 to a subset of K_1 , which has several layers of embedded surfaces following a shape of a tree. *Examples follow shortly.*

Actually, we can define this in any 3-manifold M , and there is the underlying tree map t_n that fits into

$$\begin{array}{ccc}
 \pi_0 \text{Emb}_\partial(I, M) & \xleftarrow{\partial_0} & \pi_0 \text{Grop}_n(M) \xrightarrow{t_n} \mathbb{Z}[\text{Tree}_{\pi_1 M}(n)] \\
 \downarrow & & \downarrow \text{AS, IHX, STU}^2 \\
 \pi_0 \text{Emb}_\partial(I, M) / \sim_{n+1} & \xleftarrow{\mathfrak{R}_n} & \mathcal{A}_n^t(M)
 \end{array}$$

$\xrightarrow{\text{Z}^{\text{kont}} \text{ over } \mathbb{Q} \text{ for } M=I^3}$

Examples: grope cobordisms of degree 1

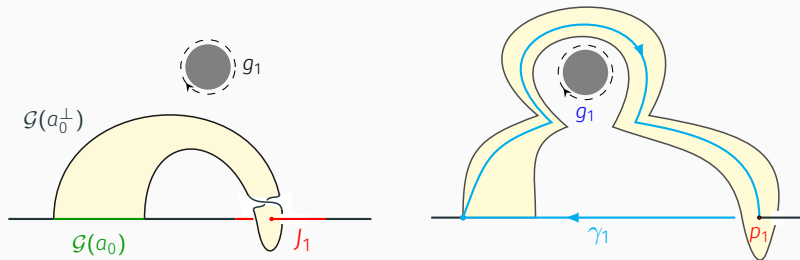


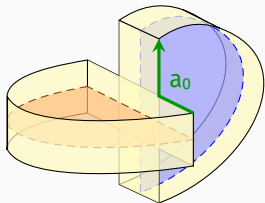
Figure 1: Two grope cobordisms of degree 1 on $K_0: I \hookrightarrow M$ (the horizontal line). In both cases the union of black and red arcs is

$$K_1 := (\mathcal{U} \setminus \mathcal{G}(a_0)) \cup \mathcal{G}(a_0^\perp)$$

The trees are given by

$$\begin{array}{c} 1 \\ | \\ \square \end{array} 1 \quad \text{and} \quad \begin{array}{c} 1 \\ | \\ \square \end{array} g_1$$

Examples: grope cobordisms of degree 2



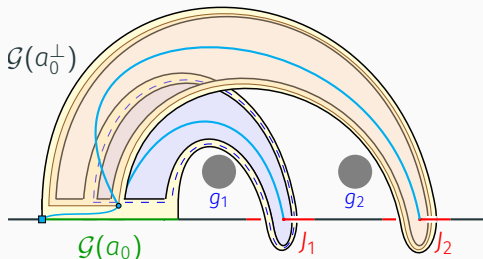
Left: Abstract grope G_Γ for $\Gamma = \begin{array}{c} 2 \\ \text{Y} \\ 1 \end{array}$ is the union of the yellow torus and two disks.

Right: A grope cobordism from U to the knot

$$\partial^\perp \mathcal{G} = (U \setminus \mathcal{G}(a_0)) \cup \mathcal{G}(a_0^\perp).$$

with underlying tree:

$$t(\mathcal{G}) = \begin{array}{c} 2 \\ \text{Y} \\ g_2 \quad g_1 \end{array}$$



$$\mathcal{G}: G_\Gamma \rightarrow I^3$$

Examples: n -equivalence of knots

Recall: $K_0 \sim_n K_1$ if there is a sequence of (capped) grope cobordisms of degree n from K_0 to K_1 . For example:

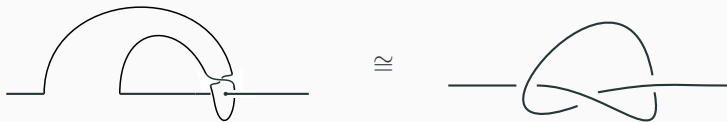


Figure 1: The right-handed trefoil RHT is 1-equivalent to the unknot.



Figure 2: RHT is 2-equivalent to the unknot. But $RHT \not\sim_3 U$.

Gropes give points in the Taylor layers

Theorem K-Shi-Teichner

There is a continuous map ψ so that the following commutes

$$\begin{array}{ccccc}
 \text{Grop}_n(M; U) & \xrightarrow{\psi} & \mathcal{H}_n(M) := \text{hofib}(\text{ev}_n) & \overset{\text{ev}_{n+1}}{\dashrightarrow} & \mathcal{F}_{n+1}(M) \\
 & \searrow \partial_1 & \downarrow & & \downarrow \\
 & & \text{Emb}_{\partial}(I, M) & \xrightarrow{\text{ev}_{n+1}} & P_{n+1}(M) \\
 & & \downarrow \text{ev}_n & & \downarrow \rho_n \\
 & & P_n(M) & \xlongequal{\quad\quad\quad} & P_n(M)
 \end{array}$$

Theorem [Kos20]

For any M of dimension $d \geq 3$ there is a homotopy equivalence

$$\mathcal{F}_{n+1}(M) \simeq \Omega^{n+1} \prod_{w \in BL(n)} \Sigma^{1+l_w(d-2)} (\Omega M^{\times l_w})_+$$

and the first non-vanishing homotopy group is

$$\pi_{n(d-3)} \mathcal{F}_{n+1}(M) \cong \text{Lie}_{\pi_1 M}(n).$$

Main theorem more precisely

Main Theorem B [[Kos20](#)]

The following diagram commutes

$$\begin{array}{ccc} \pi_0 \text{Grop}_n(M) & \xrightarrow{t_n} \twoheadrightarrow & \mathbb{Z}[\text{Tree}_{\pi_1 M}(n)] \\ \psi \downarrow & & \downarrow \text{AS, IHX} \\ \pi_0 \mathcal{H}_n(M) & \xrightarrow{\pi_0 e_{n+1}} & \text{Lie}_{\pi_1 M}(n) \end{array}$$

Corollary

The map $\pi_0 e_{n+1}$ is a surjection (of sets).

- Theorem A (that $\pi_0 \text{ev}_n$ is onto) follows from this by induction.
- Corollaries about universality follow using the work of Conant [[Con08](#)] and by considering the spectral sequence in homotopy groups of the tower of fibrations p_n .

Thank you!

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