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HIGH-FREQUENCY STABILITY OF DETONATIONS AND TURNING POINTS AT INFINITY*

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This paper is dedicated to J. J. Erpenbeck, pioneer in the study of shock and detonation stability

Abstract. The rigorous study of spectral stability of strong detonations was begun by Erpenbeck in the 1960s. Working with the Zeldovitch–von Neumann–Döring (ZND) model, he identified two fundamental classes of detonation profiles, referred to as those of decreasing (D) and increasing (I) type, which appeared to exhibit very different behavior with respect to high-frequency perturbations. Using a combination of rigorous and nonrigorous arguments, Erpenbeck concluded that type I detonations were unstable to some oscillatory perturbations for which the (vector) frequency was of arbitrarily large magnitude, while type D detonations were stable provided the frequency magnitude was sufficiently high. For type D detonations Erpenbeck's methods did not allow him to obtain a cutoff magnitude for stability that was *uniform* with respect to frequency direction. Thus, he left open the question of whether the cutoff magnitude for stability might approach $+\infty$ as certain frequency directions were approached. In this paper we show by quite different methods that for type D detonations there exists a uniform cutoff magnitude for stability independent of frequency direction. By reducing the search for unstable frequencies to a bounded frequency set, the uniform cutoff obtained here is a key step toward the rigorous validation of a number of results in the computational detonation literature. The detonation profile $P(x)$ is a stationary solution of the ZND system depending on the single spatial variable $x \in [0, +\infty)$, the reaction zone. The spectral stability of the profile is governed by a nonautonomous 5×5 system of linear ODEs in x depending on the perturbation frequency as a vector parameter. Difficulties in the analysis are caused by the existence of frequency directions ζ for which two of the eigenvalues of this system cross at a particular point $x = x(\zeta)$ in the reaction zone. Such points $x(\zeta)$ are called *turning points*. A necessary step in obtaining a uniform stability cutoff is to obtain explicit representations of the decaying solutions of the system that are uniformly valid for frequencies near turning point frequencies. The main mathematical difficulty addressed here is to produce such uniform representations for frequencies near the particular turning point frequency ζ_∞ for which the associated turning point $x(\zeta_\infty)$ is $+\infty$. As part of the stability application here, we provide methods for handling turning point problems on unbounded spatial intervals, including problems where the turning point occurs at infinity.

Key words. ZND detonations, high-frequency stability, turning points at infinity

AMS subject classifications. 80A25, 34E20

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Part I. Introduction.

The most commonly studied model of combustion is the Zeldovitch–von Neumann–Döring (ZND) system (0.1), which couples the compressible Euler equations for a reacting gas (in which pressure and internal energy are allowed to depend on the mass fraction λ of reactant) to a reaction equation that governs the finite rate at which λ changes. In three space dimensions with coordinates (x, y, z) the ZND equations for

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the unknowns $(v, \mathbf{u}, S, \lambda)$ (specific volume, particle velocity $\mathbf{u} = (u^x, u^y, u^z)$, entropy, and mass fraction of reactant) are given by the 6×6 system [E2, FD]:

$$\begin{aligned} &\partial_t v + \mathbf{u} \cdot \nabla v - v \nabla \cdot \mathbf{u} = 0, \\ &\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + v \nabla p = 0, \\ &\partial_t S + \mathbf{u} \cdot \nabla S = -r \Delta F / T := \Phi, \\ &\partial_t \lambda + \mathbf{u} \cdot \nabla \lambda = r, \end{aligned} \tag{0.1}$$

where $p = p(v, S, \lambda)$ is pressure, T is temperature, ΔF is the free energy increment, and $r(v, S, \lambda)$ is the reaction rate function. A steady planar *strong detonation profile* is a weak solution of this system depending only on x with a jump (the stationary von Neumann shock) at $x = 0$. Without loss of generality we study profiles of the form $P(x) = (v, u, 0, 0, S, \lambda)$, where $u > 0$ is the x-component of particle velocity. The solution is constant and supersonic ($u > c_0$, where c_0 is the sound speed at x) in $x < 0$, the quiescent zone,[1] and satisfies a nonlinear system of ODEs in the subsonic reaction zone $x > 0$. In order to be a weak solution of (0.1) in a neighborhood of $x = 0$, $P(x)$ must satisfy an appropriate Rankine–Hugoniot condition at $x = 0$:

(0.2)

$$\left(\frac{u}{v}\right)_+ = \left(\frac{u}{v}\right)_+ := m, \; p_+ - p_- = m^2 (v_- - v_+), \; e_+ - e_- = \frac{1}{2}(p_+ + p_-)(v_- - v_+), \\ \lambda_+ = \lambda_-,$$

where $e(v, S, \lambda)$ is the specific internal energy and $\pm$ denotes states to the right (resp., left) of the discontinuity. The jump conditions for the von Neumann shock, $(v, u, S)_\pm$, considered as a gas dynamical shock, are the same as those in (0.2). There is a well-defined limiting state $P_\infty = \lim_{x \to +\infty} P(x)$ with $\lambda(\infty) = 0$, and the range of u on $[0, \infty)$ is a compact subinterval of $(0, \infty)$.

The rigorous study of spectral stability for strong detonations was begun by Erpenbeck [E1] in the 1960s. Working with the ZND model, in [E2, E3] he identified two fundamental classes of detonation profiles, referred to as those of decreasing (D) and increasing (I) type, which appeared to exhibit very different behavior with respect to high-frequency perturbations. Using a combination of rigorous and nonrigorous arguments, Erpenbeck concluded that type I detonations were unstable to some oscillatory perturbations for which the (vector) frequency was of arbitrarily large magnitude, while type D detonations were stable provided the frequency magnitude was sufficiently high. For type D detonations Erpenbeck's methods did not allow him to obtain a cutoff magnitude for stability that was *uniform* with respect to frequency direction. Thus, he left open the question of whether the cutoff magnitude for stability might approach $+\infty$ as certain frequency directions were approached. In [LWZ1] we identified the mathematical issues left unresolved in [E2, E3] and provided proofs, together with certain simplifications and extensions, of the main conclusions of [E2, E3], in particular, high-frequency instability of type I detonations. However, the paper [LWZ1] also failed to resolve the above question for type D detonations. In this paper we show by quite different methods that for type D detonations there exists a uniform cutoff magnitude for stability independent of frequency direction, thus establishing *high-frequency stability of type D detonations*.

[1] In the quiescent zone we take $\lambda = 1$.

The spectral stability of a ZND profile is governed by a nonautonomous 5×5 system of linear ODEs in x depending on the perturbation frequency (τ, ε) as a parameter [E2, E3, LWZ1]:

$$\frac{d\theta}{dx} = -\mathcal{G}^t(x, \tau, \varepsilon)\theta \text{ on } x \geq 0. \tag{0.3}$$

Here the system defined by the matrix $\mathcal{G}(x, \tau, \varepsilon)$ is obtained by linearizing the ZND system about the profile $P(x)$ and taking the Laplace transform in time and the Fourier transform in the transverse spatial variables (y, z). The matrix $\mathcal{G}^t$ is the transpose of $\mathcal{G}$, the variable $\tau \in \mathbb{C}$ is dual to time, and $\varepsilon = \sqrt{\alpha^2 + \beta^2}$, where $(\alpha, \beta) \in \mathbb{R}^2$ is dual to (y, z).[2] We have

$$\mathcal{G}(x, \tau, \varepsilon) = -(A^x(x))^{-1}[\tau I + i\varepsilon A^y(x) + B(x)] \tag{0.4}$$

for matrices A^x, A^y, and B given in section 20. The x-dependence of these matrices enters entirely through the profile $P(x)$.[3] The reduction from a linearized system of dimension 6 to one of dimension 5 and from (α, β) to ε uses the rotational symmetry of $\mathcal{G}$ with respect to the transverse velocity components.[4]

In [E1] Erpenbeck defined a stability function $V(\tau, \varepsilon)$ whose zeros in the right half-plane $\Re\tau > 0$ ("unstable zeros") correspond to perturbations of the steady profile $P(x)$ that grow exponentially in time. The computation of V requires the evaluation within the reaction zone of the solution $\theta(x, \tau, \varepsilon)$ of (0.3) which satisfies the condition that θ remains bounded for fixed (τ, ε) with $\Re\tau \geq 0$ as $x \to \infty$, and θ decays exponentially to zero for $\Re\tau > 0$ as $x \to \infty$. As we will see, this condition determines θ uniquely up to a constant multiple. We shall refer to this solution θ as *the decaying solution*, even though it is merely bounded for certain purely imaginary τ values.

Following the notation of [E2, E3], we write τ as

$$\tau = \zeta\varepsilon, \tag{0.5}$$

where $\zeta \in \{z \in \mathbb{C} : \Re z \geq 0\}$ and $\varepsilon > 0$ is large.[5] Thus we can rewrite (0.3) as

$$\begin{aligned} &\frac{d\theta}{dx} = (\varepsilon\Phi_0 + \Phi_1)\theta, \text{ where} \\ &\Phi_0(x, \zeta) = \{(A^x(x))^{-1} \cdot (\zeta I + iA^y(x))\}^t, \\ &\Phi_1(x) = \{(A^x(x))^{-1}B(x)\}^t. \end{aligned} \tag{0.6}$$

The eigenvalues $\mu_j(x, \zeta)$ of the matrix Φ_0, which is given in (20.2), play a crucial role in all that follows. They are

$$\mu_1 = -\kappa(\kappa\zeta + s)/\eta u, \quad \mu_2 = -\kappa(\kappa\zeta - s)/\eta u, \quad \mu_3 = \mu_4 = \mu_5 = \zeta/u, \tag{0.7}$$

[2] Thus, $\varepsilon/2\pi$ is the transverse wavenumber. We briefly retain the perhaps confusing notation ε for the large quantity $\sqrt{\alpha^2 + \beta^2}$ to facilitate comparison with [E1, E2], where that notation was used. In most of the paper we work with $h = \frac{1}{\varepsilon}$; see Notation 0.1.

[3] In fact, we show in section II that $P(x)$ can be expressed as $P(x) = \mathcal{P}(\lambda(x))$.

[4] We refer to [E1] and to the introduction of [LWZ1] for the details of the derivation of (0.3) and for additional background on the ZND system.

[5] In [E2, E3] Erpenbeck used a decomposition $\tau = \varepsilon\zeta + \nu$, but ν played no role in his treatment of type D profiles and can be set equal to zero.

where with $c_0^2 = -v^2 p_v(v, S, \lambda)$

$$s(x,\zeta) = \sqrt{\zeta^2 + c_0^2\eta}, \quad \kappa(x) = \sqrt{1-\eta} = u/c_0. \tag{0.8}$$

The square root defining s, regarded as a function of ζ, is taken to be the positive branch with branch cut the segment $[-ic_0\sqrt{\eta}, ic_0\sqrt{\eta}]$ on the imaginary axis. Thus, in particular, we have

$$\begin{aligned} &s = |s| \text{ when } \zeta^2 + c_0^2\eta > 0,\\ &s = i|s| \text{ when } \zeta^2 + c_0^2\eta < 0 \text{ and } \zeta = i|\zeta|,\\ &s = -i|s| \text{ when } \zeta^2 + c_0^2\eta < 0 \text{ and } \zeta = -i|\zeta|. \end{aligned} \tag{0.9}$$

One checks that only μ_1 has, for $\Re\zeta > 0$, negative real part; consequently, for a fixed ζ with $\Re\zeta > 0$ the system (0.6) has a one-dimensional (1D) space of solutions that decay to zero as $x \to +\infty$. The eigenvectors corresponding to the μ_j, $j = 1, \dots, 5$, are the respective columns of the matrix

$$T(x,\zeta) = \begin{pmatrix} T_1 & T_2 & T_3 & T_4 & T_5 \end{pmatrix} = \begin{pmatrix} \frac{ms}{\kappa u} & -\frac{ms}{\kappa u} & -\frac{im}{1-\eta} & 0 & 0\\ \frac{\zeta}{u} & \frac{\zeta}{u} & i & 0 & 0\\ -i & -i & \frac{\zeta}{u} & 0 & 0\\ \frac{-\kappa p_S s}{um} & \frac{\kappa p_S s}{um} & 0 & 1 & 0\\ \frac{-\kappa p_\lambda s}{um} & \frac{\kappa p_\lambda s}{um} & 0 & 0 & 1 \end{pmatrix}, \tag{0.10}$$

where $m = \frac{u}{v}$ is the mass flux.

From the formulas (0.7), (0.8) we see that for any fixed value of ζ, the eigenvalues μ_1 and μ_2 are distinct except at values $x = x(\zeta)$, where $s^2(x,\zeta) = \zeta^2 + c_0^2\eta(x) = 0$; at such values the first and second columns of T are parallel. The eigenvalues μ_2 and μ_3 are distinct except at x values where $\zeta = u$, and then the second and third rows of T are clearly parallel. For all other values of x the matrix $T(x,\zeta)$ is invertible.[6]

A complex number ζ with $\Re\zeta \geq 0$ is defined in [E2] to be of Class (iii) or Class (ii), respectively, when there exists $x_* \in [0,\infty]$ such that $s(x_*,\zeta) = 0$ or $\zeta = u(x_*)$. All other ζ are said to be of Class (i). Thus we have

$$\begin{aligned} &\text{Class (iii)} = \{\zeta : \Re\zeta = 0 \text{ and } \min_x(c_0\eta^{\frac{1}{2}}) \leq |\zeta| \leq \max_x(c_0\eta^{\frac{1}{2}})\},\\ &\text{Class (ii)} = \{\zeta : \Im\zeta = 0 \text{ and } \min_x u \leq \zeta \leq \max_x u\},\\ &\text{Class (i)} = \{\text{all remaining } \zeta \in \mathbb{C} \text{ with } \Re\zeta \geq 0\}. \end{aligned} \tag{0.11}$$

Class (iii) (resp., (ii)) consists of two (resp., one) bounded closed interval(s), and the minima appearing in (0.11) are positive. In contrast to [E2, E3] we are able to treat Class (i) and Class (ii) frequencies by a single argument. The argument is based on the observation that for such ζ the eigenvalue μ_1 remains well separated from the others. So for us the important partition of $\zeta-$ space is

$$\{\zeta \in \mathbb{C} : \Re\zeta \geq 0\} = \text{(iii)} \cup \text{(iii)}^c. \tag{0.12}$$

[6]For certain types of profiles and choices of ζ, there may be more than one x-value where $T(x,\zeta)$ is singular.

NOTATION 0.1. (1) *When working with Class* (iii) *values of* ζ *we will usually suppose* $\zeta = i|\zeta|$. *The same results hold with the same proofs when* $\zeta = -i|\zeta|$, *but certain formulas change slightly. Thus, for some statements it is helpful to define*

$$\text{(0.13)} \qquad \textit{Class } (\text{iii})_+ = \textit{Class } (\text{iii}) \cap \{\zeta = i|\zeta|\}.$$

We let $(\text{iii})^o_+$ *denote the interior of the closed interval* $(\text{iii})_+$.

(2) *Henceforth, we shall use the parameter* $h = 1/\varepsilon$ *instead of* ε *and, in view of* (0.5), *we shall denote the stability function by* $V(\zeta, h)$ *instead of* $V(\tau, \varepsilon)$ *and the decaying solution of* (0.6) *by* $\theta(x, \zeta, h)$ *instead of* $\theta(x, \tau, \varepsilon)$.

DEFINITION 0.2. *A detonation profile* $P(x)$ *is said to be of type D (resp., type* I*) if the function* $c_0^2\eta = c_0^2 - u^2$ *satisfies*

$$\text{(0.14)} \qquad \frac{d}{dx}(c_0^2\eta) = \frac{d}{dx}(c_0^2 - u^2) < 0 \ \textit{(resp.,} \ > 0) \ \textit{on } [0, +\infty).$$

For profiles of type I it was shown in [E2, E3, LWZ1] that the stability function $V(\tau, \varepsilon)$ generally has zeros in $\Re\tau > 0$ (unstable zeros) for all transverse wavenumbers ε above a certain cutoff.[7] In this paper we are concerned only with profiles of type D. In the case of more general profiles, by considering intervals on which $c_0^2\eta$ is increasing or decreasing, stability and instability results can be proved by combining the results for I and D type profiles (see, for example, [LWZ1, Theorem 5.2, part e]).

We suppose from now on that $P(x)$ is a type D profile. In this case

$$\text{(0.15)} \qquad (\text{iii})_+ = \{\zeta = i|\zeta| : c_0\eta^{1/2}(\infty) \le |\zeta| \le c_0\eta^{1/2}(0)\}.$$

DEFINITION 0.3. *We refer to class* (iii) *as the set of* turning point frequencies. *For each* $\zeta \in (\text{iii})_+$ *there is a unique* $x = x(\zeta) \in [0, \infty]$ *such that* $s(x(\zeta), \zeta) = 0$. *We refer to* $x(\zeta)$ *as the* turning point *associated to* ζ. *The map* $x(\zeta) : (\text{iii})_+ \to [0, \infty]$ *is bijective. We set* $\zeta_0 = ic_0\eta^{1/2}(0)$ *and* $\zeta_\infty = ic_0\eta^{1/2}(\infty)$ *and note that* $x(\zeta_0) = 0$ *and* $x(\zeta_\infty) = +\infty$. *We refer to* $x(\zeta_\infty)$ *as the* turning point at infinity.

For $\zeta \in (\text{iii})_+$, $\zeta \neq \zeta_0$, consider an interval $[0, K]$ that does not contain the turning point $x(\zeta)$. On any such interval the matrix $T(x, \zeta)$ is invertible, and we can use WKB methods to construct[8] approximate solutions of order h^m of the system (0.6) associated to each of the eigenvalues μ_i of the form

(0.16)
$$\theta_i(x, \zeta, h) = e^{\frac{1}{h}h_i(x,\zeta) + k_i(x,\zeta)} \left[f_{i0}(x, \zeta) + h f_{i1}(x, \zeta) + \cdots + h^{(m+1)} f_{i(m+1)}(x, \zeta) \right].$$

Here

$$\text{(0.17)} \qquad h_i(x, \zeta) = \int_0^x \mu_i(x', \zeta)dx', \ k_i(0, \zeta) = 0, \ \text{and } f_{i,0} = T_i(x, \zeta).$$

We do not need explicit formulas for the other quantities appearing in (0.16). For our purposes it is only important to specify the leading term of θ_i uniquely, and the condition (0.17) does this. More generally, for a given $\zeta \in \{\Re\zeta \ge 0\}$ approximate solutions of this form can be constructed on any compact x-interval where $T(x, \zeta)$ is invertible. Classical sufficient conditions for approximate solutions of this type to

[7] Theorem 5.2 of [LWZ1] gives a precise statement.

[8] See, for example, Chapters 5 and 6 of Coddington and Levinson [CL].

be close in relative error to true exact solutions of (0.6) for h small are given, for example, in [LWZ1, Theorem 3.1].

Observing that $|\theta_1(0,\zeta,h) - T_1(0,\zeta)| \le Ch|\theta_1(0,\zeta,h)|$ we give the following definition.

DEFINITION 0.4 (Type θ_1). *Consider $\theta(x,\zeta,h)$, the decaying solution of* (0.6), *and suppose ζ lies in $\mathcal{P}(h)$, a subset of $\Re\zeta \ge 0$ that may depend on h. We say that θ is* of type θ_1 at $x = 0$ on $\mathcal{P}(h)$ *if there is a nonvanishing scalar factor $s(\zeta,h)$ so that, given any $\delta > 0$, there exists an $h_0 > 0$ such that*

$$|s(\zeta,h)\theta(0,\zeta,h) - T_1(0,\zeta)| \le \delta |T_1(0,\zeta)| \text{ for } \zeta \in \mathcal{P}(h),\ 0 < h \le h_0. \tag{0.18}$$

Remark 0.5. Let K be a subset of $\Re\zeta \ge 0$. Erpenbeck realized in [E2] that in order to show that the stability function $V(\zeta,h)$ is nonvanishing for $\zeta \in K$ for h sufficiently small, it suffices to show that the decaying solution θ is of type θ_1 at $x = 0$ on K. In [E2, E3, LWZ1] it was shown for type D profiles that θ is of type θ_1 at $x = 0$ on K when K is either a compact subset of $(\text{iii})^\circ_+$ or a compact subset of $\{\Re\zeta \ge 0\} \setminus (\text{iii})$.[9] These papers did not consider sets K containing either of the endpoints ζ_0, ζ_∞ of $(\text{iii})_+$, nor did they provide a uniform treatment of the set $|\zeta| \ge M$. Moreover, values of ζ in $(\text{iii})^\circ_+$ were treated by arguments completely different from those used to treat nearby values in $\Re\zeta > 0$. In this paper the main result will be proved by showing that θ is of type θ_1 at $x = 0$ on the full set $\Re\zeta \ge 0$. The proof here gives a uniform analysis near all points in $(\text{iii})_+$, including the endpoints, and a uniform analysis for $|\zeta| \ge M$. In section 2.1 we give a very simple example illustrating how a second-order equation with a turning point at $x = +\infty$ transforms to Bessel's equation under a suitable transformation.

1. Assumptions.

Assumption 1.1. The thermodynamic functions appearing in the ZND system (0.1), p (pressure), T (temperature), ΔF (free energy increment), and r (reaction rate) are real-analytic functions of their arguments (v, S, λ).

Assumption 1.2. The steady strong detonation profile $P(x) = (v, u, 0, 0, S, \lambda)$ is of type D. It is a real-analytic function of x in the subsonic reaction zone $[0, \infty)$. There exist constants C_i, $i = 1, \dots, 4$, such that

$$0 < C_1 \le \kappa = \frac{u}{c_0} \le C_2 < 1 \text{ and } 0 < C_3 \le u \le C_4 \text{ for all } x \in [0,\infty). \tag{1.1}$$

Assumption 1.3. The rate function satisfies

$$r|_{\lambda=0} = 0,\ \ r_\lambda < 0,\ r_v|_{\lambda=0} = 0,\ r_S|_{\lambda=0} = 0. \tag{1.2}$$

This assumption is satisfied, for example, by rate functions of the form

$$r = -k\rho\phi(T)\lambda, \tag{1.3}$$

where ρ is density and $k > 0$ is a reaction rate constant, such as the Arrhenius rate law

$$r = -k\lambda \exp(-E/RT)\ \ (E \text{ is activation energy}). \tag{1.4}$$

[9] Theorem 5.2, parts (a) (b), of [LWZ1] gives a treatment of such sets K.

Analogous to $V(\zeta,h)$, one can define a stability function $L_1(\zeta)$ for the von Neumann shock, considered as a purely gas dynamical shock. This was first done in [E4], and Erpenbeck's $L_1(\zeta)$ turned out to be a nonvanishing multiple of the Majda stability determinant for shocks defined in [M] 20 years later. The functions $V(\zeta,h)$ and $L_1(\zeta)$ are described in section 21.

Assumption 1.4. The stability function for the von Neumann shock, $L_1(\zeta)$, has no zeros in $\Re\zeta \geq 0$. This means that the equation of state of the unreacted explosive is such that the von Neumann step-shock would be stable if the reactions behind it were somehow suppressed. This assumption, which is also made in [E2, E3, LWZ1], allows us to concentrate on effects that arise solely from the reactions; it always holds, for example, for step-shocks in ideal polytropic gases.

2. Main result.

Our main result is the following theorem.

THEOREM 2.1. *Consider a strong detonation profile of type D under Assumptions* 1.1, 1.2, 1.3, *and* 1.4. *There exists an $h_0 > 0$ such that for all $\Re\zeta \geq 0$, the stability function $V(\zeta,h) \neq 0$ for $0 < h \leq h_0$.*

As explained in Remark 0.5, to prove the theorem it suffices to show that the decaying solution $\theta(x,\zeta,h)$ of (0.6) is of type θ_1 at $x = 0$ on the set $\{\Re\zeta \geq 0\}$. This property of θ is a consequence of Propositions 10.6, 10.12, and 10.18, which treat ζ_∞; Proposition 12.8, which treats points in $(\text{iii})^o_+$, and Proposition 13.1, which treats ζ_0; Corollary 11.3, which treats compact subsets of $\{\Re\zeta \geq 0\} \setminus (\text{iii})$; and Proposition 14.1, which treats $|\zeta| \geq M$ for M large. Section 21 shows how Assumption 1.4 and the fact that θ is of type θ_1 at $x = 0$ imply Theorem 2.1.

The main difficulty in proving existence of a uniform stability cutoff is to obtain explicit representations of θ at $x = 0$ that are uniformly valid for frequencies near turning point frequencies. For the finite turning point frequencies $\zeta \in (\text{iii})^o_+$, the three-step strategy is to show first that θ is of type θ_1 to the right of the turning point $x(\zeta)$,[10] then to perform a matching argument involving Airy functions with arguments depending on (ζ,h) as a parameter to show that θ is of type θ_1 just to the left of $x(\zeta)$, say, at $x(\zeta)-\delta$, and finally to match using a basis of exact solutions close to the approximate solutions $\{\theta_i\}_{i=1}^5$ (0.16) to conclude that θ is of type θ_1 on $[0, x(\zeta)-\delta]$. This strategy encounters special problems when the endpoint frequencies ζ_0 and ζ_∞ are considered, and those problems are most serious in the case of ζ_∞, to which all of Part II is devoted. For this frequency it is clear that the first step in the strategy does not even make sense since $x(\zeta_\infty) = +\infty$. The main novelty of the paper is our strategy for dealing with the turning point at $+\infty$. The case $\zeta = \zeta_0$ is distinguished by the fact that this is the only case where the point at which $\theta(x,\zeta,h)$ must be explicitly evaluated in order to compute $V(\zeta,h)$, namely, $x = 0$, is itself a turning point: $x(\zeta_0) = 0$.

In the remainder of this section we discuss our approach to analyzing the turning point at infinity. The profile $P(x)$ converges at an exponential rate to its endstate $P(+\infty)$,

$$|P(x) - P(+\infty)| \leq Ce^{-\mu x} \text{ for } \mu > 0 \text{ as in (4.7)}, \tag{2.1}$$

and we use this property in section 4 to analytically extend $P(x)$ to a half-plane of the form

$$\mathbb{W}(M_0) := \{x \in \mathbb{C} : \Re x > M_0\}. \tag{2.2}$$

[10] This is to be expected, since θ is the decaying solution and $\Re\mu_1(x,\zeta) \leq 0$

This immediately gives an analytic extension of the governing system (0.6) to $\mathbb{W}(M_0)$. For any angle θ such that $0 < \theta < \pi/2$, define the infinite wedge

$$\mathbb{W}(M_0, \theta) := \{x \in \mathbb{C} : |\arg(x - M_0)| < \theta\} \subset \mathbb{W}(M_0). \tag{2.3}$$

For $\zeta \in \{\Re\zeta \geq 0\}$ near ζ_∞ and $x \in \mathbb{W}(M_0, \theta)$ the eigenvalues $\{\mu_j(x, \zeta)\}_{j=1,2}$ are well separated from $\{\mu_j(x, \zeta)\}_{j=3,4,5}$. In section 5 we use this fact to construct a 5×5 conjugator $Y(x, \zeta, h)$ such that the map $\theta = Y(x, \zeta, h)\phi$ *exactly* transforms the system (0.6) to block diagonal form on $\mathbb{W}(M_0, \theta)$:

$$h\frac{d}{dx}\phi = \begin{pmatrix} A_{11}(x, \zeta, h) & 0 \\ 0 & A_{22}(x, \zeta, h) \end{pmatrix} \phi, \tag{2.4}$$

where the blocks A_{11} and A_{22} are 2×2 and 3×3, respectively. The conjugator has the form $Y = Y_1 Y_2$, where Y_1 gives an approximate conjugation to block form (5.8) and Y_2 solves away the error in the approximate conjugation. The matrix $Y_1 = (P_0, Q_0, T_3, T_4, T_5)$ for vectors P_0, Q_0 defined in (5.6) and satisfying

$$T_1 = P_0 + sQ_0,\ T_2 = P_0 - sQ_0 \text{ for } T_1, T_2 \text{ as in (0.10)}, \tag{2.5}$$

and the entries of the matrix Y_2 are constructed by solving certain integral equations on $\mathbb{W}(M_0, \theta)$ by a contraction argument. Unlike $T(x, \zeta)$, the matrix $Y_1(x, \zeta)$ is always invertible. The analytic extension of the system (0.6) to $\mathbb{W}(M_0)$ gives us a freedom to choose integration paths that play an important role in this and later contraction arguments.

The block A_{11} has eigenvalues close to the crossing eigenvalues $\mu_1(x, \zeta)$, $\mu_2(x, \zeta)$. Thus, for ζ near ζ_∞ we have reduced the problem of constructing the decaying solution of the governing system (0.6) on $[M_0, +\infty)$ to constructing the decaying solution of the 2×2 system $\frac{d}{dx}\phi_1 = A_{11}(x, \zeta, h)\phi_1$. In section 5 we rewrite this 2×2 system as equivalent second-order equation (5.28),

$$h^2 w_{xx} = (C(x, \zeta) + hr(x, \zeta, h))w, \text{ where } C(+\infty, \zeta_\infty) = 0, \tag{2.6}$$

and we focus on solving this equation on an infinite strip of the form $T_{M,R} := \{x \in \mathbb{C} : \Re x \geq M, |\Im x| \leq R\}$. Note that for M large enough, $T_{M,R} \subset \mathbb{W}(M_0, \theta)$. In section 6 we show that a transformation of the form $t = t(x, \zeta) = f(\zeta)e^{-\mu x/2}$ for μ as in (2.1) and some $f(\zeta)$, transforms (2.6) into an equation that is a perturbation of Bessel's equation:

$$\begin{aligned} h^2(t^2 W_{tt} + tW_t) &= (t^2 + \tilde{\alpha}^2)W \\ &\quad + [(t^2 + \alpha^2)t^2 b_1(t, \zeta) + t^3 b_2(t, \zeta) + ht^2 b_3(t, \zeta, h)]W \text{ on } \mathcal{W}, \end{aligned} \tag{2.7}$$

where $\mathcal{W}$, the image of the strip $T_{M,R}$ under the map $t = t(x, \zeta)$, is a bounded wedge in $\{\Re t \geq 0\}$ with vertex at $t = 0$ (see (6.13)).

Before returning to (2.7) we illustrate how Bessel's equation arises from a very simple model equation with a turning point at $+\infty$ by a similar transformation. This model provides motivation for introducing (2.7) and already indicates the importance of the parameter α/h which appears below in the definition of regimes I, II, and III.

2.1. Model problem: Solution in terms of Bessel functions. Consider the equation

$$h^2 \frac{d^2 w}{dx^2} = (e^{-2x} + \alpha^2)w,$$

which becomes under the change of variable $t = e^{-x}$

$$h^2(t^2 w_{tt} + t w_t) = (t^2 + \alpha^2)w.$$

Thus, the turning point at $x = +\infty$ for $\alpha = 0$ becomes a turning point at $t = 0$. Setting $z = h^{-1}t$ and observing the scale-invariance of the left side, we obtain

$$h^2 \left[z^2 \frac{d^2 w}{dz^2} + z \frac{dw}{dz} \right] = (h^2 z^2 + \alpha^2)w,$$

which suggests that the good parameter is $\beta = h^{-1}\alpha$. In this case, the equation becomes

$$z^2 u_{zz} + z u_z - (z^2 + \beta^2)u = 0.$$

We recognize this as the "modified Bessel's equation" [O].[11] One may thus deduce that $u(z) = AJ_\beta(iz) + BY_\beta(iz)$; using $z = h^{-1}e^{-x}$ one obtains

$$w(x) = AJ_{ih^{-1}\alpha}(ih^{-1}e^{-x}) + BY_{ih^{-1}\alpha}(ih^{-1}e^{-x}),$$

and classical expansions for Bessel functions can now be used to decide which solutions decay or remain bounded as $x \to \infty$ for different choices of the complex parameter α. A connection between turning points at infinity and Bessel functions is discussed in [DL].

In (2.7) the parameter $\alpha = \alpha(\zeta) \in \mathbb{C}$ satisfies $\alpha^2 = i(\zeta - \zeta_\infty)$ and $\tilde{\alpha}$ is a nonvanishing multiple of α. The strategy now is to construct solutions of (2.6) on the strip $T_{M,R}$ that decay as $\Re x \to +\infty$ by constructing solutions of (2.7) on $\mathcal{W}$ that decay as $t \to 0$.

At first it is not at all clear whether and in what sense the perturbation, given by the second line of (2.7), is "small enough" for (2.7) to be helpfully regarded as a perturbation of Bessel's equation. Bessel's equation is a very singular equation, with a regular singular point at 0, an irregular singular point at ∞, and turning points for certain choices of $(\tilde{\alpha}, h)$; it is a delicate matter to understand perturbations of such a singular object. If one ignores the perturbation in (2.7) for a moment, it can be seen that the behavior of solutions to (2.7) depends on both the phase of $\tilde{\alpha}(\zeta)$ and on the relative magnitude of $\tilde{\alpha}$ and h. Accordingly, in section 9 we identify three different parameter regimes for $(\tilde{\alpha}(\zeta), h)$. With $\tilde{\beta} = \tilde{\alpha}/h$ these are

$$\begin{aligned}
&\text{I: } |\tilde{\beta}| \geq K, 0 \leq \arg \beta \leq \frac{\pi}{2} - \delta, \text{ where } K \text{ is large and } \delta > 0,\\
&\text{II: } |\tilde{\beta}| \geq K, \frac{\pi}{2} - \delta \leq \arg \beta \leq \frac{\pi}{2},\\
&\text{III: } |\tilde{\beta}| \leq K.
\end{aligned}$$

It turns out that the perturbed Bessel problem (6.13) can be analyzed in regimes I, II, and III by using suitable transformations of dependent and independent variables to reduce (6.13) to the normal form

$$W_{\xi\xi} = (u^2 \xi^m + \psi(\xi))W, \tag{2.8}$$

where $m = 0$, 1, or -1, respectively, u, is a large parameter, and ψ depends on the perturbation in (2.7). In regimes I and II the correct choice of large parameter is $u = \tilde{\beta} = \tilde{\alpha}/h$ and a basis of solutions of (2.8) can be writtenin terms of exponentials

[11] The standard Bessel's equation is $z^2 U_{zz} + z U_z + (z^2 - \nu^2)U = 0$.

and Airy functions, respectively (see Propositions 10.2 and 10.8). In regime III the large parameter is $u = 1/h$ and solutions of (2.8) are expressed in terms of the modified Bessel functions $I_{\tilde{\beta}}$, $K_{\tilde{\beta}}$ (Proposition 10.14).

Control of the function ψ in (2.8) is gained by careful estimates of the functions b_i, $i = 1, 2, 3$, appearing in the perturbation (see, for example, Proposition 6.3). The analysis also makes use of some classical methods [O] for constructing solutions to (2.8) on *large* subdomains of $\mathbb{C}$. To complete the analysis of the turning point at ∞, one must unravel the many transformations leading from (2.6) to the normal form (2.8), identify the explicit form of the solution that decays as x (the original x) goes to ∞, and show that this solution is indeed of type θ_1 at $x = M$, the real point on the left boundary of the infinite strip $T_{M,R}$. Provided ζ lies close enough to ζ_∞, the point $x = M$ will lie to the *left* of any of the corresponding turning points $x(\zeta)$. From here it is then relatively easy to conclude that θ is of type θ_1 at $x = 0$ for ζ near ζ_∞.

Remark 2.2. We have limited the exposition here mostly to the case of an idealized single-step exothermic reaction, for which λ is scalar. In the case of a multistep reaction, the λ-equation becomes a system of ODEs, with multiple decaying modes $\sim e^{-\gamma_j x} v_j$, where γ_j is in general complex with $\Re\gamma_j > 0$. One can then apply the analytic stable manifold theorem of [LWZ2] to obtain an analytic extension of the profile to a wedge $\mathbb{W}(M_0, \theta)$ for *some* $\theta > 0$. Likewise (see [LWZ2]), we may conjugate the turning point at infinity to a 2×2 block analytic on the same wedge. In some circumstances the presence of multiple reaction steps can prevent us from recasting the 2×2 block in the form (2.7). Nonetheless, under the condition that γ_1 is real and

$$\Re\gamma_j > 2\gamma_1 \text{ for } j \neq 1 \tag{2.9}$$

the analysis of this paper gives, independently of the results of [LWZ2] just mentioned, high-frequency stability of type D detonations also in this more general case. This extension is discussed in detail in section 11. The following example is a multistep case where condition (2.9) is satisfied.

Example 2.3. In equations (4.3)–(4.5) [S, p. 8], there is described a model three-step chain-branching reaction given by F $\to$ Y; F + Y $\to$ 2Y; Y $\to$ P for a fuel F, radical species Y, and product P, corresponding to initiation, chain-branching, and chain-termination reactions, with rates

$$r_I = f e^{\theta_I(1/T_I - 1/T)}, \quad r_B = y f e^{\theta_B(1/T_B - 1/T)}, \quad r_C = y, \tag{2.10}$$

and reaction dynamics $df/dx = -r_I - r_B$, $dy/dx = r_I + r_B - r_C$, where f and y are mass fractions of F and Y; $T_I > T|_{x=0} > T_B$; $T_I > T_B > T(\infty)$; and $\theta_I >> \theta_B >> 1$. This has been proposed in [K, SD, SKQ] as a realistic model for hydrogen-oxygen detonations studied experimentally in [AT, St], wherein "a small amount of reactant is converted into chain-carriers, which may be either free radicals or atoms, by means of the slow chain-initiation reactions, while the rise in concentration of chain-radicals is retarded by chain-termination steps which occur either through absorption at the vessel walls or through three-body collisions in the interior" [SD]. Setting $\lambda = (f, y)$, the vector of reactants, and linearizing about the equilibrium $\lambda = (0, 0)$, we obtain

$$\frac{d}{dx}\lambda = \begin{pmatrix} -e^{\theta_1(1/T_I - 1/T(\infty))} & 0 \\ * & -1 \end{pmatrix} \lambda, \tag{2.11}$$

verifying the condition $\mu_1 = 1 << \mu_2 := e^{\theta_1(1/T_I - 1/T(\infty))}$ provided $\theta_I >> 1$ and $T(\infty) > T_I$.

3. Discussion and open problems. High-frequency instability was established for type I detonations in [E2, E3, LWZ1]. Hence, Theorem 2.1 completesthe program of [E2, E3] for determining the high-frequency stability behavior of ZND detonations belonging to the two main classes I and D identified by Erpenbeck. Having a uniform high-frequency cutoff for stability is of more than abstract interest. As noted in [LS, KaS], numerical stability computations are both computationally intensive and delicate, with many features difficult to resolve or extrapolate in various asymptotic limits. The use of rigorous analysis to truncate the relevant parameter regime to a closed, bounded region is thus a critical, but up to now missing, part of any numerical stability investigation. Bounding the set of possibly unstable perturbation frequencies is a step toward the rigorous validation of the existing numerical results on multidimensional stability of type D ZND detonations.

Results in one dimension corresponding to the high-frequency results given here may be found in [Z1]. However, we emphasize that the multidimensional setting is essentially different from that of one dimension, being more complicated both physically—in one dimension, high-frequency stability holds automatically for all types of detonations—and mathematically—in one dimension, nontrivial turning points do not enter, so that the analysis can be carried out using the more familiar tools of repeated diagonalization and (a useful modification of) the gap lemma, under the mild hypothesis of C^r coefficients for the eigenvalue ODE. Here, by contrast, our arguments use in important ways our assumption of analytic coefficients. This is not mathematical convenience but reflects the inherent difficulty of the problem; in a companion paper [LWZ2], we show by explicit counterexamples that the conclusions made here may fail for coefficients that are C^r or even C^∞.

The stability result of Theorem 2.1 and the instability results of [LWZ1] concern the multidimensional stability (or instability) of ZND detonations with respect to high-frequency perturbations. A fundamental open problem is to establish full multidimensional stability of ideal gas ZND detonations with one-step Arrhenius reaction rate in the small-heat release and high-overdrive limits, generalizing the 1D results of [Z1] and giving rigorous validation to the formal observations of Erpenbeck in [E1, E5]. This would represent the first complete (i.e., covering all frequencies), rigorous result on multidimensional stability of any detonation wave. Again, given the delicacy of numerical computations on this subject, any such analytical signposts are invaluable.

We note that the 1D argument of [Z1], applied word for word together with the result of Majda [M] on multidimensional stability of ideal gas shock fronts, gives already by a simple continuity argument *bounded frequency* multidimensional stability of ideal gas ZND detonations with one-step Arrhenius reaction rate in the small-heat release and high-overdrive limits. The methods of this paper provide a starting point for treating the remaining high-frequency regime.

Another possibility opened up by our analysis is the treatment of stability in the multidimensional ZND limit of reactive Navier–Stokes (rNS) detonations. Establishing a close link between ZND and rNS stability functions for small viscosity/heat conduction/diffusion would instantly give a large number of spectral stability and bifurcation results for rNS; in one dimension such results were proved in [Z2]. Solving this problem would involve giving a multiparameter extension of the turning point analysis carried out here, with viscosity, heat conduction, and species diffusion as the additional parameters. As noted in [CJLW, Z2], the nonlinear implications of spectral stability ("normal modes") analysis are far from clear for ZND, which includes all the difficulties of the nonreacting Euler equations and more. For rNS, on the other hand, which incorporates mechanisms for dissipation, there is a much better chance of translating results on multidimensional spectral stability/instability into corresponding nonlinear stability/instability results. In one dimension a number of results of this type are given in [TZ].

As a direction beyond ZND, we mention the rigorous treatment for Maxwell's equations of "hybrid resonance" or "X-mode" heating of fusion plasma at the "cutoff" frequency where light and plasma frequencies collide. This frequency corresponds to a finite but singular turning point [DIW]. For parallel electric and magnetic fields, there is exact decoupling into "ordinary" (O) modes governed by Airy's equation, and "extraordinary" (X) modes governed by a *singular* cousin which is a perturbed Bessel equation similar to our equation (2.7). Exact conjugation tools like those we have developed here may be useful for completing this singular ODE analysis.

Finally, the uniform estimates given here are potentially useful for general turning-point problems on unbounded spatial intervals, both for spectral stability analysis as here and for resolvent estimates toward linearized and nonlinear stability or instability.

Part II. The turning point at infinity.

4. Analytic extension of the profile to a half-plane.

The analytic extension of the profile $P(x)$ to a half-plane given in this section turns out to be important for the construction in the next section of the conjugator to block form near ∞. For that an extension merely to a strip like $T_{M,R}$ does not appear to suffice.

In view of Assumptions 1.1, 1.2, and 1.3, the nonzero components of the detonation profile $P(x) = (v, u, S, \lambda) := (q, \lambda)$ satisfy a 4×4 system of ODEs on $[0, \infty)$ of the form.

$$\frac{d}{dx}\begin{pmatrix} F(P) \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ h(P)\lambda \end{pmatrix}, \quad \begin{pmatrix} q(0) \\ \lambda(0) \end{pmatrix} = \begin{pmatrix} q_0 \\ \lambda_0 \end{pmatrix}, \tag{4.1}$$

where $F(P)$ and $h(P)$ are real-analytic. Here the equation $\frac{d}{dx}F(P) = 0$ expresses conservation of mass, momentum, and energy (see [FD, p. 98]) and can be integrated to give $F(q, \lambda) = F(P(+\infty))$, an equation that determines q as a function $q = Q(\lambda)$.[12] With $\mathbb{Q}(\lambda) = (Q(\lambda), \lambda)$ the system thus reduces to a scalar problem of the form

$$\frac{d}{dx}\lambda = h(\mathbb{Q}(\lambda))\lambda, \quad \lambda(0) = \lambda_0 > 0, \tag{4.2}$$

where for some constants c_1, c_2

$$-c_1 < h(\mathbb{Q}(\lambda)) < -c_2 < 0 \text{ on } [0, \lambda_0]. \tag{4.3}$$

The condition (4.3) implies

$$|\lambda(x)| \le Ce^{-c_2 x} \text{ on } [0, \infty), \tag{4.4}$$

and thus $P(x) = (Q(\lambda(x)), \lambda(x))$ satisfies

$$|P(x) - P(\infty)| \le Ce^{-c_2 x}, \text{ where } P(\infty) = (Q(0), 0). \tag{4.5}$$

The next proposition gives more precise information on the profile for x large.

PROPOSITION 4.1. *For M_0 large enough, the profile $P(x)$ extends analytically to a solution of* (4.1) *on a half-plane $\mathbb{W}(M_0) := \{x \in \mathbb{C} : \Re x > M_0\}$. The extended profile has a convergent expansion*

$$P(x) = P_0 + P_1 e^{-\mu x} + P_2 e^{-2\mu x} + \cdots \text{ on } \mathbb{W}(M_0), \tag{4.6}$$

[12] Here $Q(\lambda)$ is actually a branch of a multivalued function.

where $\mu = -h(\mathbb{Q}(0)) > 0$ *and the* P_j *are constant vectors. Thus,* $P(x)$ *satisfies*

$$|P(x) - P(\infty)| \le Ce^{-\mu \Re x} \text{ on } \mathbb{W}(M_0). \tag{4.7}$$

Proof. (1) In view of the above discussion it suffices to show that $\lambda(x)$ has an expansion like (4.6) (with $\lambda_0 = 0$) on $\mathbb{W}(M_0)$ for M_0 large. Let us set $H(\lambda) := h(\mathbb{Q}(\lambda))$, so $\mu = -H(0)$ and

$$\frac{d}{dx}\lambda = \lambda H(\lambda), \tag{4.8}$$

where $H(\lambda)$ is analytic in a neighborhood of $\lambda = 0$.

(2) We have $H(\lambda) = H(0) + \lambda K(\lambda)$, so

$$\frac{1}{H(\lambda)} - \frac{1}{H(0)} = \lambda R(\lambda) \tag{4.9}$$

for some functions $K(\lambda)$, $R(\lambda)$ analytic near $\lambda = 0$. Multiplying (4.9) by λ_x and using (4.8), we obtain

$$\frac{\lambda_x}{\lambda} + \mu = \mu R(\lambda)\lambda_x. \tag{4.10}$$

Let $V(\lambda) = \int_0^\lambda R(s)ds$ and set $\lambda(x) = e^{-\mu x}T(x)$. Noting that $\ln T$ is a primitive of the left side of (4.10), we obtain by integrating (4.10) from M_1 to x for M_1 large

$$\ln T = \mu V(e^{-\mu x}T(x)) + C_0, \tag{4.11}$$

where $C_0 = C_0(M_1)$ is a known constant. Defining $\mathcal{K}(T, b) = \ln T - \mu V(bT)$ near the basepoint $(T_0, b_0) = (e^{C_0}, 0)$, we can solve $\mathcal{K}(T, b) = C_0$ by the implicit function theorem to obtain

$$T(b) = e^{C_0} + \sum_{j=1}^{\infty} a_j b^j \text{ for } |b| < \delta \tag{4.12}$$

for some $\delta > 0$. Thus, $\tilde{\lambda}(b) := bT(b)$ is analytic for $|b| < \delta$, which implies $\lambda(x) = \tilde{\lambda}(e^{-\mu x})$ is analytic for x such that $e^{\mu \Re x} < \delta$, that is, $\Re x > -\frac{\ln \delta}{\mu} := M_0$. The expansion (4.12) implies

$$\lambda(x) = e^{C_0}e^{-\mu x} + \sum_{j=1}^{\infty} a_j e^{-(j+1)\mu x} \text{ on } \mathbb{W}(M_0), \tag{4.13}$$

so P depends analytically on $e^{-\mu x}$.[13] ∎

5. Conjugation to block form near infinity. Here we perform a conjugation, based on Proposition 5.2 below, of Erpenbeck's 5×5 system

$$h\frac{d}{dx}\theta = (\Phi_0(x, \zeta) + h\Phi_1(x))\theta := G(x, \zeta, h)\theta \tag{5.1}$$

[13] A similar analysis of profiles was given in [L].

to block form

$$h\frac{d}{dx}\phi = \begin{pmatrix} A_{11}(x,\zeta,h) & 0 \\ 0 & A_{22}(x,\zeta,h) \end{pmatrix}\phi \tag{5.2}$$

on an infinite wedge

$$\mathbb{W}(M_0,\theta) := \{x = x_r + ix_i \in \mathbb{C} : |\arg(x - M_0)| < \theta\}, \quad M_0 >> 1, \tag{5.3}$$

contained in the half-plane $\mathbb{W}(M_0)$ to which the profile $P(x) = (v, u, , S, \lambda)$ has been analytically extended. On $\mathbb{W}(M_0)$ we have

$$|P(x) - P(\infty)| \le Ce^{-\mu\Re x} \tag{5.4}$$

for $\mu > 0$ as in Proposition 4.1.

Define the 5×5 matrix

$$Y_1 = \begin{pmatrix} P_0 & Q_0 & T_3 & T_4 & T_5 \end{pmatrix}, \tag{5.5}$$

where

$$P_0 = \begin{pmatrix} 0 \\ \frac{\zeta}{u} \\ -i \\ 0 \\ 0 \end{pmatrix}, Q_0 = \begin{pmatrix} \frac{m}{\kappa u} \\ 0 \\ 0 \\ -\frac{\kappa}{mu}p_S \\ -\frac{\kappa}{mu}p_\lambda \end{pmatrix}, T_3 = \begin{pmatrix} -\frac{im}{1-\eta} \\ i \\ \frac{\zeta}{u} \\ 0 \\ 0 \end{pmatrix}, T_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, T_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{5.6}$$

Thus, we have

$$T_1 = P_0 + sQ_0, \;\; T_2 = P_0 - sQ_0, \;\; s = \sqrt{\zeta^2 + c_0^2\eta(x)}, \tag{5.7}$$

for T_1, T_2 as in (0.10). Setting $\theta = Y_1\theta^a$, we have

$$h\frac{d}{dx}\theta^a = \begin{pmatrix} A_{11}^0 & 0 \\ 0 & A_{22}^0 \end{pmatrix}\theta^a + h\begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}\theta^a, \tag{5.8}$$

where

(5.9)

$$A_{11}^0 = \begin{pmatrix} -\frac{\kappa^2\zeta}{\eta u} & -\frac{\kappa}{\eta u} \\ -\frac{s^2\kappa}{\eta u} & -\frac{\kappa^2\zeta}{\eta u} \end{pmatrix}, \; A_{22}^0 = \begin{pmatrix} \frac{\zeta}{u} & 0 & 0 \\ 0 & \frac{\zeta}{u} & 0 \\ 0 & 0 & \frac{\zeta}{u} \end{pmatrix}, \; \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} = Y_1^{-1}\Phi_1 Y_1 - Y_1^{-1}\frac{dY_1}{dx}.$$

Since the eigenvalues of A_{11}^0 are separated from those of A_{22}^0, we can apply Proposition 5.2 below to find a second conjugator, bounded and analytic in its arguments,

$$Y_2(x,\zeta,h) = \begin{pmatrix} I & h\alpha_{12} \\ h\alpha_{21} & I \end{pmatrix}, \tag{5.10}$$

such that if we set $\theta^a = Y_2\phi$, we have

$$\begin{aligned} h\frac{d}{dx}\phi &= \begin{pmatrix} A_{11}^0 + hd_{11} + h^2\beta_{11} & 0 \\ 0 & A_{22}^0 + hd_{22} + h^2\beta_{22} \end{pmatrix}, \\ \phi :&= \begin{pmatrix} A_{11}(x,\zeta,h) & 0 \\ 0 & A_{22}(x,\zeta,h) \end{pmatrix}\phi, \end{aligned} \tag{5.11}$$

where

$$\beta_{11} = d_{12}\alpha_{21}, \quad \beta_{22} = d_{21}\alpha_{12}. \tag{5.12}$$

Setting $Y = Y_1Y_2$, we conclude that θ satisfies (5.1) on the wedge $\mathbb{W}(M_0,\theta)$ if and only if $\phi = \binom{\phi_1}{\phi_2}$ defined by $\theta = Y\phi$ satisfies (5.11), where the 2×2 block

$$\begin{aligned} &A_{11}(x,\zeta,h) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \underline{a} & \underline{b} \\ \underline{c} & \underline{d} \end{pmatrix} + O(h) \text{ with} \\ &\underline{a} = -\frac{\kappa^2\zeta}{\eta u},\ \underline{b} = -\frac{\kappa}{\eta u},\ \underline{c} = -\frac{s^2\kappa}{\eta u},\ \underline{d} = -\frac{\kappa^2\zeta}{\eta u}. \end{aligned} \tag{5.13}$$

Lemma 5.1. *The functions $d_{11}(x)$ and $d_{12}(x)$ decay exponentially to* 0 *on $\mathbb{W}(M_0,\theta)$ as $\Re x \to \infty$ at the same rate as $\lambda(x)$.*

Proposition 5.2. *Let θ be any angle such that $0 < \theta < \frac{\pi}{2}$. There exist positive constants M_0, h_0 and a neighborhood $\omega \ni \zeta_\infty$ such that for $\zeta \in \omega$ and $0 < h < h_0$, the conjugator $Y_2(x,\zeta,h)$ as in* (5.10) *can be constructed on $\mathbb{W}(M_0,\theta)$ with $\alpha_{12}(x,\zeta,h)$ and $\alpha_{21}(x,\zeta,h)$ bounded and analytic in their arguments. Moreover, there exists a well-defined endstate $\alpha_{21}(\infty)$ and we have estimates*

$$|\partial_x^k\left(\alpha_{21}(x,\zeta,h) - \alpha_{21}(\infty,\zeta,h)\right)| \le C_k h^{-k} e^{-\mu\Re x} \tag{5.14}$$

for $\mu > 0$ as in Proposition 4.1.

Remark 5.3.

(1) The proofs are given in section 15. The differential equation satisfied by α_{21} is

$$h\frac{d}{dx}\alpha_{21} = A^0_{22}\alpha_{21} - \alpha_{21}A^0_{11} + d_{21} + h(d_{22}\alpha_{21} - \alpha_{21}d_{11}) - h^2\alpha_{21}d_{12}\alpha_{21}. \tag{5.15}$$

The argument uses the fact that the eigenvalues of $A^0_{11}(\infty,\zeta)$ and $A^0_{22}(\infty,\zeta)$ are separated and close to the imaginary axis for ζ near ζ_∞. Using the fact that d_{21} converges exponentially to its endstate $d_{21}(\infty)$,

$$|d_{21}(x,\zeta) - d_{21}(\infty,\zeta)| \le Ce^{-\mu\Re x}, \tag{5.16}$$

and that d_{12} satisfies a similar estimate but with $d_{12}(\infty) = 0$, one can show that α_{21} has a well-defined endstate and that the estimates (5.14) hold. These estimates are the key to the treatment of the b_3 term in the perturbation of Bessel's equation given by (2.7).

(2) The above lemma and proposition imply that the function $\beta_{11}(x,\zeta,h)$ decays exponentially to 0 on $\mathbb{W}(M_0,\theta)$ as $\Re x \to \infty$ at the same rate as $\lambda(x)$. Thus, the same holds for the $O(h)$ terms in (5.13).

A valuable tool for understanding the behavior of solutions of (5.1) as $x \to \infty$ is the conjugator $M(x,\zeta,h)$ described in the following proposition.

Proposition 5.4 ("[MZ] conjugator," [MZ, Lemma 2.6]). *Consider any $N \times N$ system $\frac{d}{dx}\theta = A(x,\zeta,h)\theta$ on $[0,\infty)$, for (ζ,h) near a fixed basepoint $(\underline{\zeta},\underline{h}) \in \{\Re\zeta \ge 0\} \times (0,1]$, where A is analytic in its arguments. Assume there is a corresponding limiting system $\frac{d}{dx}\theta_\infty = A(\infty,\zeta,h)\theta_\infty$ and that*

$$|A(x,\zeta,h) - A(\infty,\zeta,h)| \le Ce^{-\beta x} \text{ for some } \beta > 0. \tag{5.17}$$

Then there exists a neighborhood $\mathcal{O} \ni (\underline{\zeta}, \underline{h})$ and an $N \times N$ matrix $M(x, \zeta, h)$, analytic in its arguments $x \in [0, \infty]$, $(\zeta, h) \in \mathcal{O}$, and uniformly bounded together with its inverse, such that $\theta(x, \zeta, h)$ is a solution of $\frac{d}{dx}\theta = A(x, \zeta, h)\theta$ on $[0, \infty)$ if and only if θ_∞ defined by

$$\theta = M(x, \zeta, h)\theta_\infty \tag{5.18}$$

is a solution of the limiting system. Moreover, for $k \in \{0, 1, 2, \dots\}$ M satisfies $|\partial_x^k(M(x, \zeta, h) - I)| \le C_k e^{-\delta x}$ for any $0 < \delta < \beta$, uniformly for $(\zeta, h) \in \mathcal{O}$.

From the matrix formulas given in section 20, it is not hard to see that the eigenvalues of $G(x, \zeta, h)$ (5.1) are

$$\begin{aligned} &\mu_j^* := \mu_j(x, \zeta) + O(he^{-\mu x}),\ j = 1, 2, 3, 4, \\ &\mu_5^* = \mu_5(x, \zeta) - h\frac{r_\lambda}{u} + O(he^{-\mu x}), \end{aligned} \tag{5.19}$$

where $r_\lambda < 0$ and μ is as in Proposition 4.1.[14] Thus, the eigenvalues of the limiting system $G(\infty, \zeta, h)$ are $\mu_j(\infty, \zeta)$, $j = 1, 2, 3, 4$, and $\mu_5(\infty, \zeta) - h\frac{r_\lambda}{u}(\infty)$. For $\Re\zeta > 0$ only $\mu_1(\infty, \zeta)$ has negative real part, so use of the conjugator $M(x, \zeta, h)$ shows that for $\Re\zeta > 0$, the system (5.1) has a 1D space $\mathcal{D}(\zeta, h)$ of decaying solutions on $[0, \infty)$.

Lemma 5.1 implies that $A_{11}(\infty, \zeta, h) = A_{11}^0(\infty, \zeta)$, so the eigenvalues of $A_{11}(\infty, \zeta, h)$ are $\mu_j(\infty, \zeta)$, $j = 1, 2$. Use of the [MZ] conjugator again implies that for $\Re\zeta > 0$ the equation

$$h\frac{d}{dx}\phi_1 = A_{11}(x, \zeta, h)\phi_1 \tag{5.20}$$

has a 1D space of decaying solutions $\mathcal{D}_1(\zeta, h)$. Thus, we must have

$$\mathcal{D}(\zeta, h) = \{Y(x, \zeta, h)\begin{pmatrix}\phi_1 \\ 0\end{pmatrix},\ \phi_1 \in \mathcal{D}_1(\zeta, h)\}. \tag{5.21}$$

Next we reduce (5.20) to an equivalent scalar second-order equation. Letting φ_0 for the moment denote any primitive of $\frac{a+d}{2}$, and making the transformation

$$\tilde{\phi}_1 = e^{-\frac{\varphi_0}{h}}\phi_1, \tag{5.22}$$

we obtain the system

$$h\frac{d}{dx}\tilde{\phi}_1 = \begin{pmatrix}-\alpha & b \\ c & \alpha\end{pmatrix}\tilde{\phi}_1,\ \alpha = \frac{d-a}{2}. \tag{5.23}$$

Setting $\tilde{\phi}_1 = \binom{\tilde{u}}{\tilde{v}}$, we rewrite the first row of (5.23) as

$$\tilde{v} = b^{-1}(h\tilde{u}' + \alpha\tilde{u}), \quad \text{where } ' = d/dx, \tag{5.24}$$

and hence the second row of (5.23) becomes

$$h^2(b^{-1}\tilde{u}')' + h\left(\frac{\alpha}{b}\tilde{u}\right)' = \left(c + \frac{\alpha^2}{b}\right)\tilde{u} + \frac{\alpha}{b}h\tilde{u}'. \tag{5.25}$$

[14] The details are given in section 2 of [LWZ1].

Defining $w = b^{-1/2}\tilde{u}$ and[15] using the identity

$$(b^{-1}\tilde{u}')' = \left(\frac{1}{2}b^{-\frac{3}{2}}b'\right)' w + b^{-\frac{1}{2}}w'', \tag{5.26}$$

we obtain in place of (5.25)

$$h^2 w'' = (bc + \alpha^2)w - hb\left(\frac{\alpha}{b}\right)' w - h^2 b^{\frac{1}{2}}\left(\frac{1}{2}b^{-\frac{3}{2}}b'\right)' w. \tag{5.27}$$

With $A_{11}^0 = \begin{pmatrix} \underline{a} & \underline{b} \\ \underline{c} & \underline{d} \end{pmatrix}$ as in (5.9), we can rewrite (5.27) as

$$h^2 w'' = (C(x,\zeta) + hr(x,\zeta,h))w, \tag{5.28}$$

where

$$\begin{aligned} &C(x,\zeta) = \underline{bc} + \left(\frac{\underline{d}-\underline{a}}{2}\right)^2 = \left(\zeta^2 + c_0^2\eta(x)\right)\underline{b}^2(x) \text{ and} \\ &hr(x,\zeta,h) = (bc+\alpha^2) - \left(\underline{bc} + \left(\frac{\underline{d}-\underline{a}}{2}\right)^2\right) - hb\left(\frac{\alpha}{b}\right)' - h^2 b^{\frac{1}{2}}\left(\frac{1}{2}b^{-\frac{3}{2}}b'\right)'. \end{aligned} \tag{5.29}$$

PROPOSITION 5.5. *The function r in* (5.29) *satisfies* $r(\infty,\zeta,h) = 0$.

Proof. The functions appearing in the expression for r can all be expressed in terms of the components of A_{11}^0, d_{11}, and β_{11}, so the proposition follows directly from Lemma 5.1. ∎

Making the following choice of φ_0 such that $\frac{d}{dx}\varphi_0 = \frac{a+d}{2}$,

$$\varphi_0(x,\zeta,h) = \frac{a+d}{2}(\infty,\zeta,h)\cdot x + \int_\infty^x \left[\frac{a+d}{2}(s,\zeta,h) - \frac{a+d}{2}(\infty,\zeta,h)\right] ds, \tag{5.30}$$

we have shown that solutions of $h\frac{d}{dx}\phi_1 = A_{11}(x,\zeta,h)\phi_1$ are given by

$$\phi_1 = e^{\frac{\varphi_0}{h}}\begin{pmatrix} b^{1/2} & 0 \\ \alpha b^{-1/2} - h\frac{d}{dx}(b^{-1/2}) & b^{-1/2}\end{pmatrix}\begin{pmatrix} w \\ h\frac{d}{dx}w\end{pmatrix} := K(x,\zeta,h)\begin{pmatrix} w \\ h\frac{d}{dx}w\end{pmatrix}, \tag{5.31}$$

where $(w, h\frac{d}{dx}w)$ satisfies

$$h\frac{d}{dx}\begin{pmatrix} w \\ h\frac{d}{dx}w\end{pmatrix} = \begin{pmatrix} 0 & 1 \\ C(x,\zeta) + hr(x,\zeta,h) & 0\end{pmatrix}\begin{pmatrix} w \\ h\frac{d}{dx}w\end{pmatrix}. \tag{5.32}$$

Remark 5.6. Since $\frac{a+d}{2}(\infty,\zeta,h) = \frac{\mu_1+\mu_2}{2}(\infty,\zeta) = -\zeta\frac{\kappa^2}{\eta u}(\infty)$, we see that φ_0 is the sum of a term with real part ≤ 0 and, by Lemma 5.1, a term that decays exponentially to 0 as $x \to \infty$.

Using Remark 5.6, for $\Re\zeta > 0$ we obtain

$$\mathcal{D}(\zeta,h) = \text{span } Y(x,\zeta,h)\begin{pmatrix} K(x,\zeta,h)\begin{pmatrix} w \\ h\frac{d}{dx}w\end{pmatrix} \\ 0\end{pmatrix}, \tag{5.33}$$

where (w, hw_x) gives a decaying solution of (5.32). Erpenbeck's stability function $V(\zeta,h)$ is expressed in terms of $\theta(0,\zeta,h)$, where $\theta(x,\zeta,h) \in \mathcal{D}(\zeta,h)$.

Our next main task is to construct explicit asymptotic formulas for the exact solutions of (5.32) that decay to zero as $x \to \infty$.

[15]It does not matter which branch of the square root we use here, as long as we always use the same one.

6. Reduction to a perturbation of Bessel's equation in the general case. In the general case we must consider (5.28),

$$h^2 w'' = (C(x,\zeta) + hr(x,\zeta,h))\, w, \tag{6.1}$$

where $C(x,\zeta) = (\zeta^2 + c_0^2\eta(x))\underline{b}^2(x)$. Here the x-dependence enters only through the detonation profile $P(x) = (v,u,S,\lambda)$, and $\underline{b} = -\kappa/\eta u$.

Since $\zeta^2 + c_0^2\eta = (\zeta - ic_0\sqrt{\eta})(\zeta + ic_0\sqrt{\eta})$, we can write

$$C(x,\zeta) = (e(x) + \alpha^2)D(x,\zeta), \tag{6.2}$$

where

$$e(x) = c_0\sqrt{\eta}(x) - c_0\sqrt{\eta}(\infty),\ \alpha^2 = i(\zeta - \zeta_\infty),\ D(x,\zeta) = (-i\zeta + c_0\sqrt{\eta}(x))\underline{b}^2(x). \tag{6.3}$$

We have

$$D(\infty,\zeta) > 0 \text{ for } \zeta = i|\zeta|, \tag{6.4}$$

so the function $D(x,\zeta)$ is strictly bounded away from 0 for x large and ζ near ζ_∞ (the latter frequency being the endpoint of $(\mathrm{iii})_+$ corresponding to the turning point at infinity). We take ζ in a small neighborhood of ζ_∞ in $\Re\zeta \geq 0$. Note that $e(x)$ is strictly positive on $[0,\infty)$ and decreases to 0 at an exponential rate (case D).

For the wedge $\mathbb{W}(M_0,\theta)$ in Proposition 5.2 we now choose $M > M_0$ and $R > 0$ such that the strip

$$T_{M,R} := \{x = x_r + ix_i : x_r \geq M, |x_i| \leq R\} \subset \mathbb{W}(M_0,\theta), \tag{6.5}$$

and consider (6.1) on $T_{M,R}$. Proposition 4.1 implies that $e(x)$ has an expansion similar to $\lambda(x)$ (4.13) on $T_{M,R}$, so in particular

$$e(x) = ae^{-\mu x} + m(x)e^{-\mu x},\ \text{where } a > 0,\ |m(x)| \leq Ce^{-\mu\Re x}, \tag{6.6}$$

and $m(x)$ is real-valued on $[M,+\infty)$.

Setting $d(x,\zeta) = D(x,\zeta) - D(\infty,\zeta)$, the problem (6.1) can now be written

(6.7)
$$\begin{aligned} h^2 w'' &= \left(ae^{-\mu x} + \alpha^2 + m(x)e^{-\mu x}\right) D(x,\zeta)w + hr(x,\zeta,h)w \\ &= \left(ae^{-\mu x} + \alpha^2\right) D(\infty,\zeta)w + \left[\left(ae^{-\mu x} + \alpha^2\right) d(x,\zeta) + m(x)e^{-\mu x}D(x,\zeta)\right] w + hr(x,\zeta,h)w. \end{aligned}$$

Recalling that the x-dependence in $d(x,\zeta)$ enters only through the profile, using (4.6), and setting $t = \frac{2}{\mu}\sqrt{aD(\infty,\zeta)}e^{-\mu x/2}$, we can rewrite $d(x,\zeta)$ using the t variable as

$$d(x(t),\zeta) = t^2 b_1(t,\zeta), \tag{6.8}$$

where b_1 is analytic in t and $O(1)$ on the bounded wedge $\mathcal{W}(\zeta)$ with vertex at $t = 0$, which is the image of the strip $T_{M,R}$ under the change of variable $t = t(x,\zeta)$. Similarly,

$$m(x)e^{-\mu x}D(x,\zeta) = t^3 b_2(t,\zeta), \tag{6.9}$$

where $b_2(t,\zeta) = O(|t|)$ and is analytic in t on $\mathcal{W}(\zeta)$. The function $r(x,\zeta,h)$ is more complicated and must be handled carefully. Equations (5.29) and (5.11) show that the x-dependence in r enters through the components of

$$\text{(6.10)}\qquad \begin{aligned}&\text{(a) } P(x), P'(x), P''(x),\\ &\text{(b) } h\beta_{11}, h^2\beta_{11}', h^3\beta_{11}'', \text{ where } \beta_{11} = d_{12}\alpha_{21}.\end{aligned}$$

Thus, Proposition 5.5 and Lemma 5.1 allow us to write

$$\text{(6.11)}\qquad r(x(t),\zeta,h) = t^2 b_3(t,\zeta,h),$$

where b_3 is analytic in t and $O(1)$ on $\mathcal{W}(\zeta)$.

Definition 6.1. *Recall that $\mathcal{W}(\zeta)$ was defined below* (6.8) *as the image of the strip $T_{M,R}$ under the map $t = t(x,\zeta)$. It will be convenient to work on a slightly smaller wedge $\mathcal{W}$ that is independent of ζ and symmetric about the horizontal axis. So we define $\mathcal{W} := \{t \in \mathbb{C} : |\arg t| < \varepsilon_1,\ 0 < |t| < \varepsilon_2\}$, where ε_1, ε_2 are small enough positive constants so that $\mathcal{W} \subset \mathcal{W}(\zeta)$ for ζ near ζ_∞. Since* (6.4) *holds, after shrinking $\omega \ni \zeta_\infty$, we can choose ε_1 and ε_2 so that $\mathcal{W}$ still contains the image of $T_{M',R'}$ under the map $t = t(x,\zeta)$ for all $\zeta \in \omega$ for some $M' > M$, $R' < R$.*

Setting $\tilde\alpha := \frac{2}{\mu}\alpha\sqrt{D(\infty,\zeta)}$ and

$$\text{(6.12)}\qquad z = \frac{t}{h},\ \beta = \frac{\alpha}{h},\ \tilde\beta = \frac{\tilde\alpha}{h},$$

we obtain the following two equivalent forms for (6.7):

$$\text{(6.13)}\qquad \begin{aligned}&\text{(a) } h^2(t^2W_{tt} + tW_t) = (t^2 + \tilde\alpha^2)W\\ &\qquad + \left[(t^2+\alpha^2)t^2 b_1(t,\zeta) + t^3 b_2(t,\zeta) + ht^2 b_3(t,\zeta,h)\right] W \text{ on } \mathcal{W},\\ &\text{(b) } (z^2W_{zz} + zW_z) = (z^2 + \tilde\beta^2)W\\ &\qquad + \left[h^2(z^2+\beta^2)z^2 b_1(hz,\zeta) + hz^3 b_2(hz,\zeta) + hz^2 b_3(hz,\zeta,h)\right] W,\end{aligned}$$

where z lies in the wedge $\mathcal{Z}_h = \mathcal{W}/h$.[16]

Remark 6.2. For later reference we note that if we set $t = t(x)$ and $W(t(x)) = w(x)$, then the right side of (6.13)(a) is $\frac{4}{\mu^2}(C(x,\zeta) + hr(x,\zeta,h))w$.

Proposition 6.3. *The functions $b_1(t,\zeta)$, $b_2(t,\zeta)$, $b_3(t,\zeta,h)$ satisfy the following estimates. Here $b_1(0,\zeta)$, for example, denotes the limiting value of b_1 as $t \to 0$, and $k \in \{0,1,2,\ldots\}$.*

$$\text{(6.14)}\qquad \begin{aligned}&\text{(a) } \left|\partial_t^k(b_1(t,\zeta) - b_1(0,\zeta))\right| \le C_k|t|^{2-k},\\ &\text{(b) } \left|\partial_t^k b_2(t,\zeta)\right| \le C_k|t|^{1-k},\\ &\text{(c) } \left|\partial_t^k(b_3(t,\zeta,h) - b_3(0,\zeta,h))\right| \le C_k|t|^{2-k}h^{1-k}.\end{aligned}$$

The estimates are uniform for $h \in (0,1]$, ζ in a small neighborhood of ζ_∞ in $\Re\zeta \ge 0$, and $t \in \mathcal{W}$ (as in Definition 6.1*).*

Proof. (1) Recall $t = \frac{2}{\mu}\sqrt{aD(\infty,\zeta)}e^{-\mu x/2}$. Given a function $f(x)$ analytic on the strip $T_{M,R}$, let

$$\text{(6.15)}\qquad f^*(t) := f\left(-\frac{2}{\mu}\log\frac{\mu t}{2\sqrt{aD(\infty,\zeta)}}\right)$$

[16] In (6.13) the functions b_i are nonvanishing constant multiples of their former selves.

be the corresponding function on $\mathcal{W}$. Suppose

$$\left|\partial_x^k f(x)\right| \leq C_k e^{-|x|\mu},\ x \in T_{M,R}. \tag{6.16}$$

Then since $|x| \sim -\frac{2}{\mu}\ln|t|$, (6.15) implies

$$|\partial_t f^*(t)| \leq C_1 e^{(\frac{2}{\mu}\ln|t|)\mu}\frac{1}{|t|} \leq C_1|t|, \tag{6.17}$$

and by induction

$$|\partial_t^k f^*(t)| \leq C_k|t|^{2-k}. \tag{6.18}$$

(2) To estimate b_1 we use the fact that the x-dependence in $D(x,\zeta)$ enters only through the profile $P(x)$ and write $D(x,\zeta) = E(P(x),\zeta)$. We have

$$\begin{aligned} D(x,\zeta) - D(\infty,\zeta) &= (P(x) - P(\infty))\int_0^1 \partial_P E(P(\infty) + s(P(x) - P(\infty)),\zeta)ds \\ &= t^2(x)b_1(t(x),\zeta), \end{aligned} \tag{6.19}$$

where $b_1(t,\zeta) - b_1(0,\zeta) = f^*(t)$ for a function $f(x)$ which satisfies (6.16) in view of the expansion (4.6), so we obtain (6.14)(a).

(3) To estimate b_2 we use (6.9) and the fact that $m(x)$ satisfies the estimates (6.16). Thus, $f(x) := m(x)D(x,\zeta)$ satisfies the same estimates. Since $b_2(t) = \frac{f^*(t)}{t}\frac{\mu^2}{4aD(\infty,\zeta)}$, the estimate (6.14)(b) follows from (6.18).

(4) *Estimate of* b_3. Terms not involving β_{11} in the expression for b_3 can be estimated like b_1; the worst terms involve β_{11} and its derivatives, where x-dependence enters not only through the profile $P(x)$ but also through $\alpha_{21}(x,\zeta,h)$. For example, consider the term

$$h\beta_{11}\underline{b} = hd_{12}\alpha_{21}\underline{b}, \tag{6.20}$$

which appears in the expression for $\frac{1}{h}(bc - \underline{bc})$.[17] Lemma 5.1 and the argument giving (6.6) show that $d_{12} = e^{\mu x}(a + m(x))$ for some (new) a and $m(x)$ satisfying $|m(x)| \leq C^{-\mu\Re x}$. Thus

$$h\beta_{11}\underline{b} = e^{\mu x}(a + m(x))h\alpha_{21}\underline{b} := t^2(x)B_3(t(x),\zeta,h). \tag{6.21}$$

Since α_{21} satisfies the estimates (5.14), using the explicit form of $x = x(t)$ we obtain

$$|\partial_t^k(B_3(t,\zeta,h) - B_3(0,\zeta,h))| \leq C|t|^{2-k}h^{1-k} \tag{6.22}$$

by arguing as for b_1. The functions $h\frac{d}{dx}\alpha_{21}$ and $h^2\frac{d^2}{dx^2}\alpha_{21}$ also satisfy the estimates (5.14) (now with endstates 0), so the terms involving derivatives of β_{11} (recall (6.10)) can be estimated in the same way. ∎

[17] Here and in the rest of step (4) we use β_{11} to denote the appropriate entry of the 2×2 matrix β_{11}; recall (5.11). A similar remark applies to d_{12} and α_{21}.

7. Differential equations with singularities, turning points, and a large parameter. Consider equations of the form

$$w_{\sigma\sigma} = (u^2 f(\sigma) + g(\sigma))w \tag{7.1}$$

on a domain $D \subset \mathbb{C}$, where u is a large real or complex parameter, and the functions f and g are analytic except at boundary points or isolated interior points of D. Under certain conditions on f and g the problem (7.1) can be usefully transformed by a change of dependent and independent variables into one of the normal forms:

$$W_{\xi\xi} = (u^2 \xi^m + \psi(\xi))W, \tag{7.2}$$

where $m = 0$, 1, or -1, and ψ can be expressed explicitly in terms of f and g. The transformation of independent variable in these cases is, respectively,

$$\begin{aligned} &\text{(a) } \xi = \int_{\sigma_0}^{\sigma} f^{1/2}(r)dr,\\ &\text{(b) } \frac{2}{3}\xi^{3/2} = \int_{\sigma_0}^{\sigma} f^{1/2}(r)dr,\\ &\text{(c) } 2\xi^{1/2} = \int_{\sigma_0}^{\sigma} f^{1/2}(r)dr, \end{aligned} \tag{7.3}$$

where σ_0 is a zero or pole of f in (b), (c), respectively [O, Chapter 10]. With $\dot{\sigma} = \frac{d\sigma}{d\xi}$ one defines $W = \dot{\sigma}^{-1/2} w$ and then finds

$$\psi(\xi) = \dot{\sigma}^2 g(\sigma) + \dot{\sigma}^{1/2} \frac{d^2}{d\xi^2}\left(\dot{\sigma}^{-1/2}\right). \tag{7.4}$$

The problem (7.2) is easily solved in the elementary case when ψ is identically zero, so it is natural to use variation of constants and integral equations to solve the general case. This program is carried out in detail in Chapters 10, 11, and 12 of [O], which treat the respective cases $m = 0,\ 1,\ -1$. The elementary solutions are exponentials $e^{\pm u\xi}$ in the case $m = 0$ and Airy functions in the case $m = 1$.

In the case $m = -1$, it is shown in [O, Chapter 12] that if g has a simple or double pole at $\sigma = \sigma_0$, and we define ν by

$$\frac{\nu^2 - 1}{4} = (\sigma - \sigma_0)^2 g(\sigma)|_{\sigma=\sigma_0}, \tag{7.5}$$

then under the above transformations (7.1) takes the form

$$W_{\xi\xi} = \left(\frac{u^2}{\xi} + \psi(\xi)\right) W = \left(\frac{u^2}{\xi} + \frac{\nu^2 - 1}{4\xi^2} + \frac{\phi(\xi)}{\xi}\right) W, \tag{7.6}$$

where ϕ is analytic at $\xi = 0$. We now take the equation obtained by neglecting $\frac{\phi(\xi)}{\xi}$ in (7.6) as the "elementary equation"; its solutions are the modified Bessel functions $\xi^{1/2} I_\nu(2u\xi^{1/2})$ and $\xi^{1/2} K_\nu(2u\xi^{1/2})$.

8. Three parameter regimes. It is not yet clear whether and in what sense the equations (6.13) are useful perturbations of Bessel's equation. The answer turns out to depend on both the phase of $\alpha = \sqrt{i(\zeta - \zeta_\infty)}$ and the relative magnitude of

α and h.[18] Here ζ lies in a small neighborhood of ζ_∞ in $\Re\zeta \geq 0$. Let $\beta = \alpha/h$. For $K > 0$ sufficiently large and a fixed small $\delta > 0$ we distinguish the following three regimes, which exhaust the relevant α:

I: $|\beta| \geq K$, $0 \leq \arg\beta \leq \frac{\pi}{2} - \delta$, where $\delta > \varepsilon_1$ (for ε_1 as in Definition 6.1).
II: $|\beta| \geq K$, $\frac{\pi}{2} - \delta \leq \arg\beta \leq \frac{\pi}{2}$.
III: $|\beta| \leq K$.

It will turn out that the perturbed Bessel problem (6.13) can be analyzed in regimes I, II, and III by reducing to the normal form (7.2), where m is respectively 0, 1, -1.

8.1. Regime I. To get an idea of how this works in a simple setting closely related to our perturbed problem, consider the modified Bessel's equation

$$w_{zz} + \frac{1}{z}w_z = \left(1 + \frac{\beta^2}{z^2}\right)w, \tag{8.1}$$

where first we take $\beta = \alpha/h$ as in case I and $z = t/h$ for $t \in \mathcal{W}$ (Definition 6.1). So $z \in \mathcal{Z}_h = \mathcal{W}/h$.

Setting $w = \hat{w}z^{-\frac{1}{2}}$ to eliminate the first derivative, we obtain

$$\hat{w}_{zz} = \left(1 + \frac{\beta^2}{z^2}\right)\hat{w} - \frac{1}{4z^2}\hat{w} \text{ on } \mathcal{Z}_h. \tag{8.2}$$

Next set $v(\sigma) = \hat{w}(\beta\sigma)$ for σ in the rotated large wedge $\mathcal{W}/h\beta = \mathcal{W}/\alpha := \mathcal{Z}_\alpha$ to obtain

$$v_{\sigma\sigma} = \beta^2\left(1 + \frac{1}{\sigma^2}\right)v - \frac{1}{4\sigma^2}v \text{ on } \mathcal{Z}_\alpha, \tag{8.3}$$

which is a problem of the form (7.1) with

$$u = \beta,\ f(\sigma) = 1 + \frac{1}{\sigma^2},\ g(\sigma) = -\frac{1}{4\sigma^2}. \tag{8.4}$$

Note that the condition $\delta > \varepsilon_1$ in the definition of regime I implies that the points $\sigma = \pm i$, where $f(\sigma) = 0$, do not lie in $\mathcal{Z}_\alpha$ for β in regime I. As shown in [O, Chapter 10], the transformations

$$\xi = \int f^{1/2}(\sigma)d\sigma,\ \ v = f^{-1/4}(\sigma)W \tag{8.5}$$

change (8.3) into a problem satisfied by $W(\xi)$ of the normal form (7.2) with $m = 0$ and

$$\psi(\xi) = \frac{g(\sigma)}{f(\sigma)} - \frac{1}{f^{3/4}(\sigma)}\frac{d^2}{d\sigma^2}\left(\frac{1}{f^{1/4}(\sigma)}\right). \tag{8.6}$$

Remark 8.1. The problem (8.3) has a regular singularity at 0 and an irregular singularity "at ∞," but no turning points (which are points where $f(\sigma) = 0$) in $\mathcal{Z}_\alpha$. The wedge $\mathcal{Z}_\alpha$ is bounded for fixed α, but since α can be $O(h)$ for some β in regime I,

[18]The square root is positive when its argument is positive.

and since we are interested in uniform estimates as $h \to 0$, the domain $\mathcal{Z}_\alpha$ can become unbounded as $h \to 0$. Thus, we effectively have a singularity at infinity.

In our application to Erpenbeck's stability problem we study (6.1) in the original x variables on the infinite strip $T_{M,R}$ (6.5), and we need to know how the solution that decays at $x = \infty$, which corresponds to $\sigma = 0$, behaves at $x = M$, which corresponds to $\sigma = e^{-CM}/h$, for some $C > 0$. Obtaining an explicit formula for the exact decaying solution at $x = M$ is the main step before extending the solution to $x = 0$, where the stability function can be assessed. A great advantage of the method presented in Chapter 10 of [O] is that it produces an asymptotic representation of the exact solution at once on the entire (large) domain $\mathcal{Z}_\alpha$. If instead one tried, say, to use the theory of regular singularities to construct the decaying solution near $\sigma = 0$, and another method to construct a solution near infinity (i.e., for $\sigma = O(1/h)$), there would remain the difficult problem of matching up the two expansions somewhere in between.

8.2. Regime II. Next consider (8.1) again, but with large β with argument close to $\pi/2$. So $\beta = i\gamma$, where arg γ is close to 0. Rewriting (8.2) with $\beta^2 = -\gamma^2$ and setting $v(\sigma) = \hat{w}(\gamma\sigma)$ now for $\sigma \in \mathcal{W}/(-i\alpha) =: \mathcal{Z}_{-i\alpha}$, we obtain instead of (8.3)

$$v_{\sigma\sigma} = \gamma^2 \left(1 - \frac{1}{\sigma^2}\right) v - \frac{1}{4\sigma^2} v \text{ on } \mathcal{Z}_{-i\alpha}. \tag{8.7}$$

This problem has singularities at zero and infinity as before, but now there is a turning point, namely, $\sigma = 1$, in the interior of $\mathcal{Z}_{-i\alpha}$, since $\arg(-i\alpha)$ is near 0. Instead of having turning points converging to $z = 0$, or running off to infinity in the original x variables, the device of considering $v(\sigma) = \hat{w}(\gamma\sigma)$ yields a problem with a single fixed turning point and large parameter $u = \gamma$. Using the new variables ξ and W defined by

$$\left(\frac{d\xi}{d\sigma}\right)^2 = \frac{\sigma^2 - 1}{\xi\sigma^2} = \frac{f}{\xi} := \hat{f}, \quad v = \left(\frac{d\xi}{d\sigma}\right)^{-1/2} W \tag{8.8}$$

transforms (8.7) into the normal form (7.2) with $m = 1$ and

$$\psi(\xi) = \frac{g(\sigma)}{\hat{f}(\sigma)} - \frac{1}{\hat{f}^{3/4}(\sigma)} \frac{d^2}{d\sigma^2} \left(\frac{1}{\hat{f}^{1/4}(\sigma)}\right), \text{ where } g(\sigma) = -\frac{1}{4\sigma^2}. \tag{8.9}$$

The method of Chapter 11 of [O] yields an expansion of the exact solution of (8.7) valid on $\mathcal{Z}_{-i\alpha}$, a large wedge (growing as $h \to 0$) with vertex at $\sigma = 0$ and turning point $\sigma = 1$ in its interior.

8.3. Regime III. Now we consider (8.1) for $|\beta| \le K$, which includes the case corresponding to the "turning point at infinity," $\beta = 0$. Here it is best to work in the t variables on the bounded (h-independent) wedge $\mathcal{W}$. The equation in these variables is

$$w_{tt} + \frac{1}{t} w_t = \frac{1}{h^2} \left(1 + \frac{\alpha^2}{t^2}\right) w. \tag{8.10}$$

Setting $w = vt^{-1/2}$ we obtain

$$v_{tt} = \frac{1}{h^2} \left(1 + \frac{\alpha^2}{t^2}\right) v - \frac{1}{4t^2} v = \frac{1}{h^2}(v) + \left(\frac{\beta^2 - \frac{1}{4}}{t^2}\right) v \text{ on } \mathcal{W}, \tag{8.11}$$

where we have used the fact that β^2 is now comparable in size to 1/4 to group these terms together.

One might regard (8.11) as a problem that is already in the normal form (7.2) with $m = 0$, $u = \frac{1}{h}$, and $\psi = \frac{\beta^2 - \frac{1}{4}}{t^2}$ and try to apply the method of Chapter 10 of [O]. This does not work; the integrals of $|\psi|$ on paths starting at 0 need to be finite in order to solve the integral equation arising in the error estimates, but such integrals blow up. One can see from (8.6) that f must have a singularity at $t = 0$ to balance that of g at $t = 0$ in order for such integrals to be finite. Instead, one might regard (8.11) as a problem of the form (7.1) with $u = \frac{1}{h}$, $f(t) = 1 + \frac{\alpha^2}{t^2}$, and $g(t) = -\frac{1}{4t^2}$ and use transformations like (8.5) to reduce (8.11) to the normal form (7.2) with $m = 0$ and a different ψ. This also fails; the function ψ now depends on $\alpha = O(h)$, and though the integrals described above are now finite for fixed h, they blow up as $h \to 0$.

Instead we proceed as follows. Setting $t = 2s^{1/2}$ and $\hat{v}(s) = s^{1/4}v(2s^{1/2})$, we obtain

$$\hat{v}_{ss} = \left(\frac{1}{h^2 s} + \frac{\beta^2 - 1}{4s^2}\right)\hat{v} \text{ on } \frac{\mathcal{W}^2}{4}. \tag{8.12}$$

This problem already has the form of the "elementary equation" corresponding to the case $m = -1$ of section 7 and has solutions that can be expressed in terms of modified Bessel functions. There is no singularity at ∞ now, since $\mathcal{W}^2$, which is bounded, is independent of h; turning points are absent as well from (8.12).[19]

When we consider the perturbed Bessel equation in this frequency regime, we will obtain an equation like (8.12) with the same $g(s)$, but with $\frac{1}{h^2 s}$ replaced by $\frac{1}{h^2}f$, where $f = \frac{1}{s} + f_p(s)$, with f_p the perturbation given in (9.6).

9. Transformation of the perturbed Bessel's equation. Next we describe how transformations like those described above can be applied to the perturbed equations given in (6.13). Recalling the definition of $\tilde{\beta} = \tilde{\beta}(\zeta, h)$ from (6.12) and the formula for $D(\infty, \zeta)$ (6.3), we see that corresponding to ζ in the each of the parameter regimes of section 8, we have, respectively,

I: $|\tilde{\beta}| \geq K_1$, $-\delta_1 \leq \arg\tilde{\beta} \leq \frac{\pi}{2} - \delta_1$, where $\delta_1 > \varepsilon_1$ for ε_1 as in Definition 6.1,[20]
II: $|\tilde{\beta}| \geq K_1$, $\frac{\pi}{2} - \delta_2 \leq \arg\tilde{\beta} \leq \frac{\pi}{2}$,
III: $|\tilde{\beta}| \leq K_2$.

Here $0 < K_1 < K_2$, $\delta_j > 0$ is small, and K_1 can be made arbitrarily large by taking K in section 8 large.

9.1. Regime I. Applying the same transformations as in section (8.1) to the perturbed equation (6.13)(b), but with $\tilde{\beta}$ now playing the role of β, we obtain instead of (8.3) the equation

$$v_{\sigma\sigma} = \left(\tilde{\beta}^2 f(\sigma) + g(\sigma)\right) v \text{ on } \mathcal{W}/\tilde{\alpha} := \mathcal{Z}_{\tilde{\alpha}}, \tag{9.1}$$

[19] Observe, though, that turning points are present in the first equation of (8.11) for $\arg\alpha = \frac{\pi}{2}$. They converge to zero as $h \to 0$ since $\alpha = O(h)$.

[20] Since (6.4) holds, we see that after shrinking $\omega \ni \zeta_\infty$ if necessary, we can describe regime I here using a $\delta_1 > \varepsilon_1$ provided $\delta > \varepsilon_1$ for δ as in the definition of regime I in section 8.

where $g(\sigma) = -\frac{1}{4\sigma^2}$ as before and

$$f(\sigma) = f_0(\sigma) + f_p(\sigma), \text{ where}$$
$$f_0(\sigma) = 1 + \frac{1}{\sigma^2} \text{ and } f_p(\sigma) = (\tilde{\alpha}^2\sigma^2 + \alpha^2)b_1(\tilde{\alpha}\sigma, \zeta) + \tilde{\alpha}\sigma b_2(\tilde{\alpha}\sigma, \zeta) + hb_3(\tilde{\alpha}\sigma, \zeta, h). \tag{9.2}$$

Remark 9.1. It will be important later to take the perturbation $f_p(\sigma)$ sufficiently small on the relevant domain (e.g., $\mathcal{Z}_{\tilde{\alpha}}$ or $\mathcal{Z}_{-i\tilde{\alpha}}$). This will be the case provided $\tilde{\alpha}\sigma$, α, and h are small enough. One makes α small by restricting ζ to a sufficiently small neighborhood $\omega \ni \zeta_\infty$. Since $\tilde{\alpha}\sigma \in \mathcal{W}$, $\tilde{\alpha}\sigma$ is small when $\mathcal{W}$, a wedge with vertex at 0 defined in Definition 6.1, is small; more precisely, $|\tilde{\alpha}\sigma| \le \varepsilon_2$ for ε_2 as in Definition 6.1. When ε_2 is reduced, M must be increased so that $\mathcal{W}$ still contains the image of $[M, \infty)$ under the map $t = t(x, \zeta)$ for all $\zeta \in \omega$.

9.2. Regime II. Writing $\tilde{\beta}^2 = -\tilde{\gamma}^2$, where $\arg\tilde{\gamma}$ is close to zero, and applying the same transformations as in section 8.2 to the perturbed equation (6.13)(b), but with $\tilde{\gamma}$ now playing the role of γ, we obtain instead of (8.7) the equation

$$v_{\sigma\sigma} = (\tilde{\gamma}^2 f(\sigma) + g(\sigma))v \text{ on } \mathcal{W}/(-i\tilde{\alpha}) := \mathcal{Z}_{-i\tilde{\alpha}}, \tag{9.3}$$

where $g(\sigma) = -\frac{1}{4\sigma^2}$ as before and (since $h\tilde{\gamma} = -i\tilde{\alpha}$)

$$f(\sigma) = f_0(\sigma) + f_p(\sigma), \text{ where}$$
$$f_0(\sigma) = 1 - \frac{1}{\sigma^2} \text{ and } f_p(\sigma) \tag{9.4}$$
$$= \left(\alpha^2 - \tilde{\alpha}^2\sigma^2\right) b_1(-i\tilde{\alpha}\sigma, \zeta) - i\tilde{\alpha}\sigma b_2(-i\tilde{\alpha}\sigma, \zeta) + hb_3(-i\tilde{\alpha}\sigma, \zeta, h).$$

Clearly the function f_p will be small under the same conditions as described in Remark 9.1.

9.3. Regime III. Starting now with the perturbed equation in the t form (6.13)(a) and making the same transformations as in section 8.3, in place of (8.12) we obtain

$$\hat{v}_{ss} = \left(\frac{1}{h^2} f(s) + g(s)\right)\hat{v} \text{ on } \mathcal{W}^2/4, \tag{9.5}$$

where $g(s) = \frac{\tilde{\beta}^2 - 1}{4s^2}$ and

$$f(s) = f_0(s) + f_p(s) \text{ with}$$
$$f_0(s) = \frac{1}{s} \text{ and} \tag{9.6}$$
$$f_p(s) = \frac{1}{s}\left[(4s + \alpha^2)b_1(2s^{1/2}, \zeta) + 2s^{1/2}b_2(2s^{1/2}, \zeta) + hb_3(2s^{1/2}, \zeta, h)\right].$$

Note that each of the perturbations f_p as in (9.2), (9.4), or (9.6) is determined once $\mathcal{W}$, ζ, and h are specified, where $\mathcal{W}$ is the wedge defined by the choice of constants ε_1, ε_2 as in Definition 6.1. Thus, we can write $f_p(\cdot) = f_p(\cdot, \varepsilon_1, \varepsilon_2, \zeta, h)$. Remark (9.1) and the estimates of b_j, $j = 1, 2, 3$, of Proposition 6.3 directly imply the following.

PROPOSITION 9.2. *Let f_p be a perturbation as above. Set $N_p = \varepsilon_2 + |\zeta - \zeta_\infty| + h$, where ε_2 appears in the definition of $\mathcal{W}$. Then given $\delta_1 > 0$ there exists $\delta_2 > 0$ such that*

$$N_p < \delta_2 \Rightarrow |f_p|_{L^\infty(\mathcal{Z}_{\tilde\alpha})} < \delta_1 \text{ for regime I},$$
$$N_p < \delta_2 \Rightarrow |f_p|_{L^\infty(\mathcal{Z}_{-i\tilde\alpha})} < \delta_1 \text{ for regime II}, \tag{9.7}$$
$$N_p < \delta_2 \Rightarrow |sf_p|_{L^\infty(\mathcal{W}^2/4)} < \delta_1 \text{ for regime III},$$

Remark 9.3. In later arguments we will reduce the perturbation f_p by reducing N_p. We do not include ε_1 (as in Definition 6.1) as one of the summands in the definition of N_p, since in that case shrinking N_p could produce a wedge $\mathcal{W}$ that no longer contains the image of $[M, \infty)$ under the map $t = t(x, \zeta)$, even for large M. Although there are restrictions on the size of ε_1 (for example, in the definition of regime I), Proposition 9.2 implies that for almost all purposes it suffices to shrink N_p as defined above.

10. Leading term expansions. In this section we describe the form of the leading term expansions for exact solutions to the perturbed problem in each of the frequency regimes described in section 9. In each case the ξ variable is defined as in (7.3) for appropriately chosen lower limits σ_0, where f is given by (9.2), (9.4), or (9.6). In each case one achieves the normal form (7.2) by defining $W(\xi)$ and $\psi(\xi)$ as described in section 7.

The main things to check are that "progressive paths" of integration can be chosen as required by the contraction arguments and that the integrals involving $\psi(\xi)$ that arise in the error estimate converge at the singularity at zero and (in the cases of regimes I and II) at the singularity at infinity. These points are explained in the following discussion and in the proofs.

10.1. Regime I. Recall the definitions of the variables

$$t = \frac{2}{\mu}\sqrt{aD(\infty,\zeta)}e^{-\mu x/2},\ z = \frac{t}{h},\ \tilde\alpha = \frac{2}{\mu}\alpha\sqrt{D(\infty,\zeta)},\ \tilde\beta = \frac{\tilde\alpha}{h},\ \sigma = \frac{z}{\tilde\beta} \in \mathcal{W}/\tilde\alpha = \mathcal{Z}_{\tilde\alpha}. \tag{10.1}$$

When $f_p(\sigma)$ in (9.2) is neglected, the integral defining $\xi(\sigma)$ is easily evaluated by trigonometric substitution and yields

$$\xi = \int_{\sigma_0}^{\sigma} \frac{(1+s^2)^{1/2}}{s} ds = (1+\sigma^2)^{1/2} + \log \frac{\sigma}{1 + (1+\sigma^2)^{1/2}}, \tag{10.2}$$

where the branches of square root and logarithm are the principal ones. Here σ_0 is the point on the positive real axis at which the right side of (10.2) vanishes. Using, for example, the fact that for small $|\sigma|$, $\xi = \log(\frac{1}{2}\sigma) + 1 + o(1)$, while for large $|\sigma|$, $\xi = \sigma + o(1)$, it is not hard to draw a picture of the domain in the ξ−plane, $\mathcal{Z}_\xi$, that corresponds to the large wedge $\mathcal{Z}_{\tilde\alpha}$ under the map (10.2) (Figure 7.2, in [O, Chapter 10]). Progressive paths are of two types: those along which $\Re(\tilde\beta\xi)$ is nondecreasing, and those along which $\Re(\tilde\beta\xi)$ is nonincreasing. The choice of such paths is obvious in the ξ-plane and using

$$\psi(\xi) = \frac{1}{4}\sigma^2(4-\sigma^2)/(1+\sigma^2)^3, \tag{10.3}$$

one sees that in this case (f_p neglected) the integrals

$$\int_{\xi(\sigma_j)}^{\xi} |\psi(s)|\, d|s|, \text{ where } \sigma_1 = 0,\ \sigma_2 = \infty, \tag{10.4}$$

are finite along such paths.[21]

When f_p is included in the definition of f, the domain in the ξ-plane corresponding to $\mathcal{Z}_{\tilde{\alpha}}$ under the map $\xi = \int_{\sigma_0}^{\sigma} f^{1/2}(s)ds$ is a small perturbation of $\mathcal{Z}_\xi$ when f_p is small, and progressive paths are again easy to choose on an appropriate subdomain. Using the estimates of Proposition 6.3 for the functions $b_j,\ j = 1, 2, 3$, appearing in the definition of f_p, one can show that integrals (10.4) involving the redefined $\psi(\xi)$ are again finite.

To get started it is necessary to show that the map $\sigma \to \xi$ defines a good, global change of variables.

Proposition 10.1. *For f_0 and f_p as in* (9.2) *let*

$$\xi_f(\sigma) := \int_{\sigma_0}^{\sigma} (f_0 + f_p)^{1/2}(s)ds, \tag{10.5}$$

where σ_0 is (as before) the point on the positive real axis where the right side of (10.2) *vanishes. For perturbations f_p with N_p sufficiently small (recall Proposition* 9.2*) the function $\xi = \xi_f(\sigma)$, is a globally one-to-one analytic map of $\mathcal{Z}_{\tilde{\alpha}}$ onto the open set which is its range.*

In the next proposition $\mathcal{Z}_{\tilde{\alpha},s}$ is an open subdomain of $\mathcal{Z}_{\tilde{\alpha}}$ containing the image of the segment of the x-axis, $[M, \infty)$, under the map $x \to \sigma$. The domain $\mathcal{Z}_{\tilde{\alpha},s}$ is defined as $\xi_f^{-1}(\Delta_\xi)$, where Δ_ξ (described precisely in the proof given in section 16) is an open domain in ξ-space on which progressive paths can be chosen.

Proposition 10.2. *Suppose $\tilde{\beta}$ as defined in* (6.12) *lies in regime* I. *For f_p as in* (9.2) *taken sufficiently small (by the choices explained in Remark 9.1), the perturbed Bessel problem* (9.1) *has exact solutions*

$$\begin{aligned} v_1(\sigma) &= \xi_\sigma^{-1/2}(\sigma)\left(e^{\tilde{\beta}\xi(\sigma)} + \eta_1(\tilde{\beta}, \xi(\sigma))\right), \\ v_2(\sigma) &= \xi_\sigma^{-1/2}(\sigma)\left(e^{-\tilde{\beta}\xi(\sigma)} + \eta_2(\tilde{\beta}, \xi(\sigma))\right) \end{aligned} \tag{10.6}$$

on $\mathcal{Z}_{\tilde{\alpha},s}$, where the error terms satisfy

$$\left|\eta_j(\tilde{\beta}, \xi)\right|, \left|\partial_\xi \eta_j(\tilde{\beta}, \xi)\right| \le \frac{C}{|\tilde{\beta}|}\left|e^{(-1)^{j-1}\tilde{\beta}\xi}\right|. \tag{10.7}$$

Remark 10.3. The proof, given in section 16, is based on Theorem 3.1 of Chapter 10 of [O] and the estimates of Proposition 6.3. The result of [O] constructs solutions

$$W_j(\xi) = e^{(-1)^{j-1}\tilde{\beta}\xi} + \eta_j(\tilde{\beta}, \xi),\ j = 1, 2, \tag{10.8}$$

of

$$W_{\xi\xi} = (\tilde{\beta}^2 + \psi(\xi))W, \quad \text{where } \psi(\xi) = \frac{g(\sigma)}{f(\sigma)} - \frac{1}{f^{3/4}(\sigma)}\frac{d^2}{d\sigma^2}\left(\frac{1}{f^{1/4}(\sigma)}\right), \tag{10.9}$$

[21] Here "∞" should be interpreted as a point at the far right extreme of the large wedge $\mathcal{Z}_{\tilde{\alpha}}$.

by solving the integral equation satisfied by η_j (obtained by variation of parameters). When $j=1$ the equation is

$$\eta_1(\tilde\beta,\xi)=\int_{\alpha_j}^{\xi}K(\xi,v)\left[\frac{\psi(v)e^{\tilde\beta v}}{\tilde\beta}+\frac{\psi(v)\eta_1(\tilde\beta,v)}{\tilde\beta}\right]dv,\ \text{where}$$
$$K(\xi,v)=\frac{1}{2}\left(e^{\tilde\beta(\xi-v)}-e^{\tilde\beta(v-\xi)}\right), \tag{10.10}$$

and the integral is taken on progressive paths. This equation is solved on the domain $\Delta_\xi=\xi(\mathcal{Z}_{\tilde\alpha,s})$ by iteration. The progressive path property of Δ_ξ gives a useful pointwise estimate of $|K(\xi,v)|$. Together with a uniform bound on the integrals (10.4), this yields convergence of the sequence of iterates.[22] Convergence of the sequence of differentiated iterates follows from a similar pointwise estimate of $|\partial_\xi K(\xi,v)|$ and the property $K(\xi,\xi)=0$.

Consider the first-order system corresponding to (6.1):

$$h\frac{d}{dx}\begin{pmatrix}w\\ hw_x\end{pmatrix}=\begin{pmatrix}0 & 1\\ C(x,\zeta)+hr(x,\zeta,h) & 0\end{pmatrix}\begin{pmatrix}w\\ hw_x\end{pmatrix}. \tag{10.11}$$

The next proposition describes the solutions of (10.11) that are bounded for $\Re\zeta=0$ and decaying for $\Re\zeta>0$ as $x\to\infty$ in $[M,\infty)$, when $\tilde\beta(\zeta,h)$ lies in regime I. Recall that Proposition 10.2 is valid for a small enough choice of wedge $\mathcal{W}$, neighborhood $\omega\ni\zeta_\infty$, and h_0 such that $0<h\le h_0$.

PROPOSITION 10.4 (choice of decaying solution). *Choose M large enough so that $\mathcal{W}$ contains the image of $[M,\infty)$ under the map $t=t(x,\zeta)$ for all $\zeta\in\omega$. After shrinking ω and reducing h_0 if necessary, we have, for $\zeta\in\omega$ and $0<h<h_0$ such that $\tilde\beta(\zeta,h)$ lies in regime* I*, that the bounded (resp., decaying) solution of* (10.11) *on $[M,\infty)$ for $\Re\zeta=0$ (resp., $\Re\zeta>0$) is given by*

$$w(x)=z(x)^{-1/2}v_1(\sigma(x)). \tag{10.12}$$

Here v_1 is defined in (10.6) *and the maps $x\to z(x)$ and $x\to\sigma(x)$ are defined by* (10.1)*. The corresponding decaying solution of Erpenbeck's 5×5 system* (5.1) *is thus given by the formula in* (5.33) *for this choice of $w(x)$.*

Having identified the exact decaying solution of Erpenbeck's system for $\tilde\beta$ in regime I, the next step is to show that this solution is of type θ_1 at $x=M$ (recall (0.16) and Definition 0.4). Since $x=M$ is to the left of any turning point $x(\zeta)$ for $\zeta\in\omega$ when ω is small, it will then be rather easy to conclude that the exact decaying solution is of type θ_1 at $x=0$. This will allow us to deduce that the stability function $V(\zeta,h)$ is nonvanishing for $\tilde\beta$ in regime I.

PROPOSITION 10.5 (decaying solution is of type θ_1 at $x=M$). *Fix M as in Proposition* 10.4*. Let $\theta(x,\zeta,h)$ be the exact decaying solution of* (5.1) *identified in Proposition* 10.4*, and let $\theta_1(x,\zeta,h)$ be the approximate solution given by* (0.16)*. There exist $h_0>0$, a neighborhood $\omega\ni\zeta_\infty$, and a nonvanishing scalar function $H(x,\zeta,h)$ defined for x near M such that*

$$|H(x,\zeta,h)\theta(x,\zeta,h)-\theta_1(x,\zeta,h)|\le\frac{C}{|\tilde\beta(\zeta,h)|}|\theta_1(x,\zeta,h)|\ \text{for } x \text{ near } M, \tag{10.13}$$

where C is independent of x near M and of $\zeta\in\omega$, $0<h\le h_0$, such that $\tilde\beta$ lies in regime I.[23]

[22] It is not necessary to take $|\tilde\beta|$ large to obtain convergence. See Theorem 10.1 of Chapter 6 in [O].
[23] Recall that for $\tilde\beta(\zeta,h)$ in regime I we have $h<\frac{1}{|\tilde\beta|}\le\frac{1}{K_1}<<1$.

The final step in the treatment of regime I is to show that the exact decaying solution of (5.1) is of type θ_1 at $x = 0$. In fact, we show next that a multiple of θ is of type θ_1 on all of $[0, M]$. The explicit formula (21.1) for the stability function $V(\zeta, h)$ in terms of $\theta(0, \zeta, h)$ shows then that $V(\zeta, h)$ is nonvanishing for (ζ, h) in regime I when Assumption 1.4 holds.

PROPOSITION 10.6 (Decaying solution is of type θ_1 at $x = 0$). *Let $\theta(x, \zeta, h)$ be the exact decaying solution of* (5.1) *identified in Proposition* 10.4, *and let $H(x, \zeta, h)$ be the function referred to in Proposition* 10.5. *There exist $h_0 > 0$ and a neighborhood $\omega \ni \zeta_\infty$ such that*

$$|H(M, \zeta, h)\theta(x, \zeta, h) - \theta_1(x, \zeta, h)| \le \frac{C}{|\tilde{\beta}(\zeta, h)|}|\theta_1(x, \zeta, h)| \text{ on } [0, M], \tag{10.14}$$

where C is independent of $x \in [0, M]$ and of $\zeta \in \omega$, $0 < h \le h_0$ such that $\tilde{\beta}$ lies in regime I.

Proof. Let $\overline{\theta}_j(x, \zeta, h)$, $j = 1, \dots, 5$, be exact solutions of (5.1) on $[0, M]$ (constructed as in [LWZ1, Theorem 3.1], for example) such that

$$|\overline{\theta}_j - \theta_j| \le Ch|\theta_j| \text{ on } [0, M], \tag{10.15}$$

where the approximate solutions θ_j are defined on $[0, M]$, an interval with no turning points. The formulas (0.7) for the $\mu_j(x, \zeta)$ and the fact that $0 < \kappa(x) < 1$ imply that there exists a neighborhood $\omega \ni \zeta_\infty$ such that for $x \in [0, M]$ and $\zeta \in \omega$, we have

$$\Re\mu_1 < 0,\ \Re\mu_2 > 0,\ \Re\mu_j \ge 0,\ j = 3, 4, 5, \tag{10.16}$$

and thus for $h_j(x, \zeta) = \int_0^x \mu_j(x', \zeta)dx'$ we have

$$-\Re h_1(M, \zeta) := a > 0,\ \Re h_2(M, \zeta) := b > 0,\ \Re h_j(M, \zeta) = c \ge 0,\ j = 3, 4, 5. \tag{10.17}$$

Expanding the exact solution solution $H(M, \zeta, h)\theta(x, \zeta, M)$ in the given basis,

$$H(M, \zeta, h)\theta(x, \zeta, h) = c_1(\zeta, h)\overline{\theta}_1(x, \zeta, h) + \cdots + c_5(\zeta, h)\overline{\theta}_5 \text{ on } [0, M], \tag{10.18}$$

evaluating at $x = M$, and then using (10.30), (10.15), and Cramer's rule, we obtain

$$\begin{gathered} c_1(\zeta, h) = 1 + O(1/|\tilde{\beta}(\zeta, h)|),\ c_2 = O\left(e^{-\frac{a+b}{h}}/|\tilde{\beta}(\zeta, h)|\right), \\ c_j = O\left(e^{-\frac{a+c}{h}}/|\tilde{\beta}(\zeta, h)|\right),\ j = 3, 4, 5. \end{gathered} \tag{10.19}$$

In view of (10.15) the behavior of the $\overline{\theta}_j$ on $[0, M]$ is given by the explicit formulas for the θ_j. Thus, it follows from these formulas and (10.19) that the $\overline{\theta}_1$ term dominates on $[0, M]$ or, more precisely, that (10.14) holds. ∎

10.2. Regime II. Recall the definitions of the variables

$$t = \frac{2}{\mu}\sqrt{aD(\infty, \zeta)}e^{-\mu x/2},\ z = \frac{t}{h},\ \tilde{\gamma} = -i\tilde{\beta},\ \sigma = \frac{z}{\tilde{\gamma}} \in \mathcal{Z}_{-i\tilde{\alpha}}. \tag{10.20}$$

With $[0, 1]$ denoting the line segment joining 0 to 1, we define $\mathcal{Z}_{cut}(1)$ to be the simply connected subregion of $\Re\sigma > 0$ given by

$$\mathcal{Z}_{cut}(1) = \mathcal{Z}_{-i\tilde{\alpha}} \setminus [0, 1]. \tag{10.21}$$

Set $\Xi := \frac{2}{3}\xi^{3/2}$. When $f_p(\sigma)$ in (9.4) is neglected, we define $\Xi(\sigma)$ by

$$\Xi(\sigma) = \int_1^\sigma \frac{(s^2-1)^{1/2}}{s} ds = (\sigma^2-1)^{1/2} + i\log\left(\frac{1+i(\sigma^2-1)^{1/2}}{\sigma}\right), \tag{10.22}$$

where the branch of $(\sigma^2-1)^{1/2}$ on $\mathcal{Z}_{cut}(1)$ is positive for $\sigma > 1$, and the branch of $\log(\frac{1+i(\sigma^2-1)^{1/2}}{\sigma})$ takes negative (resp., positive) values in the limit as $\sigma \to a_\pm$, $0 < a < 1$, from the upper (resp., lower) half-plane.[24] The definition of $\Xi(\sigma)$ is extended by continuity to $a_\pm$ for $0 < a < 1$. Observe that

$$\Xi(a_\pm) = \mp ib, \text{ for some } b = b(a) > 0, \text{ for } 0 < a < 1, \tag{10.23}$$

and that $\mp ib(a)$ are mapped to the same point on the negative ξ axis under the map $\Xi \to \xi$. Morever, the map $\sigma \to \xi(\sigma)$ turns out to extend analytically to a map of a full neighborhood of $\sigma = 1$ onto a full neighborhood of $\xi = 0$.

In this case (f_p neglected) one can draw the domains in the Ξ and ξ planes corresponding to $\mathcal{Z}_{cut}(1)$ under (10.22).[25] Progressive paths in the ξ plane now have the property that on corresponding paths in the Ξ plane $\Re(\tilde\gamma\Xi)$ is monotonic, and it is easy to identify such paths in the Ξ-plane.

When f_p is included in the definition of f, Ξ and ξ are now defined by (7.3)(b) with f as in (9.4), where σ_0 is the point (close to 1) where $f(\sigma_0) = 0$. In the definition of Ξ we now take $\sigma \in \mathcal{Z}_{cut}(\sigma_0)$ defined by

$$\mathcal{Z}_{cut}(\sigma_0) = \mathcal{Z}_{-i\tilde\alpha} \setminus [0, \sigma_0], \tag{10.24}$$

where $[0, \sigma_0]$ is the line segment joining 0 to σ_0. We show below that, unlike $\Xi(\sigma)$, the function $\xi(\sigma)$ extends across the cut to be analytic on all of $\mathcal{Z}_{-i\tilde\alpha}$.

The domain in ξ space corresponding to $\mathcal{Z}_{-i\tilde\alpha}$ under the map $\sigma \to \xi$ is a small perturbation, when f_p is small, of the domain in the case $f_p = 0$, and progressive paths are again not hard to choose. Using the estimates of Proposition 6.3 for the functions b_j, $j = 1, 2, 3$, appearing in the definition of f_p, one can show that integrals arising in the error analysis,

$$\int_{\alpha_j}^\xi \left|\psi(s)s^{-1/2}\right| d|s| \text{ (on progressive paths)}, \tag{10.25}$$

are finite. Here $\psi(\xi)$ is given by (8.9) with f as in (9.4) and $g = -\frac{1}{4\sigma^2}$.

The next proposition, proved in section 16, is more difficult for regime II than its analogue for regime I, since $f = f_0 + f_p$ vanishes at $\sigma_0 \in \mathcal{Z}_{-i\tilde\alpha}$.

Proposition 10.7. *For perturbations f_p with N_p sufficiently small (recall Proposition 9.2) the function $\xi = \xi_f(\sigma)$, which is initially defined on $\mathcal{Z}_{cut}(\sigma_0)$, extends across the cut as a globally one-to-one analytic map of $\mathcal{Z}_{-i\tilde\alpha}$ onto the open set which is its range.*

In the next proposition we use the notation

$$Ai_0(z) = Ai(z),\ Ai_1(z) = Ai(ze^{-2\pi i/3}),\ Ai_{-1}(z) = Ai(ze^{2\pi i/3}). \tag{10.26}$$

[24] This branch takes values ib, $0 < b < \frac{\pi}{2}$, for $\sigma > 1$.

[25] Drawings for the rather different case where $f(\sigma) = \frac{1}{\sigma^2} - 1$ are given in Figures 10.1–10.4 of [O, Chapter 10].

We denote by $\mathcal{Z}_{-i\tilde{\alpha},s} \subset \mathcal{Z}_{-i\tilde{\alpha}}$ an open subdomain, chosen as explained in the proof, and containing the image of the segment of the x-axis, $[M,\infty)$, under the map $x \to \sigma$. We denote by Δ_ξ the image of $\mathcal{Z}_{-i\tilde{\alpha},s}$ under the map $\sigma \to \xi(\sigma)$.

PROPOSITION 10.8. *Suppose $\tilde{\beta}$ as defined in* (6.12) *lies in regime* II*, and set $\tilde{\beta}^2 = -\tilde{\gamma}^2$, where $\arg\tilde{\gamma}$ is near* 0*. For f_p as in* (9.4) *taken sufficiently small (by the choices explained in Remark* (9.1)*), the perturbed Bessel problem* (9.3) *has exact solutions*

$$v_j(\sigma) = \xi_\sigma^{-1/2}(\sigma)\left(Ai_j(\tilde{\gamma}^{2/3}\xi(\sigma)) + \eta_j(\tilde{\gamma},\xi(\sigma))\right),\ j = 0, 1, -1, \tag{10.27}$$

on $\mathcal{Z}_{-i\tilde{\alpha},s}$, where the error term η_j satisfies

$$\begin{aligned} &|\eta_j(\tilde{\gamma},\xi)| \le \frac{C}{|\tilde{\gamma}|}|Ai_j(\tilde{\gamma}^{2/3}\xi)|,\\ &|\partial_\xi\eta_j(\tilde{\gamma},\xi)| \le \frac{C}{|\tilde{\gamma}|}\left|\partial_\xi\left(Ai_j(\tilde{\gamma}^{2/3}\xi)\right)\right| \end{aligned} \tag{10.28}$$

for $\xi \in \Delta_\xi$ with $|\xi| >> 1$ and $\Re\xi > 0$.

Remark 10.9. Information about the error terms η_j for ξ near negative infinity is more complicated to state but is implicit in Proposition 10.10 in the case of η_1. For explicit estimates of the η_j we refer to Theorem 9.1 of Chapter 11 of [O].

The next three propositions are analogues of the last three propositions in section 10.1.

PROPOSITION 10.10 (choice of decaying solution). *After shrinking ω and reducing h_0 if necessary, we have, for $\zeta \in \omega$ and $0 < h < h_0$ such that $\tilde{\beta}(\zeta,h)$ lies in regime* II*, that the bounded (resp., decaying) solution of* (10.11) *on $[M,\infty)$ for $\Re\zeta = 0$ (resp., $\Re\zeta > 0$) is given by*

$$w(x) = z(x)^{-1/2}v_1(\sigma(x)). \tag{10.29}$$

Here v_1 is defined in (10.27) *and the maps $x \to z(x)$ and $x \to \sigma(x)$ are defined by* (10.20)*. The corresponding decaying solution of Erpenbeck's 5×5 system* (5.1) *is thus given by the formula in* (5.33)*.*

PROPOSITION 10.11 (decaying solution is of type θ_1 at $x = M$). *Let $\theta(x,\zeta,h)$ be the exact decaying solution of* (5.1) *identified in Proposition* 10.10*, and let $\theta_1(x,\zeta,h)$ be the approximate solution defined in* (0.16)*. There exist $h_0 > 0$, a neighborhood $\omega \ni \zeta_\infty$, and a nonvanishing scalar function $H(x,\zeta,h)$ defined for x near M such that*

$$|H(x,\zeta,h)\theta(x,\zeta,h) - \theta_1(x,\zeta,h)| \le \frac{C}{|\tilde{\beta}(\zeta,h)|}|\theta_1(x,\zeta,h)| \text{ for } x \text{ near } M, \tag{10.30}$$

where C is independent of x near M and of $\zeta \in \omega$, $0 < h \le h_0$ such that $\tilde{\beta}$ lies in regime II*.*

PROPOSITION 10.12 (decaying solution is of type θ_1 at $x = 0$). *Let $\theta(x,\zeta,h)$ be the exact decaying solution of* (5.1) *identified in Proposition* 10.10*, and let $H(x,\zeta,h)$ be the function referred to in Proposition* 10.11*. There exist $h_0 > 0$ and a neighborhood $\omega \ni \zeta_\infty$ such that*

$$|H(M,\zeta,h)\theta(x,\zeta,h) - \theta_1(x,\zeta,h)| \le \frac{C}{|\tilde{\beta}(\zeta,h)|}|\theta_1(x,\zeta,h)| \text{ on } [0,M], \tag{10.31}$$

where C is independent of $x \in [0,M]$ and of $\zeta \in \omega$, $0 < h \le h_0$ such that $\tilde{\beta}$ lies in regime II*.*

10.3. Regime III. Recall the definitions of the variables

$$t = \frac{2}{\mu}\sqrt{aD(\infty,\zeta)}e^{-\mu x/2},\ s = t^2/4. \tag{10.32}$$

When $f_p(s)$ in (9.6) is neglected, the integral defining $\xi(s)$ is

$$\xi^{1/2} = \int_0^s \frac{1}{2t^{1/2}}dt = s^{1/2},\ \text{so } \xi = s, \tag{10.33}$$

and the relevant domain in the ξ-plane is the bounded wedge $\mathcal{W}^2/4$. Progressive paths in the ξ-plane are now either those along which both $\Re\xi^{1/2}$ and $|\xi|$ are nondecreasing or those along which both $\Re\xi^{1/2}$ and $|\xi|$ are nonincreasing. The image of $\mathcal{W}^2/4$ under the map $s \to \xi^{1/2}$ is just $\mathcal{W}/2$, and progressive paths are easy to choose in the $\xi^{1/2}$-plane.

When f_p as in (9.6) is included in the integral defining ξ

$$2\xi^{1/2} = \int_0^s f^{1/2}(t)dt, \tag{10.34}$$

the image of $\mathcal{W}^2/4$ under the map $s \to \xi^{1/2}$ is a small perturbation of $\mathcal{W}/2$ when f_p is small, and progressive paths satisfying the above conditions are again not hard to choose.

PROPOSITION 10.13. *For perturbations f_p with N_p sufficiently small (recall Proposition* 9.2*) the function $\xi = \xi_f(s)$ is a globally one-to-one analytic map of $\mathcal{W}^2/4$ onto its image.*

In the next proposition we denote by $\mathcal{W}_s \subset \mathcal{W}^2/4$ an open subdomain, chosen as explained in the proof, and containing the image of the segment of the x-axis, $[M,\infty)$ under the map $x \to s$ given by (10.32). We let Δ_ξ denote the image of $\mathcal{W}_s$ under the map $s \to \xi(s)$. With $\hat{v}(s)$ as in (9.5) and $W(\xi)$ defined by $\hat{v} = (\frac{d\xi}{ds})^{-1/2}W$, the problem satisfied by W has the form

$$W_{\xi\xi} = \left(\frac{1}{h^2\xi} + \psi(\xi)\right)W = \left(\frac{1}{h^2\xi} + \frac{\tilde{\beta}^2-1}{4\xi^2} + \frac{\phi(\xi)}{\xi}\right)W \text{ on } \mathbb{W}_\xi, \tag{10.35}$$

where (with $g(s)$ as in (9.5))[26]

$$\phi(\xi) = \frac{1-4\tilde{\beta}^2}{16\xi} + \frac{g(s)}{f(s)} + \frac{4f(s)f''(s) - 5f'^2(s)}{16f^3(s)}. \tag{10.36}$$

Using the estimates of Proposition 6.3, one checks the finiteness of the integrals required for the error analysis of Theorem 9.1 of [O, Chapter 12]:

$$\int_{\alpha_j}^{\xi} \left|\phi(t)t^{-1/2}\right| \, d|t| \text{ (on progressive paths).} \tag{10.37}$$

In this case there is no singularity at infinity, since Δ_ξ is bounded independent of h.

PROPOSITION 10.14. *Suppose $\tilde{\beta}$ as defined in* (6.12) *lies in regime* III. *For $sf_p(s)$ as in* (9.6) *taken sufficiently small (by the choices explained in Remark* (9.1)*), the perturbed Bessel problem* (9.5) *has exact solutions on $\mathcal{W}_s$ given by*

$$\hat{v}_j(s) = \xi_s^{-1/2}(s)W_j(\xi(s)),\ j = 1, 2, \tag{10.38}$$

[26]Observe that when $f(s) = 1/s$ and $\xi = s$, we have $\phi(\xi) = 0$.

where the $W_j(\xi)$ *are exact solutions of* (10.35) *of the form*

$$\text{(10.39)} \qquad \begin{aligned} &\text{(a)} W_1(\xi) = \xi^{1/2} I_{\tilde\beta}(2\xi^{1/2}/h) + \eta_1(h,\xi),\\ &\text{(b)} W_2(\xi) = \xi^{1/2} K_{\tilde\beta}(2\xi^{1/2}/h) + \eta_2(h,\xi). \end{aligned}$$

Here the error term η_1 *satisfies*

$$\text{(10.40)} \qquad \begin{aligned} &|\eta_1(h,\xi)| \le Ch|\xi^{1/2} I_{\tilde\beta}(2\xi^{1/2}/h)|,\\ &|\partial_\xi \eta_1(h,\xi)| \le Ch \left|\partial_\xi \left(\xi^{1/2} I_{\tilde\beta}(2\xi^{1/2}/h)\right)\right| \end{aligned}$$

for $\xi \in \Delta_\xi$ *with* $|\xi^{1/2}/h|$ *large. The error* η_2 *satisfies analogous estimates.*

Remark 10.15. Information about the error terms η_j for ξ near 0 is more complicated to state but is implicit in Proposition 10.16 in the case of η_1. For explicit estimates of the η_j we refer to Theorem 9.1 of Chapter 12 of [O]. That theorem deals only with real $\tilde\beta$, but we show how the result can be extended to $\Re\tilde\beta \ge 0$ in section 16.

Proposition 10.16 (choice of decaying solution). *After shrinking* ω *and reducing* h_0 *if necessary, we have, for* $\zeta \in \omega$ *and* $0 < h < h_0$ *such that* $\tilde\beta(\zeta,h)$ *lies in regime* III*, that the bounded (resp., decaying) solution of* (10.11) *on* $[M,\infty)$ *for* $\Re\zeta = 0$ *(resp.,* $\Re\zeta > 0$*) is given by*

$$\text{(10.41)} \qquad w(x) = \frac{\sqrt{2}}{t(x)} \hat v_1(s(x)),$$

where $\hat v_1$ *is defined in* (10.38) *and the maps* $x \to t(x)$ *and* $x \to s(x)$ *are defined by* (10.32)*. The corresponding decaying solution* $\theta(x,\zeta,h)$ *of Erpenbeck's* 5×5 *system* (5.1) *is thus given by the formula in* (5.33)*.*

The next step is to show that this solution is of type θ_1 at $x = M$.

Proposition 10.17 (decaying solution is of type θ_1 at $x = M$). *Let* $\theta(x,\zeta,h)$ *be the exact decaying solution of* (5.1) *identified in Proposition* 10.16*, and let* $\theta_1(x,\zeta,h)$ *be the approximate solution defined in* (0.16)*. There exist* $h_0 > 0$*, a neighborhood* $\omega \ni \zeta_\infty$*, and a nonvanishing scalar function* $H(x,\zeta,h)$ *defined for* x *near* M *such that*

$$\text{(10.42)} \qquad |H(x,\zeta,h)\theta(x,\zeta,h) - \theta_1(x,\zeta,h)| \le Ch|\theta_1(x,\zeta,h)| \text{ for } x \text{ near } M,$$

where C *is independent of* x *near* M *and of* $\zeta \in \omega$*,* $0 < h \le h_0$ *such that* $\tilde\beta$ *lies in regime* III*.*

The proof of the next proposition is exactly like that of Proposition 10.12.

Proposition 10.18 (decaying solution is of type θ_1 at $x = 0$). *Let* $\theta(x,\zeta,h)$ *be the exact decaying solution of* (5.1) *identified in Proposition* 10.16*, and let* $\theta_1(x,\zeta,h)$ *and* $H(x,\zeta,h)$ *be as in Proposition* 10.17*. There exist* $h_0 > 0$ *and a neighborhood* $\omega \ni \zeta_\infty$ *such that*

$$\text{(10.43)} \qquad |H(M,\zeta,h)\theta(x,\zeta,h) - \theta_1(x,\zeta,h)| \le Ch|\theta_1(x,\zeta,h)| \text{ on } [0,M],$$

where C *is independent of* $x \in [0,M]$ *and of* $\zeta \in \omega$*,* $0 < h \le h_0$ *such that* $\tilde\beta$ *lies in regime* III*.*

11. Multistep reactions. The treatment of the turning point at infinity in the case of a scalar reaction equation works verbatim for type D multistep reactions provided the reactant k-vector $\lambda(x)$ is analytic and has the structure

$$\lambda(x) = Ae^{-\mu x} + m(x)e^{-\mu x}, \text{ where } A \text{ is constant}, \ \mu > 0, \text{ and } |m(x)| \leq Ce^{-\mu \Re x}, \tag{11.1}$$

on a wedge $\mathbb{W}(M_0, \theta)$ for some $\theta > 0$. With (11.1) the function $e(x)$ again has the structure in (6.6) and the proof of Proposition 6.3, giving the estimates of the functions b_j, $j = 1, 2, 3$, that appear in the perturbation of Bessel's equation (6.13), goes through unchanged. We now show that the eigenvalue separation condition (2.9) implies (11.1); thus, Example 2.3 satisfies (11.1).

We write the equation satisfied by λ as

$$\frac{d}{dx}\lambda = f(\lambda) = B\lambda + N(\lambda), \text{ where } f(0) = 0 \text{ and } B = df(0), \tag{11.2}$$

and denote by Π_{ws}, Π_{ss} the projections of $\mathbb{C}^k$ onto, respectively, the weakly stable subspace corresponding to the eigenvalue $-\mu_1$ of B, and the complementary strongly stable subspace. We can suppose λ is already given as an $\mathbb{R}^k$-valued decaying solution of (11.2) on $\mathbb{R}$. For M_0 sufficiently large and θ small enough, the problem (11.2) with initial condition $\lambda|_{x=M_0} = \lambda(M_0)$ can be solved on the wedge $\mathbb{W}(M_0, \theta)$ by a classical contraction argument applied to the integral equation

$$\lambda(x) = e^{Bx}\lambda(0) + \int_0^x e^{B(x-s)} N(\lambda(s))\, ds. \tag{11.3}$$

Here by a translation we have replaced M_0 by 0. By (2.9) the weakly stable subspace is simple with eigenvalue $-\mu := -\mu_1$, so we can rearrange (11.3) as

$$\begin{aligned} \lambda(x) = e^{-\mu x} &\left(\lambda_{ws}(0) + \int_0^x e^{\mu s} \Pi_{ws} N(\lambda(s))\, ds \right) \\ &+ \left(e^{Bx}\lambda_{ss}(0) + \int_0^x e^{B(x-s)} \Pi_{ss} N(\lambda(s))\, ds \right) =: \mathrm{I} + \mathrm{II}. \end{aligned}$$

Using $|\lambda(x)| \leq C|\lambda(0)|e^{-\mu \Re x}$, $|N| \leq C_2 |\lambda(x)|^2$, and the estimate

$$\left| e^{B(x-s)} \Pi_{ss} \right| \leq Ce^{-2\tilde{\mu}(x-s)}, \text{ where } \tilde{\mu} > 2\mu, \tag{11.4}$$

which follows from the separation condition (2.9), we find that II is bounded in modulus by $C_3|\lambda(0)|^2 e^{-2\mu \Re x}$ and so can be viewed as part of the second term on the right in (11.1). Splitting I now as

$$\mathrm{I} = e^{-\mu x} \left(\lambda_{ws}(0) + \int_0^\infty e^{\mu s} \Pi_{ws} N(\lambda(s))\, ds \right) - \int_x^\infty e^{-\mu(x-s)} \Pi_{ws} N(\lambda(s))\, ds =: \mathrm{I}_1 + \mathrm{I}_2,$$

we see that $|\mathrm{I}_2| \leq C_4 \int e^{-\mu \Re (x-s)} e^{-2\mu \Re s} d(\Re s) \leq C_5 e^{-2\mu \Re x}$, and so I_2 can be treated like II above. Setting

$$A := \lambda_{ws}(0) + \int_0^\infty e^{\mu s} \Pi_{ws} N(\lambda(s))\, ds,$$

we obtain (11.1).

The treatment of frequencies $\zeta \neq \zeta_\infty$ with $\Re\zeta \geq 0$ does not require (11.1), so for such frequencies the proofs in the scalar case work for multistep reactions as long as the assumptions of section 1 hold. Thus, our main result, Theorem 2.1, holds also for for type D multistep reactions under the additional separation condition (2.9).

Part III. Finite turning point and nonturning point frequencies.

In this part we treat nonturning point frequencies as well as frequencies $\zeta \in (\text{iii})^o_+$, for each of which there exists a turning point $x(\zeta) \in (0,\infty)$. We also study the upper endpoint frequency ζ_0 for which the corresponding turning point is the endpoint $x(\zeta_0) = 0$ of the reaction zone $[0,\infty)$.

First we give a lemma that extends the map $\zeta \to x(\zeta)$ to a neighborhood of a turning point frequency.

LEMMA 11.1. *Fix a basepoint $\underline{\zeta} \in (\text{iii})^0_+$. There exist neighborhoods $\omega \ni \underline{\zeta}$ and $\mathcal{O} \ni x(\underline{\zeta})$ and an analytic homeomorphism $x : \omega \to \mathcal{O}$, where $x(\zeta)$ is defined to be the unique root of*

$$f(x,\zeta) := \zeta^2 + c_0^2\eta(x) = 0. \tag{11.5}$$

Moreover,

$$\Im x(\zeta) \geq 0 \text{ for } \Re\zeta \geq 0 \text{ and } \Im x(\zeta) = 0 \Leftrightarrow \Re\zeta = 0. \tag{11.6}$$

Proof. The profile $P(x)$ is of type D, so $\partial_x f(x(\underline{\zeta}),\underline{\zeta}) < 0$. The fact that $x(\zeta)$ is analytic thus follows from the implicit function theorem. We have

$$\partial_x f(x(\zeta),\zeta)x_\zeta(\zeta) + 2\zeta = 0, \tag{11.7}$$

so $x_\zeta(\underline{\zeta}) \neq 0$ since $\underline{\zeta} \neq 0$. Hence we have an analytic homeomorphism of some neighborhoods ω and $\mathcal{O}$. Since $\Im\underline{\zeta} > 0$, (11.7) implies $\Im x_\zeta(\underline{\zeta}) > 0$, which yields (11.6). ∎

The frequencies $\zeta \in \{\Re\zeta \geq 0\} \setminus (\text{iii}) := \mathcal{N}$, for which there are no associated turning points, are divided into two sets:

$$\mathcal{N} = (\mathcal{N} \cap \{|\zeta| \geq M\}) \cup (\mathcal{N} \cap \{|\zeta| \leq M\}) \tag{11.8}$$

for some sufficiently large M. The unbounded set is studied in section 14. The bounded set was treated in [LWZ1] using the following theorem, which we reproduce here since it is needed for the analysis of finite turning points. In this theorem the $\mu_j(x,\zeta)$, $j = 1,\dots,5$, are the eigenvalues of $\Phi_0(x,\zeta)$ given in (0.7), and $\mu > 0$ is the constant determining the rate of profile decay in (4.7).

THEOREM 11.2 (see [LWZ1, Theorem 2.1]). (1) *Consider the system* (5.1)

$$\theta' = \frac{1}{h}\left[\Phi_0(x,\zeta) + h\Phi_1(x)\right]\theta \tag{11.9}$$

on an interval $[a,\infty)$, $a \geq 0$, and for values of ζ such that

$$|\mu_1(x,\zeta) - \mu_j(x,\zeta)| \geq C_\zeta > 0,\ j = 2,\dots,5, \text{ for } 0 < h \leq h(\zeta) \text{ small enough.} \tag{11.10}$$

Then there exists an exact solution $\theta(x,\zeta,h)$ such that for any $0 < \delta_* < \mu$

$$\left|\theta - e^{\frac{1}{h}\int_0^x \mu_1^\sharp(s,\zeta,h)ds}\left[T_1(x,\zeta) + O(h)\right]\right| \leq C_\zeta h e^{-\delta_* x}\left|e^{\frac{1}{h}\int_0^x \mu_1^\sharp(s,\zeta,h)ds}\right| \quad \text{on } [a,\infty), \tag{11.11}$$

where $T_1(x,\zeta)$ *as in* (0.10), *and*

$$\mu_1^\sharp = \mu_1(x,\zeta) + O(he^{-\mu x}). \tag{11.12}$$

(2) *Let* $K \subset \{\Re\zeta \geq 0\} \setminus$ (iii) *be compact. Then* (11.10) *and* (11.11) *hold on* $[0,\infty)$ *with constants* $h(\zeta)$, C_ζ *that can be taken independent of* $\zeta \in K$.

(3) *Let* $\underline{\zeta} \in (\text{iii})_+^o$ *and* $\delta > 0$. *There exists a neighborhood* $\omega_1 \ni \underline{\zeta}$ *in* $\Re\zeta \geq 0$ *such that*

$$x(\underline{\zeta}) - \delta < \Re x(\zeta) < x(\underline{\zeta}) + \delta \text{ for all } \zeta \in \omega_1 \tag{11.13}$$

and such that (11.10) *and* (11.11) *hold on* $[x(\underline{\zeta}) + \delta, \infty)$ *with constants* $h(\zeta)$, C_ζ *that can be taken independent of* $\zeta \in \omega_1$.

As an immediate corollary of part (2) we have the following.

Corollary 11.3 (nonturning point frequencies). *Let* $K \subset \{\Re\zeta \geq 0\} \setminus$ (iii) *be compact. The exact bounded solution* $\theta(x,\zeta)$ *of* (11.9) *given by Theorem* 11.2 *satisfies*

$$|\theta(0,\zeta,h) - T_1(0,\zeta)| \leq C_K h \text{ for } 0 \leq h \leq h_K, \tag{11.14}$$

where C_K *and* h_K *can be taken independent of* $\zeta \in K$.

The corollary implies that for $\zeta \in K$, $0 < h \leq h_K$, the solution θ is of type θ_1 at $x = 0$ and thus the stability function $V(\zeta,h)$ is nonvanishing. In view of (11.11) and (11.12), part (3) of Theorem 11.2 yields the following.

Corollary 11.4. *Let* $\omega_1 \ni \underline{\zeta}$ *and* $\delta > 0$ *be as in* (11.13). *For* $\zeta \in \omega_1$ *there is a bounded, nonvanishing function* $H(x,\zeta,h)$ *and an* $h_0 > 0$ *such that*

$$|H(x,\zeta,h)\theta(x,\zeta,h) - \theta_1(x,\zeta,h)| \leq Ch|\theta_1(x,\zeta,h)| \text{ on } [x(\underline{\zeta}) + \delta, \infty) \tag{11.15}$$

for $0 < h \leq h_0$.

In the next section we show that there is a nonvanishing scalar function $s(\zeta,h)$ such that $s(\zeta,h)\theta(x,\zeta,h)$ is of type θ_1 at $x(\underline{\zeta}) - 2\delta$ for $\zeta \in \omega_1$. This is done by matching arguments that use Airy functions to represent exact solutions on a full neighborhood of the turning points. It then follows as in Proposition 10.12 that $s(\zeta,h)\theta(x,\zeta,h)$ is of type θ_1 at $x = 0$.

12. Turning points in $(0,\infty)$.

We now fix a basepoint $\underline{\zeta} \in (\text{iii})_+^o$ with associated turning point $x(\underline{\zeta}) \in (0,\infty)$. The goal is to find a neighborhood $\omega \ni \underline{\zeta}$ and a constant $h_0 > 0$ such that the stability function $V(\zeta,h) \neq 0$ for $\zeta \in \omega$ and $0 < h \leq h_0$. The first step is to conjugate the system (11.9) to the block form (5.11):

$$\begin{aligned} h\frac{d}{dx}\phi &= \begin{pmatrix} A_{11}^0 + hd_{11} + h^2\beta_{11} & 0 \\ 0 & A_{22}^0 + hd_{22} + h^2\beta_{22} \end{pmatrix}, \\ \phi &:= \begin{pmatrix} A_{11}(x,\zeta,h) & 0 \\ 0 & A_{22}(x,\zeta,h) \end{pmatrix}\phi, \end{aligned} \tag{12.1}$$

for ζ near $\underline{\zeta}$ and x in a complex neighborhood of $x(\underline{\zeta})$.

Proposition 12.1. *Let* $\underline{\zeta} \in (\text{iii})_+^o$ *and let* $x(\underline{\zeta}) \in (0,\infty)$ *be the corresponding turning point. There exists a constant* $h_0 > 0$ *and simply connected open neighborhoods* $\omega \ni \underline{\zeta}$, $\mathcal{O} \ni x(\underline{\zeta})$ *such that for* $\zeta \in \omega$ *and* $0 < h \leq h_0$ *a conjugator* $Y(x,\zeta,h)$ *can be constructed on* $\mathcal{O}$ *with the property that* $\theta(x,\zeta,h)$ *satisfies the Erpenbeck system* (11.9) *on* $\mathcal{O}$ *if and only if* ϕ *defined by* $\theta = Y\phi$ *satisfies* (12.1). *The conjugator*

$Y(x,\zeta,h)$ is bounded and analytic in its arguments. The entries of the 2×2 block $A_{11}(x,\zeta,h)$ in (12.1) *again have the form given in* (5.13).

Proof. The conjugator is constructed as $Y = Y_1Y_2$, where $Y_1(x,\zeta)$ is given by (5.5) and $Y_2(x,\zeta,h)$ has the form (5.10). The entries of Y_2 satisfy equations like (5.15) and are constructed by a classical contraction argument; see, for example, Theorem 6.1-1 of [Wa]. The analyticity of the Y_j in x is a consequence of the fact that the profile $P(x)$ extends analytically to a complex neighborhood of $x(\underline{\zeta})$. As in Proposition 5.2, the argument uses the fact that the blocks $A^0_{11}(x,\zeta)$ and $A^0_{22}(x,\zeta)$ in (5.8) have no eigenvalues in common for $(x,\zeta)\in\mathcal{O}\times\omega$ for small enough $\mathcal{O}$, ω. ∎

Writing $\phi = (\phi_1,\phi_2)$ and letting $\varphi_0(x,\zeta,h)$ denote any function such that $\frac{d}{dx}\varphi_0 = \frac{a+d}{2}$, we obtain by the same calculations that produced (5.31) that solutions of $h\frac{d}{dx}\phi_1 = A_{11}(x,\zeta,h)\phi_1$ in $\mathcal{O}$ are given by

$$\phi_1 = e^{\frac{\varphi_0}{h}}\begin{pmatrix} b^{1/2} & 0\\ \alpha b^{-1/2} - h(b^{-1/2})_x & b^{-1/2}\end{pmatrix}\begin{pmatrix} w\\ hw_x\end{pmatrix} := K(x,\zeta,h)\begin{pmatrix} w\\ hw_x\end{pmatrix}, \tag{12.2}$$

where (w,hw_x) satisfies

$$h\frac{d}{dx}\begin{pmatrix} w\\ hw_x\end{pmatrix} = \begin{pmatrix} 0 & 1\\ C(x,\zeta)+hr(x,\zeta,h) & 0\end{pmatrix}\begin{pmatrix} w\\ hw_x\end{pmatrix}. \tag{12.3}$$

Thus, we can construct two independent solutions on $\mathcal{O}$ of the original system (11.9) of the form

$$\theta = Y(x,\zeta,h)\begin{pmatrix} K(x,\zeta,h)\begin{pmatrix} w\\ hw_x\end{pmatrix}\\ 0\end{pmatrix}, \tag{12.4}$$

using independent solutions of (12.3).

Remark 12.2. In the remainder of this section the simply connected neighborhoods $\omega\ni\underline{\zeta}$ and $\mathcal{O}\ni x(\underline{\zeta})$ may need to be reduced in size a finite number of times. These reductions will often be performed without comment.

The following propositions will allow us to construct solutions of (12.3) in terms of Airy functions. Recall that for $\zeta\in(\mathrm{iii})^0_+$ the number $x(\zeta)\in(0,\infty)$ is the unique root of $\zeta^2 + c_0^2\eta(x) = 0$.

Let us write the function $C(x,\zeta)$ in (12.3) as

$$\begin{aligned} C(x,\zeta) &= (\zeta^2 + c_0^2\eta(x))\underline{b}^2(x) = (x - x(\zeta))d(x,\zeta),\\ &\text{where } d(x,\zeta) = \int_0^1 C_x(x(\zeta)+t(x-x(\zeta)),\zeta)dt. \end{aligned} \tag{12.5}$$

Proposition 12.3. *The equation*

$$(\rho_x)^2\rho = C(x,\zeta) \tag{12.6}$$

has a solution for $(x,\zeta)\in\mathcal{O}\times\omega$ given by

$$\rho(x,\zeta) = (x(\zeta)-x)\left(\int_0^1 3u^2\sqrt{-d(x(\zeta)+u^2(x-x(\zeta)),\zeta)}\,du\right)^{2/3}, \tag{12.7}$$

where the expression inside the square root and the square root itself are positive when x is real and $\Re\zeta = 0$. The function ρ is analytic in both x and ζ and satisfies

(12.8)
(a) $\rho(x(\zeta), \zeta) = 0$ *for* $\zeta \in \omega$;
(b) *for x real and* $\Re\zeta = 0$, $\rho(x, \zeta)$ *is real and* $\rho_x(x, \zeta) < 0$;
(c) *for each* $\zeta \in \omega$, $\rho(\cdot, \zeta)$ *is an analytic homeomorphism of* $\mathcal{O}$ *onto a neighborhood* $\mathcal{O}_\zeta \ni 0$;
(d) *for* $\Re\zeta = 0$, *we have* $(-i\rho_\zeta)(x(\zeta), \zeta) > 0$.

Remark 12.4. The inequality (12.8)(d) implies that the map $\zeta \to \rho(x(\underline{\zeta}), \zeta)$ is an analytic homeomorphism of a neighborhood of $\underline{\zeta}$ onto a neighborhood of 0. From (12.8)(b) and (d) we see that for ζ near $\underline{\zeta}$, when $\Re\zeta > 0$ we have $\Im\rho(x(\underline{\zeta}), \zeta) > 0$. After shrinking ω if necessary, we conclude that for real x near $x(\underline{\zeta})$ and $\zeta \in \omega$ with $\Re\zeta > 0$, we have $\Im\rho(x, \zeta) > 0$.

The system (12.3) is equivalent to the scalar equation

$$h^2 w_{xx} = (C(x, \zeta) + hr(x, \zeta, h))w. \tag{12.9}$$

Using (12.6), the property (12.8)(c), and for each $\zeta \in \omega$ making the changes of variables

$$y = \rho(x, \zeta),\ W(y, \zeta) := w(x(y, \zeta)), \tag{12.10}$$

we find that (12.9) takes the form (suppressing some ζ arguments and setting $\rho_x = \partial_x \rho$)

$$h^2 \rho_x(x(y)) d_y\left(\rho_x(x(y)) W_y\right) = [y\rho_x^2(x(y)) + hr(x(y), \zeta, h)]W. \tag{12.11}$$

The transformation

$$f(y) = (\rho_x(x(y)))^{1/2} W(y) \tag{12.12}$$

leads to

$$h^2 f_{yy} = (y + hq(y, h))f, \text{ where } q(y, h) = r\rho_x^{-2} + h\rho_x^{-1/2} d_y^2(\rho_x^{1/2}). \tag{12.13}$$

This is a perturbation of Airy's equation that we can rewrite as

$$h\begin{pmatrix} f \\ hf_y \end{pmatrix}_y = \begin{pmatrix} 0 & 1 \\ y + hq(y, h) & 0 \end{pmatrix}\begin{pmatrix} f \\ hf_y \end{pmatrix}. \tag{12.14}$$

The following proposition is a classical result of turning point theory. A reference for the proof is [Wa, Theorem 6.5-1].

PROPOSITION 12.5 (exact conjugation to Airy's equation). *There exists $h_0 > 0$ and a conjugator $P(y, \zeta, h) = I + hQ(y, \zeta, h)$, with Q bounded and analytic in its arguments $x \in \mathcal{O}$, $\zeta \in \omega$, $0 < h \le h_0$,*[27] *such that the transformation $(f, hf_y) = PZ$ takes* (12.14) *into the equation*

$$hZ_y = \begin{pmatrix} 0 & 1 \\ y & 0 \end{pmatrix} Z. \tag{12.15}$$

[27] Recall Remark 12.2.

For any $\zeta \in \omega$ two independent solutions of (12.15) on $\mathcal{O}_\zeta$ (as in (12.8)(c)) are given by

$$Z_\pm(y) = \begin{pmatrix} Ai(h^{-2/3}e^{\pm 2\pi i/3}y) \\ h^{1/3}e^{\pm 2\pi i/3}Ai(h^{-2/3}e^{\pm 2\pi i/3}y) \end{pmatrix}. \tag{12.16}$$

We recall that the phase function φ_0 in (12.2) is required to be a primitive of $\frac{a+d}{2}$, where $a = \underline{a} + O(h)$, $d = \underline{d} + O(h)$ (see (5.11) and (5.13)). Since $\underline{a} = \underline{d}$ is defined for all $x \geq 0$ and for all ζ, and morever extends analytically to a complex neighborhood of the positive real axis, we may (and do) henceforth take φ_0 of the form

$$\varphi_0(x,\zeta,h) = \int_0^x \underline{a}(s,\zeta)ds + O(h). \tag{12.17}$$

Using the formula (12.4) for θ and retracing through the changes of variables, we obtain the next proposition.

PROPOSITION 12.6 (exact solutions of (11.9)). *For each $\zeta \in \omega$ two exact independent solutions $\theta_\pm(x,\zeta,h)$ on $\mathcal{O}$ of the original 5×5 system (11.9) are given by formula (12.4) with*

(12.18)
$$\begin{pmatrix} w \\ hw_x \end{pmatrix}_\pm = \begin{pmatrix} \rho_x^{-1/2} & 0 \\ h\rho_x d_y(\rho_x^{-1/2}) & \rho_x^{1/2} \end{pmatrix} \begin{pmatrix} f \\ hf_y \end{pmatrix}_\pm = \begin{pmatrix} \rho_x^{-1/2} & 0 \\ h\rho_x d_y(\rho_x^{-1/2}) & \rho_x^{1/2} \end{pmatrix} P(\rho,\zeta,h)Z_\pm(\rho).$$

Ignoring relative $O(h)$ errors in (12.4), we obtain by these substitutions $\theta_\pm(x,\zeta,h) \sim$

$$\begin{aligned} &e^{\varphi_0/h}\Big[b^{1/2}(\rho_x)^{-1/2}Ai(h^{-2/3}\rho e^{\pm 2\pi i/3})P_0 \\ &\qquad + b^{-1/2}h^{1/3}(\rho_x)^{1/2}e^{\pm 2\pi i/3}Ai'(h^{-2/3}\rho e^{\pm 2\pi i/3})Q_0\Big]. \end{aligned} \tag{12.19}$$

We now choose $\delta > 0$ and a neighborhood $\omega_1 \ni \underline{\zeta}$ satisfying (11.13) as in Corollary 11.4, and so that for $\mathcal{O} \ni x(\underline{\zeta})$ as Proposition 12.6 we have

$$x(\omega_1) \cup [x(\underline{\zeta}) - 2\delta, x(\underline{\zeta}) + 2\delta] \subset \mathcal{O}. \tag{12.20}$$

The next proposition shows that for H as in Corollary 11.4, a nonvanishing multiple of the exact decaying solution $H(x(\underline{\zeta}) + 2\delta, \zeta, h)\theta$ is of type θ_1 at $x(\underline{\zeta}) - 2\delta$.

PROPOSITION 12.7. *Let $x_L = x(\underline{\zeta}) - 2\delta$ and $x_R = x(\underline{\zeta}) + 2\delta$. For $\zeta \in \omega_1$ there is an $h_0 > 0$ and a nonvanishing scalar function $\alpha(\zeta,h)$ such that*

$$|\alpha(\zeta,h)H(x_R,\zeta,h)\theta(x_L,\zeta,h) - \theta_1(x_L,\zeta,h)| \leq Ch|\theta_1(x_L,\zeta,h)| \tag{12.21}$$

for $0 < h \leq h_0$.

The proof, given in section V, is based on expanding $H(x_R,\zeta,h)\theta(x,\zeta,h)$ in a basis of local exact solutions of (11.9), $\mathcal{B} = \{\theta_-, \theta_+, \overline{\theta}_3, \overline{\theta}_4, \overline{\theta}_5\}$, where the $\overline{\theta}_j$, $j = 3,4,5$, are of type θ_j for approximate solutions θ_j as in (0.16). Using the expansions (16.66) for the Airy function, we show first that appropriate multiples of θ_- and θ_+ are, respectively, of type θ_1 and θ_2 at x_R. Corollary 11.4 and the explicitly known asymptotic behavior (in h) of the elements of $\mathcal{B}$ at both x_R and x_L then allow us to conclude that (12.21) holds.

Finally, the proof of Proposition 10.6 yields the following.

PROPOSITION 12.8. *There is a neighborhood* $\omega_1 \ni \underline{\zeta}$ *and an* $h_0 > 0$ *such that for* $\zeta \in \omega_1$ *and* α *as in Proposition* 12.7*,*

$$|\alpha(\zeta,h)H(x_R,\zeta,h)\theta(x,\zeta,h) - \theta_1(x,\zeta,h)| \leq Ch|\theta_1(x,\zeta,h)| \text{ on } [0,x_L] \tag{12.22}$$

for $0 < h \leq h_0$.

13. The turning point at 0. For the boundary point frequency $\zeta_0 \in (\text{iii})_+$ the point $x = 0$, where we need explicit information about the exact decaying solution θ in order to evaluate the stability function $V(\zeta,h)$, coincides with the turning point. This fact presents some new difficulties that we sketch after stating Proposition 13.1.

The first step in treating the turning point at $x = 0$ is to extend the detonation profile $p(x)$ analytically to a complex neighborhood of $x = 0$; this allows us to study the Erpenbeck system (11.9) on a neighborhood of $x = 0$. We can then immediately extend Lemma 11.1 to obtain an analytic homeomorphism $x : \omega \to \mathcal{O}$, where now $x(\zeta_0) = 0$, $\omega \ni \zeta_0$ and $\mathcal{O} \ni 0$. Similarly, with no changes in the proofs we obtain extensions of Theorem 11.2 (3), Corollary 11.4, and Propositions 12.1, 12.3, 12.5, and 12.6 to the case where $\underline{\zeta}$ is now ζ_0. In particular, we obtain exact solutions $\theta_\pm$ on $\mathcal{O}$ satisfying (12.19).

We now choose $\delta > 0$ and a neighborhood $\omega_1 \ni \zeta_0$ satisfying (11.13) as in Corollary 11.4 and so that for $\mathcal{O} \ni 0$ as Proposition 12.6 we have

$$x(\omega_1) \cup [-2\delta, 2\delta] \subset \mathcal{O}. \tag{13.1}$$

The next proposition shows that for H as in Corollary 11.4, a nonvanishing multiple of the exact decaying solution $H(2\delta,\zeta,h)\theta$ is of type θ_1 at 0.

PROPOSITION 13.1. *Fix* $\kappa > 0$. *There exists a neighborhood* $\omega_2 \ni \zeta_0$ *with* $\omega_2 \subset \omega_1$*, an* $h_0 > 0$*, and a nonvanishing scalar function* $\alpha(\zeta,h)$ *such that*

$$|\alpha(\zeta,h)H(2\delta,\zeta,h)\theta(0,\zeta,h) - \theta_1(0,\zeta,h)| \leq \kappa|\theta_1(0,\zeta,h)| \tag{13.2}$$

for $\zeta \in \omega_2$ *and* $0 < h \leq h_0$. *Both* ω_2 *and* h_0 *depend on* κ.

As with Proposition 12.7 the proof involves working with a local basis of exact solutions $\mathcal{B} = \{\theta_-, \theta_+, \overline{\theta}_3, \overline{\theta}_4, \overline{\theta}_5\}$, and again we make use of the expressions (12.19) for $\theta_\pm$ in terms of Airy functions. However, since $\rho(0,\zeta_0) = 0$, we cannot use the Airy function expansions ((16.66), for example) when ζ is too close to ζ_0. Since the arguments of the Airy functions in (12.19) are $h^{-2/3}\rho(x,\zeta)e^{\pm 2\pi i/3}$, we see that there are two natural frequency regimes to consider:

$$\begin{aligned} &\text{regime A} = \{(\zeta,h) : |\rho(0,\zeta)h^{-2/3}| \leq M, \\ &\text{regime B} = \{(\zeta,h) : |\rho(0,\zeta)h^{-2/3}| \geq M. \end{aligned} \tag{13.3}$$

Here $\zeta \in \omega_1$ and M is chosen large enough so that standard expansions of $Ai(z)$ apply in $|z| \geq M$; thus, we are able to use those expansions in the analysis of regime B.

In the proof of Proposition 12.8 for turning points in $(0,\infty)$, it was helpful that the arguments of $\rho(x_R,\zeta)$ and $\rho(x_L,\zeta)$ were always close to π and 0, respectively, for ζ near $\underline{\zeta}$, and thus $e^{\pm 2\pi i/3}\rho(x_{R,L},\zeta)$ stayed away from the negative real axis, where the zeros of Ai and Ai' are located. Now, however, the argument of $\rho(0,\zeta)$ can take on all values in $[0,\pi]$ for ζ near ζ_0. The analysis of θ_+ is complicated by the fact that for $\arg(\rho(0,\zeta)) \sim \pi/3$, we have $\arg\left(e^{2\pi i/3}\rho(0,\zeta)\right) \sim \pi$.

The formula (0.16) for the approximate solution θ_1 shows that

$$\theta_1(0,\zeta,h) = P_0(0,\zeta) + s(0,\zeta)Q_0 + O(h). \tag{13.4}$$

The proof of Proposition 13.1 makes use of the fact that for ζ near ζ_0, the functions $s(0,\zeta)$ and $\rho(0,\zeta)$ are both close to zero. This implies that the terms involving Q_0 in both (13.4) and the expression (12.19) for θ_- are small compared to the terms involving P_0.

14. The case $|\zeta| \geq M$. We conclude Part III by treating the case $|\zeta| \geq M >> 1$. We must show that there exists $h_0 > 0$ and M such that for all $0 < h \leq h_0$ and $|\zeta| \geq M$ with $\Re\zeta \geq 0$, the decaying (or bounded when $\Re\zeta = 0$) solution $\theta(x,\zeta,h)$ of Erpenbeck's 5×5 system,

$$h\frac{d}{dx}\theta = (\Phi_0(x,\zeta) + h\Phi_1(x))\theta, \tag{14.1}$$

is of type θ_1 at $x = 0$. As noted above, this implies nonvanishing of the stability function $V(\zeta,h)$. Here there are no turning points, but the difficulty is to give a *uniform* treatment of the noncompact set of parameters ζ. This case was studied on p. 610 of [E3], but the choice of h_0 there was not uniform with respect to large ζ, and so we are not able to use this result.

PROPOSITION 14.1. *Let $\theta(x,\zeta,h)$ be as just described and let $h_0 = 1$. There exists $M > 0$ such that for $|\zeta| \geq M$ and $0 < h \leq h_0$ we have*

$$|\theta(0,\zeta,h) - T_1(0,\zeta)| \leq Ch/|\zeta|, \tag{14.2}$$

where $C > 0$ is independent of (ζ,h).

Proof. (1) First we rewrite 14.1 as

$$\frac{d}{dx}\theta = \frac{|\zeta|}{h}\left(\tilde{\Phi}_0(x,\zeta) + \frac{h}{|\zeta|}\Phi_1(x)\right), \tag{14.3}$$

where $\Phi_1(x) = ((A^x)^{-1}B(x))^t$ as before, and

$$\tilde{\Phi}_0(x,\zeta) = \frac{1}{|\zeta|}\Phi_0 = \frac{\zeta}{|\zeta|}((A^x)^{-1})^t + \frac{i}{|\zeta|}((A^x)^{-1}A^y)^t. \tag{14.4}$$

The eigenvalues of $\tilde{\Phi}_0(x,\zeta)$ are $\tilde{\mu}_j(x,\zeta) := \frac{1}{|\zeta|}\mu_j(x,\zeta)$ for μ_j as in (0.7).

(2) As in section 2 of [LWZ1], direct computation and the use of Assumption 1.3 shows that for $\mu > 0$ as in (5.4)

$$((A^x)^{-1}B)^t(x) = O(e^{-\mu x}) + \begin{pmatrix} 0 \\ \text{row } 5 \end{pmatrix}, \text{ where row 5 } = (*,*,*,*,-r_\lambda/u). \tag{14.5}$$

This implies that the eigenvalues of $\tilde{\Phi}_0(x,\zeta) + \frac{h}{|\zeta|}\Phi_1(x)$ are

$$\begin{aligned} \mu_j^* &:= \frac{1}{|\zeta|}\mu_j(x,\zeta) + O(\tilde{h}e^{-\mu x}),\ j = 1,2,3,4, \quad \tilde{h} := \frac{h}{|\zeta|}, \\ \mu_5^* &= \frac{1}{|\zeta|}\mu_3(x,\zeta) - \tilde{h}\frac{r_\lambda}{u} + O(\tilde{h}e^{-\mu x}),\ \text{where } r_\lambda < 0. \end{aligned} \tag{14.6}$$

(3) *Uniform separation of eigenvalues.* Using the fact that

$$\mu_2(x,\zeta)-\mu_1(x,\zeta)=\frac{2\kappa s}{\eta u},\ \ s(x,\zeta)=\sqrt{\zeta^2+c_0^2\eta(x)},$$
$$\mu_3(x,\zeta)-\mu_1(x,\zeta)=\frac{\zeta+\kappa s}{\eta u}, \tag{14.7}$$

and noting that $s(x,\zeta)\sim\zeta$ for $|\zeta|>>1$, we obtain for large enough M

$$|\tilde\mu_1(x,\zeta)-\tilde\mu_j(x,\zeta)|\geq C>0,\ j=2,\dots,5, \tag{14.8}$$

where C is independent of $x\in[0,\infty)$ and $|\zeta|\geq M$. Moreover, since $\Re\mu_j\geq\Re\mu_1$, $j=2,\dots,5$, we find from (14.6) that

$$\Re\mu_j^*(x,\zeta,\tilde h)-\Re\mu_1^*(x,\zeta,\tilde h)\geq O(\tilde h e^{-\mu x}),\ j=2,\dots,5, \tag{14.9}$$

uniformly for $x\in[0,\infty)$ and $|\zeta|\geq M$.

(4) *Conclusion.* As a consequence of the separation inequalities (14.8) and (14.9), we are in a position to apply (verbatim) the proof of Theorem 2.1 of [LWZ1] to the system

$$\frac{d}{dx}\theta=\frac{1}{\tilde h}\left(\tilde\Phi_0(x,\zeta)+\tilde h\Phi_1(x)\right), \tag{14.10}$$

where $\tilde\Phi_0$ and $\tilde h$ play the roles of, respectively, Φ_0 and h in the earlier proof. For a possibly larger choice of M, the argument there[28] shows that θ satisfies

(14.11)
$$\left|\theta(x,\zeta,h)-e^{\frac{1}{\tilde h}\int_0^x\mu_1^\sharp(s,\zeta,\tilde h)ds}\left[T_1(x,\zeta)+O(\tilde h)\right]\right|\leq C\tilde h e^{-\delta x}\left|e^{\frac{1}{\tilde h}\int_0^x\mu_1^\sharp(s,\zeta,\tilde h)ds}\right|\quad\text{on }[0,\infty),$$

where $0<\delta<\mu$, C is independent of $|\zeta|\geq M$, and

$$\mu_1^\sharp(s,\zeta,\tilde h)=\mu_1^*(s,\zeta,\tilde h)+O(\tilde h e^{-\mu x}). \tag{14.12}$$

Evaluating (14.11) at $x=0$ we obtain (14.2). ∎

Part IV. Proofs for Part II.

15. Conjugation to block form. In this section we prove Lemma 5.1 and Proposition 5.2.

Proof of Lemma 5.1. We give the proof for d_{12}; the proof for d_{11} is quite similar. The assertion for β_{11} then follows from $\beta_{11}=d_{12}\alpha_{21}$ and the boundedness of α_{21}.

Since the x-dependence of d_{12} enters only through the profile, it suffices to show $d_{12}(\infty)=0$. We have

$$D:=\begin{pmatrix}d_{11}&d_{12}\\d_{21}&d_{22}\end{pmatrix}=Y_1^{-1}\Phi_1Y_1-Y_1^{-1}\frac{d}{dx}Y_1, \tag{15.1}$$

so we need only show that that $(1,2)$ entry of $Y_1^{-1}\Phi_1Y_1$ is 0 at $x=\infty$. Here d_{12} is a 2×3 submatrix of the 5×5 matrix D.

[28] This argument is based on the Variable Coefficient Gap Lemma stated in Appendix A of [LWZ1] and first proved in [Z1].

On pp. 116–117 of [E2] Erpenbeck writes $\Phi_1 = W_{10} + W_{11}$, where $W_{10}(\infty) = 0$ and W_{11} is a matrix whose first four rows vanish at ∞.[29] So it suffices to consider $Y_1^{-1} W_{11} Y_1(\infty)$. Suppressing evaluations at ∞, we write

$$W_{11} = \begin{pmatrix} w_a & w_b \\ w_c & w_d \end{pmatrix}, \tag{15.2}$$

where $w_a = 0$ (w_a is 2×2), $w_b = 0$, and

$$w_c = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ * & * \end{pmatrix},\ w_d = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ * & * & * \end{pmatrix}. \tag{15.3}$$

Writing

$$Y_1 = \begin{pmatrix} t_a & t_b \\ t_c & t_d \end{pmatrix} \text{ and } Y_1^{-1} = \begin{pmatrix} s_a & s_b \\ s_c & s_d \end{pmatrix}, \tag{15.4}$$

the $(1,2)$ submatrix of $Y_1^{-1} W_{11} Y_1$ is then $s_b(w_c t_b + w_d t_d)$. From (5.6) we see that

$$t_b = \begin{pmatrix} * & 0 & 0 \\ * & 0 & 0 \end{pmatrix},\ t_d = \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix},\ s_b = \begin{pmatrix} * & 0 & 0 \\ * & 0 & 0 \end{pmatrix}. \tag{15.5}$$

Computing $s_b(w_c t_b + w_d t_d)$ we find that all entries vanish. ∎

Proof of Proposition 5.2.

(1) *Integral equation for* α_{21}. In order for Y_2 to conjugate solutions of (5.8) to solutions of (5.10), we must have

$$h \frac{d}{dx} Y_2 = \mathbb{A} Y_2 - Y_2 \mathbb{B} \text{ on the wedge } \mathbb{W} = \mathbb{W}(M_0, \theta), \tag{15.6}$$

where $\mathbb{A}$ and $\mathbb{B}$ are the coefficient matrices of (5.8) and (5.10), respectively. Direct computation shows that the functions α_{12} and α_{21} must therefore satisfy the equations

$$h \frac{d}{dx} \alpha_{12} = A_{11}^0 \alpha_{12} - \alpha_{12} A_{22}^0 + h(d_{11}\alpha_{12} - \alpha_{12} d_{22}) + d_{12} - h^2 \alpha_{12} d_{21} \alpha_{12},$$

$$h \frac{d}{dx} \alpha_{21} = A_{22}^0 \alpha_{21} - \alpha_{21} A_{11}^0 + h(d_{22}\alpha_{21} - \alpha_{21} d_{11}) + d_{21} - h^2 \alpha_{21} d_{12} \alpha_{21}. \tag{15.7}$$

Thinking of the 3×2 matrix α_{21} as an element of $\mathbb{C}^6$ and using obvious notation, we rewrite the second equation as

$$\begin{aligned} h \frac{d}{dx} \alpha_{21} = {} & \left(\mathcal{A}(\zeta, h) + O(e^{-\mu \Re x})\right) \alpha_{21} + \left(d_{21}(\infty, \zeta) + O(e^{-\mu \Re x})\right) \\ & + O(h^2 e^{-\mu \Re x})(\alpha_{21}, \alpha_{21}), \end{aligned} \tag{15.8}$$

where (with slight abuse)

$$\mathcal{A}(\zeta, h)\alpha_{21} = A_{22}^0(\infty, \zeta)\alpha_{21} - \alpha_{21} A_{11}^0(\infty, \zeta) + h\left(d_{22}(\infty, \zeta)\alpha_{21} - \alpha_{21} d_{11}(\infty, \zeta)\right). \tag{15.9}$$

Here we have used (5.4) and the fact that $d_{12}(\infty, \zeta) = 0$.

[29] Here we use Assumption 1.3.

The eigenvalues of $A^0_{11}(\infty,\zeta)$ (resp., $A^0_{22}(\infty,\zeta)$) are $\mu_j(\infty,\zeta)$, $j=1,2$ (resp., $\mu_j(\infty,\zeta)$, $j=3,4,5$). The six eigenvalues $a_j(\zeta,h)$ of $\mathcal{A}(\zeta,h)$ are differences $\lambda_2-\lambda_1$, where $\lambda_j(\zeta,h)$ is an eigenvalue of $A^0_{jj}(\infty,\zeta)+hd_{jj}(\infty,\zeta)$. From the formulas (0.7) for the μ_j we see that there exist constants $\underline{a}$ and h_0 and a neighborhood $\omega\ni\zeta_\infty$ such that all the eigenvalues $a_j(\zeta,h)$ satisfy

$$\Im a_j(\zeta,h)>\underline{a}>0 \text{ for } 0<h\le h_0,\ \zeta\in\omega. \tag{15.10}$$

Given any $\varepsilon_0>0$, after reducing h_0 and shrinking ω if necessary, we will also have for all j

$$|\Re a_j(\zeta,h)|<\varepsilon_0 \text{ for } 0<h\le h_0,\ \zeta\in\omega. \tag{15.11}$$

In view of (15.8), we will construct α_{21} as a fixed point of the map (analyzed below)

$$\begin{aligned}T\alpha_{21}(x)=h^{-1}\int_{\infty_-}^{x} e^{h^{-1}\mathcal{A}(\zeta,h)(x-y)}\Big[&O(e^{-\mu\Re y})\alpha_{21}+\big(d_{21}(\infty,\zeta)+O(e^{-\mu\Re y})\big)\\ &+O(h^2e^{-\mu\Re y})(\alpha_{21},\alpha_{21})\Big]\,dy,\end{aligned} \tag{15.12}$$

where $x\in\mathbb{W}(M_0,\theta)$, ∞_- is the point at ∞ on the lower boundary of the wedge $\mathbb{W}$, and the path of integration is a straight segment.

(2) *Estimate of* $e^{h^{-1}\mathcal{A}(\zeta,h)(x-y)}$. Write $x=x_r+ix_i$ and let $y(s)=s+iy_i(s)$ be a parametrization of the segment from ∞_- to x. If $a(\zeta,h)=a_r+ia_i$ denotes any eigenvalue of $\mathcal{A}(\zeta,h)$, we have

$$\Re(a(x-y(s)))=a_r(x_r-s)-a_i(x_i-y_i(s)):=a_r\Delta_r-a_i\Delta_i. \tag{15.13}$$

Choosing ε_0 in (15.11) such that $0<\varepsilon_0<\underline{a}\tan\theta$ and noting that $|\frac{\Delta_i}{\Delta_r}|\ge\tan\theta$, we estimate

$$\begin{aligned}&a_r\Delta_r-a_i\Delta_i\le\varepsilon_0|\Delta_r|-\underline{a}|\Delta_i|\le\varepsilon_0|\Delta_r|-\underline{a}|\Delta_r|\tan\theta=-\kappa|\Delta_r|,\\ &\quad\text{where } \kappa=\underline{a}\tan\theta-\varepsilon_0>0.\end{aligned} \tag{15.14}$$

The Jordan form of the matrix $\mathcal{A}(\zeta,h)$ can have nontrivial blocks, but the semisimplicity of the eigenvalues $\mu_j(\infty)$ $j=3,4,5$, of $A^0_{22}(\infty,\zeta)$ implies that such a block can be at most of size 4×4.[30] Thus, (15.13) and (15.14) yield the estimate

$$\left|e^{h^{-1}\mathcal{A}(\zeta,h)(x-y)}\right|\le C\left(1+\frac{|x_r-s|^3}{h}\right)e^{-\kappa\frac{|x_r-s|}{h}} \tag{15.15}$$

on the path $y(s)$.

(3) *Contraction.* Using the estimate (15.15) we can choose $K>0$ such that

$$\left|h^{-1}\int_{\infty_-}^{x} e^{h^{-1}\mathcal{A}(\zeta,h)(x-y)}d_{21}(\infty,\zeta)dy\right|\le K \text{ for } 0<h\le h_0,\ \zeta\in\omega,\ x\in\mathbb{W}(M_0,\theta). \tag{15.16}$$

[30] This is because $\mathcal{A}(\zeta,h)$ has at least three independent eigenvectors.

In fact the integral in (15.16) is independent of x, so we call it $D(\zeta, h)$. For later use we note that for $x = x_r + ix_i \in \mathbb{W}(M_0, \theta)$,

$$h^{-1} \int_{\infty}^{x_r} \left(1 + \frac{|x_r - s|^3}{h}\right) e^{-\kappa \frac{|x_r - s|}{h}} e^{-\mu s} ds \leq C e^{-\mu x_r}. \tag{15.17}$$

Denoting the set of analytic functions on $\mathbb{W}$ by $H(\mathbb{W})$, we let

$$B = \{\alpha_{21} \in H(\mathbb{W}) : |\alpha_{21}|_{L^\infty(\mathbb{W})} \leq K + 1\}. \tag{15.18}$$

After increasing M_0 if necessary, we see that (15.12), (15.15), and (15.17) imply that $T : B \to B$. Using the same facts and again increasing M_0 if necessary, we see that T gives a contraction on B. So we now have a solution α_{21} to (15.8) satisfying

$$|\alpha_{21}|_{L^\infty(\mathbb{W})} \leq K + 1. \tag{15.19}$$

The contraction argument for α_{12} is similar, and we leave it to the reader.

(4) *Decay of α_{21} to its endstate.* Recall that $D(\zeta, h)$ is the (x-independent) integral in (15.16). From (15.12) and (15.17) we see that

$$|T\alpha_{21}(x) - D(\zeta, h)| = |\alpha_{21}(x) - D(\zeta, h)| \leq C e^{-\mu x_r} \text{ for } x \in \mathbb{W}(M_0, \theta), \tag{15.20}$$

so $D(\zeta, h) = \alpha_{21}(\infty, \zeta, h)$.

(5) *Derivative estimates.* The estimates (5.14) are obtained by differentiating (15.12) and again applying (15.17). ∎

16. Regimes I and II.

Regime II is the most difficult of the regimes to treat. We give the proofs for this regime first; the proofs for regime I are generally similar but much simpler.

16.1. Proofs for regime II.

After establishing some notation, we give the proofs of Propositions 10.7, 10.8, 10.10, and 10.11.

16.1.1. The change of variable $\sigma \to \xi(\sigma)$.

An application of Rouché's theorem shows that for $|f_p|_{L^\infty(\mathcal{Z}_{-i\tilde{\alpha}})}$ sufficiently small, the function $f(\sigma) = (1 - \frac{1}{\sigma^2}) + f_p(\sigma)$ has a unique simple zero σ_0 on $\mathcal{Z}_{-i\tilde{\alpha}}$, and that $\sigma_0 \to 1$ as $|f_p|_{L^\infty} \to 0$. We set $f_0(\sigma) := 1 - \frac{1}{\sigma^2}$ and introduce subscripts to distinguish

$$\Xi_f(\sigma) = \frac{2}{3}\xi_f^{\frac{3}{2}}(\sigma) = \int_{\sigma_0}^{\sigma} \sqrt{f_0 + f_p}\,(\sigma) d\sigma,\ \sigma \in \mathcal{Z}_{cut}(\sigma_0) \text{ and}$$

$$\Xi_{f_0}(\sigma) = \frac{2}{3}\xi_{f_0}^{\frac{3}{2}}(\sigma) = \int_{1}^{\sigma} \sqrt{f_0}\,(\sigma) d\sigma, \quad \sigma \in \mathcal{Z}_{cut}(1). \tag{16.1}$$

Our analysis of $\xi_f(\sigma)$ is based on comparison with $\xi_{f_0}(\frac{\sigma}{\sigma_0})$, and for this we must carefully choose the branches of the square roots in (16.1). Write

$$f_0(\sigma) = (\sigma - 1)d_1(\sigma), \text{ where } d_1(\sigma) = \frac{\sigma + 1}{\sigma^2}, \text{ and}$$

$$f(\sigma) = (\sigma - \sigma_0)d_{\sigma_0}(\sigma) \text{ where } d_{\sigma_0}(\sigma) = \frac{\sigma^2 - 1 + \sigma^2 f_p(\sigma)}{(\sigma - \sigma_0)\sigma^2}. \tag{16.2}$$

Now define

$$\sqrt{f_0(\sigma)} = \sqrt{\sigma - 1}\sqrt{d_1(\sigma)} \text{ on } \mathcal{Z}_{cut}(1), \tag{16.3}$$

where $\sqrt{\sigma-1}$ is the branch on $\mathcal{Z}_{cut}(1)$ that is positive for $\sigma>1$, and $\sqrt{d_1(\sigma)}=\frac{\sqrt{\sigma+1}}{\sigma}$, defined on $\Re\sigma>0$, is positive for $\sigma>0$.[31] Similarly, define

$$\sqrt{f}(\sigma)=\sqrt{\sigma-\sigma_0}\sqrt{d_{\sigma_0}(\sigma)} \text{ on } \mathcal{Z}_{cut}(\sigma_0), \tag{16.4}$$

where $\sqrt{\sigma-\sigma_0}$ on $\mathcal{Z}_{cut}(\sigma_0)$ is given by

$$\sqrt{\sigma-\sigma_0}=\sqrt{\sigma_0}\sqrt{\frac{\sigma}{\sigma_0}-1} \text{ with } \sqrt{\sigma_0}\sim 1 \text{ and } \sqrt{\cdot-1} \text{ as above,} \tag{16.5}$$

and $\sqrt{d_{\sigma_0}(\sigma)}$ is close to $\sqrt{d_1(\sigma)}$ for f_p small. Other powers of $\sigma-\sigma_0$ on $\mathcal{Z}_{cut}(\sigma_0)$ are defined similarly.

Proof of Proposition 10.7.

(1) *Analyticity on* $\mathcal{Z}_{-i\tilde{\alpha}}$. In the integral

$$\frac{3}{2}\Xi_f(\sigma)=\frac{3}{2}\int_{\sigma_0}^{\sigma}\sqrt{s-\sigma_0}\sqrt{d_{\sigma_0}(s)}ds \tag{16.6}$$

we make the changes of variable $t=\sqrt{s-\sigma_0}$ and then $t=\sqrt{\sigma-\sigma_0}\,u$ to obtain

$$\begin{aligned}\frac{3}{2}\Xi_f(\sigma)&=\int_0^{\sqrt{\sigma-\sigma_0}}3t^2\sqrt{d_{\sigma_0}(t^2+\sigma_0)}\,dt\\&=(\sigma-\sigma_0)^{\frac{3}{2}}\int_0^1 3u^2\sqrt{d_{\sigma_0}((\sigma-\sigma_0)u^2+\sigma_0)}\,du.\end{aligned} \tag{16.7}$$

Estimates given below (see step (3)) imply that the second integral in (16.7) is nonvanishing on $\mathcal{Z}_{-i\tilde{\alpha}}$ for f_p sufficiently small. Thus, this integral has a well-defined analytic logarithm, which we use to define roots of the integral.[32] In particular, we obtain[33]

$$\xi_f(\sigma)=(\sigma-\sigma_0)\left(\int_0^1 3u^2\sqrt{d_{\sigma_0}((\sigma-\sigma_0)u^2+\sigma_0)}\,du\right)^{2/3}. \tag{16.8}$$

Using the above logarithm we define $\sqrt{\xi_f}$ on $\mathcal{Z}_{cut}(\sigma_0)$ and we have

$$\sqrt{\xi_f}\,\xi_f'=\sqrt{f} \text{ on } \mathcal{Z}_{cut}(\sigma_0). \tag{16.9}$$

From (16.8) it is clear that ξ_f is analytic on $\mathcal{Z}_{-i\tilde{\alpha}}$. With (16.9) it follows that ξ_f' is nonvanishing on $\mathcal{Z}_{-i\tilde{\alpha}}$. Thus ξ_f is a locally one-to-one, conformal map of $\mathcal{Z}_{-i\tilde{\alpha}}$ onto its range.

(2) *Global injectivity.* For some sufficiently small $\delta>0$ and sufficiently large $K>0$ to be chosen, we divide $\mathcal{Z}_{-i\tilde{\alpha}}$ into regions A, B, and C, where, respectively, $|\sigma|<\delta$, $\delta\le|\sigma|\le K$, and $|\sigma|>K$. The first and main step is to prove injectivity of ξ_f restricted to each of these subsets. The proof relies on the global injectivity of ξ_{f_0}, which follows from direct analysis of (10.22).

The next lemma is essential for proving the injectivity and mapping properties of ξ_f.

[31] This definition yields the branch of $\sqrt{\sigma^2-1}$ used in (10.22).

[32] The logarithm is chosen so that its argument is close to zero for z large and positive.

[33] Of course, we have a formula for $\xi_{f_0}(\sigma)$ similar to (16.8) in which σ_0 is replaced by 1.

LEMMA 16.1. *There exist constants* $\varepsilon_j = \varepsilon_j(f_p) > 0$, $j = A, B, C$, *which approach zero as* $N_p \to 0$,[34] *such that*

$$\text{(a)} \quad \left|\xi_f(\sigma) - \xi_{f_0}\left(\frac{\sigma}{\sigma_0}\right)\right| \le \varepsilon_A / \left|\xi_{f_0}\left(\frac{\sigma}{\sigma_0}\right)\right|^{\frac{1}{2}} \text{ for } \sigma \in A,$$

$$\text{(16.10)} \qquad \text{(b)} \quad \left|\xi_f(\sigma) - \xi_{f_0}\left(\frac{\sigma}{\sigma_0}\right)\right| \le \varepsilon_B \text{ for } \sigma \in B,$$

$$\text{(c)} \quad \left|\Xi_f(\sigma) - \Xi_{f_0}\left(\frac{\sigma}{\sigma_0}\right)\right| \le \varepsilon_C \left|\Xi_{f_0}\left(\frac{\sigma}{\sigma_0}\right)\right| \text{ for } \sigma \in C.$$

Proof. Estimates (a) *and* (b). We set $\underline{w} = \underline{w}(\sigma, u) = (\frac{\sigma}{\sigma_0} - 1)u^2 + 1$ and write

$$\text{(16.11)} \qquad \xi_f(\sigma) - \xi_{f_0}(\frac{\sigma}{\sigma_0}) = (\frac{\sigma}{\sigma_0} - 1)\left[\left(\sigma_0^{\frac{3}{2}} \int_0^1 3u^2 \sqrt{d_{\sigma_0}(\sigma_0 \underline{w})} du\right)^{2/3} - \left(\int_0^1 3u^2 \sqrt{d_1(\underline{w})} du\right)^{2/3}\right].$$

Thus, estimates (a) and (b) follow from the fact that given $\varepsilon > 0$, we have (for small f_p)

$$\text{(16.12)} \qquad \left|\sigma_0^{3/2} \sqrt{d_{\sigma_0}(\sigma_0 \underline{w})} - \sqrt{d_1(\underline{w})}\right| \le \varepsilon |\underline{w}| \text{ for all } (u, \sigma) \in [0, 1] \times (A \cup B).$$

To see this one computes (observing important cancellations) the difference in (16.12) to be

$$\text{(16.13)} \qquad \frac{\underline{w}\left(\sigma_0^2 - 1 + \sigma_0^2 f_p(\sigma_0 \underline{w})\right)}{\sqrt{\underline{w} - 1}\left(\sqrt{\sigma_0^2 \underline{w} - 1 + \sigma_0^2 \underline{w}^2 f_p(\sigma_0 \underline{w})} + \sqrt{\underline{w}^2 - 1}\right)}.$$

For $\underline{w}$ bounded away from 0 we can factor $\underline{w} - 1$ out of numerator and denominator (since $(f_0 + f_p)(\sigma_0) = 0$) to obtain (16.12) when (f_p, f_p') is small. When $\underline{w}$ is near 0, we obtain (16.12) since

$$\text{(16.14)} \qquad |\sigma_0^2 - 1 + \sigma_0^2 f_p(\sigma_0 \underline{w})| \le \varepsilon \text{ when } f_p \text{ is small.}$$

We remark that the individual integrals in (16.11) do blow up since $\underline{w} \to 0$ as $(\sigma, u) \to (0, 1)$. In fact it is clear from (10.22) that

$$\text{(16.15)} \qquad |\xi_{f_0}(\sigma)| \sim C(|\ln|\sigma||)^{2/3} \text{ as } \sigma \to 0.$$

Estimate (c). We use again the formula (16.13). Since $|\underline{w}| \sim |\sigma|$ for $|\sigma|$ large and $|(\sigma_0^2 - 1 + \sigma_0^2 f_p(\sigma_0 \underline{w})| \le \varepsilon$ for f_p small, we have $|(16.13)| \le \varepsilon / |\sigma|^{\frac{1}{2}}$ for $|\sigma|$ large. The estimate follows since $|\Xi_{f_0}(\frac{\sigma}{\sigma_0})| \sim |\sigma|$ for $|\sigma|$ large. ∎

(3) The nonvanishing of the second integral in (16.7) for small f_p can be deduced from the above estimates of (16.13) together with the nonvanishing of the (computable) integral

$$\text{(16.16)} \qquad \int_0^1 3u^2 \sqrt{d_1(\underline{w})} du.$$

[34] N_p occurs in Proposition 9.2.

(4) *Region B*. Writing

$$\xi_f(\sigma) - \xi_f(a) = (\sigma - a)\left(\xi_f'(a) + (\sigma - a)h(\sigma, a)\right) \text{ for } \sigma, a \in B, \tag{16.17}$$

and noting that there exist positive constants C_1, C_2 such that $|\xi_f'(\sigma)| \geq C_1$ on B and $|h(\sigma, a)| \leq C_2$ on $B \times B$, we see that there exists $\kappa > 0$ such that

$$\xi_f(\sigma) \neq \xi_f(a) \text{ for } (\sigma, a) \in B \times B, \sigma \neq a, |\sigma - a| \leq \kappa. \tag{16.18}$$

Since region B is compact, ξ_{f_0} is injective on B, and ξ_{f_0}' is everywhere nonvanishing, there exists a constant $C > 0$ such that

$$\left|\xi_{f_0}\left(\frac{\sigma_1}{\sigma_0}\right) - \xi_{f_0}\left(\frac{\sigma_2}{\sigma_0}\right)\right| \geq C|\sigma_1 - \sigma_2| \text{ for all } \sigma_1, \sigma_2 \in B. \tag{16.19}$$

Suppose now that for all $\mu > 0$ injectivity of $\xi_f|_B$ fails for some perturbation f_p with $|f_p, f_p'|_{L^\infty(B)} < \mu$. Then estimate (16.10)(b) implies that there exists a sequence of perturbations $f_{p,k}$, sequences of points $\sigma_{1,k}$, $\sigma_{2,k}$ in B, and a sequence of positive constants $\varepsilon_{B,k} \to 0$ such that

$$\begin{aligned} 0 = |\xi_{f_0 + f_{p,k}}(\sigma_{1,k}) - \xi_{f_0+f_{p,k}}(\sigma_{2,k})| \geq \Bigg|\xi_{f_0}\left(\frac{\sigma_{1,k}}{\sigma_0}\right) \\ - \xi_{f_0}\left(\frac{\sigma_{2,k}}{\sigma_0}\right)\Bigg| - \varepsilon_{B,k} \geq C|\sigma_{1,k} - \sigma_{2,k}| - \varepsilon_{B,k}. \end{aligned} \tag{16.20}$$

So $|\sigma_{1,k} - \sigma_{2,k}| \to 0$, which contradicts (16.18).

(5) Region A. For $\sigma_1, \sigma_2 \in A$ we write

$$\xi_f(\sigma_1) - \xi_f(\sigma_2) = (\sigma_1 - \sigma_2)\int_0^1 \xi_f'(\sigma_2 + s(\sigma_1 - \sigma_2))ds. \tag{16.21}$$

We claim that for δ as in step (2) small enough, the integral on the right in (16.21) is nonvanishing (and in fact very large) for small f_p. Using the explicit form of f_0 one computes directly (e.g., using partial fractions) that the dominant contribution to ξ_{f_0}' for $\sigma \in A$ is a term of the form

$$\frac{C}{\sigma(\log\sigma)^{1/3}}. \tag{16.22}$$

Since $\arg\sigma \sim 0$ for $\sigma \in A$, we deduce[35]

$$\left|\int_0^1 \xi_{f_0}'\left(\frac{\sigma_2 + s(\sigma_1 - \sigma_2)}{\sigma_0}\right) ds\right| \geq C\frac{1}{|\sigma_1||\ln|\sigma_1||^{1/3}}, \text{ where } |\sigma_1| \geq |\sigma_2|,\ \sigma_1, \sigma_2 \in A. \tag{16.23}$$

To see that (16.23) holds with ξ_f in place of ξ_{f_0}, we use the estimate

$$\left|\xi_f'(\sigma) - \frac{1}{\sigma_0}\xi_{f_0}'\left(\frac{\sigma}{\sigma_0}\right)\right| \leq \frac{\varepsilon}{|\sigma||\ln|\sigma||^{1/3}}, \tag{16.24}$$

[35] This can also be derived from the explicit formula (10.22).

where $\varepsilon \to 0$ as $N_p \to 0$. To prove this we write out the derivatives in (16.22) explicitly, forming two differences $A_1 - A_2$ and $B_1 - B_2$ in the obvious way, and use Lemma 16.1(a) to estimate

$$(16.25)\quad |A_1 - A_2| := \left| \left(\int_0^1 3u^2 \sqrt{d_{\sigma_0}} du \right)^{2/3} - \left(\int_0^1 3u^2 \sqrt{d_1} du \right)^{2/3} \right| \leq \frac{\varepsilon}{|\ln |\sigma||^{1/3}}.$$

Here and below d_{σ_0} is evaluated at $\sigma_0 \underline{w}$, and d_1 is evaluated at $\underline{w}$. Since $\sigma - \sigma_0 \sim \sigma_0 \sim 1$ we have $|B_1 - B_2| \leq$

(16.26)

$$C \left| \frac{1}{\sigma_0^2} \left(\int_0^1 u^2 \sqrt{d_1} du \right)^{-\frac{1}{3}} \left(\int_0^1 u^4 \frac{d_1'}{\sqrt{d_1}} du \right) - \left(\int_0^1 u^2 \sqrt{d_{\sigma_0}} du \right)^{-\frac{1}{3}} \left(\int_0^1 u^4 \frac{d_{\sigma_0}'}{\sqrt{\sigma_{z_0}}} du \right) \right|$$

$$\lesssim \left| \sqrt{\sigma_0} \left(\int_0^1 u^2 \sqrt{d_1} du \right)^{-\frac{1}{3}} - \left(\int_0^1 u^2 \sqrt{d_{\sigma_0}} du \right)^{-\frac{1}{3}} \right| \left| \left(\frac{1}{\sigma_0^{5/2}} \int_0^1 u^4 \frac{d_1'}{\sqrt{d_1}} du \right) \right|$$

$$+ \left| \left(\int_0^1 u^2 \sqrt{d_{\sigma_0}} du \right)^{-\frac{1}{3}} \right| \left| \left(\frac{1}{\sigma_0^{5/2}} \int_0^1 u^4 \frac{d_1'}{\sqrt{d_1}} du \right) - \left(\int_0^1 u^4 \frac{d_{\sigma_0}'}{\sqrt{d_{\sigma_0}}} du \right) \right|.$$

By (16.12) and the computation that produced (16.22) we see that the second line of (16.26) is dominated by the right side of (16.24).

Next we show that the third line of (16.26) is dominated by the right side of (16.24). We write

$$(16.27)\quad \frac{d_{\sigma_0}'}{\sqrt{d_{\sigma_0}}} - \frac{d_1'}{\sqrt{d_1}\sigma_0^{5/2}} = \left(d_{\sigma_0}' - \frac{d_1'}{\sigma_0^4} \right) \frac{1}{\sqrt{d_{\sigma_0}}} + \frac{d_1'}{\sigma_0^4} \left(\frac{1}{\sqrt{d_{\sigma_0}}} - \frac{\sigma_0^{3/2}}{\sqrt{d_1}} \right) := C + D.$$

Using (16.12) we obtain by the computation that produced (16.22) that the contribution from the term involving D to the third line of (16.26) is dominated by the right side of (16.24). To estimate the contribution from C we write after observing some cancellations

(16.28)

$$C = \frac{(1 - \sigma_0^2 - \sigma_0^2 f_p(\sigma_0 \underline{w})) + (\underline{w} - 1)\sigma_0^3 f_p'(\sigma_0 \underline{w})}{(\underline{w} - 1)^2 \sigma_0^4} \cdot \frac{\sigma_0^{3/2} \sqrt{\underline{w} - 1}\, \underline{w}}{\sqrt{\sigma_0^2 \underline{w}^2 - 1 + \sigma_0^2 \underline{w}^2 f_p(\sigma_0 \underline{w})}}.$$

We claim $|C| \leq \varepsilon$ for $(u, \sigma) \in [0,1] \times A$ when N_p is small, and thus the contribution from the term involving C to the third line of (16.26) is dominated by the right side of (16.25). To estimate C we note that when $\underline{w}$ is bounded away from 0, $(\underline{w} - 1)^2$ can be factored out of the first factor in (16.28) yielding

$$(16.29)\quad \left| d_{\sigma_0}'(\sigma_0 \underline{w}) - \frac{1}{\sigma_0^4} d_1'(\underline{w}) \right| = \left| \frac{1}{2} \int_0^1 (1 - s) f_p''(\sigma_0 + s(\sigma_0 \underline{w} - \sigma_0)) ds \right| \leq \varepsilon.$$

The second factor is treated similarly. Here we use that $f_p'(\sigma_0 \underline{w})$ and $f_p''(\sigma_0 \underline{w})$ are both small for $\underline{w}$ bounded away from 0 when N_p is small (recall Proposition 6.3). For $\underline{w}$ near 0 the smallness of C follows from the smallness of $f_p'(\sigma_0 \underline{w})\underline{w}$.

(6) *Region C.* For $\sigma \in \mathcal{Z}_{-i\tilde{\alpha}}$ with $|\sigma|$ large the correspondence $\Xi_f \leftrightarrow \xi_f$ is one-to-one, so it suffices to show Ξ_f is one-to-one on region C.

Choose $0 < \kappa < 1$ and for $\sigma, a \in C$ with $|\sigma - a| \leq \kappa|\sigma|$ write

$$\Xi_f(\sigma) - \Xi_f(a) = (\sigma - a)\left[\Xi_f'(a) + \frac{(\sigma - a)}{2}\int_0^1 (1-s)\Xi_f''(a + s(\sigma - a))ds\right]. \tag{16.30}$$

For $|\sigma|$ large we claim

$$|\Xi_f''(\sigma)| \leq \varepsilon/|\sigma|, \tag{16.31}$$

where $\varepsilon \to 0$ as $N_p \to 0$. Since $|\Xi_f'(a)| \sim 1$ for $|a|$ large, the modulus of the right side of (16.31) is $\geq \frac{1}{2}|\sigma - a|$ for ε small enough. The estimate (16.31) follows by direct computation after noting $|f_p'(\sigma)| \leq \varepsilon/|\sigma|$ for $|\sigma|$ large. For example, using Proposition 6.3 we estimate the term

$$|-\tilde{\alpha}^2\sigma b_2'(-i\tilde{\alpha}\sigma, \zeta)| \leq C|\tilde{\alpha}||\tilde{\alpha}\sigma| \leq \varepsilon/|\sigma| \text{ since } |\tilde{\alpha}\sigma| \leq 2\varepsilon_2. \tag{16.32}$$

The formula (10.22) shows that there exists $m > 0$ such that

$$\left|\Xi_{f_0}\left(\frac{\sigma}{\sigma_0}\right) - \Xi_{f_0}\left(\frac{a}{\sigma_0}\right)\right| \geq m|\sigma - a| \text{ for } \sigma, a \in C. \tag{16.33}$$

For $|\sigma - a| > \kappa|\sigma|$ ($\sigma, a \in C$) use estimate (16.10)(c) to write

(16.34)
$$|\Xi_f(\sigma) - \Xi_f(a)| \geq \left|\Xi_{f_0}\left(\frac{\sigma}{\sigma_0}\right) - \Xi_{f_0}\left(\frac{a}{\sigma_0}\right)\right| - \varepsilon_C\left|\Xi_{f_0}\left(\frac{\sigma}{\sigma_0}\right)\right| - \varepsilon_C\left|\Xi_{f_0}\left(\frac{a}{\sigma_0}\right)\right| \geq \frac{m}{2}|\sigma - a|$$

for ε_C small enough.

(7) *Adjacent regions.* Recall that the regions A, B, C are determined by the choice of parameters δ and K. The above arguments show that there exist δ_0, K_0 such that for $\delta < \delta_0$ and $K > K_0$, ξ_f is injective (for N_p small) on each of the regions A, B, C determined by the choice (δ, K). It is immediate from (10.22) and the estimates (16.10) that $\xi_f(\sigma_1) \neq \xi_f(\sigma_2)$ for $\sigma_1 \in A$, $\sigma_2 \in C$, so it remains to consider σ_j in adjacent regions.

Choose positive constants δ_j and K_j, $j = a, b$, such that $\delta_b < \delta_a < \delta_0$ and $K_b > K_a > K_0$, and which have the following additional property. There exists $M > 0$ such that if A_j, B_j, C_j are the regions determined by the choice (δ_j, K_j), we have

$$|\xi_f(\sigma)| \leq M \text{ for } \sigma \in B_a \text{ and } |\xi_f(\sigma)| > M \text{ for } \sigma \in A_b \cup C_b. \tag{16.35}$$

Suppose now that $\sigma_1 \in A_a$, $\sigma_2 \in B_a$. Considering the two cases $\sigma_1 \in A_b$, $\sigma_1 \in B_b$ and using (16.35) and the above results for single regions, we conclude $\xi_f(\sigma_1) \neq \xi_f(\sigma_2)$. The case $\sigma_1 \in B_a$, $\sigma_2 \in C_a$ is treated similarly. ∎

Proof of Proposition 10.8. In order to apply Theorem 9.1 of [O, Chapter 11], we must choose a suitable open subdomain of the ξ-plane on which to solve (16.36) below. There are three requirements:

(a) The domain should include the image of an interval $[M, \infty)$ under the map $x \to \xi$ (here, $x \in T_{M,R}$ as in (6.1)), where M can chosen independent of the parameters (ζ, h).

(b) It must be possible to choose "progressive paths" (defined below) for all points in the domain.

(c) The integrals (10.25) should all be finite; more precisely, there should be a finite upper bound independent of the choice of path and of relevant parameters such as ζ and h.

(1) *Choice of ξ-domain.* Let $v(\sigma)$ be a solution to the perturbed Bessel problem (9.3) on the dilated wedge $\mathcal{Z}_{-i\tilde{\alpha}}$. In the new variables $\xi = \xi_f(\sigma)$, $v = (\frac{d\xi}{d\sigma})^{-1/2}W$ the problem (9.3) becomes

$$W_{\xi\xi} = (\tilde{\gamma}^2\xi + \psi(\xi))W. \tag{16.36}$$

We are not able to solve (16.36) on the full open set $\xi_f(\mathcal{Z}_{-i\tilde{\alpha}})$, because of problems choosing progressive paths created by the perturbation f_p. Instead, we explain how to choose a subdomain where such paths can be chosen and which also contains the image of the segment of the x-axis, $[M,\infty)$, under the map $x \to \xi$. At first we ignore the right boundary segment of $\mathcal{Z}_{-i\tilde{\alpha}}$ and treat the wedge as if it were infinite.

We begin by specifying a domain in the Ξ-plane. For small positive constants κ, ε both less than 1, let $\Delta_\Xi(\kappa,\varepsilon) = A_\Xi \cup B_\Xi$, where A_Ξ and B_Ξ are the open subsets of $\mathbb{C}$ defined as follows. A_Ξ is the connected open set bounded by the parametrized segments

$$\{it : t \geq 0\}, \{t : t \geq 0\}, \{t - i(\kappa t + \varepsilon) : t \geq -\varepsilon\}, \{-\varepsilon + it : t \geq \kappa\varepsilon - \varepsilon\}, \tag{16.37}$$

while B_Ξ is the connected open set bounded by the segments

$$\{it : t \geq 0\}, \{t : t \geq 0\}, \{t + i(\kappa t + \varepsilon) : t \geq -\varepsilon\}, \{-\varepsilon + it : t \leq -\kappa\varepsilon + \varepsilon\}. \tag{16.38}$$

Next let $\Delta_\xi(\kappa,\varepsilon) = A_\xi \cup B_\xi \cup \mathbb{R}$, where A_ξ is the image of A_Ξ under the map $\Xi \to \xi$, where the branch of the 2/3 root is defined by taking $-\frac{3\pi}{2} < \arg\Xi < 0$ for $\Xi \in A_\Xi$, and $0 < \arg\Xi < \frac{3\pi}{2}$ for $\Xi \in B_\Xi$. Observe that $\Delta_\xi(\kappa,\varepsilon)$ is an open neighborhood of the real axis whose intersection with $\Re\xi = t$ has width $\sim t^{2/3}$ for $t > 0$ large and width $\sim |t|^{-1/2}$ for $t < 0$, $|t|$ large.

It follows from the formula (10.22) that the image of $\mathcal{Z}_{-i\tilde{\alpha}}$ under $\sigma \to \xi_{f_0}(\frac{\sigma}{\sigma_0})$ contains a subdomain of the form $\Delta_\xi(\kappa,\varepsilon)$ for some choice of κ, ε. This is because the image of $\mathcal{Z}_{cut}(\sigma_0)$ under $\sigma \to \Xi_{f_0}(\frac{\sigma}{\sigma_0})$ contains a set of the form $\Delta_\Xi(\kappa,\varepsilon)$.[36] By Proposition 10.7 and the estimates of Lemma 16.1 we deduce, after further reduction of N_p if necessary, that $\xi_f(\mathcal{Z}_{-i\tilde{\alpha}})$ must also contain a subdomain of the form $\Delta_\xi(\kappa,\varepsilon)$ for some smaller κ and ε.

Finally, we recall that the dilated wedge $\mathcal{Z}_{-i\tilde{\alpha}} = \mathcal{W}/(-i\tilde{\alpha})$ has a right boundary arc of radius $\varepsilon_2/|\tilde{\alpha}| >> 1$, where ε_2 is the radius of the right boundary arc of $\mathcal{W}$ (Definition 6.1). Thus, we define $\Delta_\Xi(\kappa,\varepsilon,\varepsilon_2)$ to be the bounded open set obtained by cutting off $\Delta_\Xi(\kappa,\varepsilon)$ with this boundary arc. With $\Delta_\xi(\kappa,\varepsilon,\varepsilon_2)$ the corresponding ξ domain, we can repeat the procedure of the previous paragraph to deduce that $\xi_f(\mathcal{Z}_{-i\tilde{\alpha}})$ contains a subdomain of the form $\Delta_\xi(\kappa,\varepsilon,\varepsilon_2')$ for some $\varepsilon_2' < \varepsilon_2$.[37]

We may now define the subdomain $\mathcal{Z}_{-i\tilde{\alpha},s}$ appearing in the statement of Proposition 10.8 as

$$\mathcal{Z}_{-i\tilde{\alpha},s} := \xi_f^{-1}\left(\Delta_\xi(\kappa,\varepsilon,\varepsilon_2')\right). \tag{16.39}$$

This domain contains the image of $[M',\infty)$ under the map $x \to \sigma$, where M' is slightly greater than M (we have $M' = M + O(|\ln(1-\varepsilon_C)|)$).

[36] We have $\Xi_{f_0}(\sigma) \sim \sigma$ for $\sigma > 0$, $|\sigma|$ large.

[37] Using estimate (16.10)(c) we can take $\varepsilon_2' = (1-\varepsilon_C)\varepsilon_2$.

(2) *Choice of progressive paths.* Define the sectors $\mathbf{S_0}$, $\mathbf{S_1}$, and $\mathbf{S_{-1}}$ by $|\arg\sigma| \le \frac{\pi}{3}$, $\frac{\pi}{3} \le \arg\sigma \le \pi$, and $-\pi \le \arg\sigma \le -\frac{\pi}{3}$, respectively. Let $\omega = \arg\tilde{\gamma}$ (recall $\tilde{\gamma} = -i\tilde{\beta}$ for $\tilde{\beta}$ as in (6.12)). From the definition of regime II we have for some small $\delta > 0$,

$$-\delta \le \arg\tilde{\gamma} \le 0. \tag{16.40}$$

DEFINITION 16.2. *Let $\Delta \subset \mathbb{C}$ be a connected open set, let $\partial\Delta$ denote its boundary, and take $j \in \{0, 1, -1\}$. We say that progressive j-paths can be chosen in Δ provided there exists a point $\alpha_j \in \partial\Delta \cap e^{-2i\omega/3}\mathbf{S_j}$, possibly at infinity, such that for all $\xi \in \Delta$ there is a path $\mathcal{P}_j$ from ξ to α_j in Δ with the following properties:*

(a) *As v traverses $\mathcal{P}_j$ from ξ to α_j, the real part of $(\tilde{\gamma}^{2/3}v)^{3/2}$ is nondecreasing. The branch of $(\tilde{\gamma}^{2/3}v)^{3/2}$ is chosen so that $\Re(\tilde{\gamma}^{2/3}v)^{3/2} \ge 0$ in $e^{-2i\omega/3}\mathbf{S_j}$ and so that this real part is ≤ 0 in $e^{-2i\omega/3}\mathbf{S_k}$, $k \ne j$.*

(b) *The path $\mathcal{P}_j$ has a parametrization $v(\tau)$ such that v'' is continuous and v' always nonvanishing or consists of a finite chain of such paths.*[38]

Remark 16.3. For example, in the case $j = 1$ the correct branch of $(\tilde{\gamma}^{2/3}v)^{3/2}$ is the one for which

$$-\frac{\pi}{3} + 2\pi - \frac{2\omega}{3} \le \arg v \le \frac{5\pi}{3} + 2\pi - \frac{2\omega}{3}. \tag{16.41}$$

The condition in part (a) of the definition is linked to the choice of weight functions $E_j(z)$ defined in section 8.3 of [O, Chapter 11]. The definition of E_j and $\mathbf{S_j}$ reflects the fact that $Ai_j(z)$ is recessive in $\mathbf{S_j}$ and dominant in $\mathbf{S_k}$, $k \ne j$.

For a given j it is easy to draw level curves of the correct branch of $\Re(\tilde{\gamma}^{2/3}\xi)^{3/2}$. A picture in the case $j = 0$, $\arg\tilde{\gamma} = 0$ is given in Figure 9.1 of [O, Chapter 11]. Aided by such a picture together with the explicit description of the (drawable) region $\Delta = \Delta_\xi(\kappa, \varepsilon, \varepsilon_2')$ given in step (1), ones sees that progressive j-paths can be chosen in Δ for $j = 0, 1, -1$. The point $\alpha_j \in \partial\Delta \cap e^{-2i\omega/3}\mathbf{S_j}$ is chosen to be a point where $\Re(\tilde{\gamma}^{2/3}\xi)^{3/2} > 0$ is maximized on $\partial\Delta \cap e^{-2i\omega/3}\mathbf{S_j}$. Depending on the value of $\omega = \arg\tilde{\gamma}$, the point α_1 or α_{-1} may need to be taken at infinity.

(3) *Finiteness of the integrals* $\int_{\alpha_j}^{\xi} |\psi(s)s^{-1/2|}\, d|s|$. By "finiteness" we mean here a finite bound that can be taken independent of the choice of j-progressive path and of the parameters ε_1, ε_2, ζ, and h appearing in the definitions of $\tilde{\gamma} = \tilde{\gamma}(\zeta, h)$ and $f_p(\sigma) = f_p(\sigma, \varepsilon_1, \varepsilon_2, \zeta, h)$ (recall Proposition 9.2). Here $\zeta \in \omega_\infty$, a neighborhood of ζ_∞, and $0 \le h \le h_0$, where ε_2, ω_∞ and h_0 were chosen in step (1) above and in the proof of Proposition 10.7 to make f_p sufficiently small; moreover, $|\tilde{\gamma}| \ge K_1$ (regime II).

Clearly, for a given fixed $N > 0$ we need only check the finiteness when at least one of $|\alpha_j|$, $|\xi|$ is $\ge N$. Observe that Δ is unbounded on the left ($\Re\xi < 0$) and, although Δ is bounded on the right for fixed h, there are choices of (ζ, h) in regime II for which the right boundary moves to infinity as $h \to 0$. Thus, Δ is effectively unbounded in both directions.

With $f = f_0 + f_p$ for $f_0(\sigma) = \frac{\sigma^2-1}{\sigma^2}$ and f_p as in (9.2), function $\psi(\xi)$ in (8.9) may be rewritten

(16.42)
$$\psi(\xi) = \frac{5}{16\xi^2} + \left[4f(\sigma)f''(\sigma) - 5f'(\sigma)^2\right]\frac{\xi}{16f^3(\sigma)} + \frac{\xi g(\sigma)}{f(\sigma)}, \text{ where } g(\sigma) = -\frac{1}{4\sigma^2}.$$

[38] This definition corrects an ambiguity in the definition given in section 9.1 of [O, Chapter 11.]

Here and in the remainder of this step $\xi = \xi_f$. Letting $\psi_0(\xi)$ denote the function obtained by replacing f by f_0 in (16.42), but leaving $\xi = \xi_f$,[39] we compute

$$\psi_0(\xi) = \frac{5}{16\xi^2} - \frac{\xi\sigma^2(\sigma^2+4)}{4(\sigma^2-1)^3}, \tag{16.43}$$

observing an important cancellation. We can write

$$\begin{aligned}\psi(\xi) &= \frac{5}{16\xi^2} + \left[4(f_0+f_p)(f_0''+f_p'') - 5(f_0'+f_p')^2\right]\frac{\xi}{16(f_0+f_p)^3} + \frac{\xi g(z)}{f_0+f_p} \\ &= \psi_0(\xi) + \psi_1(\xi),\end{aligned} \tag{16.44}$$

which defines ψ_1. First we check the finiteness of the integral

$$\int_{\alpha_j}^{\xi} |\psi_k(s)s^{-1/2}|\, d|s| \tag{16.45}$$

when $k = 0$.

Observe that when $|\sigma|$ is small or large, we have $|\xi|$ large with $\Re\xi < 0$ or > 0, respectively, and $|f_p|/|f_0| << 1$. For $|\sigma|$ large by (10.22) we have $\sigma^2 \sim \frac{4}{9}\xi^3$, so (16.43) implies $\psi_0(\xi) \sim \frac{-1}{4\xi^2}$. For $|\sigma|$ small we have $|\xi|$ large and (10.22) implies

$$\sigma \sim 2\exp\left(-\frac{2}{3}|\xi|^{3/2} - 1\right). \tag{16.46}$$

In this case (16.43) implies $\psi_0(\xi) \sim \frac{5}{16\xi^2}$, so the finiteness is again clear.

Next consider (16.45) when $k = 1$. First we write

$$\frac{\xi}{16(f_0+f_p)^3} \sim \frac{\xi}{16f_0^3}\left(1 - 3\frac{f_p}{f_0}\right) \text{ and } \frac{\xi g}{f_0+f_p} \sim \frac{\xi g}{f_0}\left(1 - \frac{f_p}{f_0}\right). \tag{16.47}$$

Now we can read off the (largest) terms appearing in ψ_1 and estimate them one by one. For example, the terms involving second derivatives are (ignoring some constant factors)

$$f_p(f_0''+f_p'')\left(\frac{\xi}{f_0^3} - 3\frac{\xi f_p}{f_0^4}\right), \quad f_0 f_p''\left(\frac{\xi}{f_0^3} - 3\frac{\xi f_p}{f_0^4}\right), \quad f_0 f_0''\frac{\xi f_p}{f_0^4}. \tag{16.48}$$

Now f_p is given by (9.4), so we can list the terms appearing in f_p'' (ignoring some constant factors):

$$\begin{aligned}&(\alpha^2 - \tilde\alpha^2\sigma^2)\tilde\alpha^2 b_1'', \;\; \sigma\tilde\alpha^3 b_1', \;\; \tilde\alpha^2 b_1,\\ &\sigma\tilde\alpha^3 b_2'', \;\; \tilde\alpha^2 b_2',\\ &\tilde\alpha^2 h b_3'',\end{aligned} \tag{16.49}$$

where the b_j derivatives are d/dt derivatives (t as in Proposition 6.3). Using Proposition 6.3 we obtain

$$\begin{aligned}&b_1' = O(|\tilde\alpha\sigma|), \; b_1'' = O(1),\\ &b_2' = O(1), \; b_2'' = O(|\tilde\alpha\sigma|^{-1}),\\ &b_3' = O(|\tilde\alpha\sigma|), \; b_3'' = O\left(\frac{1}{h}\right),\end{aligned} \tag{16.50}$$

[39]The definition of ξ_f involves f_p.

and recall that we have for $\sigma \in \mathcal{Z}_{-i\tilde{\alpha}} \supset \xi_f^{-1}(\Delta)$,

$$|\tilde{\alpha}\sigma| \leq \varepsilon_2 \text{ (for } \varepsilon_2 \text{ as in Definition 6.1).} \tag{16.51}$$

We now estimate two typical terms from (16.48):

$$\left| f_0 f'' \xi \frac{f_p}{f_0^4} \right| = 6 \left| \frac{\xi \sigma^2 f_p}{(\sigma^2 - 1)^3} \right|. \tag{16.52}$$

When $|\sigma|$ is large, the right side of (16.52) is $\leq C|\xi|/|\sigma|^4$, so the finiteness of the corresponding terms in (16.45) is clear from $|\sigma|^2 \sim \frac{4}{9}|\xi|^3$. When $|\sigma|$ is small the finiteness follows from (16.46).

Next consider one of the "worst" terms appearing in $\frac{\xi f_p''}{f_0^2}$, namely, the one corresponding to the term $\sigma\tilde{\alpha}^3 b_2''$ from (16.49). When $|\sigma|$ is large we have, using (16.50),

$$\left| \xi \frac{\sigma^4}{(\sigma^2 - 1)^2} \sigma \tilde{\alpha}^3 b_2'' \right| \leq C|\xi \sigma \tilde{\alpha}^3 (\tilde{\alpha}\sigma)^{-1}| = C|\xi||\tilde{\alpha}|^2 \leq C \frac{|\xi|}{|\sigma|^2} \leq \frac{C}{|\xi|^2}, \tag{16.53}$$

since $|\tilde{\alpha}| \leq \varepsilon_2/|\sigma|$. This gives the finiteness of the corresponding term in (16.45) at right infinity. When $|\sigma|$ is small, we write

$$\left| \xi \frac{\sigma^4}{(\sigma^2 - 1)^2} \sigma \tilde{\alpha}^3 b_2'' \right| \leq C|\xi||\sigma^5||\tilde{\alpha}^3||\tilde{\alpha}\sigma|^{-1} = C|\xi||\tilde{\alpha}|^2|\sigma|^4 \tag{16.54}$$

so the finiteness at left infinity follows from (16.46).

Next consider the term in $\frac{\xi f_p''}{f_0^2}$ corresponding to the term $\tilde{\alpha}^2 h b_3''$ in (16.49). When $|\sigma|$ is large we have

$$\left| \xi \frac{\sigma^4}{(\sigma^2 - 1)^2} \tilde{\alpha}^2 h b_3'' \right| \leq C \left| \xi \tilde{\alpha}^2 h \frac{1}{h} \right| \leq C|\xi|/|\sigma|^2 \leq C/|\xi|^2. \tag{16.55}$$

When $|\sigma|$ is small,

$$\left| \xi \frac{\sigma^4}{(\sigma^2 - 1)^2} \tilde{\alpha}^2 h b_3'' \right| \leq C \left| \xi \sigma^4 \tilde{\alpha}^2 h \frac{1}{h} \right|, \tag{16.56}$$

so finiteness at left infinity follows again from (16.46).

The estimates corresponding to the remaining terms in ψ_1 are entirely similar to those above.

(4) *Conclusion.* We have now checked that all the requirements for an application of Theorem 9.1 of [O, Chapter 11] are satisfied, so this concludes the proof of Proposition 10.8. ∎

Proposition 10.10 describes the decaying solutions of (10.11) on $[M, \infty)$. In the proof we will of course use the fact that $Ai_{\pm 1}(z)$ is recessive in the sector $\mathbf{S}_{\pm 1}$.

Proof of Proposition 10.10. The explicit formulas for $\tilde{\beta}$ and $\tilde{\gamma} = -i\tilde{\beta}$ show that for $\tilde{\beta}$ in regime II, we have

$$\arg \tilde{\gamma} \leq 0 \text{ and } \arg \tilde{\gamma} = 0 \Leftrightarrow \Re\zeta = 0. \tag{16.57}$$

The image of $[M, \infty)$ under the map $x \to \xi$ is a curve that approaches left infinity in Δ_ξ (16.39) as $x \to \infty$. The image of $[M, \infty)$ under $x \to \tilde{\gamma}^{2/3}\xi$ thus lies for x

large enough in the interior of $\mathbf{S}_1$ when $\arg\tilde{\gamma} < 0$. So Proposition 10.8 implies that $w(x) = z(x)^{-1/2}v_1(\sigma(x))$ gives a decaying solution of (10.11) on $[M, \infty)$, and thus

$$\theta(x,\zeta,h) = Y(x,\zeta,h)\begin{pmatrix} K(x,\zeta,h)\begin{pmatrix} w \\ hw_x \end{pmatrix} \\ 0 \end{pmatrix} \tag{16.58}$$

is a decaying solution of (5.1). Here we use the fact that the explicit estimates of η_1 and $\partial_\xi \eta_1$ given in Theorem 9.1 of Chapter 11 of [O] imply their contributions to $w(x)$ and $w_x(x)$ decay as well. The matrix K involves a factor of $e^{\varphi_0/h}$, so here we have used Remark 5.6. When $\Re\zeta = 0$, Remark 5.6 and the formula for w imply that θ is the desired bounded and oscillating, but not decaying, solution of (5.1). ∎

In the proof of Proposition 10.11 we will sometimes speak of "relative errors of size $O(p)$" defined as follows.

Definition 16.4 (relative error). *When a term $\eta(p)$ depending on a small parameter p (and possibly other variables) in an expression $A = B+\eta$ satisfies for some positive constant C,*

$$|\eta| \leq C|p||B|, \tag{16.59}$$

uniformly with respect to all the variables on which A, B, and η depend, we say that η is a relative error of size $O(p)$. *When $\eta = \eta_1+\cdots+\eta_N$ and η_j satisfies $|\eta_j| \leq C|p||B|$, we say that η_j* contributes a relative error of size $O(p)$.

Proof of Proposition 10.11.

(1) The proof is based on the formula (16.58), the expression for $w(x)$ given by Proposition 10.8, and a standard expansion of the Airy function.

Recall the definitions of the variables

$$t = \frac{2}{\mu}\sqrt{aD(\infty,\zeta)}e^{-\mu x/2},\ z = \frac{t}{h},\ \tilde{\gamma} = -i\tilde{\beta},\ \sigma = \frac{z}{\tilde{\gamma}}. \tag{16.60}$$

The variable z occurs in (6.13)(b), but now instead of $W(z)$ we write $w(z)$ and we will abuse notation by writing, for example, $w(z) = w(x)$ to mean $W(z(x)) = w(x)$. The variable σ occurs in (9.3) and (9.4). Recalling the tranformations that relate the dependent variables $w(z)$ of (6.13)(b) and $v(\sigma)$ of (9.3), we have

$$\begin{aligned} &v(\sigma) = \hat{w}(\tilde{\gamma}\sigma) = w(\tilde{\gamma}\sigma)(\tilde{\gamma}\sigma)^{1/2} = w(z)z^{1/2},\ z \in \mathcal{W}/h,\ \text{so} \\ &w(z) = z^{-\frac{1}{2}}v(\sigma) = z^{-\frac{1}{2}}\xi_\sigma^{-1/2}(\sigma)\left(Ai_1(\tilde{\gamma}^{2/3}\xi(\sigma)) + \eta_1(\tilde{\gamma},\xi(\sigma))\right),\ \sigma \in \mathcal{Z}_{-i\tilde{\alpha}}. \end{aligned} \tag{16.61}$$

(2) We first express the factor $C(x,\zeta) + hr(x,\zeta,h)$ appearing in the equation for $w(x)$ in terms of $\xi = \xi_f(\sigma)$, where $f = f_0 + f_p$. Using Remark 6.2 and (16.9), we obtain

$$\begin{aligned} &\frac{4}{\mu^2}(C(x,\zeta) + hr(x,\zeta,h)) = h^2z^2\left[\left(1 - \frac{\tilde{\gamma}^2}{z^2}\right) + (h^2z^2+\alpha^2)b_1(hz,\zeta) + hzb_2 + hb_3\right] \\ &\quad = -\tilde{\alpha}^2\sigma^2\left[\left(1 - \frac{1}{\sigma^2}\right) + (\alpha^2 - \tilde{\alpha}^2\sigma^2)b_1(-i\tilde{\alpha}\sigma,\zeta) - i\tilde{\alpha}\sigma b_2 + hb_3\right] \\ &\quad = -\tilde{\alpha}^2\sigma^2 f(\sigma) = -\tilde{\alpha}^2\sigma^2\xi(\xi_\sigma)^2. \end{aligned} \tag{16.62}$$

The function $\sqrt{\xi}$ was defined on $\mathcal{Z}_{cut}(z_0)$ just below (16.8), so we can use the equation

$$-\frac{\mu}{2}i\tilde{\alpha}\sigma\sqrt{\xi}\xi_\sigma = \sqrt{C(x,\zeta)+hr(x,\zeta,h)} \tag{16.63}$$

to define a branch of $\sqrt{C+hr}$ on the corresponding x-domain. Since the argument of $-i\tilde{\alpha}\sigma\sqrt{\xi}\xi'$ is close to zero for x near M,[40] we have

$$\sqrt{C+hr} = -\sqrt{\zeta^2+c_0^2\eta(x)}\,\underline{b}(x)+O(h) = -s(x,\zeta)\underline{b}(x)+O(h) \text{ for } x \text{ near } M \tag{16.64}$$

and thus

$$-\frac{\mu}{2}i\tilde{\alpha}\sigma\sqrt{\xi}\xi_\sigma = \frac{\mu}{2}hz\sqrt{\xi}\xi_\sigma = -s(x,\zeta)\underline{b}(x)+O(h) \text{ for } x \text{ near } M. \tag{16.65}$$

(3) *Preliminaries.* We will use the standard asymptotic expansions valid for $|z|$ large on $|\arg z|\leq \pi-\delta$:

$$\begin{aligned} &Ai(z) \sim \frac{e^{-\chi}}{2\sqrt{\pi}z^{1/4}}\sum_0^\infty (-1)^s\frac{u_s}{\chi^s}, \\ &Ai'(z) \sim -\frac{z^{1/4}e^{-\chi}}{2\sqrt{\pi}}\sum_0^\infty(-1)^s\frac{v_s}{\chi^s}, \text{ where } \chi=\frac{2}{3}z^{3/2},\ u_0=v_0=1. \end{aligned} \tag{16.66}$$

In the expression for the approximate solution θ_1,

$$\theta_1(x,\zeta,h) = e^{\frac{1}{h}h_1(x,\zeta)+k_1(x,\zeta)}T_1(x,\zeta), \tag{16.67}$$

we have

(16.68)

$$\begin{aligned} &T_1 = P_0+sQ_0 \text{ and, with } \mu_1(x,\zeta)=\underline{a}+s\underline{b}, \text{ where } \underline{a}=-\frac{\kappa^2\zeta}{\eta u},\ \underline{b}=-\frac{\kappa}{\eta u}, \\ &h_1(x,\zeta) = \int_0^x \mu_1(x',\zeta)dx' = \int_0^x \underline{a}(x',\zeta)dx' + \int_0^x s(x',\zeta)\underline{b}(x')dx' := h_{1a}+h_{1b}. \end{aligned}$$

Since $\frac{d}{dx}\varphi_0(x,\zeta,h) = \frac{a+d}{2} = \frac{\underline{a+d}}{2}+O(h) = \underline{a}+O(h)$, we obtain

$$\varphi_0 - h_{1a} = O(h)+C_a(\zeta,h) \text{ near } x=M, \text{ where } C_a(\zeta,h)=O(1). \tag{16.69}$$

(4) *Approximations.* Using the formula (16.61) for $w(z)$ and the expansions (16.66), and setting $\psi = e^{-\frac{2\pi i}{3}}\tilde{\gamma}^{2/3}\xi$, for x near M we approximate[41]

$$\begin{aligned} &\text{(a) } w(z) \sim z^{-1/2}\xi_\sigma^{-1/2}Ai_1(\tilde{\gamma}^{2/3}\xi) \sim \frac{1}{2\sqrt{\pi}}\,z^{-1/2}\xi_\sigma^{-1/2}e^{-\frac{2}{3}\psi^{3/2}}\psi^{-1/4}, \\ &\text{(b) } w_z(z) \sim z^{-1/2}\xi_\sigma^{-1/2}Ai_1'(\tilde{\gamma}^{2/3}\xi)\tilde{\gamma}^{2/3}\xi_\sigma\frac{1}{\tilde{\gamma}} \\ &\qquad \sim -\frac{1}{2\sqrt{\pi}}\,z^{-1/2}\xi_\sigma^{-1/2}e^{-\frac{2\pi i}{3}}e^{-\frac{2}{3}\psi^{3/2}}\psi^{1/4}\tilde{\gamma}^{-1/3}\xi_\sigma. \end{aligned} \tag{16.70}$$

[40] This is because z is large with $\arg z\sim 0$ for x near M.
[41] Here the roots of ψ are defined for $|\arg\psi|\leq\pi-\delta$.

In the first "$\sim$" of (16.70)(a) we have ignored the η_1 contribution to w, while in the second "$\sim$" we have ignored contributions from terms in the expansion of $Ai(z)$ corresponding to $s \geq 1$. The computations below will make it clear that these approximations contribute relative errors of size $O(1/\tilde{\beta})$ in our approximation of $\theta(x, \zeta, h)$. In the approximation (16.70)(b) we have ignored similar terms contributing relative errors of the same size. In addition, we have ignored the term $d_z(z^{-1/2}\xi_\sigma^{-1/2})Ai_1(\tilde{\gamma}^{2/3}\xi)$, which contributes a relative error of size $O(h)$. Thus, we obtain

$$h\frac{d}{dx}w = -\frac{\mu}{2}hzw_z \sim \frac{\mu}{2}h\frac{1}{2\sqrt{\pi}}\, z^{1/2}\xi_\sigma^{-1/2}e^{-\frac{2\pi i}{3}}e^{-\frac{2}{3}\psi^{3/2}}\psi^{1/4}\tilde{\gamma}^{-1/3}\xi_\sigma \tag{16.71}$$

for x near M.

(5) Using the formula (16.58) for the exact decaying solution θ, we find

$$\theta(x, \zeta, h) \sim e^{\frac{\varphi_0}{h}}[b^{1/2}wP_0 + b^{-1/2}(hw_x)Q_0]. \tag{16.72}$$

Here we have ignored relative errors of size $O(h)$ by ignoring the $O(h)$ entries in Y_2 (recall $Y = Y_1Y_2$) and the (2,1) entry of K, which is of size $O(h)$. Plugging in (16.70)(a) and (16.71) we obtain[42]

(16.73)
$$\begin{aligned}\theta &\sim e^{\frac{\varphi_0}{h}-\frac{2}{3}\psi^{3/2}}\left(\frac{1}{2\sqrt{\pi}}b^{1/2}z^{-1/2}\xi_\sigma^{-1/2}\psi^{-1/4}\right)\left[P_0 + \frac{\mu}{2}b^{-1}hz\psi^{1/2}e^{-\frac{2\pi i}{3}}\tilde{\gamma}^{-1/3}\xi_\sigma Q_0\right]\\ &= e^{\frac{\varphi_0}{h}+\frac{2}{3}\tilde{\gamma}\xi^{3/2}}\left(\frac{1}{2\sqrt{\pi}}b^{1/2}z^{-1/2}\xi_\sigma^{-1/2}\psi^{-1/4}\right)\left[P_0 - \frac{\mu}{2}b^{-1}hz\sqrt{\xi}\xi_\sigma Q_0\right].\end{aligned}$$

From (16.65) and $b = \underline{b} + O(h)$ we find

$$\begin{aligned}&-\frac{\mu}{2}b^{-1}hz\sqrt{\xi}\xi_\sigma = s(x,\zeta) + O(h),\\ &\frac{d}{dx}\left(\frac{2}{3}\tilde{\gamma}\xi^{3/2}\right) = -\sqrt{\xi}\xi_\sigma\frac{\mu}{2}z = \frac{s\underline{b}}{h} + O(1) = \frac{1}{h}\frac{d}{dx}h_{1b} + O(1) \Rightarrow \frac{2}{3}\tilde{\gamma}\xi^{3/2}\\ &= \frac{h_{1b}}{h} + \frac{C_b(\zeta,h)}{h} + O(1)\end{aligned} \tag{16.74}$$

near $x = M$. With (16.69) we obtain

$$\frac{\varphi_0}{h} + \frac{2}{3}\tilde{\gamma}\xi^{3/2} = \frac{h_1(x,\zeta)}{h} + g(x,\zeta,h); \tag{16.75}$$

here $g = \frac{g_1(\zeta,h)}{h} + g_2(x,\zeta,h)$ with $g_1 = O(1)$ and $g_2 = O(1)$ near $x = M$. Using (16.74) and ignoring another $O(h)$ relative error, we can now rewrite (16.73)

(16.76)
$$\theta \sim e^{\frac{h_1(x,\zeta)}{h}+g}\left(\frac{1}{2\sqrt{\pi}}b^{1/2}z^{-1/2}\xi_\sigma^{-1/2}\psi^{-1/4}\right)T_1 = G(x,\zeta,h)\theta_1(x,\zeta,h) \text{ near } x = M,$$

where the nonvanishing scalar function

$$G(x,\zeta,h) = e^g e^{-k_1}\left(\frac{1}{2\sqrt{\pi}}b^{1/2}z^{-1/2}\xi_\sigma^{-1/2}\psi^{-1/4}\right). \tag{16.77}$$

[42]Here we use $\sqrt{\psi} = e^{-\frac{\pi i}{3}}\tilde{\gamma}^{1/3}\sqrt{\xi}$.

Setting

$$H(x,\zeta,h) = G^{-1}(x,\zeta,h), \tag{16.78}$$

we obtain the estimate of Proposition 10.11. □

16.2. Proofs for regime I. This subsection gives the proofs of Propositions 10.1, 10.2, 10.4, and 10.5. We begin by examining the change of variable $\sigma \to \xi_f(\sigma)$.

Proof of Proposition 10.1. The proof is parallel to that of Proposition 10.7 for regime II, but simpler.

(1) The analyticity of ξ_f follows immediately from the fact that $f + f_p$ is nonvanishing on $\mathcal{Z}_{\tilde{\alpha}}$ for N_p sufficiently small. This nonvanishing makes regime I much easier to treat than regime II.

(2) *Estimates of* $\xi_f - \xi_{f_0}$. Here we provide the analogue of Lemma 16.1 for regime I. For N_p small we have

$$\sqrt{f_0 + f_p} = \sqrt{f_0} + O(f_p/\sqrt{f_0}) \text{ on } \mathcal{Z}_{\tilde{\alpha}}. \tag{16.79}$$

Thus, given $K >> 1$, there exists a positive constant $\varepsilon = \varepsilon(N_p)$, which can be taken to approach 0 as $N_p \to 0$, such that

$$\begin{aligned} &|\xi_f(\sigma) - \xi_{f_0}(\sigma)| \leq \varepsilon \text{ for } |\sigma| \leq K, \\ &|\xi_f(\sigma) - \xi_{f_0}(\sigma)| \leq \varepsilon |\xi_{f_0}(\sigma)| \text{ for } |\sigma| \geq K. \end{aligned} \tag{16.80}$$

(3) *Injectivity.* Parallel to the proof of Proposition 10.7, we divide $\mathcal{Z}_{\tilde{\alpha}}$ into subregions A, B, and C consisting of σ with respectively small, medium, and large modulus, and first prove injectivity on each subregion. The arguments used to treat regions B and C in the case of regime II can be repeated (almost) verbatim here. The treatment of region A is much the same as before, but easier. Again, one starts with (16.21) and shows that the integral has large modulus. The case of adjacent regions can be treated as in regime II to finish the proof. □

Proof of Proposition 10.2. In order to apply Theorem 3.1 in [O, Chapter 10], there are three requirements:

(a) We must choose a suitable subdomain Δ_ξ of the ξ plane on which to solve (10.9). The domain should include the image of an interval $[M, \infty)$ under the map $x \to \xi$ (here, $x \in T_{M,R}$ as in (6.1)), where M can be chosen independent of the parameters (ζ, h).

(b) It must be possible to choose "progressive paths" (defined below) for all points in the domain.

(c) The integrals (10.4) should all be finite, with bounds independent of the choice of path and the parameters ζ and h.

(1) *Definition of progressive paths.* Let $\Delta \subset \mathbb{C}$ be an open, connected set and let $\partial\Delta$ denote its boundary.

(a) We say that progressive 1-paths can be chosen in Δ provided there exists a point $\alpha_1 \in \partial\Delta$, possibly at infinity, such that any point $\xi \in \Delta$ can be linked to α_1 by a path $\mathcal{P}_1$ in Δ such that as v traverses $\mathcal{P}_1$ from α_1 to ξ, the quantity $\Re(\tilde{\beta} v)$ is nondecreasing.

(b) We say that progressive 2-paths can be chosen in Δ provided there exists a point $\alpha_2 \in \partial\Delta$, possibly at infinity, such that any point $\xi \in \Delta$ can be linked to α_2 by a path $\mathcal{P}_2$ in Δ such that as v traverses $\mathcal{P}_2$ from α_2 to ξ, the quantity $\Re(\tilde{\beta} v)$ is nonincreasing.

The paths are assumed to have a parametrization with the same regularity as described in Definition 16.2(b).

(2) *Choice of the domain* Δ_ξ*.* At first we ignore the right boundary segment of $\mathcal{Z}_{\tilde\alpha}$ and treat this wedge as if it were infinite.

For small positive constants κ, ε define a domain $\Delta_\xi(\kappa,\varepsilon)$ to be the open set whose boundary consists of the segments

$$\{t+i\varepsilon : t\leq 0\},\ \{t-i\varepsilon : t\leq 0\},\ \{t+i(\kappa t+\varepsilon) : t\geq 0\},\ \{t-i(\kappa t+\varepsilon) : t\geq 0\}. \tag{16.81}$$

Recall that we have

$$\xi_{f_0}(\sigma)=\begin{cases}\log(\frac{\sigma}{2})+1+o(1) \text{ for } |\sigma| \text{ small},\\ \sigma+o(1) \text{ for } |\sigma| \text{ large.}\end{cases} \tag{16.82}$$

Together with the formula (10.2) for ξ_{f_0}, this implies that when $\arg\tilde\alpha\sim 0$, the open set $\xi_{f_0}(\mathcal{Z}_{\tilde\alpha})$ contains a set of the form $\Delta_\xi(\kappa,\varepsilon)$ for some choice of κ, ε. Proposition 10.1 and the estimates (16.80) then imply, after further reduction of N_p if necessary, that the perturbed domain $\xi_f(\mathcal{Z}_{\tilde\alpha})$ also contains a subdomain of the form $\Delta_\xi(\kappa,\varepsilon)$ for some smaller κ and ε.[43]

Recall that the dilated wedge $\mathcal{Z}_{\tilde\alpha}$ has a right boundary arc of radius $\varepsilon_2/|\tilde\alpha| >> 1$ for ε_2 as in Definition 6.1. We define $\Delta_\xi(\kappa,\varepsilon,\varepsilon_2)$ to be the bounded open set obtained by cutting off $\Delta_\xi(\kappa,\varepsilon)$ with this boundary arc. We then repeat the procedure above to conclude that $\xi_f(\mathcal{Z}_{\tilde\alpha})$ contains a subdomain of the form $\Delta_\xi=\Delta_\xi(\kappa,\varepsilon,\varepsilon_2')$ for some $\varepsilon_2'<\varepsilon_2$ (but close to ε_2). Finally, we define the subdomain $\mathcal{Z}_{\tilde\alpha,s}$ appearing in the statement of Proposition 10.2 as

$$\mathcal{Z}_{\tilde\alpha,s}:=\xi_f^{-1}(\Delta_\xi(\kappa,\varepsilon,\varepsilon_2')). \tag{16.83}$$

Provided N_p is small enough, this domain contains the image of $[M',\infty)$ under the map $x\to\sigma$, where M' is slightly greater than M.

Next consider the other extreme case where $\arg\tilde\alpha=\frac{\pi}{2}-\delta$. The wedge $\mathcal{Z}_{\tilde\alpha}=\mathcal{W}/\tilde\alpha$ then consists of points σ with

$$-\varepsilon_1-\frac{\pi}{2}+\delta<\arg\sigma<\varepsilon_1-\frac{\pi}{2}+\delta,\ 0<|\sigma|<\varepsilon_2/|\tilde\alpha| \tag{16.84}$$

for $\varepsilon_1<\delta$ as in Definition 6.1. Using (16.82) and the formula (10.2) for ξ_{f_0}, we see that $\xi_{f_0}(\mathcal{Z}_{\tilde\alpha})$ contains a domain, call it $\Delta_\xi(\rho_1,\rho_2,\varepsilon_2)$, similar to $\Delta_\xi(\kappa,\varepsilon,\varepsilon_2)$ above, *except* that the part of $\Delta_\xi(\rho_1,\rho_2,\varepsilon_2)$ corresponding to small (resp., large) $|\sigma|$ consists of points satisfying[44]

$$\rho_1<\Im\xi<\rho_2,\ \text{ respectively, }\ \rho_1<\arg\xi<\rho_2, \tag{16.85}$$

for constants ρ_j such that

$$-\varepsilon_1-\frac{\pi}{2}+\delta<\rho_1<\rho_2<\varepsilon_1-\frac{\pi}{2}+\delta. \tag{16.86}$$

[43]Helpful drawings of the range of ξ_{f_0} are given in Figures 7.1 and 7.2 of Chapter 10 of [O].

[44]There is a sharp bend in the domain, downward and to the right, which occurs near points $\xi_{f_0}(\sigma)$ for σ close to $-i$, since $f_0(-i)=0$. However, note that $-i\notin\mathcal{Z}_{\tilde\alpha}$.

As above the estimates (16.80) imply that for N_p small the perturbed domain $\xi_f(\mathcal{Z}_{\tilde{\alpha}})$ contains a set $\Delta_\xi = \Delta_\xi(\rho_1, \rho_2, \varepsilon_2)$ of the same form for a slightly different choice of $(\rho_1, \rho_2, \varepsilon_2)$, and we define

$$\mathcal{Z}_{\tilde{\alpha},s} := \xi_f^{-1}(\Delta_\xi(\rho_1, \rho_2, \varepsilon_2')). \tag{16.87}$$

As before this set can be chosen to include the image of $[M', \infty)$ under the map $x \to \sigma$, where M' is slightly greater than M.

Domains Δ_ξ corresponding to other choices of $\tilde{\beta}$ in regime I are chosen by the method just described. If we write $\tilde{\alpha} = (a_1 + ia_2)$, a progressive 1-path is characterized by the property that its tangent vector $v_1 + iv_2$ at any given point satisfies $v_1 a_1 - v_2 a_2 \geq 0$; that is, the vector (v_1, v_2) makes an angle $\leq \frac{\pi}{2}$ with $(a_1, -a_2)$. A sketch of the range of admissible tangent vectors shows that progressive 1-paths can be chosen in the domain Δ_ξ described above if we take α_1 to be any point at left infinity in Δ_ξ. Similar considerations show that progressive 2-paths can be chosen if α_2 is taken to be a point on the right boundary arc of Δ_ξ where $\Re(\tilde{\beta}\xi)$ is maximized.

(3) *Finiteness of the integrals* $\int_{\alpha_j}^{\xi} |\psi(r)|\, d|r|$. The argument is much like that for regime II, so here we focus on the main differences. First observe that since $\xi(\sigma) = \int_{\sigma_0}^{\sigma} \sqrt{f(s)} ds$,

$$\int_{\mathcal{P}} |\psi(\xi)|\, d|\xi| = \int_{\xi^{-1}(\mathcal{P})} |\psi(\xi(\sigma))\sqrt{f(\sigma)}|\, d|\sigma|, \tag{16.88}$$

for a given path $\mathcal{P}$ in Δ_ξ. So we must check the finiteness of the integral on the right at 0 and ∞.

We have $f = f_0 + f_p$, where f_0 and f_p are now defined in (9.2), and

$$\psi(\xi_f(\sigma)) = \frac{g(\sigma)}{f(\sigma)} + \frac{4f(\sigma)f'' - 5f'^2}{16f^3}, \text{ where } g(\sigma) = -\frac{1}{4\sigma^2} \text{ and } f' = d_\sigma f. \tag{16.89}$$

Observe that for N_p small,

$$\sqrt{f}(\sigma) \sim \sqrt{f_0}(\sigma) \sim \begin{cases} \frac{1}{\sigma} \text{ for } |\sigma| \text{ small,} \\ 1 \text{ for } |\sigma| \text{ large.} \end{cases} \tag{16.90}$$

Letting $\psi_0(\sigma)$ denote the function obtained by setting $f_p = 0$ on the right in (16.89), we have

$$\psi_0(\sigma) = \frac{1}{4}\frac{\sigma^2(4-\sigma^2)}{(1+\sigma^2)^3}, \tag{16.91}$$

so the integral on the right in (16.88), with $\psi(\xi(\sigma))$ replaced by $\psi_0(\sigma)$, is integrable at 0 and at ∞. We note that in the computation of $\psi_0(\sigma)$, a bad term of order $O(1)$ near $\sigma = 0$ cancels out.

Next define $\psi_1(\sigma)$ by

$$\psi(\xi_f(\sigma)) = \psi_0(\sigma) + \psi_1(\sigma). \tag{16.92}$$

Writing

$$\frac{1}{f_0 + f_p} = \frac{1}{f_0}\left(1 - \frac{f_p}{f_0} + \cdots\right), \; \frac{1}{(f_0 + f_p)^3} = \frac{1}{f_0^3}\left(1 - 3\frac{f_p}{f_0} + \cdots\right), \tag{16.93}$$

we see that the main contribution of g/f to ψ_1 is

$$-\frac{gf_p}{f_0^2} = \begin{cases} O(\sigma^2) \text{ near } \sigma = 0, \\ O(\frac{1}{\sigma^2}) \text{ near } \infty, \end{cases} \tag{16.94}$$

so the corresponding contributions to (16.88) are finite.

It remains to consider the contribution of $(4ff'' - 5f'^2)/16f^3$ to ψ_1. The terms involving second derivatives have the same form as the terms in (16.48) *after* setting the factor of ξ there equal to one. The terms in f_p'' have the same form as (16.49), and the estimates (16.50) still apply. We estimate the contribution of one of the "worst terms" appearing in $\frac{f_p''}{f_0^2}$, namely, the one corresponding to the term $\tilde{\alpha}^2 h b_3''$ in (16.49). When $|\sigma|$ is large we have

$$\left|\frac{\sigma^4}{(\sigma^2+1)^2}\tilde{\alpha}^2 h b_3''\right| \leq C\left|\tilde{\alpha}^2 h \frac{1}{h}\right| \leq C/|\sigma|^2. \tag{16.95}$$

The corresponding contribution of (16.95) to (16.88) is thus integrable near infinity. When $|\sigma|$ is small,

$$\left|\frac{\sigma^4}{(\sigma^2+1)^2}\tilde{\alpha}^2 h b_3''\right| \leq C\left|\sigma^4\tilde{\alpha}^2 h \frac{1}{h}\right| = |\sigma^4\tilde{\alpha}^2| \leq |\sigma|^4, \tag{16.96}$$

so the corresponding contribution to (16.88) is integrable near $\sigma = 0$.

The estimates corresponding to the remaining terms in ψ_1 are similar to those above.

(4) *Conclusion.* We have now checked that all the requirements for an application of Theorem 3.1 of [O, Chapter 10] are satisfied, so this concludes the proof of Proposition 10.2. ∎

Proof of Proposition 10.4. The image of $[M, +\infty)$ under the map $x \to \xi(\sigma(x))$ is a curve that remains close to the real axis and approaches left infinity in Δ_ξ as $x \to \infty$. Thus, $\Re(\tilde{\beta}\xi(\sigma(x))) \to -\infty$ as $x \to \infty$ for $\tilde{\beta}$ in regime I. Since $\xi_\sigma(\sigma) = O(\frac{1}{\sigma})$ for σ near 0 and $z = \sigma\tilde{\beta}$, we have

$$z^{-1/2}(x)\xi_\sigma^{-1/2}(\sigma(x)) = O(1/|\tilde{\beta}|^{1/2}) \text{ for large } |x|. \tag{16.97}$$

Together with the estimates for η_1 in Proposition 10.2, the above statements imply that for $w(x)$ given by (10.12), (w, hw_x) is a decaying solution of (10.11). ∎

Proof of Proposition 10.5. (1) The proof is parallel to that of Proposition 10.11, so we focus on the main differences. Recall the definitions of the variables

$$t = \frac{2}{\mu}\sqrt{aD(\infty,\zeta)}e^{-\mu x/2},\ z = \frac{t}{h},\ \sigma = \frac{z}{\tilde{\beta}}. \tag{16.98}$$

With notation similar to (16.61) we write $w(z)$ for the unknown function $W(z)$ in (6.13)(b) and

$$w(z) = z^{-\frac{1}{2}}v_1(\sigma) = z^{-\frac{1}{2}}\xi_\sigma^{-1/2}(\sigma)\left(e^{\tilde{\beta}\xi(\sigma)} + \eta_1(\tilde{\beta},\xi(\sigma))\right),\ \sigma \in \mathcal{Z}_{\tilde{\alpha}}. \tag{16.99}$$

(2) Using Remark 6.2 and $\xi_\sigma = \sqrt{f}$, we obtain

(16.100)
$$\frac{4}{\mu^2}(C(x,\zeta) + hr(x,\zeta,h)) = \tilde{\alpha}^2\sigma^2\left[\left(1+\frac{1}{\sigma^2}\right) + (\alpha^2+\tilde{\alpha}^2\sigma^2)b_1(\tilde{\alpha}\sigma,\zeta) + \tilde{\alpha}\sigma b_2 + hb_3\right]$$
$$= \tilde{\alpha}^2\sigma^2 f(\sigma) = \tilde{\alpha}^2\sigma^2\xi_\sigma^2.$$

Thus,

(16.101)
$$\frac{\mu}{2}\tilde{\alpha}\sigma\xi_\sigma = \frac{\mu}{2}hz\xi_\sigma = \sqrt{C(x,\zeta)+hr(x,\zeta,h)} = -s(x,\zeta)\underline{b}(x) + O(h) \text{ for } x \text{ near } M.$$

(3) *Approximations.* Using the formula (16.99), for x near M we approximate

$$\text{(a) } w(z) \sim z^{-1/2}\xi_\sigma^{-1/2}e^{\tilde{\beta}\xi},$$
$$\text{(b) } w_z(z) \sim z^{-1/2}\xi_\sigma^{-1/2}e^{\tilde{\beta}\xi}\tilde{\beta}\xi_\sigma\frac{1}{\tilde{\beta}} = z^{-1/2}\xi_\sigma^{1/2}e^{\tilde{\beta}\xi}. \tag{16.102}$$

In (16.102)(a) we have ignored an $O(1/|\tilde{\beta}|)$ relative error coming from the η_1 contribution to w. In the approximation (16.102)(b) we have ignored a similar term contributing a relative error of the same size. In addition, we have ignored the term $d_z(z^{-1/2}\xi_\sigma^{-1/2})e^{\tilde{\beta}\xi}$, which contributes a relative error of size $O(h)$. Thus, we obtain

$$hw_x = -\frac{\mu}{2}hzw_z \sim -\frac{\mu}{2}hz^{1/2}\xi_\sigma^{1/2}e^{\tilde{\beta}\xi} \tag{16.103}$$

for x near M.

(4) Using the formula (16.58) for the exact decaying solution θ, we find as before

$$\theta(x,\zeta,h) \sim e^{\frac{\varphi_0}{h}}[b^{1/2}wP_0 + b^{-1/2}(hw_x)Q_0]. \tag{16.104}$$

Plugging in (16.102)(a) and (16.103) we obtain

$$\theta \sim e^{\frac{\varphi_0}{h}+\tilde{\beta}\xi}\left(b^{1/2}z^{-1/2}\xi_\sigma^{-1/2}\right)\left[P_0 - \frac{\mu}{2}b^{-1}hz\xi_\sigma Q_0\right]. \tag{16.105}$$

From (16.101) and $b = \underline{b} + O(h)$ we find

$$-\frac{\mu}{2}b^{-1}hz\xi_\sigma = s(x,\zeta) + O(h),$$
$$\frac{d}{dx}(\tilde{\beta}\xi) = -\xi_\sigma\frac{\mu}{2}z = \frac{s\underline{b}}{h} + O(1) = \frac{1}{h}\frac{d}{dx}h_{1b} + O(1) \text{ near } x = M \tag{16.106}$$

for h_{1b} as in (16.68). As in (16.75) we obtain

$$\frac{\varphi_0}{h} + \tilde{\beta}\xi = \frac{h_1(x,\zeta)}{h} + g(x,\zeta,h) \text{ near } x = M \tag{16.107}$$

for a function g as in (16.75). Using (16.106) and ignoring another $O(h)$ relative error, we can now rewrite (16.105) as

$$\theta \sim e^{\frac{h_1(x,\zeta)}{h}+g}\left(b^{1/2}z^{-1/2}\xi_\sigma^{-1/2}\right)T_1 = G(x,\zeta,h)\theta_1(x,\zeta,h) \text{ near } x = M, \tag{16.108}$$

where the nonvanishing scalar function

$$G(x,\zeta,h) = e^g e^{-k_1}\left(b^{1/2}z^{-1/2}\xi_\sigma^{-1/2}\right). \tag{16.109}$$

Setting

$$H(x,\zeta,h) = G^{-1}(x,\zeta,h), \tag{16.110}$$

we obtain the estimate of Proposition 10.5. ∎

17. Regime III. In this section we prove Propositions 10.13, 10.14, 10.16, and 10.17. Recall that $f = f_0 + f_p$, where

$$f_0(s) = \frac{1}{s} \text{ and } f_p(s) = \frac{1}{s}\left[(4s+\alpha^2)b_1(2s^{1/2},\zeta) + 2s^{1/2}b_2(2s^{1/2},\zeta) + hb_3(2s^{1/2},\zeta,h)\right]. \tag{17.1}$$

First we prove Proposition 10.13, which concerns the change of variable defined by

$$2\xi^{1/2}(s) = \int_0^s f^{1/2}(r)dr \text{ for } s \in \mathcal{W}^2/4. \tag{17.2}$$

Proof of Proposition 10.13. For N_p small we have

$$\sqrt{f} = \frac{1}{\sqrt{s}}(1+\varepsilon_1(s)) \text{ where } |\varepsilon_1(s)| << 1; \tag{17.3}$$

thus, $\xi(s)$ is analytic on $\mathcal{W}^2/4$. From (17.3) and (17.2) we obtain

$$\sqrt{\xi(s)} = \sqrt{s}(1+\varepsilon_2(s)), \quad |\varepsilon_2(s)| << 1, \tag{17.4}$$

and thus, since $\xi^{-1/2}\xi_s = \sqrt{f}$, we have

$$\xi_s(s) = 1 + \varepsilon_3(s), \text{ where } |\varepsilon_3(s)| << 1. \tag{17.5}$$

This implies injectivity on $\mathcal{W}^2/4$ since

$$|\xi(s_1) - \xi(s_2)| = \left|(s_1 - s_2)\int_0^1 \xi_s(s_2 + r(s_1 - s_2))dr\right| \geq \frac{1}{2}|s_1 - s_2|. \tag{17.6}$$ ∎

The proof of Proposition 10.14 can be based on Theorem 9.1 of Chapter 12 of [O] in the case where $\tilde{\beta} \geq 0$. However, the latter theorem does not treat the case of $\tilde{\beta}$ nonreal needed here, and the proof given in [O] fails in that case.[45] We show next how the proof of this theorem can be modified to treat the case $\Re\tilde{\beta} \geq 0$.

Proof of Theorem 9.1 *of Chapter* 12 *of* [O] *for* $\Re\tilde{\beta} \geq 0$. (1) The modified argument uses the following estimates for the Bessel functions I_ν, K_ν proved in section 16 of [O2]. Let $\mathbf{M}$ denote a bounded subset of the half-plane $\Re\nu \geq 0$. For $\nu \in \mathbf{M}$ and $|\arg z| \leq \pi/2$ we have

$$|I_\nu(z)| \leq kV_\nu(z), \quad |K_\nu(z)| \leq kX_\nu(z), \tag{17.7}$$

where

$$\begin{aligned} &V_\nu(z) = \frac{|z^\alpha e^z|}{1+|z|^{\alpha+\frac{1}{2}}}, \quad X_\nu(z) = \ell_\nu(z)\frac{1+|z|^\alpha}{1+|z|^{\frac{1}{2}}}\frac{e^{-z}}{|z|^\alpha}, \\ &\ell_\nu(z) = \ln\frac{1+2|z|}{|z|} \ (|\nu| < \delta), \quad \ell_\nu(z) = 1 \ (|\nu| \geq \delta), \end{aligned} \tag{17.8}$$

where $\alpha = \Re\nu \geq 0$ and δ is an arbitrary number in the range $0 < \delta < \frac{1}{2}$. The constant k is independent of μ and z but depends on δ.

[45] For example, the properties of the weight function $\mathfrak{E}_\nu(z)$ defined in (8.08) of Chapter 12 of [O] are derived using the fact that when $\nu \geq 0$, the modified Bessel function $K_\nu(z)$ does not vanish in $|\arg z| \leq \pi/2$. But when $\nu = i|\nu| \neq 0$, for example, K_ν has infinitely many zeros on the positive real axis [FS].

(2) Next, in place of the weight function $\mathfrak{E}_\nu$ defined in (8.08) of [O, Chapter 12], we redefine $\mathfrak{E}_\nu$ as

$$\mathfrak{E}_\nu(z) := \left(\frac{V_\nu(z)}{X_\nu(z)}\right)^{1/2} \text{ for } \nu \in M,\ |\arg z| \leq \pi/2. \tag{17.9}$$

It is easy to check that for $\nu \in \mathbf{M}$

$$\begin{aligned} &\mathfrak{E}_\nu(z) \sim \begin{cases} |e^z|, & |z| \text{ large} \\ |z|^\alpha, & |z| \text{ small} \end{cases} \quad \text{for } |\nu| \geq \delta, \\ &\mathfrak{E}_\nu(z) \sim \begin{cases} \ln 2\,|e^z|, & |z| \text{ large} \\ (\ln \frac{1}{|z|})^{-\frac{1}{2}}|z|^\alpha, & |z| \text{ small} \end{cases} \quad \text{for } |\nu| < \delta. \end{aligned} \tag{17.10}$$

For $|z|$ of intermediate size $\mathfrak{E}_\nu(z)$ is continuous and bounded away from 0 for each $\nu \in \mathbf{M}$; positive upper and lower bounds can be chosen independently of $\nu \in \mathbf{M}$, $|\arg z| \leq \pi/2$.

Following [O] we next define functions $\mathfrak{M}_\nu(z)$ and $\vartheta(z)$ by the equations

(17.11)
$$|I_\nu(z)| = \mathfrak{E}_\nu(z)\mathfrak{M}_\nu(z)\cos\vartheta(z), \quad |K_\nu(z)| = \mathfrak{E}_\nu^{-1}(z)\mathfrak{M}_\nu(z)\sin\vartheta(z), \text{ for } |\arg z| \leq \pi/2.$$

Thus,

$$\mathfrak{M}_\nu(z) = [\mathfrak{E}_\nu^{-2}(z)|I_\nu(z)|^2 + \mathfrak{E}_\nu^2(z)|K_\nu(z)|^2]^{1/2}. \tag{17.12}$$

Using (17.7) and (17.10) one readily verifies

$$\mathfrak{M}_\nu(z) \leq C \begin{cases} \frac{1}{|z|^{1/2}}, & |z| \text{ large },\\ 1, & |z| \text{ small}, |\nu| \geq \delta,\\ (\ln \frac{1}{|z|})^{1/2}, & |z| \text{ small}, |\nu| < \delta, \end{cases} \tag{17.13}$$

where C can be chosen independent of $\nu \in M$. One can now define bounded constants μ_j, $j = 1, \ldots, 4$, as in (8.26), (8.27) of [O, Chapter 12]; they can now be chosen independent of $\nu \in \mathbf{M}$.

(3) With these definitions the remainder of the proof of Theorem 9.1 in [O, Chapter 12] goes essentially as before. For example, in the error estimate for the solution expressed in terms of I_ν, progressive paths are those along which both $\Re t^{1/2}$ and $|t|$ are nondecreasing as t passes from 0 to ξ. It follows from this and the properties of $\mathfrak{E}_\nu$ given in and below (17.10) that $\mathfrak{E}_\nu^{-1}(u\xi^{1/2})\mathfrak{E}_\nu(ut^{1/2}) \leq N$, for some N that can be chosen independently of t, ζ and the particular progressive path being considered. Here $u > 0$ is a large parameter, taken to be $\frac{2}{h}$ in our application to Proposition 10.14. Thus, the key estimate (9.08) of [O, Chapter 12] of the kernel $K(\xi, v)$ in the integral equation for the error term still holds, but with 2 replaced by a larger constant.[46] □

[46] The estimate of $K(\xi, v)$ just above (9.08) in [O, Chapter 12] (ζ is used in place of ξ there) is incorrect, but a slightly modified estimate of $|K(\xi, v)|$ leading to (9.08) is easily given.

Proof of Proposition 10.14. In order to apply this version of Theorem 9.1 in [O, Chapter 12], there are three requirements:

(a) We must choose a suitable subdomain Δ_ξ of the ξ plane on which to solve (10.35). The domain should include the image of an interval $[M,\infty)$ under the map $x \to \xi$ (here, $x \in T_{M,R}$ as in (6.1)), where M can be chosen independent of the parameters (ζ, h).

(b) It must be possible to choose "progressive paths" (defined below) for all points in the domain.

(c) The integrals (10.37) should all be finite, with bounds independent of the choice of path and the parameters ζ and h.

(1) *Definition of progressive paths.* Let Δ be an open, connected subset of $\{\xi : |\arg\xi| < \pi/2\}$ and let $\partial\Delta$ denote its boundary. We suppose $0 \in \partial\Delta$.

(a)We say that progressive 1-paths can be chosen in Δ provided that any point $\xi \in \Delta$ can be linked to the origin by a path $\mathcal{P}_1$ in Δ such that as v traverses $\mathcal{P}_1$ from 0 to ξ, both $\Re v^{1/2}$ and $|v|$ are nondecreasing.

(b)We say that progressive 2-paths can be chosen in Δ provided there exists a point $\alpha \in \partial\Delta$ with the following property: any point $\xi \in \Delta$ can be linked to α by a path $\mathcal{P}_2$ in Δ such that as v traverses $\mathcal{P}_2$ from α to ξ, both $\Re v^{1/2}$ and $|v|$ are nonincreasing.

The paths are assumed to have a parametrization with the same regularity as described in Definition 16.2(b).

(2) *Choice of the domain* Δ_ξ. Recall the definition of $\mathcal{W}$ from Definition 6.1, we see that

$$\mathcal{W}^2/4 = \{s \in \mathbb{C} : |\arg s| < 2\varepsilon_1,\ |s| < \varepsilon_2^2/4\}. \tag{17.14}$$

The estimate (17.4) implies

$$|\xi(s) - s| \le \varepsilon_0 |s|,\ \text{where } \varepsilon_0 << 1, \tag{17.15}$$

and therefore the image of $\mathcal{W}^2/4$ under the map $s \to \xi(s)$ will contain

$$\Delta_\xi := \left\{ \xi \in \mathbb{C} : |\arg\xi| < \frac{3}{2}\varepsilon_1,\ \ |\xi| < (1-\varepsilon_0)\frac{\varepsilon_1^2}{4} \right\}. \tag{17.16}$$

If we take α to be the point on the right boundary arc of Δ_ξ where $\Re\xi^{1/2}$ is maximized, it is obvious that progressive 1- and 2-paths can be chosen in Δ_ξ. For example, in the $\xi^{1/2}$ plane one can choose these paths to be line segments. Moreover, the domain Δ_ξ contains the image of $[M', \infty)$ under the map $x \to \xi$, where M' is slightly greater than M (we have $M' = M + O(|\ln(1-\varepsilon_0)|)$. We define the domain $\mathcal{W}_s$ appearing in the statement of Proposition 10.14 to be

$$\mathcal{W}_s := \xi^{-1}(\Delta_\xi). \tag{17.17}$$

(3) *Finiteness of the integrals* $\int_0^\xi |\phi(r) r^{-1/2}|d|r|$. Since Δ_ξ is bounded independent of h (and ζ), we need only consider behavior of the integrals near the origin. Recall that

$$\phi(\xi) = \frac{1-4\tilde{\beta}^2}{16\xi} + \frac{g(s)}{f(s)} + \frac{4f(s)f''(s) - 5f'^2(s)}{16f^3(s)}, \tag{17.18}$$

where $f = f_0 + f_p$ as in (17.1). Clearly, we must look for some cancellation of the singularity of ϕ due to the vanishing of ξ at $s = 0$ and the singularity of f at $s = 0$.

Let us first rewrite f as $f(s) = \frac{a}{s} + f_2(s)$, where

$$\begin{aligned} a &:= 1 + \alpha^2 b_1(0,\zeta) + h b_3(0,\zeta,h) \\ f_2(s) &= \frac{\alpha^2\left(b_1(2s^{1/2},\zeta) - b_1(0,\zeta)\right)}{s} + \frac{h\left(b_3(2s^{1/2},\zeta,h) - b_3(0,\zeta,h)\right)}{s} \\ &\quad + \left(4b_1 + \frac{2b_2(2s^{1/2},\zeta)}{s^{1/2}}\right). \end{aligned} \tag{17.19}$$

The estimates of Proposition 6.3 for the b_j imply that $f_2(s) = \frac{O(s)}{s}$, and thus

$$f(s) = \frac{a}{s}(1 + O(s)) \Rightarrow \sqrt{f} = \sqrt{\frac{a}{s}}\,(1 + O(s)) \Rightarrow \xi^{1/2} = \sqrt{as} + O(s^{\frac{3}{2}}). \tag{17.20}$$

This gives $\xi(s) = as + O(s^2)$, and thus

$$\frac{1 - 4\tilde{\beta}^2}{16\xi} = \frac{1 - 4\tilde{\beta}^2}{as}\,(1 + O(s)) = \frac{1 - 4\tilde{\beta}^2}{as} + O(1) := A(s) + B(s). \tag{17.21}$$

Set $\tilde{f}_0(s) = \frac{a}{s}$. A short computation shows

$$A(s) + \frac{g(s)}{\tilde{f}_0(s)} + \frac{4\tilde{f}_0(s)\tilde{f}_0''(s) - 5\tilde{f}_0'^2(s)}{16\tilde{f}_0^3(s)} = 0. \tag{17.22}$$

Since the contribution of $B(s)$ to

$$\int_0^\xi |\phi(r) r^{-1/2}| d|r| \tag{17.23}$$

is finite,[47] it just remains to examine the contribution of

$$\left(\frac{g(s)}{f(s)} + \frac{4f(s)f''(s) - 5f'^2(s)}{16f^3(s)}\right) - \left(\frac{g(s)}{\tilde{f}_0(s)} + \frac{4\tilde{f}_0(s)\tilde{f}_0''(s) - 5\tilde{f}_0'^2(s)}{16\tilde{f}_0^3(s)}\right). \tag{17.24}$$

Recall $f = \tilde{f}_0 + f_2$. Thus, the terms in (17.24) involving second derivatives are (ignoring some constant factors)[48]

$$f_2(\tilde{f}_0'' + f_2'')\left(\frac{1}{\tilde{f}_0^3} - 3\frac{f_2}{\tilde{f}_0^4}\right), \quad \tilde{f}_0 f_2''\left(\frac{1}{\tilde{f}_0^3} - 3\frac{f_2}{\tilde{f}_0^4}\right), \quad \tilde{f}_0\tilde{f}_0''\frac{f_2}{\tilde{f}_0^4}. \tag{17.25}$$

Setting $q(2s^{1/2},\zeta,h) = b_3(2s^{1/2},\zeta,h) - b_3(0,\zeta,h)$, we consider for example the contribution of $\tilde{f}_2 := hq/s$ to $f_2''/\tilde{f}_0^2$ (one of the "worst" terms in (17.25)). We compute

$$\frac{\tilde{f}_2''}{\tilde{f}_0^2} = \frac{s^2}{a^2}\tilde{f}_2'' = \frac{s^2}{a^2}h\left[q_{tt}s^{-2} - \frac{5}{2}q_t s^{-5/2} + 2qs^{-3}\right] = \frac{1}{a^2}h\left[q_{tt} - \frac{5}{2}q_t s^{-1/2} + 2qs^{-1}\right]. \tag{17.26}$$

[47] Here we use $d\xi = \xi_s ds$ and (17.5).
[48] Compare (16.48).

The estimates of Proposition 6.3 show that the right side of (17.26) is $O(1)$, so its contribution to the integrand of (17.23) is $O(s^{-\frac{1}{2}})$, which is integrable near 0. The terms in (17.24) involving first derivatives are estimated similarly.

(4) *Conclusion.* We have now checked that all the requirements for an application of Theorem 9.1 of [O, Chapter 12] are satisfied, so this concludes the proof of Proposition 10.14. ∎

Next we show that for $\Re\zeta > 0$ the decaying solution of (10.11) is given by

$$w(x) = \frac{\sqrt{2}}{t(x)}\hat{v}_1(s(x)) = \frac{\sqrt{2}}{2s^{1/2}}\xi_s^{-1/2}\left(\xi^{1/2}I_{\tilde{\beta}}(2\xi^{1/2}/h) + \eta_1(\tilde{\beta},\xi)\right). \tag{17.27}$$

Proof of Proposition 10.16. As $x \to \infty$ we have $s \to 0$ and $\xi(s) \to 0$. Recall from (17.4) and (17.5) that

$$\xi(s) = s + \varepsilon_a(s),\ \xi_s(s) = 1 + \varepsilon_b(s), \quad \text{where } |\varepsilon_j(s)| << 1, \tag{17.28}$$

so in estimating $w(x)$ we can ignore the factors multiplying $I_{\tilde{\beta}}$ and η_1. For $|z|$ small with $|\arg z| \le \frac{\pi}{2}$ we have

$$|I_{\tilde{\beta}}(z)| \le k|z|^{\Re\tilde{\beta}}. \tag{17.29}$$

Since $\Re\tilde{\beta} > 0$ for $\Re\zeta > 0$, this implies decay of the term involving $I_{\tilde{\beta}}$ as $x \to \infty$. The estimate of η_1 in Theorem 9.1 of Chapter 12 of [O] implies that this contribution decays to zero as well. Differentiating (17.27) and arguing as above we obtain that w_x also decays to 0 as $x \to \infty$. ∎

We now show that the exact decaying solution θ of Erpenbeck's system (5.1) identified in Proposition 10.16 is of type θ_1 at $x = M$.

Proof of Proposition 10.17. The proof runs parallel to that of Proposition 10.11. The variables are

$$t = \frac{2}{\mu}\sqrt{aD(\infty,\zeta)}e^{-\mu x/2},\ t = 2s^{1/2},\ 2\xi^{1/2} = \int_0^s \sqrt{f(r)}\,dr. \tag{17.30}$$

(1) We recall from Remark 6.2 that

$$\frac{4}{\mu^2}(C(x,\zeta) + hr(x,\zeta,h)) = t^2\left[\left(1 + \frac{\tilde{\alpha}^2}{t^2}\right) + (t^2+\alpha^2)b_1(t,\zeta) + tb_2 + hb_3\right]. \tag{17.31}$$

Using (17.30) and recalling the definition of $f(s)$ (17.1), we rewrite this as

$$\frac{4}{\mu^2}(C(x,\zeta) + hr(x,\zeta,h)) = \tilde{\alpha}^2 + 4s^2 f(s) = \tilde{\alpha}^2 + 4s^2\xi^{-1}\xi_s^2. \tag{17.32}$$

For x near M and $\zeta \in \omega$ we have $|C(x,\zeta)| > k > 0$, $\arg C(x,\zeta) \sim 0$. We have $\tilde{\alpha} = O(h)$ in regime III, so (17.32) implies

$$\mu s\xi^{-1/2}\xi_s = \sqrt{C + hr} + O(h) = -s(x,\zeta)\underline{b}(x) + O(h) \text{ for } x \text{ near } M. \tag{17.33}$$

(2) For $|z|$ large with $|\arg z| < \frac{\pi}{2}$ we have asymptotic expansions [AS, Chapter 9]

$$\begin{aligned} I_{\tilde{\beta}}(z) &\sim \frac{e^z}{\sqrt{2\pi z}}(1 + O(1/z)),\\ I'_{\tilde{\beta}}(z) &\sim \frac{e^z}{\sqrt{2\pi z}}(1 + O(1/z)), \end{aligned} \tag{17.34}$$

where $(1 + O(1/z))$ can be expanded explicitly in powers of z^{-1}.

(3) *Approximations.* Using the formula (17.27) for w, the expansions (17.34), and the fact that

$$d_t(\xi^{1/2}(s(t))) = \xi^{-1/2}\xi_s \frac{t}{4}, \tag{17.35}$$

we approximate for x near M

$$\begin{aligned} w &\sim \frac{\sqrt{2}}{t}\xi_s^{-1/2}\xi^{1/2}I_{\tilde{\beta}}(2\xi^{1/2}/h) \sim \frac{1}{\sqrt{2\pi}}\, t^{-1}\xi_s^{-1/2}\xi^{1/4}h^{1/2}e^{2\xi^{1/2}/h}, \\ hw_t &\sim h\frac{\sqrt{2}}{t}\xi_s^{-1/2}\xi^{1/2}I'_{\tilde{\beta}}(2\xi^{1/2}/h)\frac{1}{h}\xi^{-1/2}\xi_s\frac{t}{2} \sim \sqrt{\frac{2}{\pi}}\frac{1}{4}\xi_s^{1/2}h^{1/2}\xi^{-1/4}e^{2\xi^{1/2}/h}. \end{aligned} \tag{17.36}$$

Here we have ignored relative errors of size $O(h)$ associated with higher-order terms in the expansions (17.34), with η_1, and with other terms in the expression for hw_t. This gives

$$hw_x = -\frac{\mu}{2}htw_t \sim -\sqrt{\frac{2}{\pi}}\frac{\mu}{8}\, t\xi_s^{1/2}h^{1/2}\xi^{-1/4}e^{2\xi^{1/2}/h}. \tag{17.37}$$

(4) Using the formula (16.58) for θ and ignoring $O(h)$ relative errors as in (16.72), we obtain

$$\theta(x,\zeta,h) \sim e^{\frac{\varphi_0}{h}}[b^{1/2}wP_0 + b^{-1/2}(hw_x)Q_0]. \tag{17.38}$$

Substituting in the expressions for w and hw_x and using (17.33), we find

$$\begin{aligned} \theta &\sim e^{\frac{\varphi_0}{h}+\frac{2\xi^{1/2}}{h}}\left(b^{1/2}\sqrt{\frac{h}{2\pi}}\frac{1}{t}\xi_s^{-1/2}\xi^{1/4}\right)\left[P_0 - \frac{\mu}{4}b^{-1}t^2\xi_s\xi^{-1/2}Q_0\right] \\ &\sim e^{\frac{\varphi_0}{h}+\frac{2\xi^{1/2}}{h}}\left(b^{1/2}\sqrt{\frac{h}{2\pi}}\frac{1}{t}\xi_s^{-1/2}\xi^{1/4}\right)\left[P_0 + s(x,\zeta)Q_0\right]. \end{aligned} \tag{17.39}$$

We have

$$\frac{d}{dx}(2\xi^{1/2}) = -\xi^{-1/2}\xi_s\mu s = bs(x,\zeta) = \underline{b}s(x,\zeta) + O(h) \text{ near } x = M \tag{17.40}$$

for h_{1b} as in (16.68). As in (16.75) we obtain

$$\frac{\varphi_0}{h} + \frac{2\xi^{1/2}}{h} = \frac{h_1(x,\zeta)}{h} + g(x,\zeta,h) \text{ near } x = M \tag{17.41}$$

for a function g as in (16.75). Thus, we can now rewrite (17.39)

(17.42)

$$\theta \sim e^{\frac{h_1(x,\zeta)}{h}+g}\left(b^{1/2}\sqrt{\frac{h}{2\pi}}\frac{1}{t}\xi_s^{-1/2}\xi^{1/4}\right)T_1 = G(x,\zeta,h)\theta_1(x,\zeta,h) \text{ near } x = M,$$

where the nonvanishing scalar function

$$G(x,\zeta,h) = e^g e^{-k_1}\left(b^{1/2}\sqrt{\frac{h}{2\pi}}\frac{1}{t}\xi_s^{-1/2}\xi^{1/4}\right). \tag{17.43}$$

Setting

$$H(x,\zeta,h) = G^{-1}(x,\zeta,h), \tag{17.44}$$

we obtain the estimate of Proposition 10.17. ▯

Part V. Proofs for Part III.

18. Turning points in $(0, \infty)$.

Here we prove Propositions 12.3 and 12.7.

Proof of Proposition 12.3. For $\Re\zeta = 0$ and $x < x(\zeta)$ we take

$$\rho^{3/2}(x,\zeta) = \frac{3}{2}\int_{x(\zeta)}^{x} \sqrt{x(\zeta) - y}\sqrt{-d(y,\zeta)}\,dy, \tag{18.1}$$

where the square roots are taken to be positive.[49] Making the changes of variable $t = \sqrt{x(\zeta) - y}$ and then $t = u\sqrt{x(\zeta) - x}$, we obtain

$$\rho^{3/2}(x,\zeta) = -(x(\zeta) - x)^{3/2}\int_0^1 3u^2\sqrt{-d(x(\zeta) + (x - x(\zeta))u^2, \zeta)}\,du, \tag{18.2}$$

which implies (12.7). The analyticity of ρ in x and ζ and the properties (12.8)(a)–(c) are evident from the formula (12.7). Property (12.8)(d) is proved by differentiating $\rho_x^2\rho = C(x,\zeta)$ with respect to ζ and evaluating at $x = x(\zeta)$. The analyticity of both sides of (12.6) implies that ρ is a solution on $\mathcal{O} \times \omega$. ∎

Proof of Proposition 12.7. (1) First we show that appropriate multiples of θ_- and θ_+ are, respectively, of type θ_1 and θ_2 at x_R. For $\zeta \in \omega_1$ and x near x_R, $\rho(x,\zeta)$ takes values near the negative real axis. Noting that we must take

$$-\pi < \arg(h^{-2/3}\rho e^{\pm 2\pi i/3}) < \pi \tag{18.3}$$

in order to use the expansions to rewrite the expressions in (12.19), we obtain for x near x_R

$$\theta_- \sim e^{\frac{\varphi_0}{h} - \frac{2}{3}h^{-1}i(-\rho)^{3/2}}\left(\frac{1}{2\sqrt{\pi}}b^{1/2}\rho_x^{-1/2}(h^{-2/3}\rho e^{-2\pi i/3})^{-1/4}\right)[P_0 + s(x,\zeta)Q_0]. \tag{18.4}$$

Here we have used[50]

$$-\frac{2}{3}(h^{-2/3}\rho e^{-2\pi i/3})^{3/2} = -\frac{2}{3}ih^{-1}(-\rho)^{3/2} \text{ and } i\rho_x b^{-1}(-\rho)^{1/2} = s(x,\zeta) + O(h). \tag{18.5}$$

Similarly, we obtain for x near x_R

$$\theta_+ \sim e^{\frac{\varphi_0}{h} + \frac{2}{3}h^{-1}i(-\rho)^{3/2}}\left(\frac{1}{2\sqrt{\pi}}b^{1/2}\rho_x^{-1/2}(h^{-2/3}\rho e^{2\pi i/3})^{-1/4}\right)[P_0 - s(x,\zeta)Q_0]. \tag{18.6}$$

For x near x_R and $\zeta \in \omega_1$ we have

$$-\frac{2}{3}i(-\rho)^{3/2}(x,\zeta) = \int_{x_R-\delta}^{x} s\underline{b}(y,\zeta)\,dy - \frac{2}{3}i(-\rho)^{3/2}(x_R - \delta,\zeta) \text{ and so}$$
$$\int_0^x s\underline{b}(y,\zeta)\,dy = -\frac{2}{3}i(-\rho)^{3/2}(x,\zeta) + \int_0^{x_R-\delta} s\underline{b}(y,\zeta)\,dy + \frac{2}{3}i(-\rho)^{3/2}(x_R - \delta,\zeta). \tag{18.7}$$

[49] Here we use the fact that $d(y,\zeta)$ is negative for real y near $x(\zeta) \in \mathbb{R}$.

[50] Recall that $\rho_x^2\rho = C(x,\zeta) = s^2\underline{b}^2$ and that for $\zeta = i|\zeta| \in \omega_1$ and real x near x_R, we have $s = i|s| = i\sqrt{|\zeta|^2 - c_0^2\eta(x)}$.

Since $\mu_1(x,\zeta) = \underline{a} + s(x,\zeta)\underline{b}$ and $T_1(x,\zeta) = P_0 + sQ_0$, (18.4) implies

$$\begin{aligned}(18.8)\quad \theta_-(x,\zeta,h) &\sim \theta_1(x,\zeta,h)G_-(x,\zeta,h) \text{ for } x \text{ near } x_R \text{ where } G_-(x,\zeta,h)\\ &= \left(\frac{1}{2\sqrt{\pi}} b^{1/2}\rho_x^{-1/2}(h^{-2/3}\rho e^{-2\pi i/3})^{-1/4}\right)\\ &\times \exp\left[-\frac{1}{h}\left(\int_0^{x_R-\delta} s\underline{b}(y,\zeta)\,dy + \frac{2}{3}i(-\rho)^{3/2}(x_R-\delta,\zeta)\right)\right] e^{k_-(x,\zeta,h)}\end{aligned}$$

for a function $k_- = O(1)$. Similarly, we obtain from (18.6)

$$\begin{aligned}(18.9)\quad \theta_+(x,\zeta,h) &\sim \theta_2(x,\zeta,h)G_+(x,\zeta,h) \text{ for } x \text{ near } x_R \text{ where } G_+(x,\zeta,h)\\ &= \left(\frac{1}{2\sqrt{\pi}} b^{1/2}\rho_x^{-1/2}(h^{-2/3}\rho e^{2\pi i/3})^{-1/4}\right)\\ &\times \exp\left[\frac{1}{h}\left(\int_0^{x_R-\delta} s\underline{b}(y,\zeta)\,dy + \frac{2}{3}i(-\rho)^{3/2}(x_R-\delta,\zeta)\right)\right] e^{k_+(x,\zeta,h)}\end{aligned}$$

for a function $k_+ = O(1)$.

From (18.8) and (18.9) we see that the functions

$$(18.10)\qquad \overline{\theta}_1 := G_-^{-1}(x_R,\zeta,h)\theta_-(x,\zeta,h) \text{ and } \overline{\theta}_2 := G_+^{-1}(x_R,\zeta,h)\theta_+(x,\zeta,h)$$

are exact solutions of (11.9) on $\mathcal{O}$, which are respectively of type θ_1 and θ_2 at x_R.[51] For later use we note that the growth rates in h of the factors $G_{\mp}^{-1}(x_R,\zeta,h)$ are

(18.11)
$$R_{\mp}(\zeta,h) := h^{-1/6}\exp\left[\pm\frac{1}{h}\Re\left(\int_0^{x_R-\delta} s\underline{b}(y,\zeta)\,dy + \frac{2}{3}i(-\rho)^{3/2}(x_R-\delta,\zeta)\right)\right].$$

(2) Computations like those that produced (18.4) and (18.6) show that for x near x_L we have

$$\begin{aligned}(18.12)\quad \theta_- &\sim e^{\frac{\varphi_0}{h}+\frac{2}{3}h^{-1}\rho^{3/2}}\left(\frac{1}{2\sqrt{\pi}} b^{1/2}\rho_x^{-1/2}(h^{-2/3}\rho e^{-2\pi i/3})^{-1/4}\right)[P_0 + s(x,\zeta)Q_0],\\ \theta_+ &\sim e^{\frac{\varphi_0}{h}+\frac{2}{3}h^{-1}\rho^{3/2}}\left(\frac{1}{2\sqrt{\pi}} b^{1/2}\rho_x^{-1/2}(h^{-2/3}\rho e^{2\pi i/3})^{-1/4}\right)[P_0 + s(x,\zeta)Q_0],\end{aligned}$$

since $b^{-1}\rho_x\sqrt{\rho} = s(x,\zeta) + O(h)$ for x near x_L and $\zeta \in \omega_1$.[52] From (18.12) and the fact that $T_1 = P_0 + sQ_)$ it is evident that

$$(18.13)\qquad \theta_-(x,\zeta,h) \sim \theta_1(x,\zeta,h)K_-(x,\zeta,h) \text{ for } x \text{ near } x_L$$

for a nonvanishing scalar function K_-.

[51] Caution: It is not necessarily true that $\overline{\theta}_1$, for example, is of type θ_1 for $x \neq x_R$.

[52] Recall that $\rho(x,\zeta) > 0$ for real x near x_L and for $\zeta \in \omega_1$ such that $\zeta = i|\zeta|$.

(3) *Exact solutions* $\overline{\theta}_i,\ i=3,4,5$. After shrinking the neighborhoods $\mathcal{O}$ and ω_1 and reducing $\delta>0$ if necessary, we choose an open ball $B(\underline{\zeta},R)$ centered at $x(\underline{\zeta})$ such that

$$x(\omega_1)\cup[x_L,x_R]\subset\mathcal{O}\subset B(x(\underline{\zeta}),R/2) \tag{18.14}$$

and such that the profile $p(x)$ has an analytic extension to $B(\underline{\zeta},R)$. The exact solutions $\overline{\theta}_i(x,\zeta,h)$ are constructed for $\zeta\in\omega_1$ from approximate solutions θ_i of the form (0.16), which are defined initially on $[0,x_L]$, and then extended to a simply connected neighborhood of $\{x_L,x_R\}$ by analytic continuation in

$$\mathcal{S}:=B(x(\underline{\zeta}),R)\cap\{x:\Im x\geq 0\}\setminus x(\omega_2),\ \text{ where } \omega_1\subset\subset\omega_2 \tag{18.15}$$

and ω_2 is a slight enlargement of ω_1. As explained in section 4.2 of [LWZ1], the $\overline{\theta}_i$ are exact solutions of (11.9) and satisfy[53]

$$|\overline{\theta}_i(x,\zeta,h)-\theta_i(x,\zeta,h)|\leq Ch|\theta_i(x,\zeta,h)| \text{ in } \mathcal{S} \text{ for } \zeta\in\omega_1. \tag{18.16}$$

Like $\overline{\theta}_i,\ i=1,2$, the functions $\overline{\theta}_i,\ i=3,4,5$, are solutions of (11.9) in a full neighborhood of $x(\zeta)$ for $\zeta\in\omega_1$; however, the asymptotic behavior (18.16) is known only in $\mathcal{S}$.

(4) *Growth rates.* From the expressions (0.16) for the θ_i we can read off the growth rates with respect to h of the $\overline{\theta}_i(x,\zeta,h),\ i=1,\dots,5$ at x_R for $\zeta\in\omega_1$:

$$\begin{aligned}
&\overline{\theta}_1(x_R,\zeta,h):\ e^{\frac{1}{h}\Re\int_0^{x_R}[\underline{a}(y,\zeta)+s(y,\zeta)\underline{b}(y)]\,dy}:=e^{A(\zeta)/h},\\
&\overline{\theta}_2(x_R,\zeta,h):\ e^{\frac{1}{h}\Re\int_0^{x_R}[\underline{a}(y,\zeta)-s(y,\zeta)\underline{b}(y)]\,dy}:=e^{B(\zeta)/h},\\
&\overline{\theta}_i(x_R,\zeta,h),i\geq 3:\ e^{\frac{1}{h}\Re\int_0^{x_R}\frac{\zeta}{u(y)}\,dy}:=e^{C(\zeta)/h}.
\end{aligned} \tag{18.17}$$

(5) *Expand* $H(x_R,\zeta,h)\theta(x,\zeta,h)$. The exact bounded (or decaying) solution $H(x_R,\zeta,h)\theta$ on $[x_R,\infty)$ extends analytically to a complex neighborhood of $[0,\infty]$. On $\mathcal{S}$ we can expand it as

$$H(x_R,\zeta,h)\theta(x,\zeta,h)=c_1(\zeta,h)\overline{\theta}_1+\cdots+c_5(\zeta,h)\overline{\theta}_5 \text{ for } \zeta\in\omega_1. \tag{18.18}$$

Corollary 11.4 implies that $H(x_R,\zeta,h)\theta(x,\zeta,h)$ is of type θ_1 at x_R. Evaluating (18.18) at x_R and using Cramer's rule and (18.17), we determine the growth rates of the coefficients in (18.18):

$$c_1(\zeta,h)=1+O(h),\ c_2=O(he^{(A(\zeta)-B(\zeta))/h}),\ c_i=O(he^{(A(\zeta)-C(\zeta))/h}),i\geq 3, \tag{18.19}$$

where

$$\begin{aligned}
A(\zeta)-B(\zeta)&=2\Re\int_0^{x_R}s(y,\zeta)\underline{b}(y)\,dy \text{ and } A(\zeta)-C(\zeta)\\
&=\Re\int_0^{x_R}\left(s(y,\zeta)\underline{b}(y)-\frac{\zeta}{\eta u(y)}\right)dy.
\end{aligned} \tag{18.20}$$

[53]The proof by a contraction argument is based on being able to choose "progressive paths" in $\mathcal{S}$; see Theorem 3.1 of [LWZ1].

(6) *Conclusion.* Using (18.10), (18.11), (18.12), and (18.19), we can now read off the growth rates at x_L of the individual terms in the expansion (18.18):

$$\begin{aligned}
&\text{(a)}\ \ c_1(\zeta,h)\overline{\theta}_1(x_L,\zeta,h):\ (1+O(h))\cdot R_-(\zeta,h)\\
&\qquad \times h^{1/6}\exp\left[\frac{1}{h}\Re\left(\int_0^{x_L}\underline{a}(y,\zeta)\,dy+\frac{2}{3}\rho^{3/2}(x_L,\zeta)\right)\right],\\
&\text{(b)}\ \ c_2(\zeta,h)\overline{\theta}_2(x_L,\zeta,h):\ he^{(A(\zeta)-B(\zeta))/h}\cdot R_+(\zeta,h)\\
&\qquad \times h^{1/6}\exp\left[\frac{1}{h}\Re\left(\int_0^{x_L}\underline{a}(y,\zeta)\,dy+\frac{2}{3}\rho^{3/2}(x_L,\zeta)\right)\right],\\
&\text{(c)}\ \ c_i(\zeta,h)\overline{\theta}_i(x_L,\zeta,h),\ i\geq 3:\ he^{(A(\zeta)-C(\zeta))/h}\cdot e^{\Re\frac{1}{h}\int_0^{x_L}\frac{\zeta}{u(y)}\,dy}.
\end{aligned}\tag{18.21}$$

First we compare the rates in (18.21)(a),(b). Recalling the expressions (18.11) for $R_\pm$, and noting from (18.20) that

$$e^{(A(\zeta)-B(\zeta))/h}\cdot\exp\left(-\frac{1}{h}\Re\int_0^{x_R-\delta}s(y,\zeta)\underline{b}\,dy\right)\leq\exp\left(\frac{1}{h}\Re\int_0^{x_R-\delta}s(y,\zeta)\underline{b}\,dy\right)\tag{18.22}$$

and from Remark 12.4 that

$$\Im(-\rho)^{3/2}(x_R-\delta,\zeta)\leq 0\text{ for }\zeta\in\omega_1,\tag{18.23}$$

we obtain

$$|c_2(\zeta,h)\overline{\theta}_2(x_L,\zeta,h)|/|c_1(\zeta,h)\overline{\theta}_1(x_L,\zeta,h)|\leq Ch.\tag{18.24}$$

Next we compare the rates in (18.21)(a),(c). From (18.23), the fact that $\Re\rho^{3/2}(x_L,\zeta)>0$, and

$$e^{\frac{1}{h}\left(A(\zeta)-\Re\int_0^{x_R}\frac{\zeta}{u(y)}\,dy+\Re\int_0^{x_L}\frac{\zeta}{u(y)}\,dy\right)}\leq e^{\frac{1}{h}\Re\int_0^{x_R-\delta}s(y,\zeta)\underline{b}\,dy}\cdot e^{\frac{1}{h}\Re\int_0^{x_L}\underline{a}(y,\zeta)\,dy},\tag{18.25}$$

we see that

$$|c_3(\zeta,h)\overline{\theta}_3(x_L,\zeta,h)|/|c_1(\zeta,h)\overline{\theta}_1(x_L,\zeta,h)|\leq Ch.\tag{18.26}$$

Thus, $c_1(\zeta,h)\overline{\theta}_1(x_L,\zeta,h)$ is, for small h, the dominant term in the expansion (18.18) evaluated at x_L. Since $c_1(\zeta,h)=1+O(h)$, we see from (18.13) and (18.10) that the estimate of Proposition 12.7 holds with $\alpha(\zeta,h):=G_-(x_R,\zeta,h)K_-^{-1}(x_L,\zeta,h)$. ∎

19. The turning point at 0. This section gives the proof of Proposition 13.1.

Proof of Proposition 13.1.

(1) *Basis of exact solutions near* 0. As noted before the statement of Proposition 13.1, we have exact solutions $\theta_\pm$ on $\mathcal{O}\ni 0$ satisfying $\theta_\pm(x,\zeta,h)\sim$

$$e^{\varphi_0/h}\Big[b^{1/2}(\rho_x)^{-1/2}Ai(h^{-2/3}\rho e^{\pm 2\pi i/3})P_0+b^{-1/2}h^{1/3}(\rho_x)^{1/2}e^{\pm 2\pi i/3}\times Ai'(h^{-2/3}\rho e^{\pm 2\pi i/3})Q_0\Big]\tag{19.1}$$

modulo $O(h)$ errors. Exact solutions $\overline{\theta}_1$ and $\overline{\theta}_2$, which are respectively of type θ_1 and θ_2 at $x_R=2\delta$, are again given by the formulas (18.10). Here the functions $G_\mp^{-1}(2\delta,\zeta,h)$ have growth rates in h, $R_\mp(\zeta,h)$, given by (18.11).

To construct exact solutions $\overline{\theta}_j$, $j = 3, 4, 5$, near $x = 0$, we use the block diagonal form provided by our extension of Proposition 12.1 to a neighborhood of $x = 0$. The 3×3 block $A_{22}(x, \zeta, h)$ in (12.1) has semisimple eigenvalues

$$\mu_j^*(x,\zeta,h) = \mu_j(x,\zeta) + O(h) = \frac{\zeta}{h} + O(h),\ j = 3,4,5. \tag{19.2}$$

Since this block has no turning points, we can apply standard results (for example, Theorem 3.1 of [LWZ1]) to construct exact solutions $\phi_{2,j}(x, \zeta, h)$ of $\frac{d}{dx}\phi_{2,j} = A_{22}(x, \zeta, h)\phi_{2,j}$ on $[0, 3\delta]$ satisfying

$$\left|\phi_{2,j}(x,\zeta,h) - e^{\frac{1}{h}\int_0^x \mu_j(s,\zeta)ds} a_j(x,\zeta,h)\right| \le Ch\left|e^{\frac{1}{h}\int_0^x \mu_j(s,\zeta)ds}\right|,\ j = 3,4,5, \tag{19.3}$$

for appropriate $a_j = O(1)$. We then obtain exact solutions $\overline{\theta}_j$ of type θ_j on $[0, 3\delta]$ by setting

$$\overline{\theta}_j = Y(x,\zeta,h)\begin{pmatrix} 0 \\ \phi_{2,j} \end{pmatrix},\ j = 3,4,5, \tag{19.4}$$

where Y is the conjugator of Proposition 12.1.

We note that the elements of the basis $\mathcal{B} = \{\overline{\theta}_1 \ldots, \overline{\theta}_5\}$ have the growth rates at $x_R = 2\delta$ given by (18.17).

(2) *Expand* $H(2\delta, \zeta, h)\theta$. As in (18.18) we expand the exact solution $H(2\delta, \zeta, h)\theta$ in the basis $\mathcal{B}$ and, after evaluating at $x_R = 2\delta$, we again obtain the growth rates (18.19) for the coefficients $c_j(\zeta, h)$, $j = 1, \ldots, 5$.

(3) *Regime* A. We show that for (ζ, h) in regime A, the term $c_1\overline{\theta}$ is the dominant term in the expansion (18.18) at $x = 0$. Observe that for *all* $\zeta \in \omega_1$ we have

$$\begin{aligned} &\text{(a)}\ \arg\rho(0,\zeta) \in [0,\pi],\ \text{and thus} \\ &\text{(b)}\ \arg(e^{-2\pi i/3}\rho(0,\zeta)) \in [-2\pi/3, \pi/3],\ \text{while} \\ &\text{(c)}\ \arg(e^{2\pi i/3}\rho(0,\zeta)) \in [2\pi/3, 5\pi/3]. \end{aligned} \tag{19.5}$$

In case (b) the zeroes of $Ai(z)$, which all lie on the negative real axis, are avoided; thus, there exist positive constants A_i such that

$$A_1 \le |Ai(h^{-2/3}\rho(0,\zeta)e^{-2\pi i/3})| \le A_2 \text{ for } (\zeta,h) \text{ in regime A}. \tag{19.6}$$

Ignoring an error of size $h^{-1/3}$, we have

$$\theta_-(0,\zeta,h) \sim b^{1/2}(\rho_x)^{-1/2}Ai(h^{-2/3}\rho e^{-2\pi i/3})P_0 := q(0,\zeta,h)P_0 \text{ in regime A}. \tag{19.7}$$

With (18.10), (18.11), and (18.19) this gives for some positive constant C,

$$\begin{aligned} |c_1(\zeta,h)\overline{\theta}_1(0,\zeta,h)| &\ge CR_-(\zeta,h) = Ch^{-1/6} \\ &\times \exp\left[\frac{1}{h}\Re\left(\int_0^\delta s\underline{b}(y,\zeta)\,dy + \frac{2}{3}i(-\rho)^{3/2}(\delta,\zeta)\right)\right]. \end{aligned} \tag{19.8}$$

Similarly, we obtain

$$\begin{aligned} &|c_2(\zeta,h)\overline{\theta}_2(0,\zeta,h)| \le Che^{\frac{1}{h}2\Re\left(\int_0^{2\delta} s\underline{b}(y,\zeta)\,dy\right)}R_+(\zeta,h),\ \text{where} \\ &R_+(\zeta,h) = h^{-1/6}\exp\left[-\frac{1}{h}\Re\left(\int_0^\delta s\underline{b}(y,\zeta)\,dy + \frac{2}{3}i(-\rho)^{3/2}(\delta,\zeta)\right)\right], \\ &|c_j(\zeta,h)\overline{\theta}_j(0,\zeta,h)| \le Che^{\frac{1}{h}\Re\int_0^{2\delta}\left(s(y,\zeta)\underline{b}(y) - \frac{\zeta}{\eta u(y)}\right)dy},\ j = 3,4,5, \end{aligned} \tag{19.9}$$

where in the last estimate we have used $|\overline{\theta}_j(0, \zeta, h)| = O(1)$, $j = 3, 4, 5$.

Recalling (18.23), from (19.8) and (19.9) we see that in regime A at $x = 0$

$$|c_2\overline{\theta}_2/c_1\overline{\theta}_1| \le Ch \text{ and } |c_j\overline{\theta}_j/c_1\overline{\theta}_1| \le Ch^{7/6},\ j = 3,4,5, \tag{19.10}$$

and thus $c_1\overline{\theta}_1$ is the dominant term in the expansion (18.18) at $x = 0$. Using (18.10) and (19.7) we obtain

$$\overline{\theta}_1(0,\zeta,h) = G_-^{-1}(2\delta,\zeta,h)q(0,\zeta,h)P_0. \tag{19.11}$$

Since $\theta_1(0,\zeta,h) = P_0 + s(0,\zeta)Q_0 + O(h)$, for fixed $\kappa > 0$ we therefore obtain (13.2) with $\alpha(\zeta,h) := G_-(2\delta,\zeta,h)q^{-1}(0,\zeta,h)$ provided (ζ,h) lies in regime A, $\zeta \in \omega_2$, and $0 < h \le h_0$ for small enough $\omega_2 \ni \zeta_0$ and h_0.

(4) *Regime* B *with* $\arg\rho(0,\zeta)$ *away from* $\pi/3$. First we determine the size of $c_1\overline{\theta}_1(0,\zeta,h)$ in regime B. By (19.5)(b) we can use the expansions (16.66) of $Ai(z)$ and $Ai'(z)$ for all (ζ,h) in regime B to obtain, modulo $O(h)$ relative errors,

$$\begin{aligned} &c_1\overline{\theta}_1(0,\zeta,h) \sim k(\zeta,h)R_-(\zeta,h)e^{-\frac{2}{3}(h^{-2/3}\rho(0,\zeta)e^{-2\pi i/3})^{3/2}} \\ &\quad\times b^{1/2}\rho_x^{-1/2}(h^{-2/3}\rho(0,\zeta)e^{-2\pi i/3})^{-1/4} \\ &\qquad\times\left[P_0 - b^{-1}\rho_x e^{-2\pi i/3}h^{1/3}(h^{-2/3}\rho(0,\zeta)e^{-2\pi i/3})^{1/2}Q_0\right], \end{aligned} \tag{19.12}$$

where $k(\zeta,h) = O(1)$ and is bounded away from 0. Letting $\arg\rho(0,\zeta) = \beta \in [0,\pi]$ and noting that for ζ near ζ_0 the second term inside the brackets is small compared to the first, we obtain for some positive constant K

$$c_1\overline{\theta}_1(0,\zeta,h) \ge Ke^{\frac{1}{h}\Re\left(\int_0^\delta s\underline{b}(y,\zeta)\,dy+\frac{2}{3}i(-\rho)^{3/2}(\delta,\zeta)\right)}|\rho(0,\zeta)|^{-1/4}e^{\frac{1}{h}\frac{2}{3}|\rho(0,\zeta)|^{3/2}\cos(\frac{3\beta}{2})}. \tag{19.13}$$

For a small positive ε_0 we note that for $\beta \in [0,\frac{\pi}{3}-\varepsilon_0]$ (resp., $\beta \in [\frac{\pi}{3}+\varepsilon_0,\pi]$), $c_2\overline{\theta}_2$ has an expansion similar to (19.12), except that c_1 is replaced by c_2, R_- by R_+, and all factors of $e^{-2\pi i/3}$ are replaced by $e^{2\pi i/3}$ (resp., $e^{-4\pi i/3}$). Thus, for $\beta \in [0,\frac{\pi}{3}-\varepsilon_0]$ we obtain

$$\begin{aligned} c_2\overline{\theta}_2(0,\zeta,h) &\le Che^{\frac{1}{h}2\Re\left(\int_0^{2\delta} s\underline{b}(y,\zeta)\,dy\right)}e^{-\frac{1}{h}\Re\left(\int_0^\delta s\underline{b}(y,\zeta)\,dy+\frac{2}{3}i(-\rho)^{3/2}(\delta,\zeta)\right)} \\ &\quad\times|\rho(0,\zeta)|^{-1/4}e^{\frac{1}{h}\frac{2}{3}|\rho(0,\zeta)|^{3/2}\cos(\frac{3\beta}{2})}. \end{aligned} \tag{19.14}$$

while for $\beta \in [\frac{\pi}{3}+\varepsilon_0,\pi]$ we obtain

$$\begin{aligned} c_2\overline{\theta}_2(0,\zeta,h) &\le Che^{\frac{1}{h}2\Re\left(\int_0^{2\delta} s\underline{b}(y,\zeta)\,dy\right)}e^{-\frac{1}{h}\Re\left(\int_0^\delta s\underline{b}(y,\zeta)\,dy+\frac{2}{3}i(-\rho)^{3/2}(\delta,\zeta)\right)} \\ &\quad\times|\rho(0,\zeta)|^{-1/4}e^{-\frac{1}{h}\frac{2}{3}|\rho(0,\zeta)|^{3/2}\cos(\frac{3\beta}{2})}, \end{aligned} \tag{19.15}$$

From (19.14) and (19.13) we see that at $x = 0$

$$|c_2\overline{\theta}_2/c_1\overline{\theta}_1| \le Ch \text{ for } \beta \in \left[0,\frac{\pi}{3}-\varepsilon_0\right]. \tag{19.16}$$

The same estimate holds for $\beta \in [\frac{\pi}{3}+\varepsilon_0,\pi]$, but this is much less clear since now $\cos(3\beta/2) \le 0$! Inspection of (19.13) and (19.15) shows that the estimate holds for this range of β provided

$$\Re\left(i(-\rho)^{3/2}(\delta,\zeta)\right) \ge |\rho(0,\zeta)|^{3/2}|\cos(3\beta/2)| \text{ for } \zeta \in \omega_2, \tag{19.17}$$

where $\omega_2 \subset \omega_1$ is a neighborhood of ζ_0. Writing $\zeta = \zeta_r + i\zeta_i$ and setting $\gamma = \arg(-\rho(\delta,\zeta))$, we have

$$\Im(-\rho^{3/2}(\delta,\zeta)) = |\rho(\delta,\zeta)|^{3/2}\sin(3\gamma/2), \tag{19.18}$$

and (12.8) implies $\gamma \leq 0$. Now γ is close to zero and, by (12.8)(d), we have $|\Im\rho(\delta,\zeta)| \geq C|\zeta_r|$, so

$$|\sin 3\gamma/2| \sim |3\gamma/2| \sim \frac{3}{2}|\tan\gamma| = \frac{3}{2}\left|\frac{\Im\rho(\delta,\zeta)}{\Re\rho(\delta,\zeta)}\right| \geq C_1|\zeta_r|/|\rho(\delta,\zeta)|. \tag{19.19}$$

With (19.18) this implies

$$|\Im(-\rho^{3/2}(\delta,\zeta))| \geq C_1|\zeta_r||\rho(\delta,\zeta)|^{1/2}. \tag{19.20}$$

To have (19.17) it now suffices to choose ω_2 so that

$$|\cos(3\beta/2)| \leq C_1\frac{|\zeta_r||\rho(\delta,\zeta)|^{1/2}}{|\rho(0,\zeta)|^{3/2}} \text{ for } \zeta\in\omega_2. \tag{19.21}$$

Using (12.8) again, we have

$$\rho(0,\zeta) = \rho(0,\zeta_0)+\rho_\zeta(\zeta_0)(\zeta-\zeta_0)+O(|\zeta-\zeta_0|^2) \sim C|\zeta-\zeta_0| = C|\zeta_r,\Im(\zeta-\zeta_0)|. \tag{19.22}$$

When $|\rho(0,\zeta)| \sim |\zeta-\zeta_0| \sim |\zeta_r|$, we can choose ω_2 so that $|\rho(\delta,\zeta)|/|\rho(0,\zeta)|$ is large and thereby arrange to have (19.21). When $|\zeta_r| \leq \kappa|\Im(\zeta-\zeta_0)|$ for κ small, we must have β close to π. Setting $\alpha = \pi-\beta$ we have

$$|\cos(3\beta/2)| = |\sin(3\alpha/2)| \sim |\tan\alpha| \sim \frac{|\Im\rho(0,\zeta)|}{|\Re\rho(0,\zeta)|} \sim \frac{|\zeta_r|}{|\rho(0,\zeta)|}. \tag{19.23}$$

We can now shrink ω_2 if necessary, so that $|\rho(\delta,\zeta)|/|\rho(0,\zeta)|$ is large for $\zeta\in\omega_2$, thereby arranging to have (19.21). This establishes (19.17),[54] and thus for (ζ,h) in regime B we have at $x=0$,

$$|c_2\overline{\theta}_2/c_1\overline{\theta}_1| \leq Ch \text{ for } \beta\in\left[\frac{\pi}{3}+\varepsilon_0,\pi\right],\ \zeta\in\omega_2. \tag{19.24}$$

The estimate (19.9) for $j=3,4,5$ still holds for regime B, so (19.13) and (19.17) imply that at $x=0$

$$|c_j\overline{\theta}_j/c_1\overline{\theta}_1| \leq Ch \text{ for } \ \zeta\in\omega_2 \tag{19.25}$$

and (ζ,h) in Regime B, when $\beta\in[0,\frac{\pi}{3}-\varepsilon_0]\cup[\frac{\pi}{3}+\varepsilon_0,\pi]$. With (19.16) and (19.24), we obtain (13.2) as before for these (ζ,h).

(5) *Regime* B *with* $\arg\rho(0,\zeta)$ *near* $\pi/3$. To treat $\overline{\theta}_2$ we now use the fact that for large $|z|$ with $|\arg z| \leq 2\pi/3$

$$\begin{aligned} Ai(-z) &\sim \pi^{-1/2}z^{-1/4}\left[\sin\left(\gamma+\frac{\pi}{4}\right)\sum_0^\infty a_k\gamma^{-2k} - \cos\left(\gamma+\frac{\pi}{4}\right)\sum_0^\infty b_k\gamma^{-2k-1}\right] \\ Ai'(-z) &\sim -\pi^{-1/2}z^{1/4}\left[\cos\left(\gamma+\frac{\pi}{4}\right)\sum_0^\infty c_k\gamma^{-2k} + \sin\left(\gamma+\frac{\pi}{4}\right)\sum_0^\infty d_k\gamma^{-2k-1}\right], \end{aligned} \tag{19.26}$$

[54]The argument shows that (19.17) holds for $\zeta\in\omega_2$ when $\beta\in[\varepsilon_0,\pi]$.

where $\gamma := \frac{2}{3}z^{3/2}$ [AS, 10.4.60, 10.4.62]. We write, for example,

$$Ai(h^{-2/3}\rho(0,\zeta)e^{2\pi i/3}) = Ai(-h^{-2/3}e^{-i\pi/3}\rho(0,\zeta)), \tag{19.27}$$

where now $\arg(e^{-i\pi/3}\rho(0,\zeta)) := \theta$ is close to 0. We have

$$\left|\cos\left(\frac{2}{3}(h^{-2/3}e^{-\pi i/3}\rho(0,\zeta))^{3/2} + \frac{\pi}{4}\right)\right| \le Ce^{\frac{1}{h}\frac{2}{3}|\rho(0,\zeta)|^{3/2}|\sin(\frac{3\theta}{2})|}, \tag{19.28}$$

and (19.15) now holds with the exponential on the right in (19.28) replacing that on the far right in (19.15). Since $\arg(\rho(0,\zeta))$ is near $\pi/3$, we have $|\zeta_r| \sim |\zeta-\zeta_0| \sim |\rho(0,\zeta)|$, so we can arrange to have (19.21), with $|\sin(3\theta/2)|$ now in place of $|\cos(3\beta/2)|$, by choosing ω_2 so that $|\rho(\delta,\zeta)|/|\rho(0,\zeta)|$ is large for $\zeta \in \omega_2$. Thus, we can obtain the estimates (19.24) and (19.25) for this range of β, and the estimate (13.2) is a consequence of these as before. □

Part VI. Appendices.

20. Coefficients appearing in the linearized systems.

The matrix coefficients appearing in the reduced system (0.4) are

$$\begin{aligned} A^x &= \begin{pmatrix} u & -v & 0 & 0 & 0 \\ vp_v & u & 0 & vp_S & vp_\lambda \\ 0 & 0 & u & 0 & 0 \\ 0 & 0 & 0 & u & 0 \\ 0 & 0 & 0 & 0 & u \end{pmatrix}, \quad A^y = \begin{pmatrix} 0 & 0 & -v & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ vp_v & 0 & 0 & vp_S & vp_\lambda \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\ B &= \begin{pmatrix} -u' & v' & 0 & 0 & 0 \\ p' - v(c_0^2/v^2)' & u' & 0 & vp'_S & vp'_\lambda \\ 0 & 0 & 0 & 0 & 0 \\ -\Phi_v & S' & 0 & -\Phi_S & -\Phi_\lambda \\ -r_v & \lambda' & 0 & -r_S & -r_\lambda \end{pmatrix}, \end{aligned} \tag{20.1}$$

where $(')$ denotes differentiation with respect to x and $c_0^2 = -v^2 p_v(v,S,\lambda)$.

The matrix $\Phi_0(x,\zeta)$ in the system (0.6) is computed in [E3, p. 112] to be

$$\Phi_0(x,\zeta) = \begin{pmatrix} -\frac{(1-\eta)\zeta}{\eta u} & -\frac{m\zeta}{\eta u} & -\frac{im}{1-\eta} & 0 & 0 \\ -\frac{(1-\eta)\zeta}{\eta m u} & -\frac{(1-\eta)\zeta}{\eta u} & 0 & 0 & 0 \\ \frac{i(1-\eta)}{\eta m} & \frac{i}{\eta} & \frac{\zeta}{u} & 0 & 0 \\ \frac{(1-\eta)p_S\zeta}{\eta m^2 u} & \frac{(1-\eta)p_S\zeta}{\eta m u} & \frac{ip_S}{m} & \frac{\zeta}{u} & 0 \\ \frac{(1-\eta)p_\lambda\zeta}{\eta m^2 u} & \frac{(1-\eta)p_\lambda\zeta}{\eta m u} & \frac{ip_\lambda}{m} & 0 & \frac{\zeta}{u} \end{pmatrix}. \tag{20.2}$$

This computation can be done using (0.6) and (20.1).

21. The stability function $V(\zeta, h)$.

The stability function $V(\zeta,h)$ is given by

$$V(\zeta,h) = \theta(0,\zeta,h)\cdot P(0+) - \theta(0,\zeta,h)\cdot\frac{1}{h}(\zeta H^t + iH^y). \tag{21.1}$$

Here $m = u/v$, the mass flux, is a constant independent of x,

$$H^t = \frac{v_- - v_+}{v_- T_+ \eta_+}\begin{pmatrix} 2(1-\eta_+)g_+/m \\ T_+\eta_+ + 2(1-\eta_+)g_+ \\ 0 \\ -m(v_- - v_+)\eta_+ \\ 0 \end{pmatrix}, \tag{21.2}$$

and H^y has the single nonzero component $(H^y)_3 = m(v_- - v_+)$. By $v_\pm$, for example, we denote components of the profile states $P_\pm := P(0\pm)$ just to the right and left of the von Neumann shock, and

$$g_+ = T_+ - \frac{1}{2}(v_- - v_+)p_{S+}. \tag{21.3}$$

The expression (21.1) for $V(\zeta, h)$, found in [CJLW], is simpler than the expression derived in [E1] and used in [E2, E3]. The equality of the two forms of V was proved in section 4 of [CJLW].

The stability function for the von Neumann shock, $L_1(\zeta)$, which appears in Assumption 1.4, is given explicitly in [E2] as

$$L_1(\zeta) = -\frac{u_-(1-\chi_v)}{\eta_+}\left[\frac{\ell_+\zeta(\zeta+\kappa_+ s_+)}{u_+u_-} + \eta_+\left(1 - \frac{\zeta^2}{u_+u_-}\right)\right],$$
$$\ell = 2 - (1-\eta)(1-\chi_v)v_-p_S/T,\ \chi_v = v_+/v_-. \tag{21.4}$$

From the expression (21.1) and the fact that

$$L_1(\zeta) = -T_1(0,\zeta)\cdot(\zeta H^t + iH^y), \tag{21.5}$$

it is clear that Assumption 1.4 implies that $V(\zeta, h)$ is nonvanishing for small h when $\theta(0,\zeta,h)$ is of type θ_1.

22. Classical asymptotic ODE results used. Here we state the theorems from [O] that are used in this paper. To keep this section brief, we state the results only in the simplified form that we actually use; also, we refer to earlier parts of this paper for definitions of some terms that appear below. We note that Theorem 22.3 below is an extension of Theorem 9.1 of [O, Chapter 12], to the case where the parameter ν satisfies $\Re\nu \geq 0$ instead of just $\nu \geq 0$. The extension was proved in section 17.

For a parameter $u \in \mathbb{C}$ with $|u|$ large, we consider equations of the form

$$W_{\xi\xi} = (u^2\xi^m + \psi(\xi))W, \text{ where } m = 0, 1, -1, \tag{22.1}$$

on a simply connected, open subset Δ, possibly unbounded, of the complex ξ-plane. The function ψ is analytic on Δ but may have singularities at isolated points on its boundary. The following three theorems deal respectively with the cases $m = 0, 1, -1$.

THEOREM 22.1 (Theorem 3.1 of [O, Chapter 10]). *Let $m = 0$ in* (22.1) *and suppose $|\arg u| < \pi/2$. For $j = 1, 2$ let $\alpha_j \in \partial\Delta$ and suppose that for any $\xi \in \Delta$ a progressive j-path can be chosen in Δ from α_j to ξ.*[55] *Suppose also that there is an upper bound for the integrals*

$$\int_{\alpha_j}^{\xi} |\psi(s)|d|s| \text{ on progressive } j\text{-paths}, \tag{22.2}$$

which is independent of $\xi \in \Delta$. Then (22.1) *has solutions W_j on Δ satisfying*

$$W_j(\xi) = e^{(-1)^{j-1}u\xi} + \eta_j(u,\xi),\ j = 1, 2, \tag{22.3}$$

where the errors η_j satisfy the estimates (10.7).

[55] Such paths are defined in step (1) of the proof of Proposition 10.2.

With $Ai(z)$ the standard Airy function, we set

$$Ai_0(z) = Ai(z),\ Ai_1(z) = Ai(ze^{-2\pi i/3}),\ \ Ai_{-1}(z) = Ai(ze^{2\pi i/3}). \tag{22.4}$$

THEOREM 22.2 (Theorem 9.1 of [O, Chapter 11]). *Let $m = 1$ in (22.1) and for small $\delta > 0$, suppose $|\arg u| < \delta$. For $j = 0, 1, -1$ let $\alpha_j \in \partial\Delta$ and suppose that for any $\xi \in \Delta$ a progressive j-path can be chosen in Δ from α_j to ξ.*[56] *Suppose also that there is an upper bound for the integrals*

$$\int_{\alpha_j}^{\xi} |\psi(s)s^{-1/2}|d|s| \text{ on progressive } j\text{-paths}, \tag{22.5}$$

which is independent of $\xi \in \Delta$. Then (22.1) has solutions W_j on Δ satisfying

$$W_j(\xi) = Ai_j(u^{2/3}\xi) + \eta_j(u,\xi),\ j = 0, 1, -1, \tag{22.6}$$

where the errors η_j satisfy the estimates (10.28).

THEOREM 22.3 (Theorem 9.1 of [O, Chapter 12],). *Let $m = -1$ in (22.1) and suppose $u > 0$. We now assume $\Delta \subset \{\xi : |\arg \xi| < \pi/2\}$, $0 \in \partial\Delta$, and that $\psi(\xi)$ has the form*

$$\psi(\xi) = \frac{\nu^2 - 1}{4\xi^2} + \frac{\phi(\xi)}{\xi}, \tag{22.7}$$

where ϕ is analytic at $\xi = 0$. Let $\alpha_1 = 0$, $\alpha_2 \in \partial\Delta$ and suppose that for $j = 1, 2$ and any $\xi \in \Delta$ a progressive j-path can be chosen in Δ from α_j to ξ.[57] *Suppose also that there is an upper bound for the integrals*

$$\int_{\alpha_j}^{\xi} |\phi(s)s^{-1/2}|d|s| \text{ on progressive } j\text{-paths}, \tag{22.8}$$

which is independent of $\xi \in \Delta$. Then (22.1) has solutions W_j on Δ satisfying

$$\begin{aligned} &(a) W_1(\xi) = \xi^{1/2} I_\nu(2u\xi^{1/2}) + \eta_1(u,\xi), \\ &(b) W_2(\xi) = \xi^{1/2} K_\nu(2u\xi^{1/2}) + \eta_2, \end{aligned} \tag{22.9}$$

where the errors η_j satisfy the estimates (10.40).

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[56] Such paths are defined in step (2) of the proof of Proposition 10.8.

[57] Such paths are defined in step (1) of the proof of Proposition 10.14.

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