Universiteit Leiden
Mathematisch Instituut

Institut de Mathématiques de Bordeaux

Collège Sciences et Technologie
Institut de Mathématiques de Bordeaux
M2 Algèbre, Géométrie & Théorie des Nombres

Mémoire de Master 2

presented by

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Unramified cohomology, cycles, and integral Hodge classes

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Algebra, Geometry and Number Theory
ALGANT Erasmus Mundus

Academic Year 2022-2023

Contents

Introduction

— Let X be a smooth algebraic variety over a field k. Among the deepest conjectures in (arithmetic) algebraic geometry, the Weil conjectures and the Hodge and Tate conjectures are essentially attempts to calculate the «arithmetic filtration» on a suitable cohomology theory $\mathrm{H}^*(X)$. This filtration, which is given by

$$F^p\,\mathrm{H}^*(X) = \bigcup_{\substack{Z\subset X \text{ closed} \\ \mathrm{codim}_X(Z)=p}} \ker[\mathrm{H}^*(X) \to \mathrm{H}^*(X\setminus Z)],$$

is usually called the *filtration by coniveau* (or *filtration by codimension of support*). These conjectures assert that this mysterious filtration is equal to (or at least contained in) another filtration which can «actually be computed». The filtration by coniveau is the filtration of a natural spectral sequence, whose E_1-page was written down by Grothendieck, and one has

$$E_1^{p,q} = \bigoplus_{x\in X^{(p)}} \mathrm{H}^{q-p}(\kappa(x)).$$

While the Weil conjectures have been proven by Deligne in 1974, the Hodge and Tate conjectures are still far from being solved, unless we consider specific dimensions or families of varieties. In [BO74], Bloch and Ogus showed by mimicking Quillen's proof of the Gersten conjecture in algebraic K-theory that when $\mathrm{H}^*(X)$ is étale cohomology with torsion coefficients, then the coniveau spectral sequence on X computes, as its E_2-page, the cohomology of the Zariski sheaf $\mathcal{H}^*$ given by the sheafification of the presheaf that sends an open subset $U\subset X$ to $\mathrm{H}^*(U)$. It was soon realised that this approach could provide an interesting point of view on a stable birational invariant of smooth and projective varieties called *unramified cohomology*, which was introduced by Colliot-Thélène and Ojanguren in [CTO89]. The unramified cohomology of a smooth and connected variety over a field may indeed be alternatively defined, thanks to the results of Bloch-Ogus, as the subgroup of the cohomology of the generic point given by all classes that have trivial residues at all codimension one points.

Over the complex numbers, unramified cohomology with torsion coefficients proved to be highly useful in the study of the Lüroth problem, that is the study of unirational varieties which are not rational. In fact, an invariant used by Artin-Mumford in this regard, which is the torsion in the Betti cohomology group $\mathrm{H}^3_B(X,\mathbb{Z})$, is equal for rationally connected varieties to the unramified Brauer group $\mathrm{Br}_{\mathrm{nr}}(X) := \mathrm{H}^2_{\mathrm{nr}}(X,\mathbb{Q}/\mathbb{Z})$. In [CTO89], the authors exhibited unirational sixfolds with trivial unramified Brauer group but non vanishing group $\mathrm{H}^3_{\mathrm{nr}}(X,\mathbb{Q}/\mathbb{Z})$. A particular instance where this notion has been extremely successful is Saltman's paper [Sal84], where the author showed that some function fields (invariant fields of a linear action of a finite group G) are not purely transcendental over the ground field, thus settling Noether's problem over an algebraically closed field in the negative, and providing a new perspective on the inverse Galois problem. Saltman used the unramified Brauer group $\mathrm{Br}_{\mathrm{nr}}(\Bbbk(X)/k)$ of the quotient variety $X = \mathbf{SL}_{n,k}/G$ (for $n\geq 1$ sufficiently large), which may be shown to be equal to the Brauer-Grothendieck group $\mathrm{Br}(\widetilde{X}) := \mathrm{H}^2_{\text{ét}}(\widetilde{X},\mathbb{G}_m)$ of a smooth and projective compactification $\widetilde{X}$. In concrete cases, it is quite unclear how to construct such a model for a given function field. A key aspect of Saltman's paper is that the unramified point of view enables one to dispense with the construction of an explicit model, and even with the existence of such a model.

Nowadays essentially all deep conjectures in the theory of algebraic cycles on smooth complex projective varieties are formulated rationally. For instance, Hodge originally formulated his famous conjecture integrally, but when Atiyah and Hirzebruch showed that it fails for torsion cycles, it became clear that one should phrase it rationally. Nonetheless, investigating instances where the Hodge conjecture may hold integrally remained an active field of research. Similarly,

it is natural to investigate to which extent other cycle conjectures (such as the Tate conjecture) may hold integrally, or on torsion cycles. It is well-known that the finiteness of the Brauer group (which coincides for smooth, proper and connected varieties with its unramified counterpart) implies the (rational) Tate conjecture for codimension 1 cycles on surfaces over finite fields. On the other hand, Bloch-Ogus showed in [BO74] that over the complex numbers, unramified cohomology in degree 3 is related to the Griffiths group of codimension 2 cycles, that is, the kernel of the cycle class map $\mathrm{CH}^2(X)/\mathrm{alg} \to \mathrm{H}^4_B(X, \mathbb{Z}(2))$. Colliot-Thélène and Voisin computed in [CTV12] the failure of the integral Hodge conjecture for codimension 2 cycles on smooth complex projective varieties in terms of unramified cohomology in degree 3 ; a similar statement holds for the integral Tate conjecture thanks to Kahn in [Kah12]. These observations suggest together that unramified cohomology in higher degree might play a critical role in these various conjectures.
This thesis is organised as follows :

- Chapter I is devoted to the prerequisites. We provide a rather long survey on algebraic cycles and their various relations to (co)homology theories. More precisely, we discuss generalities on Chow rings (§I.1) such as localisation, intersection products and homotopy invariance, as well as cycle classes (§I.2) (in the ℓ-adic and complex case). We also discuss some more specific declinations (equivariant intersection theory and decompositions of the diagonal à la Bloch-Srinivas). We provide a detailed account on K-theory (§1.3) in the sense of Quillen (K-theory of schemes and Gersten's conjecture) and Milnor's K-theory (tame and Galois symbols, the Bloch-Kato conjecture) ; we dedicate a whole section (§I.4) to the proof of Gersten's conjecture in étale cohomology, due to Bloch-Ogus. We finish with some applications to unramified cohomology (§I.5), together with some general properties (stable-birational invariance, codimension one purity, *etc.*).

- Chapter II provides a detailed treatment of the proof of the main result of the paper [Pey07] by Peyre, which quite precisely describes the unramified cohomology group $\mathrm{H}^3_{\mathrm{nr}}(\mathbb{C}(W)^G/\mathbb{C}, \mathbb{Q}/\mathbb{Z}(2))$ attached to a finite group G endowed with a faithful complex representation W.

- Chapter III deals with the main results of the paper [CTV12] by Colliot-Thélène and Voisin. In particular, we explain (§§III.2.1–2.2) how the authors -using the Bloch-Kato conjecture and arguments related to the decomposition of the diagonal- derived an isomorphism between the groups $\mathrm{H}^3_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z})$ and $Z^4(X) = \mathrm{Hdg}^4(X, \mathbb{Z})/\mathrm{Im}[c\ell_2 : \mathrm{CH}^2(X) \to \mathrm{H}^4_B(X, \mathbb{Z})]$ (thus measuring the failure of the integral Hodge conjecture in terms of unramified cohomology) for codimension 2 cycles on a smooth and projective connected complex variety X such that the group $\mathrm{CH}_0(X)$ is supported on a surface. We also describe how the authors used this identification to obtain some new results for 0-cycles on varieties over a field of cohomological dimension at most 1.

- Chapter IV explains the main ideas of a recent paper [Sca21] by Scavia, which determines the non-vanishing of some motivic classes of classifying stacks in the Grothendieck ring of stacks $K_0(\underline{\mathbf{Stck}}_{\mathbb{C}})$ over the complex numbers. We first provide a small account on the general theory of algebraic stacks (§IV.1.1) and the basic properties of $K_0(\underline{\mathbf{Stck}}_{\mathbb{C}})$ (§IV.1.2), and then explain how a general result due to the author (§IV.2.1) allows one, building upon the results of Peyre and Colliot-Thélène-Voisin (that are discussed in the previous chapters), to give an example of a finite group G whose classifying stack has a non-trivial class in the Grothendieck ring, despite the vanishing of $\mathsf{Br}_{\mathrm{nr}}(\mathbb{C}(W)^G/\mathbb{C})$ for any faithful complex representation W of G.

Acknowledgements

— This work owes a great deal to my advisor, Olivier Wittenberg. His advice was invaluable and his intuitions always truly inspirational. I thank him for his patience, his mathematical generosity and constant clarity. I wish to express my deepest respect for him, and thank him for giving me the opportunity to be his student.

I am also indebted to Qing Liu, who largely contributed to my inclination towards arithmetic and algebraic geometry. I wish to thank him for his support and the various interesting discussions we had, as well as for his thoroughness when dealing with mathematical content, which I often lack. I also thank him for providing me with a detailed argument regarding some torsion questions on Chow groups and allowing me to use some notes he wrote for the purposes of this thesis (they constitute Appendix C, §C.2).

I will also use this space to thank all the people (professors, advisors, and others) who contributed, over the past few years, to my studies and more generally to my mathematical development, thus making me able to develop this work.

To all the friends I made during my time in Leiden and Bordeaux, and to those I already had before - thank you, I learned a lot also from you (both humanly and mathematically speaking).

Finally, my deepest gratitude goes to my family, for their unconditional love and support, and in particular for their inexhaustible patience (and they know why I put the emphasis on this word).

Notation

Algebra and arithmetic

— If E is a set, we denote by $\#E$ its cardinality. If M is a (left) G-set for an arbitrary group G, we write the corresponding action by $M \times G \ni (m, g) \mapsto g.m$. If G is a group and $n \in \mathbb{Z}_{\geq 1}$, then G^n denotes (unless explicitely stated) the set of elements of the form ng form $g \in G$, and $\mathcal{Z}(G)$ denotes the center of G. If A is an abelian group, and if n and ℓ are respectively a non negative integer and a prime integer, then $A[n]$ denotes the subgroup of n-torsion elements of A and $A\{\ell\}$ denotes the subgroup of ℓ-primary torsion elements of A.

The characteristic of a field k is denoted by $\mathrm{char}(k)$. A field $k \subset L$ is denoted by L/k. We write k_s and $\overline{k}$ for a separable and algebraic closure of k, respectively.

The absolute Galois group $\mathsf{Gal}(k_s/k)$ of k is denoted by Γ_k. If M is a set together with a Γ_k-action, we usually denote it by $M \times \Gamma_k \ni (m, \sigma) \mapsto {}^{\sigma}m$. If k is a global field, then Ω_k denotes the set of its places, $\Omega_\infty \subset \Omega_k$ its archimedean places and $\Omega_f \subset \Omega_k$ its finite places. The maximal unramified extension of a global (or local) field k is denoted k_{nr}. The completion of k at a place $v \in \Omega_k$ is denoted by k_v. We denote by $\mathrm{cd}(k)$ the cohomological dimension of Γ_k.

We denote by $\underline{\mathbf{Sets}}$ the category of sets and maps of sets, $\underline{\mathbf{Ab}}$ the category of abelian groups and morphims of groups. If R is a commutative ring, then $\underline{\mathbf{Alg}}_R$ denotes the category of R-algebras and morphism of algebras, and $\underline{\mathbf{Mod}}_R$ denotes the category of R-modules and morphisms of modules.

If G is an arbitrary group, then $\underline{\mathbf{Mod}}_G$ denotes the category of G-modules, that is, the category of abelian groups together with a (left) G-action compatible with their $\mathbb{Z}$-module structure, and the morphisms of G-modules are the morphisms of abelian groups that are G-equivariants. If H is a subgroup of G and A is an H-module, we denote by $\mathsf{M}_H^G(A) := \mathsf{Hom}_{\underline{\mathbf{Mod}}_H}(\mathbb{Z}[G], A)$ the induced G-module.

Algebraic geometry

— If X is a scheme and $x \in X$, then $\mathcal{O}_X$ is the structure sheaf on X and $\mathcal{O}_{X,x}$ is the stalk of $\mathcal{O}_X$ at x, which is a local ring. We denote its maximal ideal by $\mathfrak{m}_x$ and its residue field by $\kappa(x)$. If $c \geq 1$, then $X^{(c)}$ is the set of codimension c points of X. In particular, $X^{(1)}$ can alternatively denote the set of irreducible prime divisors of X. If X is integral, we write $\mathbb{k}(X)$ for its function field and $\mathbb{k}_X$ for the associated constant sheaf.

If k is a field and X is a k-scheme, and if L/k is a field extension, then we denote by X_L the fibre product $X \times_k L = X \times_{\mathsf{Spec}\, k} \mathsf{Spec}\, L$. In particular, $\overline{X} := X \times_k \overline{k}$ and $X^s := X \times_k k_s$. A k-variety is a separated k-scheme of finite type. If X, Y are two k-varieties where X is quasi-projective and Y is projective, then we denote by $\mathrm{Mor}(X, Y)$ the (locally noetherian) scheme that parametrizes morphisms from X to Y in the sense of [Deb01, Chap. 2, §2.2].

We say that X satisfies a property geometrically if it is satisfied over $\overline{X}$, *e.g.* X is geometrically irreducible, integral, *etc.* A scheme X is purely of dimension d if each of its irreducible components has dimension d. If U is an open subset of a scheme X, the Zariski closure of U in X is denoted by $\mathsf{Cl}_{\mathrm{Zar}}(U)$.

If $f : X \to Y$ is a morphism of S-schemes, we call f an X-point of Y, and $Y(X)$ is the set $\mathsf{Hom}_S(X, Y)$ of all the morphisms of S-schemes $X \to Y$. In the particular case where $X = \mathsf{Spec}\, R$ is the spectrum of a ring R, we write $Y(R) := Y(\mathsf{Spec}\, R)$.

The formal completion of a local ring R is denoted by $\widehat{R}$. A noetherian ring R is a G-ring («G» stands for Grothendieck) if for every prime ideal $\mathfrak{p} \subset R$, the fibres of the morphism $\mathsf{Spec}\, \widehat{R_\mathfrak{p}} \to \mathsf{Spec}\, R_\mathfrak{p}$ induced by the inclusion $R_\mathfrak{p} \hookrightarrow \widehat{R_\mathfrak{p}}$ are geometrically regular. Such a ring R is said to be excellent if it is also universally catenary [Mat89, §5] and such that for every R-algebra A of finite type, the set of regular point of $\mathsf{Spec}\, A$ is dense in the latter. Every finite type ring extension of either a field, $\mathbb{Z}$, a complete noetherian local ring or a Dedekind ring of characteristic zero is an excellent ring. A scheme is said to be excellent if it admits a covering by spectra of excellent rings.

If X is a scheme and $n \geq 1$, then $\mathbb{P}^n_X := \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} X$ and $\mathbb{A}^n_X := \mathbb{A}^n_{\mathbb{Z}} \times_{\mathbb{Z}} X$ where $\mathbb{P}^n_{\mathbb{Z}} := \mathsf{Proj}(\mathbb{Z}[x_0, \dots, x_n])$ and $\mathbb{A}^n_{\mathbb{Z}} := \mathsf{Spec}(\mathbb{Z}[x_1, \dots, x_n])$.

We denote by $\underline{\mathbf{Sch}}$ the category of schemes and morphisms of schemes. If k is a field, we denote by $\underline{\mathbf{Var}}_k$ the category of k-varieties and morphisms of varieties and by $\underline{\mathbf{Sm}}_k$ the category of smooth k-varieties and morphisms of k-varieties. If S is a scheme, then $\underline{\mathbf{Sch}}_S$ denotes the category of S-schemes and morphims of S-schemes.

If X is a scheme, we denote by $X_{\text{ét}}$ the small étale site of X, $X_{\text{Ét}}$ the big étale site of X and X_{Zar} the big Zariski site of X. If X_E is any site on X, we denote by $\mathrm{Sh}(X_E)$ the category of sheaves of abelian groups on X_E. If X is a scheme over a field k and $n \geq 1$ is an integer that is invertible on k, then for $j \in \mathbb{Z}$, we put

$$\mathbb{Z}/n(j) := \begin{cases} \mu_n^{\otimes(j-1)} \otimes \mu_n & \text{if } j \geq 1, \\ \mathbb{Z}/n & \text{if } j = 0, \\ \mathsf{Hom}_{\mathrm{Sh}(X_{\text{ét}})}(\mu_n^{\otimes(-j)}, \mathbb{Z}/n) & \text{if } j < 0, \end{cases}$$

where $\mu_n = \mu_{n,X}$ is the étale sheaf of n^{th} roots of unity on X.

Chapter I

Preliminaries

I.1. Chow rings

— We review the necessary background material about intersection theory that we will use extensively in the rest of this text. This includes Chow groups and correspondences, as well as the decomposition of the diagonal (which proved to be a very powerful tool from a birational point of view, for instance in papers of Bloch-Srinivas, Voisin, or Colliot-Thélène), and their equivariant counterpart.

1.1. Recollection on Chow groups

— Let X be a scheme over a field k and $\mathcal{Z}_i(X)$ the group of i-dimensional algebraic cycles on X, that is, the free abelian group formally generated by the reduced and irreducible closed i-dimensional k-subvarieties of X. If $Y \subset X$ is a closed subscheme of dimension $\leq i$, then one can associate a cycle $c(Y) \in \mathcal{Z}_i(X)$:

$$c(Y) := \sum_W n_W [W]$$

where the sum is taken over the i-dimensional reduced irreducible components of Y and the integer n_W is the length of the local ring $\mathcal{O}_{Y,\eta_W}$ (which is Artinian).

If $\phi : Y \to X$ is a proper morphism between quasi-projective schemes, then one has a pushforward map

$$\phi_* : \mathcal{Z}_i(Y) \to \mathcal{Z}_i(X)$$

by sending the class of any irreducible closed subscheme $Z \subset Y$ to the cycle $[\Bbbk(Z) : \Bbbk(Z')] \cdot [Z']$ where $Z' := \phi(Z)$ if $\phi : Z \to Z'$ is generically finite and to 0 otherwise (by the properness of ϕ, the subscheme Z' is guaranted to be closed in X).

Suppose W is a normal algebraic k-variety. Then the local rings at points of codimension one are discrete valuation rings, hence one can define the divisor $\operatorname{div}(\phi)$ of any nonzero rational function $\phi \in \Bbbk(W)^\times$ by setting

$$\operatorname{div}(\phi) := \sum_{D \in W^{(1)}} \nu_D(\phi)[D]$$

where ν_D is the normalised valuation of the local ring at the generic point of D. This gives a cycle of dimension $\dim W - 1$. If we now suppose that $W \subset X$ is a closed subvariety and $\tau : \widetilde{W} \to X$ is the normalisation morphism of W, then one has a natural pushforward map

$$\tau_* : \mathcal{Z}_i(\widetilde{W}) \longrightarrow \mathcal{Z}_i(X)$$

for any $i \geq 0$.

Definition 1.1.1. For $i \geq 0$ we define the subgroup $\mathcal{Z}_i(X)_{\mathrm{rat}}$ of *i-dimensional cycles rationally equivalent to* 0 as the subgroup of $\mathcal{Z}_i(X)$ generated by the cycles of the form

$$\tau_* \mathrm{div}(\phi), \;\dim W = i+1, \; \phi \in \Bbbk(W)^\times$$

where $\tau : \widetilde{W} \to W \hookrightarrow X$ is the normalisation of the closed subvariety W of X. The *Chow group* of i-dimensional cycles is the quotient

$$\mathrm{CH}_i(X) := \mathcal{Z}_i(X)/\mathcal{Z}_i(X)_{\mathrm{rat}}.$$

When X has pure dimension n, we can define the Chow group $\mathrm{CH}^i(X) := \mathrm{CH}_{n-i}(X)$ of i-codimensional cycles.

If X is n-dimensional, reduced and irreducible, then one has a natural morphism

$$\mathsf{Pic}(X) \longrightarrow \mathrm{CH}_{n-1}(X)$$

that sends an invertible sheaf $\mathcal{L}$ to the cycle $\tau_*\mathrm{div}(\sigma)$ where $\tau : \widetilde{X} \to X$ is the normalisation morphism and σ is a nonzero meromorphic section of the pullback invertible sheaf $\tau^*\mathcal{L}$.

If X is now smooth over k, or more generally locally factorial, then $\mathcal{Z}_{n-1}(X)$ is nothing more than the group of Cartier divisors, and we have an isomorphism

$$\mathsf{Pic}(X) \xrightarrow{\sim} \mathrm{CH}_{n-1}(X),$$

see for instance [Voi14, Chap. 2, §2.1.1].

1.2. Some intersection theory

1.2.1. Localisation on Chow groups

— Let X be a quasi-projective scheme and $\iota : Z \hookrightarrow X$ be the inclusion of a closed subscheme. Let $j : U := X \setminus Z \hookrightarrow X$ be the inclusion of the complement. The morphism ι is finite hence proper. By restricting cycles to the open subset U, on defines a pullback morphism j^* on Chow groups. It is clear that $j^* \circ \iota_* = 0$ since any cycle with support on Z cannot meet U. Actually, we can say something better about these morphisms :

Lemma 1.1.2. *For any $i \geq 0$, there exists a localisation exact sequence :*

$$\mathrm{CH}_i(Z) \xrightarrow{\iota_*} \mathrm{CH}_i(X) \xrightarrow{j^*} \mathrm{CH}_i(U) \longrightarrow 0.$$

Proof. If $Z' \subset U$ is any i-dimensional subvariety, then its Zariski closure is an i-dimensional subvariety whose intersection with U is exactly Z, so the surjectivity on the right follows. If $Z' \in \mathcal{Z}_i(X)$ is such that $j^*(Z') \in \mathcal{Z}_i(U)_{\mathrm{rat}}$, then there exist subvarieties $W_l \subset U$ with $\dim W_l = i+1$ and $\phi_l \in \Bbbk(W_l)^\times$ and some integers n_l such that

$$Z' \cap U = \sum_l n_l (\tau_l)_*(\mathrm{div}(\phi_l))$$

where $\tau_l : \widetilde{W_l} \to U$ denotes the normalisation of W_l. If we write $\overline{W_l}$ for the Zariski closure of W_l in X and $\overline{\tau_l}$ for its normalisation (in X), then one has $\phi_l \in \Bbbk(\overline{W_l})^\times$ and the decomposition of $Z' \cap U$ gives

$$Z'' := Z' - \sum_l n_l (\overline{\tau_l})_*(\mathrm{div}(\phi_l)) \in \mathcal{Z}_i(Z),$$

so this implies the exactness of the sequence in the middle (the cycle Z'' being rationally equivalent to Z' in X). $\square$

1.2.2. Intersection products

1.2.2.1. Naive intersection products. In [Ful98], Fulton defines an intersection theory on Chow groups

$$\mathrm{CH}_i(X) \times \mathrm{CH}_l(X) \longrightarrow \mathrm{CH}_{i+l-n}(X)$$

for any smooth n-dimensional variety X. If Z, Z' are two irreducible and reduced subschemes of X of respective dimensions i and l which intersect properly and generically transversally, *i.e.* such that

$$\dim Z \cap Z' = i + l - n,$$

and generically along $Z \cap Z'$, Z and Z' are smooth and have transverse intersection, then one defines $Z \cdot Z'$ as the cycle associated to the scheme $Z \cap Z'$ (note that this scheme has its components of multiplicity 1 given the assumptions). Extending this definition bilinearly, one defines the intersection $Z \cdot Z'$ for any pair of cycles whose supports intersect properly and generically transversally. For more details, see [Ful98, Chap. 6].

If we now suppose that Z and Z' do not intersect properly, then the classical (and now obsolete) theory replaces Z by a cycle $\widetilde{Z}$ that is rationally equivalent to Z and intersects Z' properly. Such a cycle exists when X is quasi-projective by the so called *Chow moving lemma*, and the idea would then be to define $Z \cdot Z'$ as the class of $\widetilde{Z} \cap Z$ in $\mathrm{CH}_{i+l-n}(X)$. There are several drawbacks to this method: the moving lemma is problematic in the sense that it does not allow to choose cycles that meet generically transversally, so this leads to some issues with respect to the well-definedness of intersection products (it requires a substantial amount of work to check that this product is well-defined). Moreover, it has the bad property of not respecting supports (it would be reasonable to expect that the intersection product of two cycles should be a cycle supported on the set-theoretic intersection of the support of those two cycles). This leads us to Fulton's approach.

1.2.2.2. Refined intersection products. Let Z be any i-dimensional cycle and Z' be an l-dimensional cycle. Fulton bypasses the issues of Chow's moving lemma by using the deformation to the normal cone [Ful98, Chap. 5], and defines a «refined» intersection product as a cycle

$$Z \cdot Z' \in \mathrm{CH}_{i+l-n}(|Z| \cap |Z'|),$$

where $|Z|$ and $|Z'|$ denote the support of Z and Z' respectively. This product has the nice property that it naturally maps to the classical intersection product $Z \cdot Z'$ in $\mathrm{CH}_{i+l-n}(X)$, but it provides the «right» intersection product that deals with the problems of excess, that is, when Z and Z' do not intersect properly. For more details about this construction, see [Ful98, Chap. 6, §6.3].

We can therefore endow the direct sum

$$\mathrm{CH}_*(X) := \bigoplus_{l \geq 0} \mathrm{CH}_l(X)$$

with the structure of a commutative graded ring with unit given by the class $[X] \in \mathrm{CH}_{\dim X}(X)$ of X. Actually, one can show that Fulton's intersection theory gives a contravariant graded ring structure on $\mathrm{CH}_*(X)$ that agrees with flat pullbacks, intersection with Cartier divisors, and admits a projection formula, and that it is unique for these properties, see [Gil05, Thm. 23].

1.3. Functoriality

1.3.1. Homotopy invariance, projection formulæ

— In this section, we discuss a few cases where certain morphisms of schemes induce pushforwards that descend to Chow groups.

1.3.1.1. Proper pushforwards. Let $f : Y \to X$ be a proper morphism between quasi-projective schemes and $f_* : \mathcal{Z}_i(Y) \to \mathcal{Z}_i(X)$ the induced pushforward.

Lemma 1.1.3. *The morphism f_* sends $\mathcal{Z}_i(Y)_{rat}$ to $\mathcal{Z}_i(X)_{rat}$, and thus descends to a morphism*

$$f_* : \mathrm{CH}_i(Y) \longrightarrow \mathrm{CH}_i(X).$$

Proof. Fix a subvariety W of Y of dimension $i+1$ and let $\tau : \widetilde{W} \to W$ be the normalisation morphism such that $f \circ \tau$ is generically finite. Let $\phi \in \Bbbk(W)^\times$, $W' := (f \circ \tau)(\widetilde{W})$ and $\tau' : \widetilde{W}' \to W'$ the normalisation, so that we have a commutative diagram :

$$\begin{tikzcd}
\widetilde{W} \arrow[r, "\tau"] \arrow[d, "\tilde{f}"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d, "f"] \\
\widetilde{W}' \arrow[r, "\tau'"'] & X
\end{tikzcd}.$$

We thus obtain that $f_* \circ \tau_* = \tau'_* \circ \tilde{f}_* : \mathcal{Z}_i(\widetilde{W}) \to \mathcal{Z}_i(X)$. On the other hand, one can assume that the field extension $\Bbbk(\widetilde{W})/\Bbbk(\widetilde{W}')$ is algebraic (see the proof of [Ful98, Prop. 1.4]), so that we have a norm morphism $N : \Bbbk(\widetilde{W})^\times \to \Bbbk(\widetilde{W}')^\times$, and if $\phi \in \Bbbk(\widetilde{W})^\times$ we have $\tilde{f}_*(\mathrm{div}(\phi)) = \mathrm{div}(N(\phi))$. Hence, f_* sends cycles rationally equivalent Y to 0 to cycles rationally equivalent to 0 on X, as desired. □

1.3.1.2. Flat pullbacks. Let now $f : Y \to X$ be flat of relative dimension d, and Z a reduced and irreducible i-dimensional subscheme of X. Then $f^{-1}(Z)$ is a subscheme of Y of dimension $i+d$; we therefore obtain pullback cycle $f^*Z \in \mathcal{Z}_{i+d}(Y)$, and extending linearly, this defines a pullback morphism $f^* : \mathcal{Z}_i(X) \to \mathcal{Z}_{i+d}(Y)$. Once again, this pullback commutes with rational equivalence :

Lemma 1.1.4 ([Ful98, Thm. 1.7])**.** *The morphism f^* sends $\mathcal{Z}_i(X)_{rat}$ to $\mathcal{Z}_{i+d}(Y)_{rat}$, and thus descends to a morphism*

$$f^* : \mathrm{CH}_i(X) \longrightarrow \mathrm{CH}_{i+d}(Y).$$

In particular, if X is irreducible, then flat pullback sends $\mathrm{CH}^i(X)$ to $\mathrm{CH}^i(Y)$.

1.3.1.3. Extending to the smooth case. One can easily see why flatness is a much too restrictive condition when one wants to define a pullback morphism. But with flat pullbacks at our disposal, we can bypass this issue, at least in a smooth setting. Suppose indeed that $f : Y \to X$ is a morphism of varieties where X is smooth over k. Since the latter is in particular flat over the base (a k-algebra is always flat), then by base change the projection $\mathrm{pr}_2 : Y \times_k X \to X$ is flat as well, so it induces well-defined pullbacks on Chow groups. On the other hand, the smoothness of X shows that the image of $j_f = (\mathrm{Id}, f) : Y \to Y \times_k X$ is a local complete intersection, see *e.g.* [Liu02, Chap. 6, §6.3, Prop. 3.20]. As shown in [Ful98, Chap. 6, §6.6], one can define a restriction map on Chow groups for local complete intersections morphisms, so in our case a map $j_f^* : \mathrm{CH}^k(Y \times_k X) \to \mathrm{CH}^k(Y)$. This yields a well-defined pullback morphism $f^* : \mathrm{CH}^k(X) \to \mathrm{CH}^k(Y)$ given by the composition

$$\mathrm{CH}^k(X) \xrightarrow{\mathrm{pr}_2^*} \mathrm{CH}^k(Y \times_k X) \xrightarrow{j_f^*} \mathrm{CH}^k(Y).$$

1.3.1.4. Projection formulæ. We would now like to highlight how these pullback and pushforward morphisms behave we dealing with intersection products. Suppose indeed that $f : Y \to X$ is a morphism of smooth k-varieties. We have the following results :

Proposition 1.1.5 ([Ful98, Prop. 1.7]).

(i) *For $Z, Z' \in \mathrm{CH}^*(X)$, we have :*

$$f^*(Z \cdot Z') = f^*Z \cdot f^*Z' \in \mathrm{CH}^*(Y).$$

(ii) *If f is moreover proper, then for any $Z \in \mathrm{CH}^*(Y)$ and $Z' \in \mathrm{CH}^*(X)$, we have :*

$$f_*(f^*Z \cdot Z') = Z \cdot f_*Z' \in \mathrm{CH}^*(X).$$

Corollary 1.1.6. *If $f : Y \to X$ is a proper morphism between smooth k-varieties of the same dimension, then for each $Z \in \mathrm{CH}^*(X)$, we have :*

$$f_*f^*Z = \deg f \cdot Z.$$

Proof. By the projection formula, if we put $Z' := [Y] \in \mathrm{CH}_{\dim Y}(Y)$, then by definition of the proper pushforward, we have $f_*Z' = \deg f \cdot [X] \in \mathrm{CH}^*(X)$. Thus,

$$f_*f^*Z = f_*(f^*Z \cdot [Y]) = Z \cdot (\deg f \cdot [X]) = \deg f \cdot Z.$$

□

1.3.2. Correspondences

Definition 1.1.7. A *correspondence* of dimension i between two smooth k-varieties X and Y is a cycle $\Gamma \in \mathrm{CH}_i(X \times_k Y)$[1]. A 0-*correspondence* between X and Y is a cycle $\Gamma \in \mathrm{CH}^{\dim X}(X \times_k Y)$.

Under adequate assumptions, correspondences act naturally on Chow rings. Indeed, if the variety X is proper, then in particular $\mathsf{pr}_2 : X \times_k Y \to Y$ is proper by base change, so a correspondence $\Gamma \in \mathrm{CH}^i(X \times_k Y)$ yields a natural morphism

$$\Gamma_* : \mathrm{CH}^l(X) \longrightarrow \mathrm{CH}^{l+i-\dim X}(Y)$$

given by

$$Z \longmapsto \mathsf{pr}_{2*}(\mathsf{pr}_1^*(Z) \cdot \Gamma).$$

If in particular Γ is a 0-correspondence, then the induced morphism preserves degrees, that is, $\Gamma_* : \mathrm{CH}^l(X) \to \mathrm{CH}^l(Y)$. If Y is also projective, then one can also consider a pullback morphism

$$\Gamma^* : \mathrm{CH}^l(Y) \longrightarrow \mathrm{CH}^l(X)$$

given by

$$Z \longrightarrow \mathsf{pr}_{1*}(\mathsf{pr}_2^*(Z) \cdot \Gamma).$$

1.3.2.1. Composition of correspondences. Assume now that X, Y and W are smooth varieties where X and Y are proper. Let $\Gamma \in \mathrm{CH}^l(X \times_k Y)$ and $\Gamma' \in \mathrm{CH}^{l'}(Y \times_k W)$ be correspondences. We can define their *composition* $\Gamma \circ \Gamma' \in \mathrm{CH}^{l+l'-\dim Y}(X \times_k W)$ as follows :

$$\Gamma \circ \Gamma' := \mathsf{pr}_{1,3*}(\mathsf{pr}_{1,2}^*(\Gamma) \cdot \mathsf{pr}_{2,3}^*(\Gamma')),$$

where $\mathsf{pr}_{i,j}$ denotes the projection from $X \times_k Y \times_k W$ onto the product of the i^{th} and j^{th} factors for $1 \leq i < j \leq 3$. It is quite straightforward to show that the composition of correspondences is an associative operation. In particular, it endows $\mathrm{CH}^*(X \times_k X) := \bigoplus_{l \geq 0} \mathrm{CH}^l(X \times_k X)$ with the structure of a (non-necessarily commutative) ring.

An important remark is that the action of correspondences on Chow groups commutes with composition, thanks to the projection formula :

[1] Sometimes, one may write $X \vdash Y$ in order to denote a correspondence from X to Y (this notation is for instance used extensively in [Ful98, Chap. 16], where it becomes more relevant when one refers to a correspondence of a given degree, as the dimension of the first variety plays a role here).

Proposition 1.1.8 ([Voi03, Prop. 9.17]). *Let Γ, Γ' be as above, and*

$$\Gamma_* : \mathrm{CH}^*(X) \longrightarrow \mathrm{CH}^*(Y) \quad \textit{and} \quad \Gamma'_* : \mathrm{CH}^*(Y) \longrightarrow \mathrm{CH}^*(W)$$

their associated morphisms. Then we have :

$$(\Gamma' \circ \Gamma)_* = \Gamma'_* \circ \Gamma_* : \mathrm{CH}^*(X) \longrightarrow \mathrm{CH}^*(W).$$

1.4. Decomposition of the diagonal

— The following notion was first introduced by Bloch (based on an idea of Colliot-Thélène) and Bloch-Srinivas in [BS83]. Here we mainly follow the expositions provided in [Voi14, Chap. 3] and [Sch21, §7].

Definition 1.1.9. A variety X of pure dimension n over a field k admits a *decomposition of the diagonal* if

$$[\Delta_X] = [X \times z] + [Z_X] \in \mathrm{CH}_n(X \times_k X),$$

where $\Delta_X \subset X \times_k X$ is the diagonal, $z \in \mathcal{Z}_0(X)$ is a 0-cycle on X and Z_X is a cycle on $X \times_k X$ which does not dominate any component of the first factor.

Examples 1.1.10.

- $X = \mathbb{P}^n_k$ admits a decomposition of the diagonal because $\mathrm{CH}_n(X \times_k X)$ is generated by $[\mathbb{P}^n_k \times \{x\}]$ for some k-rational point $x \in \mathbb{P}^n_k$, together with cycles that do not dominate the first factor, namely any $Y^{n-i} \times Y^i$ for $i \in [\![1, n]\!]$ where $Y^i \subset \mathbb{P}^n_k$ denotes any linear i-dimensional subspace.
- The glueing $X = \mathbb{P}^n_k \cup_H \mathbb{P}^n_k$ along a hypersurface $H \subset \mathbb{P}^n_k$ admits a decomposition of the diagonal if H admits a k-rational point. Indeed, if we write $X = X_1 \cup X_2$ where X_1, X_2 are the irreducible components of X, then $X \times_k X$ has 4 irreducible components $X_i \times_k X_j$ for $i, j \in \{1, 2\}$. We can write

$$[\Delta_X] = [\Delta_{X_1}] + [\Delta_{X_2}] \in \mathrm{CH}_n(X \times_k X)$$

where $\Delta_{X_i} \subset X_i \times_k X_i$ is the diagonal for $i = 1, 2$. By proper pushforward, we have two maps

$$\mathrm{CH}_n(X_i \times_k X_i) \longrightarrow \mathrm{CH}_n(X \times_k X), \quad i = 1, 2.$$

As in the previous example, we have a decomposition of the diagonal for X_1 and X_2, so we obtain the following :

$$[\Delta_X] = [X_1 \times \{x_1\}] + [X_2 \times \{x_2\}] + [Z_X] \in \mathrm{CH}_n(X \times_k X)$$

where $x_1 \in X_1(k), x_2 \in X_2(k)$ and Z_X is a cycle on $X \times_k X$ which does not dominate any component of the first factor. Now since H contains a k-rational point, then any two k-rational points of X are rationally equivalent (*e.g.* they can be joined by a chain of two lines), so $[X_2 \times \{x_1\}] = [X_2 \times \{x_2\}] \in \mathrm{CH}_n(X \times_k X)$. Hence,

$$[\Delta_X] = [X \times \{x_1\}] + [Z_X] \in \mathrm{CH}_n(X \times_k X),$$

and we have a decomposition of the diagonal.

1.4.1. Decomposition of the diagonal and zero-cycles

Lemma 1.1.11. *Let X be a proper variety of pure dimension n over a field k. If X admits a decomposition of the diagonal, then :*

(i) *the 0-cycle z in Definition (1.1.9) has degree 1 ;*

(ii) *X is geometrically connected.*

Proof. By the properness of X, the projection $\mathsf{pr}_1 : X \times_k X \to X$ gives a pushforward

$$\mathsf{pr}_{1*} : \mathrm{CH}_n(X \times_k X) \longrightarrow \mathrm{CH}_n(X).$$

If $[\Delta_X]$ admits a decomposition, then we have

$$\mathsf{pr}_{1*}([\Delta_X]) = \mathsf{pr}_{1*}([X \times z]) + \mathsf{pr}_{1*}([Z_X]) = \deg(z)[X] \in \mathrm{CH}_n(X)$$

since Z_X does not dominate any component of the first factor. On the other hand, $\mathsf{pr}_{1*}([\Delta_X]) = [X]$, and since this class is torsion-free, then we must have $\deg(z) = 1$, hence the first claim.

Now since any field extension L/k is flat over k, then by flat pullback we obtain a decomposition of the diagonal for $X \times_k L$ as well. It thus suffices to show that X is connected if it admits a decomposition of the diagonal. Suppose by contradiction that

$$X = \bigsqcup_{i=1}^{r} X_i$$

is a disjoint union of finitely many varieties X_i with $r \geq 2$. Write the 0-cycle z as $z = \sum_{i=1}^r z_i$ where each z_i is supported on X_i respectively. Since the $X_i \times_k X_j$ for $i \neq j$ are open subschemes of $X \times_k X$ and open immersions are flat, then we can pull back the decomposition of the diagonal to each $X_i \times_k X_j$. We obtain that each

$$[X_i \times z_j] \in \mathrm{CH}_n(X_i \times_k X_j)$$

is rationally equivalent to a cycle that does not dominate X_i *via* the projection onto the first factor. Now, pushing the identity forward to the first factor, we obtain that z_j has degree 0. This holds for all j since $r \geq 2$, so actually z has degree 0, which contradicts the first claim, as desired. □

If we further assume the variety X to be integral, then the following result holds :

Proposition 1.1.12. *An integral variety X of dimension n over a field k admits a decomposition of the diagonal if and only if there is a 0-cycle $z \in \mathcal{Z}_0(X)$ on X such that*

$$[\delta_X] = [z_{\Bbbk(X)}] \in \mathrm{CH}_0(X_{\Bbbk(X)}),$$

where δ_X denotes the 0-cycle on $X_{\Bbbk(X)}$ induced by flat pullback by the diagonal Δ_X.

Proof. The universal property of the generic point provides an isomorphism

$$\varinjlim_{U \subset X} \mathrm{CH}_n(X \times_k U) = \varinjlim_{U \subset X} \mathrm{CH}^n(X \times_k U) \xrightarrow{\sim} \mathrm{CH}_0(X_{\Bbbk(X)})$$

where U ranges among the non-empty open subsets of X. Indeed, $X^{(n)}_{\Bbbk(X)} = \varprojlim_{U \subset X} (X \times_k U)^{(n)}$ and

$$\mathrm{CH}^n(X_{\Bbbk(X)}) = \operatorname{coker}\left[\bigoplus_{x \in X^{(n-1)}_{\Bbbk(X)}} \kappa(x)^\times \xrightarrow{\oplus \mathrm{div}} \bigoplus_{x \in X^{(n)}_{\Bbbk(X)}} \mathbb{Z} \right]$$

and

$$\mathrm{CH}^n(X \times_k U) = \operatorname{coker}\left[\bigoplus_{x \in (X \times_k U)^{(n-1)}} \kappa(x)^\times \xrightarrow{\oplus \mathrm{div}} \bigoplus_{x \in (X \times_k U)^{(n)}} \mathbb{Z} \right].$$

If we therefore suppose that X admits a decomposition of the diagonal, then the above isomorphism yields the desired identity of the proposition. Conversely, if we are provided with this identity, then applying the localisation exact sequence and passing to the limit we obtain an exact sequence :

$$\varinjlim_{U \subset X} \mathrm{CH}_n((X \times_k X) \setminus (X \times_k U)) \longrightarrow \mathrm{CH}_n(X \times_k X) \longrightarrow \mathrm{CH}_0(X_{\Bbbk(X)}) \longrightarrow 0.$$

Since X has only one irreducible component, then the identity from the proposition must come from a decomposition of the diagonal *via* the second map. □

1.4.2. The Bloch-Srinivas principle

1.4.2.1. A general principle. Here we state a quite general statement relating the torsion of algebraic cycles to the generic 0-cycle on a connected variety. This is part of a series of arguments due to Bloch-Srinivas in [BS83, Thm. 1] (see also [Bl010, Appendix to Lecture 1] and [Voi03, Thm. 10.19]) :

> **Proposition 1.1.13 (Bloch-Srinivas Principle [Voi19, Thm. 3.1]).** *Let $\phi : Y \to B$ be a flat morphism of varieties over a field k where B is smooth and connected, and let Z be a cycle on Y. Suppose that $\Omega \supset k$ is a universal domain*[2] *and that for any point $b \in B(\Omega)$, the restricted cycle $Z|_{Y_b}$ is rationally equivalent to 0. Then there exists an integer $N > 0$ and a dense Zariski open subset $U \subset B$ such that $NY|_{\phi^{-1}(U)} = 0 \in \mathrm{CH}^*(\phi^{-1}(U))$.*

Remark 1.1.14. The flatness condition ensures that the restricted cycles $Z|_{Y_b}$ are well-defined. Note also that the smoothness assumption on B is not too restrictive since the conclusion only concerns a dense open subset.

Keeping the notations of the above proposition, let us first prove the following intermediary result :

Lemma 1.1.15. *Let $k \subset L \subset \Omega$ be field extensions. Then for each $i \geq 0$, the kernel of the natural map $\mathrm{CH}^i(Y_L) \to \mathrm{CH}^i(Y_\Omega)$ is torsion.*

Proof. If the degree $[\Omega : L]$ is finite, then one always has a norm map $\mathrm{CH}^i(Y_L) \to \mathrm{CH}^i(Y_\Omega)$, so one can do a restriction-corestriction argument (by projection formula) and the lemma immediately follows. If Ω is an arbitrary algebraic extension of L, then one restricts to the finite subextensions and passes to the limit. Otherwise, up to enlarging L and Ω, one can assume that L is algebraically closed, so in this case $\mathrm{CH}^i(Y_\Omega)$ is a limit of Chow groups of the form $\mathrm{CH}^i(Y \times_L U)$, where U is an L-variety. As the latter is in particular of finite type, up to shrinking U and fixing a closed immersion into $\mathbb{A}^d_L$ one can find a closed point on U, which provides a section of $\mathrm{CH}^i(Y) \to \mathrm{CH}^i(Y \times_L U)$, hence the claim in the general case (for further details, see Appendix C, §C.2). ◻

Proof of Proposition (1.1.13). Fix an embedding $\Bbbk(B) \subset \Omega$. We may apply the assumptions to the generic point η_B of B, which is therefore defined over Ω, so that we get a cartesian diagram

$$\begin{tikzcd} (Y_{\eta_B})_\Omega \arrow[r] \arrow[d] & Y_{\eta_B} \arrow[d] \\ \mathsf{Spec}\,\Omega \arrow[r] & \mathsf{Spec}\,\Bbbk(B) \end{tikzcd} .$$

Since Z vanishes in $\mathrm{CH}^*((Y_{\eta_B})_\Omega)$ (as it vanishes in every fibre), then we can apply the previous lemma to $k \subset \Bbbk(B) \subset \Omega$, which shows that Z must be torsion in $\mathrm{CH}^*(Y_{\eta_B})$. By genericity, there must exist a dense open subset $U \subset B$ such that Z actually is torsion in $\mathrm{CH}^*(\phi^{-1}(U))$, as desired. ◻

One can also derive these arguments to obtain the following auxiliary statement :

> **Proposition 1.1.16 ([Voi19, Thm. 3.2]).** *Under the same assumptions as in the above theorem, there exists a dense Zariski open subset $U \subset B_{reg}$ of the regular locus of B and a finite cover $U' \to U$ such that $Z_{U'} = 0 \in \mathrm{CH}^*(U' \times_U \phi^{-1}(U))$, and $Z_{U'}$ is the pullback of the cycle $Z|_{\phi^{-1}(U)}$ to $U' \times_U \phi^{-1}(U)$.*

1.4.2.2. The complex case. If we specialise to complex varieties, then the situation is even more comfortable. Indeed, if X is a variety over $\mathbb{C}$, then it is also defined over a field k of finite transcendence degree over $\mathbb{Q}$, and $\mathbb{C}$ is a universal domain with respect to k. We therefore get :

[2] That is, an algebraically closed field of infinite transcendence degree over k. This condition ensures that Ω contains any finitely generated extension of k (this allows us to package generic information of k-varieties universally).

Theorem 1.1.17. *Let $\phi : Y \longrightarrow B$ be a morphism of complex varieties where B is smooth and connected and Z a cycle on Y such that for any $b \in B(\mathbb{C})$, the cycle $Z|_{Y_b}$ is rationally equivalent to* 0. *Then there exists an integer $N > 0$ and a dense Zariski open subset $U \subset B$ such that $NZ|_{\phi^{-1}(U)} = 0 \in \mathrm{CH}^*(\phi^{-1}(U))$.*

This theorem naturally leads to the famous classical *Bloch-Srinivas decomposition of the diagonal* :

Theorem 1.1.18 (Bloch-Srinivas [BS83, Prop. 1]). *Let X be a smooth and connected complex variety of dimension n, $V \subset X$ a subvariety (possibly reducible) such that the proper pushforward $\mathrm{CH}_0(V) \to \mathrm{CH}_0(X)$ is surjective. Then there exists an integer $N > 0$, a divisor D on X and two correspondences $\Gamma_1, \Gamma_2 \in \mathrm{CH}_n(X \times_{\mathbb{C}} X)$ such that* $\mathrm{Supp}(\Gamma_1) \subset D \times_{\mathbb{C}} X$, $\mathrm{Supp}(\Gamma_2) \subset X \times_{\mathbb{C}} V$ *and*

$$N[\Delta_X] = \Gamma_1 + \Gamma_2 \in \mathrm{CH}_n(X \times_{\mathbb{C}} X).$$

Proof. Applying the localisation exact sequence for Chow groups

$$\mathrm{CH}_0(V) \xrightarrow{j_*} \mathrm{CH}_0(X) \longrightarrow \mathrm{CH}_0(X \setminus V) \longrightarrow 0,$$

we see as before that the assumption is equivalent to the vanishing of $\mathrm{CH}_0(X \setminus V)$. By the previous theorem, we thus know that there exists a dense open subset $U \subset X$ and an integer $N > 0$ such that

$$N[\Delta_X]|_{U \times_{\mathbb{C}} (X \setminus V)} = 0 \in \mathrm{CH}^*(U \times_{\mathbb{C}} (X \setminus V)).$$

In particular, without loss of generality we can let $D := X \setminus U$, and the localisation exact sequence shows that the equality above is equivalent to the decomposition in the theorem. □

Remark 1.1.19. Suppose that X is a rationally connected variety over a field k. Then its Chow group of 0-cycles is trivial when passing to an algebraically closed field, as all points are rationally equivalent. We thus see that in the above Bloch-Srinivas decomposition of the diagonal (that is, over $\mathbb{C}$), we can take $V = \{x\}$ where x is a $\mathbb{C}$-point of X, and the theorem provides a decomposition of the diagonal in the sense of Definition (1.1.9) up to a rational factor.

This is in particular true if X is smooth, projective and unirational : let indeed $V \subset X$ be a dense open subset and $\varphi : U \to V$ a surjective morphism where U is an open subscheme of some affine space. If $x, y \in V(\mathbb{C})$, then there exist $p, q \in U(\mathbb{C})$ mapping to x and y under φ respectively. As p and q are rationally connected and X is proper, then we find a morphism $\mathbb{P}^1_{\mathbb{C}} \to X$ whose image contains both x and y, so any two 0-cycles of degree 1 on V are rationally equivalent. Now if $x \in X(\mathbb{C})$, a moving lemma shows that this point is rationally equivalent to a 0-cycle supported on U, see [CT05, Complément, p. 599], so we fall in the previous case.

1.5. Equivariant Chow groups

— The Chow ring of the classifying space of an algebraic group was originally defined independently by Morel and Voevodsky. Edidin and Graham later generalized their approach to define the equivariant Chow ring and (more generally) equivariant motivic cohomology. Here, we follow the approach of Totaro [Tot14] and Edidin and Graham [EG98, §2.2]. Let X be a k-algebraic space of dimension n and G a k-algebraic group of dimension g. Fix an m-dimensional representation V of G over k and an open subset U of V on which G acts freely and such that $\mathsf{codim}_V(V \setminus U) \geq m$. The diagonal action of G on $X \times_k U$ is free, so there exists a quotient object $X \times_k U \to (X \times_k U)/G = [(X \times_k U)/G]$ in the category of k-algebraic spaces[3] which is a principal G-bundle, see *e.g.* [EG98, Prop. 22] for a proof of this claim. We denote this quotient object by X_G. We also have some (rather mild) conditions for X_G to be a scheme :

[3] See Chapter IV, §1.1.2.1 for the definition of an algebraic space.

[4] Here we mean that if we denote this action by $\sigma : G \times_k X \to X$, then given any line bundle $\pi : \mathcal{L} \to X$, there exists an extension $\tilde{\sigma} : G \times_k \mathcal{L} \to \mathcal{L}$ that commutes with σ *via* π.

Theorem 1.1.20 ([EG98, Prop. 23]). *Let G be a k-algebraic group, U a k-scheme on which G acts freely and suppose that a principal bundle quotient $U \longrightarrow U/G$ exists. Let X be a k-scheme equipped with a G-action, and assume that one of the following conditions holds :*

(i) *X is quasi-projective with a linearised G-action*[4] *;*

(ii) *G is connected and X is equivariantly embedded as a closed subscheme of a proper k-variety ;*

(iii) *every principal G-bundle is locally trivial for the Zariski topology.*

Then a principal bundle quotient X_G exists in $\underline{\mathbf{Sch}}_k$.

From now on we will assume that X_G is a scheme. When X is a k-scheme endowed with an action of a k-algebraic group G, Totaro showed in [Tot14, Thm. 2.5] that the so called *G-equivariant* (or simply *equivariant*) Chow groups of X are well defined objects if we are provided with some conditions on the representation category of G. More precisely, we have the following statement :

Proposition 1.1.21. *Let G be an affine group scheme of finite type over a field k, $i \geq 0$ an integer, V a representation of G defined over k, S a G-invariant closed subset of V such that :*

(i) *G acts freely on $U := V \setminus S$;*

(ii) *the quotient U/G exists in $\underline{\mathbf{Var}}_k$;*

(iii) $\mathsf{codim}_V(S) \geq i$.

Then the Chow groups $\mathrm{CH}^j(U/G)$ do not depend on the choice of V and S for $j \leq i$.

Proof. We first show that the choice of S is superfluous given the codimension condition. Let S' be an *a priori* larger G-invariant closed subset of V of codimension $\geq i$. Since the action of G on U is free, then $\mathsf{codim}_{U/G}((S' \setminus S)/G) \geq i$, so applying the localisation exact sequence for Chow groups to U/G and $(S' \setminus S)/G$ we obtain

$$\mathrm{CH}_*((S' \setminus S)/G) \longrightarrow \mathrm{CH}_*(U/G) \longrightarrow \mathrm{CH}_*((V \setminus S')/G) \longrightarrow 0.$$

Since $\mathrm{CH}^j((S' \setminus S)/G)$ vanishes for $j > \dim(U/G) - i$, we obtain that the groups for U/G and $(V \setminus S')/G$ are isomorphic for $j < i$.

We now prove the claim about the independence of V. Suppose that W is another representation of G over k satisfying the same conditions as V and let S', U' be the corresponding closed subset and complement in W. The quotient $(U \times_k W)/G$ exists in $\underline{\mathbf{Var}}_k$ since it is an algebraic vector bundle over U/G[5] By the same argument the quotient $(U' \times_k V)/G$ exists as a k-variety. Now we use the independence of S for the representation $V \oplus W$: this yields that the respective total spaces of the two vector bundles defined above have isomorphic Chow groups in degree $\leq i$. Since these vector bundles are affine and pure dimensional, then by Lemma (1.1.4) the homotopy invariance property for Chow groups by flat pullbacks shows that we have isomorphisms

$$\mathrm{CH}^j((U \times_k W)/G) \xrightarrow{\sim} \mathrm{CH}^j(U/G)$$

for $j \leq i$ (and similarly for $(U' \times_k V)/G$ and U'/G), hence we obtain isomorphisms $\mathrm{CH}^j(U/G) \xrightarrow{\sim} \mathrm{CH}^j(U'/G)$

[5] Indeed, by faithfully flat descent and Grothendieck's version of Hilbert 90, and since $U \to U/G$ is a principal G-bundle, we have an equivalence between the category of G-equivariant vector bundles on U and the category of vector bundles on U/G induced by the functor $Y \mapsto Y/G$, see *e.g.* [Wat12, §17.2] and [Tot14, §2.2]. Given the assumption on W, we know that $U \times W$ is a G-equivariant vector bundle on U, hence the claim.

in degree at most i as given by :

$$\begin{tikzcd}
\mathrm{CH}^j((U \times_k W)/G) \arrow[r, "\sim"] \arrow[d, "\sim"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{CH}^j((U' \times_k V)/G) \arrow[d, "\sim"] \\
\mathrm{CH}^j(U/G) \arrow[r, dashed, "\sim"'] & \mathrm{CH}^j(U'/G)
\end{tikzcd}.$$

□

Remark 1.1.22. Suppose G is a finite group. Then for every representation V of G over k, the quotient V/G exists as an affine k-variety. In particular, for every closed subset S of V such that G acts freely on $U := V \setminus S$, the quotient U/G is a quasi-projective variety over k. We claim that the Chow groups of U/G are all independent of the choice of V and S.

Indeed, if G is an affine group scheme over k, and $i \geq 0$ is an integer, then there always exists a representation V of G over k together with a closed subset S of codimension $\geq i$ in V such that G acts freely on $V \setminus S$ and U/G is a quasi-projective variety over k. Indeed, if we fix a faithful representation W of G of dimension $n \geq 1$, then for any $N \geq 1$ we have a representation $V := \mathsf{Hom}(\mathbb{A}_k^{n+N}, W) \simeq W^{\oplus(N+n)}$. If we take S to be the closed subset of V given by non-surjective linear maps $\mathbb{A}_k^{n+N} \to W$, then $\mathsf{codim}_V(S) = N+1$, and taking N arbitrary large yields the result.

Definition 1.1.23. Let X be a smooth and geometrically integral k-variety together with a G-action defined over k. If $i \geq 0$ is an integer and V, U are given as before (with $\mathsf{codim}_V(V \setminus U) \geq i$), the i^{th} *G-equivariant Chow group* of X is the group

$$\mathrm{CH}^i_G(X) := \mathrm{CH}^i((X \times_k U)/G).$$

If in particular $X = \mathsf{Spec}\, k$, then we write $\mathrm{CH}^i_G(k) := \mathrm{CH}^i(U/G)$.

I.2. Cycle classes

— Although Chow groups satisfy a few nice functorial properties, they are usually not enough when one deals with more «concrete» computations (for instance, we can extend the localisation sequence which stops too abruptly by means of «higher Chow groups», a construction due to Bloch, but despite the fact that these groups form a good cohomology theory with supports in the sense of [CTHK97, Def. 5.1.1 a)] and enjoy a motivic cohomological interpretation thanks to *loc. cit.*, Thm. 7.5.2, they are not easy to describe in practice). But there exist several homomorphisms (known as cycle maps) from Chow groups to more computable (co)homology theories. In this section, we describe the construction of two well-known cycle classes, the étale ℓ-adic cycle class map on the one hand (defined by Grothendieck with a view towards SGA5 and written down by Deligne in SGA4 1/2) and the Betti cycle class on the other hand, with values in singular cohomology for varieties over the complex numbers.

2.1. Étale cycle classes

— Here we mainly follow the construction of the ℓ-adic étale cycle class map as in [Del77, Cycle] and [Mil80, Chap. VI, §5–§9]. As in the notes of Deligne, we assume for simplicity that all schemes are noetherian and separated.

2.1.1. The class associated to a divisor

— It is quite easy to define the cycle map attached a Cartier divisor. Let indeed X be a scheme and D be a Cartier divisor on X. Outside of D, the associated invertible sheaf $\mathcal{O}_X(D)$ is trivialised by the unit section.

We have a canonical associated $\mathbb{G}_m$-torsor given by $\mathsf{Isom}(\mathcal{O}_X, \mathcal{O}_X(D))$ (where $\mathbb{G}_m$ acts by $(\lambda, f) \mapsto f \circ (\lambda \cdot)$). The *class* $c\ell(D)$ of D in $\mathrm{H}^1_{\text{ét},D}(X, \mathbb{G}_m)$ is defined as the class of this $\mathbb{G}_m$-torsor, which is trivialised on $X \setminus D$. By construction of étale cohomology with support, we have for each $i \geq 0$ a connecting morphism

$$\partial_D : \mathrm{H}^i_{\text{ét}}(X \setminus D, \mathbb{G}_m) \to \mathrm{H}^{i+1}_{\text{ét},D}(X, \mathbb{G}_m),$$

see [Del77, Cycle, §1.1.4]. If D admits a global equation f, then multiplication by f yields an isomorphism from $\mathcal{O}_X(D)$ which is trivialised by the unit section on $X \setminus D$ to $\mathcal{O}_X$ which is trivialised by f on $X \setminus D$. This implies that $c\ell(D) = \partial f$ (the class of the trivial torsor $\mathbb{G}_m$ trivialised on $X \setminus D$ by f).

Let $n \in \mathbb{Z}_{\geq 1}$ be an integer that is invertible on X. The Kummer sequence on $X_{\text{ét}}$ yields a long exact cohomology sequence ; for $i \geq 0$, let $\partial_i : \mathrm{H}^i_{\text{ét},D}(X, \mathbb{G}_m) \longrightarrow \mathrm{H}^{i+1}_{\text{ét},D}(X, \mathbb{Z}/n(1))$ be the connecting morphism.

Definition 1.2.24. The *cycle class* $c\ell_n(D)$ of D in $\mathrm{H}^2_{\text{ét},D}(X, \mathbb{Z}/n(1))$ is defined as $\partial_1(c\ell(D))$.

Proposition 1.2.25. *Let $\iota : D \hookrightarrow X$ be the inclusion. If both D and X are regular, then the cohomology sheaves with support $\mathcal{R}^i\iota^!\mathbb{Z}/n(1)$ are zero for $i \neq 2$, and $\mathcal{R}^2\iota^!\mathbb{Z}/n(1) = \mathbb{Z}/n$.*

Proof. Without loss of generality we can assume that X is strictly local and that D is defined by a global regular parameter f. By Gabber's absolute purity theorem (see [ILO14, Exposé XVI, Thm. 3.1.1]), to prove the first claim we just have to show that for $i = 0, 1$ we have $\mathcal{R}^i\iota^!\mathbb{Z}/n(1) = 0$. We have isomorphisms

$$\widetilde{\mathrm{H}}^{i-1}_{\text{ét}}(X \setminus D, \mathbb{Z}/n(1)) \xrightarrow{\sim} \mathrm{H}^i_{\text{ét},D}(X, \mathbb{Z}/n(1)),$$

where $\widetilde{\mathrm{H}}$ denotes reduced cohomology. The claim for $i = 0, 1$ becomes equivalent to the fact that $\widetilde{\mathrm{H}}^0_{\text{ét}}(X \setminus D, \mathbb{Z}/n(1)) = 0$, that is, X is not disconnected by D, which is precisely the case here. For $i = 2$, as X and D are regular, Abhyankar's lemma [GR71, Exposé XIII, Prop. 5.2] applies, so that $\mathrm{H}^2_{\text{ét},D}(X, \mathbb{Z}/n(1)$ is cyclic of order n generated by $c\ell_n(D)$. □

2.1.2. The class associated to a cycle of higher codimension

— Let $\iota : Y \hookrightarrow X$ be an immersion of schemes of local complete intersection and of codimension c. We want to define a local fundamental class $c\ell_n(Y)$ which is a global section of $\mathcal{R}^{2c}\iota^!\mathbb{Z}/n(c)$ on Y.

Since Y is locally given by the intersection of c divisors $D_1, \ldots, D_c$ in X, one can naturally define $c\ell(Y)$ as the cup-product of the $c\ell(D_i)$'s. Since each $c\ell(D_i)$ is supported on D_i, we therefore obtain that $c\ell(Y)$ is supported on Y. It can be checked that $c\ell(Y)$ is well-defined in the sense that this cup-product does not depend locally of the choice of the D_i's, see [Del77, Cycle, §2.2.3]. One can also generalise this construction to any Y that is locally definable by c equations. If X comes with some good purity properties with respect to Y, then one can actually see this local fundamental class as a global one, namely, an element of $\mathrm{H}^{2c}_{\text{ét},Y}(X, \mathbb{Z}/n(c))$:

Proposition 1.2.26 ([Del77, Cycle, Prop. 2.2.6]). (i) *If $\mathcal{R}^i\iota^!\mathbb{Z}/n = 0$ for every $i < 2c$, then we have*

$$\mathrm{H}^{2c}_{\acute{e}t,Y}(X, \mathbb{Z}/n(c)) \xrightarrow{\sim} \mathrm{H}^0(Y, \mathcal{R}^{2c}\iota^!\mathbb{Z}/n(c)).$$

(ii) *(Excision.) Let $j : Z \hookrightarrow Y$ be a closed subset, $V := Y \setminus Z$ and $k : Z \hookrightarrow X$ the inclusion. If $\mathcal{R}^i j^!\mathbb{Z}/n = 0$ for every $i \leq 2c$, then we have an injection*

$$\mathrm{H}^{2c}_{\acute{e}t,Y}(X, \mathbb{Z}/n(c)) \hookrightarrow \mathrm{H}^{2c}_{\acute{e}t,V}(X \setminus Z, \mathbb{Z}/n(c)).$$

If we further assume that $\mathcal{R}^i j^!\mathbb{Z}/n = 0$ for every $i \leq 2c+1$, then this arrow is an isomorphism.

2.1.2.1. A reduction step. We now discuss some sufficient conditions for X and Y to satisfy the semi-purity hypotheses of Proposition (1.2.26). Let S be a scheme and $f : Y \longrightarrow X$ be a separated morphism of S-schemes of finite type with $g : X \longrightarrow S$ smooth of pure relative dimension N and $Y \longrightarrow S$ of relative dimension $\leq d$ and let $c := N - d$:

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "h"'] \arrow[rr, phantom, "\circlearrowleft", shift right=4ex] & & X \arrow[dl, "g"] \\ & S & \end{tikzcd} .$$

Proposition 1.2.27. (i) *For any abelian torsion étale sheaf $\mathcal{F}$ on S, we have $\mathcal{R}^i f^! g^* \mathcal{F} = 0$ for $i < 2c$; in particular, this holds for $g^*\mathcal{F} = \mathbb{Z}/n$.*

(ii) *If moreover the fibres of Y over a dense open subset U of S have dimension strictly less than d, then $\mathcal{R}^{2c} f^! \mathbb{Z}/n = 0$. If in addition $Z := S \setminus U$ does not disconnect S locally, then $\mathcal{R}^{2c+1} f^! \mathbb{Z}/n = 0$.*

Proof. We have $\mathcal{R} g^! \mathcal{F} = g^*\mathcal{F}(N)[2N]$ by Poincaré duality, see [Del77, Arcata, IV, §4], so the commutativity of the above diagram and the Grothendieck spectral sequence of composed functors for $f^!$ and $g^!$ yield the transition formula $\mathcal{R} f^! \mathcal{R} g^! = \mathcal{R} h^!$. Therefore, we have that $\mathcal{R}^{2c+i} f^!(g^*\mathcal{F}) = 0$ if and only if $\mathcal{R}^{-2d+i} f^! \mathcal{F} = 0$, so we can suppose without loss of generality that $X = S$ and $f = g$. As f is separated of finite type and our schemes are assumed to be noetherian, Nagata's compactification theorem applies, so f is compactifiable. In particular, it satisfies the conditions of [DA73, Exposé XVIII, Prop. 3.1.7], which proves (i).

Now let Y' be the inverse image of Z in Y, so that we have a commutative diagram

$$\begin{tikzcd} Y \arrow[d, "f"'] & Y' \arrow[l, "v"'] \arrow[d, "f"] \\ S \arrow[ur, phantom, "\circlearrowleft"] & Z \arrow[l, "u"] \end{tikzcd}$$

By (i), the sheaves $\mathcal{R}^i f^! \mathbb{Z}/n$ have support in Y' for $i \leq -2d+1$. The Grothendieck spectral sequence

$$\mathcal{R}^p v^! \mathcal{R}^q f^! \Rightarrow \mathcal{R}^{p+q}(fv)^!$$

shows that they actually coincide with the sheaves $\mathcal{R}^i (fv)^! \mathbb{Z}/n = \mathcal{R}^i (uf)^! \mathbb{Z}/n$. Applying (i) to $Y' \to Z$ and the spectral sequence $\mathcal{R}^p f^! \mathcal{R}^q u^! \Rightarrow \mathcal{R}^{p+q}(uf)^!$ shows that $\mathcal{R}^i u^! \mathbb{Z}/n = 0$ for $i = 0$ (or $i \in \{0, 1\}$ in the second assumption). This implies (ii). ◻

2.1.2.2. Cycle classes. One can therefore define the class attached to any cycle of codimension $c \geq 0$ on a smooth scheme X over a field (this is a particular case of the situation described at the beginning of §2.1.2.1). Indeed, if we write

$$Y := \sum_{i=1}^{n} d_i [Y_i] \in \mathrm{CH}^c(X)$$

where the Y_i's are reduced and irreducible, then for each $i = 1, \dots, n$, one can choose some open subset $U_i \subset Y_i$ such that $\mathrm{codim}_X(X \setminus U_i) > c$ in X. In particular, U_i is of local complete intersection in X (it is regular since X is smooth over k, and a morphism of finite type between regular noetherian schemes is always locally of complete intersection, see [Liu02, Chap. 6, §6.3.2, Ex. 3.18]), so it admits a local fundamental class. By the codimension assumption and applying successively Proposition (1.2.27) and Proposition (1.2.26, (ii) and (i)), we obtain that this class comes from a unique class $c\ell(Y_i) \in \mathrm{H}^{2c}_{\text{ét}, Y_i}(X, \mathbb{Z}/n(c))$.

Definition 1.2.28. With Y as above, the *cycle class of Y* is defined as

$$c\ell(Y) := \sum_{i=1}^{n} d_i \cdot c\ell(Y_i) \in \mathrm{H}^{2c}_{\text{ét}, Y}(X, \mathbb{Z}/n(c)).$$

It is actually possible to descend the cycle class to rational equivalence. More precisely :

Proposition 1.2.29 ([Del77, Cycle, Rmq. 2.3.10]). *If two cycles of codimension c on a smooth k-scheme X are algebraically equivalent*[6]*, then they have the same class in $\mathrm{H}^{2c}_{\acute{e}t}(X, \mathbb{Z}/n(c))$. In particular, we have a well defined cycle class*

$$c\ell : \mathrm{CH}^c(X) \longrightarrow \mathrm{CH}^c(X)/alg \longrightarrow \mathrm{H}^{2c}_{\acute{e}t}(X, \mathbb{Z}/n(c)).$$

[6] Recall that the algebraic equivalence relation is generated by the deformation relation : two closed algebraic subsets of X of codimension c are deformation equivalent if they are the fibers, over two points, of a codimension c closed algebraic subset $Z \subset C \times_k X$, parameterized by a smooth connected curve C over k.

2.2. The Betti cycle class

— Let X be a complex algebraic variety. We consider the site X_{cl} given by local isomorphisms $f : U \to X(\mathbb{C})$, that is, the continuous maps of topological spaces (where $X(\mathbb{C})$ is endowed with its usual topology) such that for any $x \in U$, there exists an open neighborhood U_x of x such that $f|_{U_x}$ is a homeomorphism onto an open neighborhood of $f(x)$ (for more details, see [DA73, Exposé XI, (4.0)]). There is a natural morphism of sites $\partial : X_{\mathrm{cl}} \to X(\mathbb{C})$[7], and their associated topoï are equivalent [DA73, Exposé XI, (4.1)]. Moreover, there is a commutative diagram of morphisms of sites :

$$\begin{tikzcd} X_{\mathrm{cl}} \arrow[r, "\partial"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & X(\mathbb{C}) \arrow[d, "h"] \\ X_{\text{ét}} \arrow[r, "g"'] & X_{\mathrm{Zar}} \end{tikzcd}$$

(where g and h are induced by the identity on X and f is obtained by remarking that an étale morphism $Y \to X$ induces a local isomorphism $Y(\mathbb{C}) \to X(\mathbb{C})$ by the Jacobian criterion and the implicit functions theorem). We let $\pi = h \circ \partial = g \circ f : X_{\mathrm{cl}} \to X_{\mathrm{Zar}}$ be the obtained morphism of sites.

Let us restrict ourselves to the case of a smooth and quasi-projective variety X. For a given abelian group A, we denote by

$$\mathrm{H}_{*,B}(X, \mathbb{Z}) := \mathrm{H}_*(X(\mathbb{C}), A)$$

and

$$\mathrm{H}^*_B(X, A) := \mathrm{H}^*(X(\mathbb{C}), A) \simeq \mathrm{H}^*(X_{\mathrm{cl}}, A)$$

the Betti homology and cohomology groups of X with coefficients in A respectively, that is, the singular (co)homology groups of the topological space $X(\mathbb{C})$ with coefficients in A (or in the second case, the cohomology groups of X_{cl} with coefficients in the constant sheaf A, equivalently). Let $\iota : Z \hookrightarrow X$ be a reduced and irreducible subvariety of codimension i in X. By Hironaka's theorem, we have a resolution of singularities

$$\tilde{\iota} : \widetilde{Z} \longrightarrow Z$$

for Z, and therefore one can consider the composition of natural maps

$$\mathrm{H}_{2n-2i,B}(\widetilde{Z}, \mathbb{Z}) \xrightarrow{\sim} \mathbb{Z} \xrightarrow{\tilde{\iota}} \mathrm{H}_{2n-2i,B}(X, \mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^{2i}_B(X, \mathbb{Z}),$$

where the first isomorphism comes the fact that $\widetilde{Z}(\mathbb{C})$ is a connected compact complex manifold (so it admits a canonical orientation) and the last isomorphism comes from Poincaré duality for singular cohomology. The image of $1 \in \mathrm{H}_{2n-2i,B}(\widetilde{Z}, \mathbb{Z})$ under this composite map gives a class

$$c\ell(Z) \in \mathrm{H}^{2i}_B(X, \mathbb{Z})$$

called the *integral Betti cycle class* of Z. One can also consider rational coefficients instead. Extending this cycle class bilinearly defines a cycle class for any cycle of codimension i in X. Actually, this yields a well defined cycle class on Chow groups :

Lemma 1.2.30 ([Voi03, Lem. 9.18]). *If Z is rationally equivalent to 0, then $c\ell(Z) = 0$ in $\mathrm{H}^{2i}_B(X, \mathbb{Z})$. In particular, this yields a cycle class map*

$$c\ell : \mathrm{CH}_i(X) \longrightarrow \mathrm{H}^{2n-2i}_B(X, \mathbb{Z}).$$

Remark 1.2.31. Although the Betti cycle class resembles the étale cycle class, they should not be mistaken, as the former is constructed almost purely by topological means. On the other hand, one could be tempted to consider a Betti cycle class with finite coefficients, but this would not be so relevant : indeed, for any locally constant torsion sheaf $\mathscr{F}$ on X_{cl} with finite fibres, there is a canonical isomorphism $\mathrm{H}^i_{\text{ét}}(X, \mathscr{F}) \xrightarrow{\sim} \mathrm{H}^i(X_{\mathrm{cl}}, \mathscr{F})$, see [DA73, Exposé XI, Thm. 4.4].

[7] An open immersion is a local isomorphism, so we get an inclusion of the category of open subsets of $X(\mathbb{C})$ into X_{cl}, hence a reverse morphism between sites.

This cycle class map is naturally compatible with intersection products on the side of cycles, and the cup-product on the cohomological side :

Proposition 1.2.32 (**[Voi03, Prop. 9.20]**). *For $k, l \geq 0$ and $Z \in \mathrm{CH}^l(X)$, $Z' \in \mathrm{CH}^k(X)$, we have :*

$$c\ell(Z \cdot Z') = c\ell(Z) \smile c\ell(Z') \in \mathrm{H}_B^{k+l}(X, \mathbb{Z}).$$

Moreover, it is compatible with pullbacks and proper pushforwards in the smooth setting :

Proposition 1.2.33 (**[Voi03, Prop. 9.21]**). *Let $f : Y \to X$ be a morphism of smooth varieties. Then :*

(i) *If $Z \in \mathrm{CH}^k(X)$, then*
$$f^* c\ell(Z) = c\ell(f^* Z) \in \mathrm{H}_B^{2k}(Y, \mathbb{Z}).$$

(ii) *If f is moreover proper and $Z \in \mathrm{CH}^k(Y)$, then*

$$c\ell(f_* Z) = f_* c\ell(Z) \in \mathrm{H}_B^{2k-2\dim Y+2\dim X}(X, \mathbb{Z}).$$

Remark 1.2.34. In particular, the formation of the Betti cycle class commutes with the action of correspondences. More precisely, if X, Y are smooth and proper varieties and $\Gamma \in \mathrm{CH}^l(X \times_{\mathbb{C}} Y)$, then for every cycle $Z \in \mathrm{CH}^k(X)$, we have
$$c\ell(\Gamma_*(Z)) = [\Gamma]_* c\ell(Z),$$
where $[\Gamma]_* : \mathrm{H}_B^{2k}(X, \mathbb{Z}) \to \mathrm{H}_B^{2(l+k-\dim X)}(Y, \mathbb{Z})$ is given by

$$\alpha \longmapsto \mathrm{pr}_{2*}(\mathrm{pr}_1^* \alpha \smile [\Gamma]).$$

I.3. K-theoretic methods

— Algebraic K-theory has its origins in Grothendieck's formulation (and proof) of the celebrated Riemann-Roch Theorem in the mid-1950's. While K-theory now plays a significant role in many diverse branches of mathematics, Grothendieck's original focus on the interplay of algebraic vector bundles and algebraic cycles on algebraic varieties is much reflected in current research. In this section, we would like to present the classical (yet fruitful) approaches to higher K-theory, due to Quillen on the one hand and to Milnor on the other hand.

3.1. Generalities on K-theories

3.1.1. Quillen's K-theory

— In the early 1970's, Quillen provided the now accepted definition of higher algebraic K-theory and established remarkable properties of «Quillen's K-groups», thereby advancing the formalism of the algebraic side of K-theory and enabling various computations. An important application of Quillen's theory has been the identification by Merkurjev and Suslin, for a field k, of $K_2(k) \otimes_{\mathbb{Z}} \mathbb{Z}/n$ with the n-torsion in the Brauer group $\mathrm{H}^2(k, \mathbb{Z}/n(2))$ (when n is invertible on k). Others soon recognized that many of Quillen's techniques could be applied to rings with additional structure. Conjectures by Bloch and Beilinson concerning algebraic K-theory and arithmetic algebraic geometry were also formulated during the 1970's; these conjectures prepared the way for many current developments.

3.1.1.1. K_0 of an exact category, the Q-Construction. In this section, we very briefly describe the construction of (higher) algebraic K-theory in the sense of Quillen. The main reference is [Qui73], see also [Sri96] for an updated treatment. For further details and omitted definitions, see Appendix B.

Definition 1.3.35. An *exact category* is an additive category $\mathscr{C}$ which can be embedded as a full subcategory of an abelian category $\mathscr{A}$ in such a way that it is closed under extensions in $\mathscr{A}$, that is, for any exact sequence

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

in $\mathscr{A}$, where A' and A'' are isomorphic to objects of $\mathscr{C}$, then A is isomorphic to an object of $\mathscr{C}$.

Example 1.3.36. If R is a commutative ring with unit, consider the category $\mathscr{P}(R)$ of finitely generated projective R-modules. It is a full subcategory of the abelian category $\underline{\mathbf{Mod}}_R$, and closed under extensions. The short exact sequences in $\mathscr{P}(R)$ are the split exact sequences in $\underline{\mathbf{Mod}}_R$:

$$0 \longrightarrow P \longrightarrow P \oplus Q \longrightarrow Q \longrightarrow 0.$$

Definition 1.3.37. Suppose that $\mathscr{C}$ is moreover skeletally small, that is, its isomorphism classes of objects form a set. The group $K_0(\mathscr{C})$ is the abelian group freely generated by the isomorphism classes of objects of $\mathscr{C}$ modulo the relations $[M] - [M'] - [M''] = 0$ for any

$$0 \to M' \to M \to M'' \to 0$$

exact in $\mathscr{C}$.

Remark 1.3.38. The example one should have in mind is where $\mathscr{C} = \mathscr{P}(X)$ is the category of locally free $\mathscr{O}_X$-modules of finite rank over a scheme X (in other words, vector bundles) ; the group $K_0(X) := K_0(\mathscr{P}(X))$ is the famous *Grothendieck group* of X.

Given a diagram

$$0 \longrightarrow M' \xrightarrow{i} M \xrightarrow{j} M'' \longrightarrow 0$$

that is exact in $\mathscr{A}$, Quillen refers to i as an *admissible monomorphism* and j as an *admissible epimorphism* (see Appendix B for details). A functor $F : \mathscr{C} \to \mathscr{C}'$ between exact categories is exact if it preserves exact sequences.

An exact category admits a zero object, so its classifying space $\mathbf{B}\mathscr{C} = |N(\mathscr{C})|$ is contractible (here $N(\mathscr{C})$ denotes the *nerve* of $\mathscr{C}$ and $|\cdot|$ is the *geometric realisation* functor). The idea of Quillen is to build from $\mathscr{C}$ a new category $Q\mathscr{C}$ in a functorial way and such that

$$\pi_1^{\mathrm{top}}(\mathbf{B}Q\mathscr{C}, 0) \simeq K_0(\mathscr{C}),$$

where $0 \in \mathrm{Ob}(\mathscr{C})$ is a zero object. Roughly explained, the objects of $Q\mathscr{C}$ are the same as the ones of $\mathscr{C}$, but the morphisms are different. A morphism $M_1 \to M_2$ in $Q\mathscr{C}$ is defined as an isomorphism (in $\mathscr{C}$) of M_1 with a subquotient of M_2, that is a filtration $M' \subset M'' \subset M''$ by subobjects with M_2/M'', M_2/M' and M''/M' in $\mathscr{C}$ and an isomorphism $M_1 \simeq M''/M'$. In other words, a morphism in $Q\mathscr{C}$ is a diagram

$$M_1 \overset{j}{\twoheadleftarrow} M'' \overset{i}{\hookrightarrow} M_2 \ ,$$

where i is an admissible monomorphism and j is an admissible epimorphism. Quillen then naturally defines the higher K-groups of $\mathscr{C}$ as follows :

Definition 1.3.39. For $n \geq 0$, the n^{th} *algebraic K-theory group* of the category $\mathscr{C}$ is defined as

$$K_n(\mathscr{C}) := \pi_{n+1}^{\mathrm{top}}(\mathbf{B}Q\mathscr{C}, 0)$$

where $0 \in \mathrm{Ob}(\mathscr{C})$ is a zero object.

Remark 1.3.40. One can also find another (slightly modified) definition of $K_n(\mathscr{C})$ in the literature : if $\Omega\mathbf{B}Q\mathscr{C}$ denotes the loop space of $\mathbf{B}Q\mathscr{C}$, then the usual loop-suspension adjunction in homotopy theory (see *e.g.* [Hat02, Thm. 4J.1]) provides a canonical isomorphism $\pi_{n+1}^{\mathrm{top}}(\mathbf{B}Q\mathscr{C}, 0) \cong \pi_n^{\mathrm{top}}(\Omega\mathbf{B}Q\mathscr{C}, 0)$, so one can define $K_n(\mathscr{C}) := \pi_n^{\mathrm{top}}(\Omega\mathbf{B}Q\mathscr{C}, 0)$ (which is more convenient for tracking indexes).

The point of this construction should now appear more clearly. Indeed, we know a lot about functorial properties of homotopy groups of topological spaces, especially in the case of CW-complexes[8] : for example, we know how to relate homotopy groups of pairs of CW-complexes through homotopy fibrations. Quillen's definition of higher algebraic K-theory should therefore allow a finer understanding of K_0 and its higher analogues. We now list some of the main results proved by Quillen in this regard.

[8] Actually, it was quickly realized that although Quillen's theory initially revolved around a functor K from the category of rings (or schemes) to the category of topological spaces, K in fact takes its values in the category of infinite loop spaces and infinite loop maps. More, K is best thought of as a functor not to topological spaces, but to the category of *spectra* (recall that a spectrum is a family of based topological spaces $\{X_i\}_{i\geq 0}$, together with bonding maps $\sigma_i : X_i \to \Omega X_{i+1}$, which can be taken to be homeomorphisms). There is a great deal of value to this refinement of the functor K, see *e.g.* [Car05] for a more detailed account.

3.1.1.2. The main K-theoretic results of Quillen's paper. Let $\mathscr{S}$ be the category of short exact sequences in $\mathscr{C}$. For a given sequence $S \in \mathrm{Ob}(\mathscr{S})$, write it as

$$0 \longrightarrow sS \longrightarrow tS \longrightarrow qS \longrightarrow 0$$

where s, t, q are viewed as functors $\mathscr{S} \to \mathscr{C}$. Since $\mathscr{C}$ is an exact category, then it is clear that $\mathscr{S}$ is exact as well (a sequence $0 \longrightarrow S' \longrightarrow S \longrightarrow S'' \longrightarrow 0$ is exact if and only if the diagrams obtained in $\mathscr{C}$ after applying s, t, q are exact).

Theorem 1.3.41 (Characteristic sequence theorem [Sri96, Thm. 4.1]). *The functor*

$$(s, q) : Q\mathscr{S} \longrightarrow Q\mathscr{C} \times Q\mathscr{C}$$

is a homotopy equivalence of categories and therefore induces a homotopy equivalence $\mathbf{B}Q\mathscr{S} \longrightarrow \mathbf{B}Q\mathscr{C} \times \mathbf{B}Q\mathscr{C}$.

Corollary 1.3.42. *Let $F, F', F'' : \mathscr{C}_1 \to \mathscr{C}_2$ be three exact functors between exact categories and suppose that there exist natural transformations $F' \to F$ and $F \to F''$ such that for any $M \in \mathrm{Ob}(\mathscr{C}_1)$, the sequence*

$$0 \longrightarrow F'(M) \longrightarrow F(M) \longrightarrow F''(M) \longrightarrow 0$$

is exact. Then for any $n \geq 0$ we have an equality of pushforwards :

$$F_* = F'_* + F''_* : K_n(\mathscr{C}_1) \longrightarrow K_n(\mathscr{C}_2).$$

Proof. The data given above amounts to giving a functor $G : \mathscr{C}_1 \to \mathscr{S}_2$ where $\mathscr{S}_2$ is the category of exact sequences in $\mathscr{C}_2$. By the previous theorem, we have an induced homotopy equivalence between classifying spaces, which therefore yields an equality of pushforwards on homotopy groups. □

The two following results are the so-called *resolution* and *dévissage* theorems, which allow us (in various situations) to replace an exact category by another, without changing the K-groups.

Theorem 1.3.43 (Resolution theorem [Sri96, Thm. 4.6]). *Let $\mathscr{P}$ be a full subcategory of an exact category $\mathscr{C}$ which is closed under extensions and such that :*

(i) *for any extension $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ in $\mathscr{C}$, if $M, M'' \in \mathrm{Ob}(\mathscr{P})$ then $M' \in \mathrm{Ob}(\mathscr{P})$;*

(ii) *for any $M \in \mathscr{C}$, there exists a finite resolution $0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$ in $\mathscr{C}$ where $P_i \in \mathrm{Ob}(\mathscr{P})$ for each $i \in [\![0, n]\!]$.*

Then the natural map $\mathbf{B}Q\mathscr{P} \to \mathbf{B}Q\mathscr{C}$ is a homotopy equivalence.

An important particular case of application of this theorem is the following :

Corollary 1.3.44. *Let R be a regular ring, $\underline{\mathbf{Coh}}(R)$ the category of coherent (or equivalently here, finitely generated) R-modules and $\mathscr{P}(R)$ the full subcategory of finitely generated projective R-modules. Then for each $n \geq 1$, we have an isomorphism*

$$K_n(\mathscr{P}(R)) \xrightarrow{\sim} K_n(\underline{\mathbf{Coh}}(R)).$$

Proof. The condition (ii) is satisfied since R is regular, see *e.g.* [Mat89, Thm. 19.2]. On the other hand, consider an exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

where M and M'' are projective. Since M'' is projective, we obtain that this sequence splits, so that $M \simeq M' \oplus M''$. But M is projective, so M' must be projective as well (a direct sum of modules is projective if and only if each of its summands is projective). This shows that (i) holds as well. □

Let now $\mathscr{A}$ be an abelian category and $\mathscr{B}$ a full abelian subcategory of $\mathscr{A}$ that is closed under taking subobjects, quotient objects and finite products. Denote by $\mathscr{A}/\mathscr{B}$ the quotient category (which is abelian, given the hypotheses on $\mathscr{B}$; for details about the construction of a quotient category, we refer to [Sri96, Appendix B, §B.2]). If we further assume that $\mathscr{B}$ is stable under extensions in $\mathscr{A}$, then we call it a *Serre subcategory*. Quillen then provided two ways of relating the K-groups of an exact category with those of a subcategory, by considering a suitable filtration on the one hand, or by means of the K-theory of the quotient category when the considered subcategory is a Serre subcategory on the other hand :

Theorem 1.3.45 (Devissage theorem [Sri96, Thm. 4.8]). *Suppose that every object A of $\mathscr{A}$ admits a finite filtration*

$$0 = A_0 \subset A_1 \subset \ldots \subset A_r = A$$

such that $A_j/A_{j-1} \in \mathrm{Ob}(\mathscr{B})$ for all $j \geq 1$. Then for each $n \geq 0$, there exists an isomorphism

$$K_n(\mathscr{B}) \xrightarrow{\sim} K_n(\mathscr{A}).$$

Theorem 1.3.46 (Localisation theorem [Sri96, Thm. 4.9]). *Suppose that $\mathscr{B}$ is a Serre subcategory of $\mathscr{A}$. Then the natural exact functors $\mathscr{B} \hookrightarrow \mathscr{A} \to \mathscr{A}/\mathscr{B}$ induce a homotopy fibration on classifying spaces*

$$\mathbf{B}Q\mathscr{B} \longrightarrow \mathbf{B}Q\mathscr{A} \longrightarrow \mathbf{B}Q\mathscr{A}/\mathscr{B},$$

hence a long exact sequence in algebraic K-theory :

$$\cdots \longrightarrow K_n(\mathscr{B}) \longrightarrow K_n(\mathscr{A}) \longrightarrow K_n(\mathscr{A}/\mathscr{B}) \longrightarrow K_{n-1}(\mathscr{B}) \longrightarrow \cdots$$

3.1.2. Milnor's K-theory

— A few years before Quillen, in 1970, another definition of higher algebraic K-theory was suggested by Milnor, at least in the case of fields. Milnor's definition was originally motivated by Matsumoto's presentation of K_2 of a field k. While he stressed that his definition is purely ad hoc, and although Quillen's approach has been almost unanimously recognised as the right way to define higher K-theory in the algebraic setting, Milnor's K-groups are actually fundamental objects which enjoy very deep connections with Galois cohomology. Let us first define these groups :

Definition 1.3.47. If k is a field and $n \geq 0$, the n^{th} *Milnor K-group* is defined as

$$K_n^M(k) := \frac{(k^\times)^{\otimes n}}{\langle a_1 \otimes \cdots \otimes a_n \mid a_i + a_j = 1 \text{ for some } 1 \leq i < j \leq n \rangle}.$$

Remark 1.3.48. One can also extend this definition to an arbitrary commutative ring A with unit : for $n \geq 0$, we put $K_n^M(A) := (A^\times)^{\otimes n}/\langle a_1 \otimes \cdots \otimes a_n \mid a_i + a_j = 1 \text{ for some } 1 \leq i < j \leq n\rangle$.

The relation $a_i + a_j = 1$ above is often referred to as the *Steinberg relation*. We write $\{a_1, \ldots, a_n\}$ for the image of $a_1 \otimes \ldots \otimes a_n$ in $K_n^M(k)$. When $n \in \{0, 1, 2\}$, the K-groups in the sense of Quillen coincide with those in the sense of Milnor (the statement for $n = 0, 1$ is clear, for $n = 2$ see *e.g.* [Sri96, Cor. 2.6]). Milnor's K-groups are easily seen to be functorial with respect to field extensions : if $\phi : k \hookrightarrow L$ is any inclusion of fields, then there are natural maps $i_{L/k} : K_n^M(k) \to K_n^M(L)$ for $n \geq 0$ induced by ϕ. The tensor product pairing $(k^\times)^{\otimes n} \times (k^\times)^{\otimes m} \to (k^\times)^{\otimes n+m}$ preserves the Steinberg relation, so it induces a natural product structure :

$$K_n^M(k) \times K_m^M(k) \longrightarrow K_{n+m}^M(k).$$

In particular, the direct sum

$$K_*^M(k) = \bigoplus_{n \geq 0} K_n^M(k)$$

admits a graded ring structure, which is graded commutative, see [GS17, Prop. 7.1.1].

3.1.2.1. The Tame symbol. There is an analogue of residue maps in Galois cohomology in a K-theoretic setting. Suppose indeed that k is a discretely valued field, with valuation $\nu : k^\times \to \mathbb{Z}$, and let R be its valuation ring and κ its residue field. Fixing a uniformiser $\pi \in R$, we know that each $x \in k^\times$ can be uniquely written as a product $u\pi^i$ where $u \in R^\times$ and $i \in \mathbb{Z}$. It follows that for each $n \geq 0$, the group $K_n^M(k)$ is generated by elements of the form $\{\pi, u_2, \dots, u_n\}$ and $\{u_1, \dots, u_n\}$ where the u_i's are units in R. With this description of $K_n^M(k)$ in our hands, it is possible to construct a residue map $K_n^M(k) \to K_{n-1}^M(\kappa)$ called the *Tame symbol* :

Proposition 1.3.49 ([GS17, Prop. 7.1.4]). *For each $n \geq 1$, there exists a unique homomorphism*

$$\partial_n^M : K_n^M(k) \longrightarrow K_{n-1}^M(\kappa)$$

satisfying

$$\partial_n^M(\{\pi, u_2, \dots, u_n\}) \longmapsto \{\overline{u_2}, \dots, \overline{u_n}\},$$

where $\pi \in R$ is a uniformiser, $u_2, \dots, u_n \in R^\times$ and $\overline{u_i}$ denotes the image of u_i in κ for $i = 2, \dots, n$.

Example 1.3.50. The tame symbol $\partial_1^M : K_1(k) \to K_0(\kappa)$ is just the valuation map $\nu : k^\times \to \mathbb{Z}$. The tame symbol $\partial_2^M : K_2(k) \to K_1(\kappa)$ is given by the formula

$$\partial_2^M(\{a, b\}) = (-1)^{\nu(a)\nu(b)}\overline{a^{-\nu(b)}b^{\nu(a)}},$$

see *e.g.* [GS17, Lem. 7.1.2] for a proof (this easily follows from the Kummer sequence).

3.1.2.2. The Galois symbol. For any integer $n \geq 1$ that is invertible on k, there is a natural map from $K_*^M(k)/n$ to the Galois cohomology of k. Indeed, the Kummer sequence

$$1 \longrightarrow \mu_n \longrightarrow k_s^\times \xrightarrow{x \mapsto x^n} k_s^\times \longrightarrow 1$$

provides an exact portion

$$k^\times \xrightarrow{x \mapsto x^n} k^\times \longrightarrow \mathrm{H}^1(k, \mathbb{Z}/n(1)) \longrightarrow \mathrm{H}^1(k, k_s^\times),$$

and the right term vanishes by Hilbert 90, which yields an isomorphism $k^\times/k^{\times n} \simeq \mathrm{H}^1(k, \mathbb{Z}/n(1))$. Taking cup-products, we obtain for $i \geq 0$ a map

$$(k^\times/k^{\times n})^{\otimes i} \longrightarrow \mathrm{H}^i(k, \mathbb{Z}/n(i)),$$

which, as remarked by Tate, descends to a map (see [GS17, Prop. 4.6.1]) :

$$K_i^M(k)/n \xrightarrow{h_{k,n}^i} \mathrm{H}^i(k, \mathbb{Z}/n(i))$$

called the *Galois symbol*. Suppose now that k is a discretely valued field, so that the tame symbols are well-defined. A non-trivial verification involving K-theoretic reciprocity laws (see [GS17, Chap. 7, §7.4]) shows that the tame symbols and the residue maps in Galois cohomology agree *via* the Galois symbol :

Proposition 1.3.51 ([GS17, Prop. 7.5.1]). *Let k be a discretely valued field with residue field κ, and $n \geq 1$ an integer that is invertible on k. Then for each $i \geq 1$, if $\partial_\nu^n : \mathrm{H}^i(k, \mathbb{Z}/n(i)) \to \mathrm{H}^{i-1}(k, \mathbb{Z}/n(i-1))$ denotes the residue in Galois cohomology, then the following natural diagram commutes:*

$$\begin{tikzcd} K_i^M(k)/n \arrow[r, "\partial_i^M"] \arrow[d, "h_{k,n}^i"'] & K_{i-1}^M(\kappa)/n \arrow[d, "h_{\kappa,n}^{i-1}"] \\ \mathrm{H}^i(k, \mathbb{Z}/n(i)) \arrow[r, "\partial_\nu^n"'] & \mathrm{H}^{i-1}(k, \mathbb{Z}/n(i-1)) \end{tikzcd} .$$

3.1.2.3. The Bloch-Kato conjecture. The famous Bloch-Kato conjecture asks whether the Galois symbol is an isomorphism. The general case has recently been proved by Rost and Voevodsky. It would be impossible to sketch their proof or even provide an idea of it, as it relies on very deep results in motivic cohomology. We simply state their main theorem :

Theorem 1.3.52 (Rost-Voevodsky [Voe11, Thm. 6.1]). *For any field k, for any $i \geq 1$ and any integer $n \geq 1$ that is invertible on k, the Galois symbol yields an isomorphism :*

$$K_i^M(k)/n \xrightarrow{\sim} \mathrm{H}^i(k, \mathbb{Z}/n(i)).$$

As announced at the beginning of this section, the special case where $i = 2$ has been treated in 1982 by Merkurjev and Suslin. It has the virtue that the arguments involved in its proof are far less sophisticated : they rely on a version of Hilbert 90 for K_2, and an observation due to Bloch on the homology of certain Gersten complexes, which we are going to investigate in §3.2. We refer to [GS17, Chap. 8] for a complete treatment :

Theorem 1.3.53 (Merkurjev-Suslin [GS17, Thm. 8.6.1]). *For any field k and any integer $n \geq 1$ that is invertible on k, the Galois symbol induces an isomorphism :*

$$K_2^M(k)/n \xrightarrow{\sim} \mathrm{H}^2(k, \mathbb{Z}/n(2)).$$

3.2. K-theory of schemes

3.2.1. Filtration by coniveau

— Let X be a noetherian scheme of finite dimension. There are two natural abelian categories that one can associate to X, namely, the category $\underline{\mathbf{Coh}}(X)$ of coherent sheaves of $\mathcal{O}_X$-modules, and the full subcategory $\mathcal{P}(X)$ of locally free $\mathcal{O}_X$-modules of finite rank (or vector bundles on X, equivalently). For $p \geq 0$, we let $\mathcal{T}_X^p$ be the full subcategory of $\underline{\mathbf{Coh}}(X)$ whose objects are sheaves with support in codimension $\geq p$. Note that these categories are stable under flat pullbacks, that is, if $f : X \to Y$ is a flat morphism, then the induced pullback functor $f^* : \underline{\mathbf{Coh}}(Y) \to \underline{\mathbf{Coh}}(X)$ descends to functors $f_p^* : \mathcal{T}_Y^p \to \mathcal{T}_X^p$ (this is a statement about local rings, on which flat morphisms preserve dimensions, see [Qui73, §7, (2.1)]).

3.2.1.1. A dévissage. It is quite straightforward to check that for each $p \geq 0$, the category $\mathcal{T}_X^{p+1}$ is a Serre subcategory of $\mathcal{T}_X^p$, see [Sri96, §5.19], and by the commutativity of homotopy groups with direct limits of filtered systems (*cf.* [Sri96, Lem. 3.8, Prop. 5.14]), we obtain that for each $n \geq 0$ we have :

$$K_n(\mathcal{T}_X^p) \simeq \varinjlim_{Z \subset X} K_n(\underline{\mathbf{Coh}}(Z)),$$

where Z runs over the closed subschemes of X of codimension $\geq p$. We also have an equivalence of categories :

$$\mathcal{T}_X^p/\mathcal{T}_X^{p+1} \xrightarrow{\sim} \coprod_{x \in X^{(p)}} \bigcup_{r \geq 0} \underline{\mathbf{Coh}}(\mathcal{O}_{X,x}/\mathfrak{m}_{X,x}^r).$$

This is actually quite non-trivial. We provide a rather detailed explanation of this decomposition. First remark that the category of finite length modules on a noetherian local ring $(R, \mathfrak{m})$ is equivalent to the direct limit of the categories of finitely generated modules on $R/\mathfrak{m}^r$ for $r \geq 1$ (any finite length module is noetherian, so it is annihilated by a power of the maximal ideal). We can rephrase this as an equivalence of categories (when R is artinian) :

$$\underline{\mathbf{Coh}}(R) \xrightarrow{\sim} \bigcup_{r \geq 0} \underline{\mathbf{Coh}}(R/\mathfrak{m}^r).$$

We first show the above statement in the case where X is irreducible and $p = 0$. Let $\eta \in X$ be the generic point ; since $\mathcal{O}_{X,\eta}$ is artinian, the natural restriction map

$$\mathcal{T}_X^0/\mathcal{T}_X^1 = \underline{\mathbf{Coh}}(X)/\mathcal{T}_X^1 \longrightarrow \underline{\mathbf{Coh}}(\mathcal{O}_{X,\eta})$$

is an equivalence of categories. To see this, we first use genericity and spreading out (see *e.g.* [Gro67, III, §8]), and we obtain an equivalence :

$$\varinjlim_{\emptyset\neq U\subset X} \underline{\mathbf{Coh}}(U) \xrightarrow{\sim} \underline{\mathbf{Coh}}(\mathcal{O}_{X,\eta}).$$

We are thus reduced to showing that the faithful natural functor

$$\underline{\mathbf{Coh}}(X)/\mathcal{T}^1_X \longrightarrow \varinjlim_{\emptyset\neq U\subset X} \underline{\mathbf{Coh}}(U)$$

is an equivalence[9]. Fix some non-empty open subset $U \subset X$ and let $Z := X \setminus U$. The restriction functor $\underline{\mathbf{Coh}}(X) \to \underline{\mathbf{Coh}}(U)$ is exact with kernel equal to the Serre subcategory $\mathcal{M}_Z(X)$ of coherent sheaves supported on Z. Actually, the induced faithful functor

$$\underline{\mathbf{Coh}}(X)/\mathcal{M}_Z(X) \longrightarrow \underline{\mathbf{Coh}}(U)$$

is an equivalence. Indeed, it is essentially surjective by [DJ+22, Tag 01PF] : for any quasi-coherent sheaf $\mathcal{F}$ on X and any quasi-coherent subsheaf $\mathcal{G}$ of $\mathcal{F}|_U$, there exists a quasi-coherent subsheaf $\widetilde{\mathcal{G}}$ of $\mathcal{F}$ which restricts to $\mathcal{G}$ on U. If $\mathcal{G}$ is actually coherent on U, let $j : U \hookrightarrow X$ be the open immersion and take $\mathcal{F} := j_*\mathcal{G}$, so that $\mathcal{F}|_U = \mathcal{G}$. We get that any coherent sheaf on U is the restriction of a coherent sheaf on X.

Now, to see that this functor is full, let $\varphi : \mathcal{G}_1 \to \mathcal{G}_2$ be any morphism of sheaves and $\widetilde{\mathcal{G}_1} \subset j_*\mathcal{G}_1$ as above. We have an induced morphism $j_*\varphi|_{\widetilde{\mathcal{G}_1}} : \widetilde{\mathcal{G}_1} \to j_*\mathcal{G}_2$, whose image is a coherent subsheaf $\mathcal{H}$ of $j_*\mathcal{G}_2$. Applying the previous argument to $j_*\mathcal{G}_2/\mathcal{H}$, we obtain a coherent subsheaf $\widetilde{\mathcal{G}_2}$ of $j_*\mathcal{G}_2$ that contains $\mathcal{H}$ and such that $\widetilde{\mathcal{G}_2}|_U = \mathcal{G}_2$. Hence $j_*\varphi$ restricts to a morphism $\tilde{\varphi} : \widetilde{\mathcal{G}_1} \to \widetilde{\mathcal{G}_2}$, which itself restricts to φ, hence the claim about fullness. This allows us to reduce ourselves to showing that the natural faithful functor

$$\underline{\mathbf{Coh}}(X)/\mathcal{T}^1_X \longrightarrow \varinjlim_{\emptyset\neq U\subset X} \underline{\mathbf{Coh}}(X)/\mathcal{M}_{X\setminus U}(X),$$

(which comes from the identity functor on $\underline{\mathbf{Coh}}(X)$ by the compatibility of restrictions) is an equivalence. (Remark that $\mathcal{T}^1_X = \bigcup_{\emptyset\neq U\subset} \mathcal{M}_{X\setminus U}(X)$ is the full Serre subcategory of $\underline{\mathbf{Coh}}(X)$ consisting of sheaves supported on $X \setminus U$ for some non-empty open subset $U \subset X$.) If $\mathcal{A}$ is an arbitrary abelian category and $(\mathcal{B}_i)_{i\in I}$ is a direct system of full Serre subcategories, then the natural functor

$$\mathcal{A}/(\cup_{i\in I}\mathcal{B}_i) \longrightarrow \varinjlim_{i\in I} \mathcal{A}/\mathcal{B}_i$$

induced by the identity on $\mathcal{A}$ is an equivalence, see *e.g.* [Sri96, Appendix B, §B.11] (this essentially follows from the construction of quotient categories as localisations). This proves the desired equivalence in the case where X is irreducible and $p = 1$.

We now treat the general case. If $Z \subset X$ is an irreducible closed subset of codimension $p \geq 0$ with generic point $\eta_Z \in Z$, let $\mathcal{M}^1_Z(X) := \mathcal{M}_Z(X) \cap \mathcal{T}^{p+1}_X$ be the subcategory of coherent sheaves on X supported on a proper closed subset of Z. The restriction morphisms induces a faithful exact functor $\mathcal{M}_Z(X)/\mathcal{M}^1_Z(X) \to \underline{\mathbf{Coh}}(\mathcal{O}_{X,\eta_Z})$. Let $\iota_n : Z_{\mathrm{red},n} \hookrightarrow X$ be the n^{th}-infinitesimal neighborhood of Z_{red} in X, that is, the closed subscheme of X defined by the exact sequence of sheaves

$$0 \longrightarrow \mathcal{I}^n_{Z_{\mathrm{red}}} \longrightarrow \mathcal{O}_X \longrightarrow \iota_{n*}\mathcal{O}_{Z_{\mathrm{red,n}}} \longrightarrow 0$$

(where $\mathcal{I}_{Z_{\mathrm{red}}}$ is the ideal sheaf of Z_{red}). By construction we have that $Z_{\mathrm{red},n}$ has a unique generic point, and $\mathcal{O}_{Z_{\mathrm{red},n},\eta_Z} = \mathcal{O}_{X,\eta_Z}/\mathfrak{m}^n_{X,\eta_Z}$. Applying the case $p = 0$ (treated earlier) to each $Z_{\mathrm{red,n}}$, we obtain an equivalence of categories :

$$\varinjlim_{n\geq 1} \underline{\mathbf{Coh}}(Z_{\mathrm{red,n}})/\mathcal{T}^1_{Z_{\mathrm{red,n}}} \xrightarrow{\sim} \varinjlim_{n\geq 1} \underline{\mathbf{Coh}}(\mathcal{O}_{Z_{\mathrm{red,n}},\eta_Z}) = \underline{\mathbf{Coh}}(\mathcal{O}_{X,\eta_Z}).$$

[9] Note that the coherent sheaves sent to zero under the functor $\underline{\mathbf{Coh}}(X) \to \varinjlim_{\emptyset\neq U\subset X} \underline{\mathbf{Coh}}(U)$ are precisely those that vanish on some non-empty open subset, hence the well-definedness (X is irreducible, so any proper open subset has codimension at least 1).

On the other hand, the inclusion functors $\underline{\mathbf{Coh}}(Z_{\mathrm{red,n}}) \to \mathcal{M}_Z(X)$ induce a natural functor :

$$\underline{\mathbf{Coh}}(\mathcal{O}_{X,\eta_Z}) \longrightarrow \mathcal{M}_Z(X)/\mathcal{M}^1_Z(X),$$

which we readily conclude to be the a quasi-inverse functor to the one constructed above, and *vice versa*. Finally, since $\mathcal{M}_Z(X) \cap \mathcal{T}_X^{p+1} = \mathcal{M}^1_Z(X)$, we have a faithful exact functor $\mathcal{M}_Z(X)/\mathcal{M}^1_Z(X) \to \mathcal{T}_X^p/\mathcal{T}_X^{p+1}$ (induced by the obvious inclusion $\mathcal{M}_Z(X) \subset \mathcal{T}_X^p$). Varying the closed subschemes $Z \subset X$ and taking direct sums of coherent sheaves, we thus define a functor

$$\coprod_{x \in X^{(p)}} \underline{\mathbf{Coh}}(\mathcal{O}_{X,x}) \longrightarrow \mathcal{T}_X^p/\mathcal{T}_X^{p+1}$$

which is quasi-inverse to the natural restriction $\mathcal{T}_X^p/\mathcal{T}_X^{p+1} \to \coprod_{x\in X^{(p)}} \underline{\mathbf{Coh}}(\mathcal{O}_{X,x})$ (indeed, if $\mathcal{F} = \bigoplus_{i=1}^n \mathcal{F}_i$ is a coherent sheaf on X where each $\mathcal{F}_i$ is supported on the closure $\overline{\{x_i\}}$ of some point $x_i \in X^{(p)}$, then $\mathcal{F}_{x_i} \simeq (\mathcal{F}_i)_{x_i}$ for $j = 1, \dots, n$).

Now that we identified the successive quotients in the coniveau filtration more precisely, remark that for any $x \in X^{(p)}$, any $\mathcal{O}_{X,x}/\mathfrak{m}^r_{X,x}$-module admits a finite filtration whose successive quotients are $\mathcal{O}_{X,x}/\mathfrak{m}_{X,x} = \kappa(x)$-modules. Applying devissage as in Theorem (1.3.45) (note that for a ring R and for all $n \geq 0$, we have a canonical isomorphism $K_n(R) \simeq K_n(\mathcal{P}(R))$), we obtain isomorphisms :

$$K_n(\kappa(x)) \simeq K_n\left(\bigcup_{r\geq 0} \underline{\mathbf{Coh}}(\mathcal{O}_{X,x}/\mathfrak{m}^r_{X,x})\right).$$

3.2.1.2. Coniveau exact couples. With this decomposition at our disposal, we may apply Theorem (1.3.46), so that we obtain exact sequences :

$$\cdots \longrightarrow K_n(\mathcal{T}_X^{p+1}) \longrightarrow K_n(\mathcal{T}_X^p) \longrightarrow \coprod_{x\in X^{(p)}} K_n(\kappa(x)) \longrightarrow K_{n-1}(\mathcal{T}_X^{p+1}) \longrightarrow \cdots,$$

and up to re-indexing (this is more or less a technical convenience here), we get diagrams of the form :

$$\begin{tikzcd}
\cdots \arrow[r] & K_{-p-q}(\mathcal{T}_X^{p+1}) \arrow[r, "i^{p+1,q-1}"] & K_{-p-q}(\mathcal{T}_X^{p}) \arrow[r, "j^{p,q}"] & \coprod_{x\in X^{(p)}} K_{-p-q}(\kappa(x)) \arrow[r, "k^{p,q}"] \arrow[d, dashed] & K_{-p-q-1}(\mathcal{T}_X^{p+1}) \arrow[r] \arrow[dll, "\sim"'] & \cdots \\
\cdots \arrow[r] & K_{-p-q-1}(\mathcal{T}_X^{p+2}) \arrow[r, "i^{p+2,q-1}"] & K_{-p-q-1}(\mathcal{T}_X^{p+1}) \arrow[r, "j^{p+1,q}"] & \coprod_{x\in X^{(p+1)}} K_{-p-q-1}(\kappa(x)) \arrow[rr] & & \cdots
\end{tikzcd}$$

Following *e.g.* [Sri96, Appendix C, §C.2], we define an exact couple $C(D, E, i, j, k)$ as the datum of the objects $D^{p,q} := K_{-p-q}(\mathcal{T}_X^p)$ and $E^{p,q} := \coprod_{x\in X^{(p)}} K_{-p-q}(\kappa(x))$, so that we have exact triangles :

$$\begin{tikzcd}
D^{p+1,q-1} \arrow[rr, "i^{p+1,q-1}"] & \arrow[d, phantom, "\circlearrowleft"] & D^{p,q} \arrow[dl, "j^{p,q}"] \\
& E^{p,q} \arrow[ul, "k^{p,q}"] &
\end{tikzcd}.$$

Note here that we make the convention that $\mathcal{T}_X^p = \underline{\mathbf{Coh}}(X)$ and $X^{(p)} = X^{(0)}$ for $p < 0$ and $K_n(\mathcal{T}_X^p) = 0$ for $n < 0$ and p arbitrary.

It is a well known fact that exact couples provide spectral sequences, see *e.g.* [Sri96, Appendix C, §C.1] or [Wei94, Prop. 5.9.2] for a detailed explanation (note that in the latter, the statement is of homological type). Since $D^{p,q} = 0$ for $p + q > 0$ and $D^{p,q} = D^{p-1,q+1} = K_{-p-q}(\underline{\mathbf{Coh}}(X))$ for $p < 0$ and q arbitrary, we see that the abutment terms of the obtained spectral sequence are the $K_{-n}(\underline{\mathbf{Coh}}(X))$ for $n \in \mathbb{Z}$, and the associated (topological) filtration is

$$F^p := \mathrm{Im}\big[K_{-n}(\mathcal{T}_X^p) \to K_{-n}(\underline{\mathbf{Coh}}(X))\big],$$

where the terms on the first page are the $E_1^{p,q} = E^{p,q}$ and the differentials $d_1^{p,q} : E^{p,q} \to E^{p+1,q}$ are the composites

$$\coprod_{x \in X^{(p)}} K_{-p-q}(\kappa(x)) \xrightarrow{k} K_{-p-q-1}(\mathcal{T}_X^{p+1}) \xrightarrow{j} \coprod_{x \in X^{(p+1)}} K_{-p-q-1}(\kappa(x)).$$

We thus have shown that :

Theorem 1.3.54 (Brown-Gersten-Quillen, [Qui73, §7, Thm. 5.4]). *There is a convergent spectral sequence of cohomological type :*

$$E_1^{p,q} = \coprod_{x \in X^{(p)}} K_{-p-q}(\kappa(x)) \Rightarrow K_{-p-q}(\underline{\mathbf{Coh}}(X)),$$

whose associated coniveau filtration is given by :

$$F^p K_{-p-q}(\underline{\mathbf{Coh}}(X)) = \mathrm{Im}\big[K_{-p-q}(\mathcal{T}_X^p) \to K_{-p-q}(\underline{\mathbf{Coh}}(X))\big].$$

This spectral sequence is often referred to as the *Brown-Gersten-Quillen spectral sequence* (or more simply *Gersten-Quillen spectral sequence*), also abreviated as BGQ spectral sequence.

Remarks 1.3.55.

- The formation of the BGQ spectral sequence is functorially contravariant with respect to flat morphisms. If we consider a filtered inverse system of noetherian schemes $(X_i)_{i \in I}$ with affine and flat transition morphisms whose limit X is noetherian, then the BGQ spectral sequence for X is the direct limit of the spectral sequences for the X_i's, see *e.g.* [Qui73, §7, (5.2), (5.3)] for details.
- For $n \geq 0$, let $\mathcal{G}_n$ be the sheafification of the presheaf $U \mapsto K_n(\underline{\mathbf{Coh}}(U))$ on X_{Zar}. The datum of the BGQ spectral sequences with respect to the open subsets of X thus yields a spectral sequence of sheaves :

$$\mathcal{E}_1^{p,q} = \coprod_{x \in X^{(p)}} \iota_{x*} K_{-p-q}(\kappa(x)) \Rightarrow \mathcal{G}_{-p-q},$$

  where $\iota_x : \mathsf{Spec}\,\kappa(x) \hookrightarrow X$ denotes the inclusion of the point $x \in X$ and $K_{-p-q}(\kappa(x))$ is viewed as a constant sheaf on $\mathsf{Spec}\,\kappa(x)$.

3.2.2. Gersten's conjecture

3.2.2.1. Gersten's conjecture for Quillen's K-theory.

Lemma 1.3.56. *The following conditions are equivalent :*

(i) *for all $p \geq 0$, the natural functor $\mathcal{T}_X^{p+1} \to \mathcal{T}_X^p$ induces the zero map on K-groups ;*

(ii) *for all $q \leq 0$, we have $E_2^{p,q} = 0$ for $p \neq 0$, otherwise the edge map $K_{-q}(\underline{\mathbf{Coh}}(X)) \to E_2^{0,q}$ is an isomorphism ;*

(iii) *For all $q \geq 0$, the associated Cousin complex yields an exact sequence*

$$0 \longrightarrow K_q(\underline{\mathbf{Coh}}(X)) \xrightarrow{e} \coprod_{x \in X^{(0)}} K_q(\kappa(x)) \xrightarrow{d_1^{0,q}} \coprod_{x \in X^{(1)}} K_{q-1}(\kappa(x)) \xrightarrow{d_1^{1,q}} \cdots$$

where $d_1^{i,q}$ denotes the differential $E_1^{i,q} \to E_1^{i+1,q}$ and e is induced by functoriality for the flat morphisms $\mathsf{Spec}\,\mathcal{O}_{X,x} \to X$ for $x \in X^{(0)}$ and the isomorphisms $K_q(\underline{\mathbf{Coh}}(\mathcal{O}_{X,x})) \simeq K_q(\underline{\mathbf{Coh}}(\kappa(x))) \simeq K_q(\kappa(x))$.

Proof. For $p \geq 0$ and $q \geq 1$, if (i) holds then the localisation exact sequences break up into short exact sequences :

$$0 \longrightarrow K_q(\mathcal{T}_X^p) \longrightarrow \coprod_{x \in X^{(p)}} K_q(\kappa(x)) \longrightarrow K_{q-1}(\mathcal{T}_X^{p+1}) \longrightarrow 0.$$

Glueing them together, we obtain the exact sequence in (iii). On the other hand, since the sequences in (iii) are constructed from the complexes given by the E_1-terms, whose cohomology groups are the E_2-terms, then (ii) follows. Thus we have shown that (i) $\Rightarrow$ (iii) $\Leftrightarrow$ (ii).

For the implication (iii) $\Rightarrow$ (i), we proceed by induction on $p \geq 0$. First note that the injection

$$e : K_q(\underline{\mathbf{Coh}}(X)) \hookrightarrow \coprod_{x \in X^{(0)}} K_q(\kappa(x))$$

fit into the localisation exact sequence

$$\cdots \longrightarrow K_q(\mathcal{T}_X^1) \longrightarrow K_q(\underline{\mathbf{Coh}}(X)) \xrightarrow{e} \coprod_{x \in X^{(0)}} K_q(\kappa(x)) \longrightarrow K_{q-1}(\mathcal{T}_X^1) \longrightarrow \cdots,$$

so that this sequence breaks up into short exact sequences

$$0 \longrightarrow K_q(\mathcal{T}_X^1) \longrightarrow K_q(\underline{\mathbf{Coh}}(X)) \xrightarrow{e} \coprod_{x \in X^{(0)}} K_q(\kappa(x)) \longrightarrow K_{q-1}(\mathcal{T}_X^1) \longrightarrow 0$$

so that $K_q(\mathcal{T}_X^1) \to K_q(\underline{\mathbf{Coh}}(X))$ is zero.

Now let $p \geq 2$. If we suppose that for all $p' \in [\![0, p]\!]$, the localisation sequence for the pair $(\mathcal{T}_X^{p'+1}, \mathcal{T}_X^{p'})$ splits into short exact sequences

$$0 \longrightarrow K_q(\mathcal{T}_X^{p'}) \longrightarrow \coprod_{x \in X^{(p')}} K_q(\kappa(x)) \longrightarrow K_{q-1}(\mathcal{T}_X^{p'+1}) \longrightarrow 0,$$

then the map $K_q(\mathcal{T}_X^{p'+1}) \to K_q(\mathcal{T}_X^{p'})$ is zero. Hence the differential

$$d_1^{p-2,q} : \coprod_{x \in X^{(p-2)}} K_{q+1}(\kappa(x)) \longrightarrow \coprod_{x \in X^{(p-1)}} K_q(\kappa(x))$$

factors as a composite map

$$\coprod_{x \in X^{(p-2)}} K_{q+1}(\kappa(x)) \twoheadrightarrow K_q(\mathcal{T}_X^{p-1}) \hookrightarrow \coprod_{x \in X^{(p-1)}} K_q(\kappa(x))$$

whose cokernel is precisely

$$\coprod_{x \in X^{(p-1)}} K_q(\kappa(x)) \longrightarrow K_{q-1}(\mathcal{T}_X^p).$$

As a consequence, in the factorisation of $d_1^{p-1,q}$ we see by the exactness of the sequence in (iii) that the map

$$K_{q-1}(\mathcal{T}_X^p) \longrightarrow \coprod_{x \in X^{(p)}} K_{q-1}(\kappa(x))$$

must be injective for all q. Therefore, the localisation sequence for the pair $(\mathcal{T}_X^{p+1}, \mathcal{T}_X^p)$ breaks into short exact sequences which show that the maps $K_q(\mathcal{T}_X^{p+1}) \to K_q(\mathcal{T}_X^p)$ must be zero for all q (the argument for $p = 0$ is similar). This establishes the desired induction, from which the claim follows. ◻

Proposition 1.3.57. *If for each $x \in X$, the stalk $\mathcal{O}_{X,x}$ satisfies one of the equivalent conditions of the previous lemma, then the E_2-terms of the Brown-Gersten-Quillen spectral sequence are*

$$E_2^{p,q} = \mathrm{H}^p_{Zar}(X, \mathcal{G}_{-q}).$$

Proof. For each $n \geq 0$, the data of the complexes in Lemma (1.3.56), (iii) with respect to all the open subsets of X is equivalent to the datum of a complex of Zariski sheaves :

$$0 \longrightarrow \mathcal{G}_n \longrightarrow \coprod_{x \in X^{(0)}} \iota_{x*} K_n(\kappa(x)) \longrightarrow \coprod_{x \in X^{(1)}} \iota_{x*} K_{n-1}(\kappa(x)) \longrightarrow \cdots$$

To a given point $x \in X$, taking stalks yields a corresponding complex for $\mathcal{O}_{X,x}$ (we have an isomorphism $K_n(\underline{\mathbf{Coh}}(\mathcal{O}_{X,x})) \simeq \varinjlim K_n(\underline{\mathbf{Coh}}(U))$ where $U \subset X$ runs over the open subsets containing x, and as stated in Remark (1.3.55) the BGQ spectral sequence commutes with filtered inverse limits of schemes with flat and affine transition maps). Since these local complexes are exact by assumption, then actually the complex of sheaves given above is itself exact, and therefore provides a flasque resolution of $\mathcal{G}_n$. Since flasque sheaves are Zariski-acyclic, then taking global sections gives a complex of E_1-terms which computes the cohomology groups $\mathrm{H}^i_{\mathrm{Zar}}(X, \mathcal{G}_n)$. On the other hand, the cohomology groups of the E_1-terms are by definition the E_2-terms of the spectral sequence, hence the claim. □

One can therefore wonder under which sufficient conditions the stalks $\mathcal{O}_{X,x}$ on a noetherian scheme X of finite dimension all satisfy the conditions of the previous lemma. In 1973, Gersten formulated the following conjecture :

Conjecture 1.3.58 (Gersten's Conjecture for Quillen's K-theory)**.** *The equivalent conditions of Lemma* (1.3.56) *hold whenever X is the spectrum of a regular local ring R.*

3.2.2.2. Quillen's main result in the semi-local case. This conjecture is still far from being known in the general case. However, Quillen managed to prove it in the case of semi-local rings of varieties over a field :

Theorem 1.3.59 (Quillen [Qui73, §7, Thm. 5.11]). *Let R be a regular semi-local ring which is a localisation of a finitely generated algebra over a field k. Then the equivalent conditions of Lemma* (1.3.56) *hold for R.*

Let us first give a slightly strengthened version of Noether's normalisation lemma, which is a crucial step in the proof of Quillen's result (in the modern terminology, the following result is called a *presentation lemma*):

Lemma 1.3.60 (Quillen's presentation lemma)**.** *Let R be a smooth algebra of finite type and of finite Krull dimension d over a field k. Let t be a regular element of R and S a finite set of points of* $\operatorname{Spec} R$. *Then there exist elements $x_1, \ldots, x_{d-1} \in R$ algebraically independent over k such that $B := k[x_1, \ldots, x_{d-1}] \subset R$ and :*

(i) *R/tR is finite over B, and*

(ii) *R is smooth over B at every point of S.*

Proof. Up to choosing for each prime in S a maximal ideal containing it, we can assume that S consists only of closed points. Given the assumptions on R, we know that the R-module $\Omega^1_{R/k}$ of Kähler differentials is projective of rank d, see [Liu02, Chap. 6, §6.2, Cor. 2.6], and R is smooth over $B = k[x_1, \ldots, x_{d-1}]$ at the points of S if and only if the $\mathrm{d}x_i \in \Omega^1_{R/k}$ are independent at these points. Let $\mathfrak{M}$ be the intersection of the ideals in S. For each $n \geq 1$, we have that $R/\mathfrak{M}^n \simeq \prod_{\mathfrak{m} \in S} R/\mathfrak{m}^n$; since this is a finite dimensional k-vector space, one can find a k-vector subspace V of R such that for each $\mathfrak{m} \in S$, there exist $v_1, \ldots, v_d \in V$ such that $(\mathrm{d}v_1, \ldots, \mathrm{d}v_d)$ is a basis for $\Omega^1_{R_\mathfrak{m}/\kappa(\mathfrak{m})}$ and vanishing at the other points of S. Without loss of generality one can also assume that V generates R as a k-algebra.

We now define an increasing filtration $(F_n(R/tR))_{n \geq 0}$ of R/tR by letting $F_n(R/tR)$ be the subspace spanned by the monomials in the elements of V of degree $\leq n$. The induced graded ring

$$\mathrm{gr}(R/tR) = \bigoplus_{n \geq 0} F_n(R/tR)$$

has dimension $d-1$: indeed, $\mathsf{Proj}(\mathrm{gr}(R/tR))$ is the part at infinity of the projective closure of $\mathsf{Spec}(R/tR)$ (which has dimension $d-1$ since t is regular) seen as a subscheme of $\mathsf{Spec}(\mathsf{Sym}(V))$, so it has dimension $d-2$.

If we let $z_1, \dots, z_{d-1}$ be a system of parameters for $\mathrm{gr}(R/tR)$ such that each z_i is homogeneous of degree ≥ 2, then the latter is finite over $k[z_1, \dots, z_{d-1}]$; lifting the z_i's to some elements $x'_1, \dots, x'_{d-1} \in R$, we obtain that R/tR is finite over $k[x'_1, \dots, x'_{d-1}]$.

Now given our choice of V, we can take $v_1, \dots, v_{d-1} \in V$ such that $x_i = x'_i + v_i$ for $i = 1, \dots, d-1$, and the x_i's have independent differentials at the points of S, as desired. Since each x_i has leading term z_i in $\mathrm{gr}(R/tR)$, we obtain that R/tR must be finite over $k[x_1, \dots, x_{d-1}]$. This finishes the proof. □

Remark 1.3.61. In the case where the field k is infinite, one can actually deduce this lemma more directly using Bertini's theorem (such as in [Har77, Chap. III, Cor. 10.9]), see *e.g.* [Sri96, Lem. 5.25].

Proof of Theorem (1.3.59). Let us first restrict ourselves to the case where the algebra from which R is a localisation, is smooth over an infinite field k. As R is of finite type, there exists a subfield k' of k that is finitely generated over its prime subfield, a k'-algebra R' of finite type and a finite subset S' of $\mathsf{Spec}\, R'$ such that $R = k \otimes_{k'} R'$, and such that the primes in S are obtained from the ones in S' by base extension. If A' is the semi-local ring of R' at S', then it is regular, and $A = A' \otimes_{k'} k$. If we let K run over the subfields of k containing k' and finitely generated over the prime subfield, then $A = \varinjlim A' \otimes_{k'} K$ and since field extensions are flat, then $K_n(\mathcal{T}^p_A) \simeq \varinjlim K_n(\mathcal{T}^p_{A'\otimes_k K})$ for all $n \geq 0$. This shows that we can restrict ourselves to the case where k is finitely generated over its prime subfield. We then have that A is a localisation of a finitely generated algebra over the prime subfield, which shows that up to changing R, we can assume that k itself is prime, hence perfect. Over a perfect field, regularity implies that R is smooth over k at the points of S, hence also in a neighborhood, so we may choose a function $f \in R$ not vanishing at the points of S such that R_f is smooth over k. Therefore, up to replacing R again by R_f, we can suppose that R is smooth over k.

Once again, as localisations are flat, we know that for each $n \geq 0$ we have :

$$K_n(\mathcal{T}^{p+1}_A) \simeq \varinjlim_{f \in R} K_n(\mathcal{T}^{p+1}_{R_f})$$

where f runs over the functions that do not vanish at S, so up to replacing R by R_f, we are reduced to showing that the functor $\mathcal{T}^{p+1}_R \to \mathcal{T}^p_A$ induces the zero map on K-groups. On the other hand,

$$K_n(\mathcal{T}^{p+1}_R) \simeq \varinjlim_{f \in R} K_n(\mathcal{T}^p_{R/tR})$$

where t runs over the regular functions in R, so we only have to show that for some fixed t, there exists a function f not vanishing at S such that the functor $\mathcal{T}^p_{R/tR} \to \mathcal{T}^p_R$ induced by localising at f induces zero on K-groups. We thus fall in the right conditions for applying the above lemma. Let $B' := R/tR$ and $R' := R \otimes_B B'$, so that there is a surjective map of B'-algebras $s : R' \twoheadrightarrow B'$ providing a commutative square

$$\begin{tikzcd} R' \arrow[r] \arrow[d, shift right, "s"'] & R \\ B' \arrow[r] \arrow[u, shift right, "u'"'] & B \arrow[u, "u"'] \end{tikzcd}$$

with finite horizontal arrows. Let S' be the (finite) set of points of $\mathsf{Spec}\, R'$ over S. Since u is smooth of relative dimension 1 at the points of S, we deduce by commutativity of the diagram that u' is also smooth of relative dimension 1 at S'. By [GR71, Exposé II, Thm 4.15], this implies that $I := \ker s$ is locally principal at the points of S' (Srinivas provides an alternative -more elementary- explanation in [Sri96, Proof of Thm. 5.24] by reducing to the complete local rings in an appropriate neighborhood), so it is principal in a neighborhood. On the other hand, since R' is finite over R, then this neighborhood must contain the inverse image of a neighborhood of S in $\mathsf{Spec}\, R$. One can thus find some $f \in R$ not vanishing at any point of S such that $I_f \simeq R'_f$ as an R'_f-module and R'_f is smooth (hence flat) over B'.

Now, for any B'-module M, consider the exact sequence

$$0 \longrightarrow I_f \otimes_{B'} M \longrightarrow R'_f \otimes_{B'} M \longrightarrow M_f \longrightarrow 0,$$

and by the flatness of R'_f, we get that if $M \in \mathrm{Ob}(\mathcal{T}^p_{B'})$, then $R'_f \otimes_{B'} M$ is an object of $\mathcal{T}^p_{R'_f}$. This means that, viewed as an R'_f-module, we have that $R'_f \otimes_{B'} M \in \mathrm{Ob}(\mathcal{T}^p_{R_f})$. This gives an exact sequence of exact functors $\mathcal{T}^p_{B'} \to \mathcal{T}^p_{R_f}$. Finally, since $I_f \simeq R'_f$, then $M \mapsto M \otimes_{B'} I_f$ and $M \mapsto M \otimes_{B'} R'_f$ yield isomorphic functors, hence they induce the same maps on K-groups. This shows that $M \mapsto M_f$ induces the zero map on K-groups, which concludes the proof. □

3.2.2.3. Bloch's K-theoretic formula. A spectacular consequence of Quillen's proof of the Gersten conjecture is that one can obtain a cohomological formula for groups of cycles modulo rational equivalence (actually, one could argue on the fact that this result is the starting point of the formalism of motivic cohomology à la Voevodsky-Suslin). Let us first identify the image of the extremal differentials in the K-theoretic Cousin complexes :

> **Proposition 1.3.62** (**[Sri96, Prop. 5.26]**). *If X is a scheme of finite type over a field k, then in the Cousin complex associated to the E_1-terms of the BGQ spectral sequence, the image of the differential*
>
> $$d_1^{p-1,-p} : \bigoplus_{x \in X^{(p-1)}} K_1(\kappa(x)) \longrightarrow \bigoplus_{x \in X^{(p)}} K_0(\kappa(x))$$
>
> *is precisely the group of cycles of codimension p in X that are rationally equivalent to zero.*

We only give some idea of the proof (due to Quillen) of this statement. We also refer to [Qui73, §7, Prop. 5.14] for details. The point is to rewrite this differential as :

$$\bigoplus_{x \in X^{(p-1)}} \kappa(x)^\times \xrightarrow{d_1^{p-1,-p}} \bigoplus_{x \in X^{(p)}} \mathbb{Z} \simeq \mathcal{Z}^p(X).$$

We thus want to show that $d_1^{p-1,-p}$ and the divisor map $\oplus\mathrm{div} : \bigoplus_{x \in X^{(p-1)}} \kappa(x)^\times \to \bigoplus_{x \in X^{(p)}} \mathbb{Z}$ have the same image. Let $y \in X^{(p-1)}$, $x \in X^{(p)}$, $Y := \overline{\{y\}}$ and $(d_1^{p-1,p})_{x,y}$ be the (x,y)-component of $d_1^{p-1,p}$. The closed immersion $Y \hookrightarrow X$ defines an exact functor $\underline{\mathbf{Coh}}(Y) \to \underline{\mathbf{Coh}}(X)$ such that $\mathcal{T}^i_Y \subset \mathcal{T}^{p-1+i}_Y(X)$ for all $i \geq 0$. We thus obtain a natural map of BGQ spectral sequences :

$$E_r^{i,j}(Y) \longrightarrow E_r^{i+p-1,j+1-p}(X)$$

which increases the filtration degree by $p-1$. In particular, we get a commutative diagram :

$$\begin{tikzcd}
 & E_1^{p-1,-p}(X) \arrow[r] & E_1^{p,-p}(X) \arrow[r, equal] & \coprod_{x \in X^{(p)}} \mathbb{Z} \\
K_1(\kappa(y)) \arrow[r, equal] & E_1^{0,-1}(X) \arrow[r] \arrow[u] \arrow[ur, phantom, "\circlearrowleft"] & E_1^{1,-1}(X) \arrow[r, equal] \arrow[u] & \coprod_{x \in Y^{(p)}} \mathbb{Z} \arrow[u, hook]
\end{tikzcd}$$

so that $(d_1^{p-1,p})_{x,y} = 0$ unless $x \in Y$. If we fix $x_0 \in Y$, then the flat morphism $\mathsf{Spec}\,\mathcal{O}_{Y,x_0} \to Y$ induces a contravariant morphism of spectral sequences by the previous discussion, so that we get a diagram of the form :

$$\begin{tikzcd}
K_1(\kappa(y)) \arrow[r, equal] \arrow[d, equal] & E_1^{0,-1}(Y) \arrow[r, "d_1"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & E_1^{1,-1}(Y) \arrow[r, equal] \arrow[d, "\mathsf{pr}_{x_0}"] & \coprod_{x \in Y^{(1)}} \mathbb{Z} \\
K_1(\kappa(y)) \arrow[r, equal] & E_1^{0,-1}(\mathcal{O}_{Y,x_0}) \arrow[r, "d_1"'] & E_1^{1,-1}(\mathcal{O}_{Y,x_0}) \arrow[r, equal] & \mathbb{Z}
\end{tikzcd}$$

where pr_{x_0} is the projection onto the summand corresponding to x_0. We are thus reduced to proving the following result, which we will only state here :

Lemma 1.3.63 ([Sri96, Lem. 5.28])**.** *Let R be an equicharacteristic noetherian local domain of dimension 1, κ its residue field, and let*

$$\cdots \longrightarrow K_1(\underline{\mathbf{Coh}}(R)) \longrightarrow K_1(\mathscr{P}(\mathsf{Frac}\, R)) \longrightarrow K_0(\mathscr{P}(\kappa)) \longrightarrow K_0(\underline{\mathbf{Coh}}(R))$$

be the localisation sequence attached to the closed immersion $\mathsf{Spec}\,\kappa \hookrightarrow \mathsf{Spec}\, R$. Then $K_1(\mathscr{P}(\mathsf{Frac}\, R)) \to K_0(\mathscr{P}(\kappa))$ coincides with the divisor map $\mathrm{div} : \mathsf{Frac}\, R^{\times} \to \mathbb{Z}$.

Corollary 1.3.64 (Bloch's formula[10] for Quillen's K-theory). *Let X be a regular scheme of finite type over a field k. Then for every $p \geq 0$, there is a canonical isomorphism :*

$$\mathrm{H}^p_{Zar}(X, \mathscr{K}_p) \xrightarrow{\sim} \mathrm{CH}^p(X),$$

where $\mathscr{K}_p$ is the sheafification of the presheaf $U \mapsto K_p(\mathscr{P}(U))$ on X_{Zar}. Moreover, we have a flasque resolution of finite length :

$$0 \longrightarrow \mathscr{K}_p \longrightarrow \coprod_{x\in X^{(0)}} \iota_{x*}K_p(\kappa(x)) \longrightarrow \coprod_{x\in X^{(1)}} \iota_{x*}K_{p-1}(\kappa(x)) \longrightarrow \cdots \longrightarrow \coprod_{x\in X^{(p)}} \iota_{x*}K_0(\kappa(x)) \longrightarrow 0,$$

where $\iota_x : \mathsf{Spec}\,\kappa(x) \hookrightarrow X$ denotes the inclusion of the point $x \in X$.

Proof. First note that since X is regular, then Corollary (1.3.44) applies to each open subset $U \subset X$, so that we have an isomorphism $K_p(\mathscr{P}(U)) \simeq K_p(\underline{\mathbf{Coh}}(U))$, hence an isomorphism of sheaves $\mathscr{G}_p \simeq \mathscr{K}_p$. Now, by Theorem (1.3.59) and Proposition (1.3.57), we obtain isomorphisms

$$E_2^{p,q} \xrightarrow{\sim} \mathrm{H}^p_{\mathrm{Zar}}(X, \mathscr{K}_{-q})$$

for each q. The proof of Proposition (1.3.57) then provides the desired flasque resolution. Finally, since the complex given by the global sections of this resolution computes the cohomology groups of $\mathscr{K}_p$, we obtain in particular by the previous lemma that

$$\mathrm{H}^p_{\mathrm{Zar}}(X, \mathscr{K}_p) \simeq \frac{\bigoplus_{x\in X^{(p)}} K_0(\kappa(x))}{\mathrm{Im}\left[\bigoplus_{x\in X^{(p-1)}} K_1(\kappa(x)) \to \bigoplus_{x\in X^{(p)}} K_0(\kappa(x))\right]} \simeq \mathcal{Z}^p(X)/\mathcal{Z}^p(X)_{\mathrm{rat}} = \mathrm{CH}^p(X).$$

□

3.2.2.4. Gersten's conjecture for Milnor's K-theory. One can also make sense of Gersten's conjecture in the case of Milnor's K-theory. Indeed, for a given scheme X, one can similarly define a Zariski sheaf $\mathscr{K}_i^M$ as the sheafification of the presheaf $U \mapsto K_i^M(\Gamma(U, \mathscr{O}_X))$. Assuming that X is noetherian of finite dimension, one can once again define a coniveau spectral sequence and the associated Gersten complex (actually, to define the filtration by coniveau, it would make more sense, for a field k, to identify $K_i^M(k)$ with the Zariski motivic cohomology group $\mathrm{CH}^i(X, i) \simeq \mathrm{H}^i(k, \mathbb{Z}(i))$, see [Kah12, Thm. 2.3] and [Ker09, §7]). It has been shown by Kerz that under the same conditions as before, that is, if X is smooth of finite type over an infinite field, then Gersten's conjecture holds. More precisely :

Theorem 1.3.65 (Kerz [Ker09, Thm. 7.1]). *Let X be a smooth and connected variety over an infinite field k. Then for any integers $i \geq 1$ and $n \geq 0$, the Zariski sheaf $\mathscr{K}_i^M/n$ admits a flasque resolution*

$$0 \longrightarrow \mathscr{K}_i^M/n \longrightarrow \iota_{\eta_X *}K_i^M(\Bbbk(X))/n \longrightarrow \coprod_{x\in X^{(1)}} \iota_{x*}K_{i-1}^M(\kappa(x))/n \longrightarrow \cdots \longrightarrow \coprod_{x\in X^{(0)}} \iota_{x*}K_0(\kappa(x))/n \longrightarrow 0,$$

where η_X is the generic point of X and $\iota_x : \mathsf{Spec}\,\kappa(x) \hookrightarrow X$ denotes the inclusion of the point $x \in X$.

[10] The famous special case $p = 2$ was also discovered by Bloch in [Blo74] by different methods, using notably the second universal Chern class.

Actually, one can say a bit more. Indeed, Hoobler remarked that one can use the Gersten conjecture to prove a generalisation of the Bloch-Kato conjecture for semi-local rings over an infinite field (also known as *Levine's Bloch-Kato conjecture*), without assuming any smoothness condition ; that is, if A is a semi-local ring that contains an infinite field and $n \geq 1$ is an integer invertible over k, then for each $i \geq 1$ there is an isomorphism :

$$K_i^M(A)/n \xrightarrow{\sim} \mathrm{H}^i_{\text{ét}}(A, \mathbb{Z}/n(i)).$$

We give a very brief explanation of the proof (a complete treatment is given in [Ker09, §7, Thm. 7.8]). As Milnor's K-theory and étale cohomology are (almost by definition) locally of finite presentation, one can actually assume that A is of geometric type over k. One can then find a surjective local morphism $B \to A$ of semi-local rings with kernel I such that (B, I) is a henselian pair and B is ind-smooth over k. Under these conditions, Gabber proved an affine version of the proper base change theorem in étale cohomology, so that the natural map $\mathrm{H}^i_{\text{ét}}(B, \mathbb{Z}/n(i)) \to \mathrm{H}^i_{\text{ét}}(A, \mathbb{Z}/n(i))$ is an isomorphism (see *e.g.* [DJ+22, Tag 09ZE] for details). We thus reduce ourselves to the smooth case, where Kerz's proof of the Gersten conjecture provides the desired isomorphism thanks to the Bloch-Kato conjecture for fields.

I.4. Bloch-Ogus theory

— In this section, we give an overview of the theory initially developed by Bloch and Ogus in [BO74]. Roughly stated, Bloch-Ogus theory arose from the study of the Gersten conjecture in algebraic K-theory. Their goal was to translate Quillen's arguments (a filtration by coniveau and a strengthened version of Noether's normalisation lemma) to the setting of étale cohomology. The main accomplishment of their paper is the so-called Bloch-Ogus theorem, which can be described as follows. Given any smooth variety X over a field k and a «suitable»[11] cohomology theory H^* on X (the prototypical example being étale cohomology with torsion coefficients), the filtration by codimension of support on X yields *Cousin complexes* which provide the E_1-terms of a coniveau spectral sequence converging to $\mathrm{H}^*(X)$. On the other hand, restricting these complexes to open subsets of X yields complexes of flasque sheaves on the big Zariski site of X. The Bloch-Ogus theorem then asserts that these complexes are acyclic except in degree 0 where their cohomology is precisely the sheafification $\mathscr{H}^*$ of the presheaf $U \mapsto \mathrm{H}^*(U)$ on X_{Zar}. In particular, this identifies the E_2-terms of the coniveau spectral sequence to $\mathrm{H}^*_{\mathrm{Zar}}(X, \mathscr{H}^*)$; when H^* is étale cohomology with torsion coefficients, this notably allows a refined study of the Leray spectral sequence attached to $\mathrm{Id} : X_{\text{ét}} \to X_{\mathrm{Zar}}$.

4.1. Filtration by coniveau revisited

— Let X be an arbitrary scheme and $\mathscr{F}$ a sheaf of abelian groups on $X_{\text{ét}}$. Consider a chain of closed subsets of X as follows :

$$\vec{Z} : \varnothing \subset Z_d \subset \ldots \subset Z_0 = X.$$

By convention we assume that $Z_i = \varnothing$ for $i > d$ and $Z_i = X$ for $i < 0$. For each pair (Z_{p+1}, Z_p), the long exact cohomology sequence with support reads

$$\cdots \longrightarrow \mathrm{H}^{p+q}_{\text{ét},Z_{p+1}}(X, \mathscr{F}) \xrightarrow{i^{p+1,q-1}} \mathrm{H}^{p+q}_{\text{ét},Z_p}(X, \mathscr{F}) \xrightarrow{j^{p,q}} \mathrm{H}^{p+q}_{\text{ét},Z_p \setminus Z_{p+1}}(X \setminus Z_{p+1}, \mathscr{F}) \xrightarrow{k^{p,q}} \mathrm{H}^{p+q+1}_{\text{ét},Z_{p+1}}(X, \mathscr{F}) \longrightarrow \cdots$$

4.1.1. An exact couple

— Applying this process inductively, we thus obtain diagrams of the form :

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{H}^{p+q}_{\text{ét},Z_{p+1}}(X, \mathscr{F}) \arrow[r, "i^{p+1,q-1}"] & \mathrm{H}^{p+q}_{\text{ét},Z_p}(X, \mathscr{F}) \arrow[r, "j^{p,q}"] & \mathrm{H}^{p+q}_{\text{ét},Z_p \setminus Z_{p+1}}(X \setminus Z_{p+1}, \mathscr{F}) \arrow[r, "k^{p,q}"] \arrow[d, dashed] & \mathrm{H}^{p+q+1}_{\text{ét},Z_{p+1}}(X, \mathscr{F}) \arrow[r] \arrow[dll, "\sim"] & \cdots \\
\cdots \arrow[r] & \mathrm{H}^{p+q+1}_{\text{ét},Z_{p+2}}(X, \mathscr{F}) \arrow[r, "i^{p+2,q-1}"] & \mathrm{H}^{p+q+1}_{\text{ét},Z_{p+1}}(X, \mathscr{F}) \arrow[r, "j^{p+1,q}"] & \mathrm{H}^{p+q+1}_{\text{ét},Z_{p+1} \setminus Z_{p+2}}(X \setminus Z_{p+2}, \mathscr{F}) \arrow[r] & \cdots &
\end{tikzcd}$$

[11] Actually, any «cohomology theory with supports» satisfies the right conditions for Bloch-Ogus theory to function ; for details about the axioms defining such a cohomology theory, see [CTHK97, §§5.1–6.2].

Following [CTHK97, §1.1], we define an exact couple $C_{\vec{Z}}(D, E, i, j, k)$ as the datum of the objects $D^{p,q} := \mathrm{H}^{p+q}_{\text{ét},Z_p}(X, \mathcal{F})$ and $E^{p,q} := \mathrm{H}^{p+q}_{\text{ét},Z_p \setminus Z_{p+1}}(X \setminus Z_{p+1}, \mathcal{F})$, so that we have exact triangles :

$$\begin{tikzcd} D^{p+1,q-1} \arrow[rr, "i^{p+1,q-1}"] & & D^{p,q} \arrow[dl, "j^{p,q}"] \\ & E^{p,q} \arrow[ul, "k^{p,q}"] \arrow[u, phantom, "\circlearrowleft"] & \end{tikzcd}.$$

As explained in §3.2.1.1, such an exact couple provides a spectral sequence of cohomological type ; in our case, it converges to $D^{0,n} = \mathrm{H}^n_{\text{ét}}(X, \mathcal{F})$ with respect to the filtration

$$F^p := \mathrm{Im}\big[\, \mathrm{H}^n_{\text{ét},Z_p}(X, \mathcal{F}) \to \mathrm{H}^n_{\text{ét}}(X, \mathcal{F})\big] = \ker\big[\, \mathrm{H}^n_{\text{ét}}(X, \mathcal{F}) \to \mathrm{H}^n_{\text{ét}}(X \setminus Z_p, \mathcal{F})\big],$$

where the terms on the first page are the $E_1^{p,q} = E^{p,q}$ and the differentials $d_1^{p,q} : E^{p,q} \to E^{p+1,q}$ are the composites

$$\mathrm{H}^{p+q}_{\text{ét},Z_p \setminus Z_{p+1}}(X \setminus Z_{p+1}, \mathcal{F}) \xrightarrow{k} \mathrm{H}^{p+q+1}_{\text{ét},Z_{p+1}}(X, \mathcal{F}) \xrightarrow{j} \mathrm{H}^{p+q+1}_{\text{ét},Z_{p+1} \setminus Z_{p+1}}(X \setminus Z_{p+2}, \mathcal{F}).$$

Suppose now that X is noetherian of pure dimension d and that for all $p \geq 0$, $\mathsf{codim}_X(Z_p) \geq p$. One can define an ordering on the set of $(d+1)$-tuples $\vec{Z}$ by setting

$$\vec{Z} \leq \vec{Z'} \iff \forall p \geq 0,\ Z_p \subseteq Z'_p.$$

The formation of the associated exact couple $C_{\vec{Z}}$ is covariant with respect to this ordering. Passing to the limit over $(d+1)$-tuples in X yields an exact couple $\underset{\to}{C}$ satisfying

$$\underset{\to}{D}^{p,q} = \varinjlim \mathrm{H}^{p+q}_{\text{ét},Z_p}(X, \mathcal{F}) =: \mathrm{H}^{p+q}_{X^{(p)}}(X, \mathcal{F})$$

and

$$\underset{\to}{E}^{p,q} = \varinjlim \mathrm{H}^{p+q}_{\text{ét},Z_p \setminus Z_{p+1}}(X \setminus Z_{p+1}, \mathcal{F}).$$

Together with some additional data on X, one can describe the second limit more easily :

Lemma 1.4.66. (i) *If $Y_1, \ldots, Y_n$ are pairwise disjoint closed subsets of X, then for all $p \geq 0$ we have*

$$\mathrm{H}^p_{\acute{e}t,\cup_i Y_i}(X, \mathcal{F}) \xrightarrow{\sim} \bigoplus_{i=1}^{n} \mathrm{H}^p_{\acute{e}t,Y_i}(X, \mathcal{F}).$$

(ii) *We have*

$$E^{p,q} \xrightarrow{\sim} \coprod_{x \in X^{(p)}} \mathrm{H}^{p+q}_{\acute{e}t,x}(X, \mathcal{F}),$$

where for $x \in X^{(p)}$, $\mathrm{H}^{p+q}_{\acute{e}t,x}(X, \mathcal{F}) := \varinjlim \mathrm{H}^{p+q}_{\acute{e}t,\overline{\{x\}} \cap U}(X, \mathcal{F})$[12].

Proof. By an immediate induction on $n \geq 2$, one can only consider two disjoint closed subsets Y and Y' of X. Applying excision for étale cohomology [Mil80, Chap. III, Prop. 1.27] and the long exact cohomology sequences with support for $Y' \cup Y$ and Y on the one hand and $Y' \cup Y$ and Y' on the other hand, we obtain a commutative diagram with exact row and column

$$\begin{tikzcd} & \mathrm{H}^p_{\text{ét},Y}(X, \mathcal{F}) \arrow[d] \arrow[dr, "\sim"] & \\ \mathrm{H}^p_{\text{ét},Y'}(X, \mathcal{F}) \arrow[r] \arrow[dr, "\sim"'] & \mathrm{H}^p_{\text{ét},Y' \cup Y}(X, \mathcal{F}) \arrow[r] \arrow[d] & \mathrm{H}^p_{\text{ét},Y}(X \setminus Y', \mathcal{F}) \\ & \mathrm{H}^p_{\text{ét},Y'}(X \setminus Y, \mathcal{F}) & \end{tikzcd},$$

[12] If $x \in X^{(p)}$ is a closed point, then $\mathrm{H}^{p+q}_{\text{ét},x}(X, \mathcal{F}) = \mathrm{H}^{p+q}_{\text{ét},\{x\}}(X, \mathcal{F})$.

so the claim (i) follows. Now writing the irreducible components of codimension p of Z_p as $Y_1, \dots, Y_r$, then if Z_{p+1} contains the intersections $Y_i \cap Y_j$ for $i, j \in [\![1, r]\!]$ and the higher codimensional components of Z_p, we have

$$Z_p \setminus Z_{p+1} = \coprod_{i=1}^{r} (Y_i \setminus Z_{p+1}).$$

Applying (i) to the obtained partition of $Z_p \setminus Z_{p+1}$ and passing to the limit, we obtain (ii). □

4.1.2. Coniveau spectral sequence

Definition 1.4.67. The spectral sequence associated to the exact couple $\underset{\rightarrow}{C}$ converges to $\mathrm{H}^*_{\text{ét}}(X, \mathscr{F})$, and it is called the *coniveau spectral sequence* associated to X and $\mathscr{F}$:

$$E_1^{p,q} = \coprod_{x \in X^{(p)}} \mathrm{H}^{p+q}_{\text{ét},x}(X, \mathscr{F}) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(X, \mathscr{F}).$$

The associated filtration

$$F^p\, \mathrm{H}^n_{\text{ét}}(X, \mathscr{F}) = \mathrm{Im}\big[\, \mathrm{H}^n_{\text{ét},X^{(p)}}(X, \mathscr{F}) \to \mathrm{H}^n_{\text{ét}}(X, \mathscr{F})\big]$$

is called the *coniveau filtration*. Its E_1-terms yield *Cousin complexes* :

$$0 \longrightarrow \coprod_{x \in X^{(0)}} \mathrm{H}^q_{\text{ét},x}(X, \mathscr{F}) \xrightarrow{d_1^{0,q}} \coprod_{x \in X^{(1)}} \mathrm{H}^{1+q}_{\text{ét},x}(X, \mathscr{F}) \xrightarrow{d_1^{1,q}} \cdots \xrightarrow{d_1^{p,q}} \coprod_{x \in X^{(p)}} \mathrm{H}^{p+q}_{\text{ét},x}(X, \mathscr{F}) \xrightarrow{d_1^{p,q}} \cdots \tag{I.1}$$

Lemma 1.4.68. *For $n, p \geq 0$, the presheaf*

$$U \longmapsto \coprod_{x \in X^{(p)}} \mathrm{H}^n_{\acute{e}t,x}(U, \mathscr{F})$$

is a flasque sheaf on X_{Zar} which can be identified with the constant sheaf

$$\coprod_{x \in X^{(p)}} \iota_{x*}\, \mathrm{H}^n_{\acute{e}t,x}(X, \mathscr{F}),$$

where $\iota_x : x \hookrightarrow X$ is the inclusion and $\mathrm{H}^n_{\acute{e}t,x}(X, \mathscr{F})$ is considered as a constant sheaf on $\{x\}$ for the Zariski topology.

Proof. For $x \in X^{(p)}$, let $\mathscr{F}_x$ be the Zariski sheaf on X given by

$$U \longmapsto \begin{cases} \mathrm{H}^n_{\text{ét},x}(X, \mathscr{F}) & \text{if } x \in U \\ 0 & \text{if } x \notin U \end{cases}.$$

By definition of $\mathrm{H}^n_{\text{ét},x}(X, \mathscr{F})$ and the universal property of the direct limit, we have $\mathscr{F}_x(U) = \mathscr{F}_x(X)$ if $x \in U$, so that $\mathscr{F}_x$ coincides with $\iota_{x*}\, \mathrm{H}^n_{\text{ét},x}(X, \mathscr{F})$, which is flasque as the pushforward of a constant sheaf on the irreducible space $\{x\}$. This proves the claim. □

4.1.2.1. Purity in Cousin complexes. Now for convenience let us assume that X is a smooth and irreducible variety over a field k, that $n \geq 1$ is an integer that is invertible on k, and that $\mathscr{F}$ is locally constant and constructible (see *e.g.* [Del77, Arcata, §IV.3] for the definition of a constructible sheaf) with n-torsion stalks. For $i \in \mathbb{Z}$, let

$$\mathscr{F}(i) := \mathscr{F} \otimes \mathbb{Z}/n(i).$$

Let $Z \subset X$ be a smooth irreducible closed subvariety of codimension p. By Gabber's absolute purity theorem[13], for every $n \geq 2p$ we have canonical isomorphisms

$$\mathrm{H}^n_{\text{ét},Z}(X, \mathcal{F}) \xleftarrow{\sim} \mathrm{H}^{n-2p}_{\text{ét}}(Z, \mathcal{F}(-p)).$$

If Z is an arbitrary closed subvariety of X and k is perfect, then $Z \cap U$ is a smooth subvariety for $U \subset X$ open small enough, so this yields smooth pairs $Z \cap U \subset U$. We therefore fall in the previous case, which provides isomorphisms

$$\mathrm{H}^{p+q}_{\text{ét},x}(X, \mathcal{F}) \xleftarrow{\sim} \mathrm{H}^{q-p}_{\text{ét}}(\kappa(x), \mathcal{F}(-p))$$

for each $x \in X^{(p)}$. If k is imperfect, then the isomorphisms still hold after passing to the perfect closure. Since étale cohomology is invariant under purely inseparable extensions, then the isomorphisms given above hold in full generality. Taking these identifications into account, the Cousin complexes become

$$0 \longrightarrow \mathrm{H}^q(\Bbbk(X), \mathcal{F}) \longrightarrow \coprod_{x \in X^{(1)}} \mathrm{H}^{q-1}(\kappa(x), \mathcal{F}(-1)) \longrightarrow \cdots \longrightarrow \coprod_{x \in X^{(p)}} \mathrm{H}^{q-p}(\kappa(x), \mathcal{F}(-q)) \longrightarrow \cdots$$

In particular, we obtain that the nontrivial E_1-terms of the coniveau spectral sequence are concentrated in the subdiagonal $\{E_1^{p,q} \mid 0 \leq p \leq q\}$.

4.2. Effaceable sheaves

Definition 1.4.69. Let X be a k-variety and $t_1, \dots, t_r \in X$ be finitely many points contained in an affine open subset of X. An étale sheaf $\mathcal{F}$ on X is said to be *effaceable at* $t_1, \dots, t_r$ if for any $p \geq 0$, for any suitable (small enough) open subset W of X containing $t_1, \dots, t_r$ and any closed subset $Z \subseteq W$ of codimension larger than $p+1$, there exists a smaller open neighborhood $U \subseteq W$ of $t_1, \dots, t_r$ and a closed subset $Z' \subset U$ containing $Z \cap U$ such that :

(i) $\mathrm{codim}_U(Z') \geq p$;

(ii) the composite morphism $\mathrm{H}^n_{\text{ét},Z}(W, \mathcal{F}) \to \mathrm{H}^n_{\text{ét},Z\cap U}(U, \mathcal{F}) \to \mathrm{H}^n_{\text{ét},Z'\cap U}(U, \mathcal{F})$ is zero for each $n \geq 0$.

The sheaf $\mathcal{F}$ is said to be *effaceable* if it is effaceable at any set of points $t_1, \dots, t_r$ contained in an affine open subset of X.

Proposition 1.4.70 (Compare Lemma (1.3.56)). *Let $t_1, \dots, t_r$ be as above and $R := \mathcal{O}_{X,(t_1,\dots,t_r)}$ be the semi-local ring of X at $(t_1, \dots, t_r)$ and $Y := \mathrm{Spec}\, R$. Suppose $\mathcal{F}$ is effaceable at $t_1, \dots, t_r$. Then, in the exact couple defining the coniveau spectral sequence for Y and $\mathcal{F}$, the map $i^{p,q}$ is zero for every $p > 0$. In particular, we have that*

$$E_2^{p,q} = \begin{cases} \mathrm{H}^q_{\acute{e}t}(Y, \mathcal{F}) & \text{if } p = 0 \\ 0 & \text{if } p \neq 0 \end{cases} .$$

Moreover the associated Cousin complex yields an exact sequence

$$0 \longrightarrow \mathrm{H}^q_{\acute{e}t}(Y, \mathcal{F}) \xrightarrow{e} \coprod_{x \in Y^{(0)}} \mathrm{H}^q_{\acute{e}t,x}(Y, \mathcal{F}) \xrightarrow{d_1^{0,q}} \coprod_{x \in Y^{(1)}} \mathrm{H}^{q+1}_{\acute{e}t,x}(Y, \mathcal{F}) \xrightarrow{d_1^{1,q}} \cdots$$

Proof. Consider the natural diagram :

$$\begin{array}{ccccc}
\mathrm{H}^n_{\text{ét},Z}(W, \mathcal{F}) & \longrightarrow & \mathrm{H}^n_{\text{ét},Z\cap U}(U, \mathcal{F}) & \longrightarrow & \mathrm{H}^n_{\text{ét},Z'\cap U}(U, \mathcal{F}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{H}^n_{\text{ét},W^{(p+1)}}(W, \mathcal{F}) & \longrightarrow & \mathrm{H}^n_{\text{ét},U^{(p+1)}}(U, \mathcal{F}) & \longrightarrow & \mathrm{H}^n_{\text{ét},U^{(p)}}(U, \mathcal{F}) \\
 & & \downarrow & & \downarrow \\
 & & \mathrm{H}^n_{\text{ét},Y^{(p+1)}}(Y, \mathcal{F}) & \longrightarrow & \mathrm{H}^n_{\text{ét},Y^{(p)}}(Y, \mathcal{F})
\end{array} .$$

[13] This actually follows from a weaker relative purity result proved in [Del77, Arcata, §V.3, Thm. 3.4].

Since $\mathcal{F}$ is effaceable at $t_1, \ldots, t_r$, then the composition of the arrows on the first row is zero for any $n \geq 0$. Thus the compositions $\mathrm{H}^n_{\text{ét},Z}(W, \mathcal{F}) \to \mathrm{H}^n_{\text{ét},Y^{(p+1)}}(Y, \mathcal{F}) \to \mathrm{H}^n_{\text{ét},Y^{(p)}}(Y, \mathcal{F})$ are also zero. Taking the direct limit over Z, we obtain together with the assumptions on W that the compositions

$$\mathrm{H}^n_{\text{ét},W^{(p+1)}}(W, \mathcal{F}) \to \mathrm{H}^n_{\text{ét},Y^{(p+1)}}(Y, \mathcal{F}) \to \mathrm{H}^n_{\text{ét},Y^{(p)}}(Y, \mathcal{F})$$

are zero for every $n \geq 0$. Now passing to the limit over W, we deduce that the map

$$\mathrm{H}^n_{\text{ét},Y^{(p+1)}}(Y, \mathcal{F}) \xrightarrow{i^{p+1,n-p-1}} \mathrm{H}^n_{\text{ét},Y^{(p)}}(Y, \mathcal{F})$$

is zero as desired. □

Corollary 1.4.71 (The Bloch-Ogus theorem, compare Proposition (1.3.57)). *If $\mathcal{F}$ is an effaceable sheaf on X, then the E_2-terms of the coniveau spectral sequence associated with X and $\mathcal{F}$ are*

$$E_2^{p,q} = \mathrm{H}^p_{Zar}(X, \mathcal{H}^q(\mathcal{F})),$$

where $\mathcal{H}^q_{\acute{e}t}(\mathcal{F}) = \mathcal{R}^q\mathrm{Id}_*\mathcal{F}$ is the sheafification of the presheaf $U \mapsto \mathrm{H}^q_{\acute{e}t}(U, \mathcal{F})$ on X_{Zar}.

Proof. Consider the complex of flasque Zariski sheaves given by the Cousin complexes :

$$0 \longrightarrow \coprod_{x \in X^{(0)}} \iota_{x*}\,\mathrm{H}^q_{\text{ét},x}(X, \mathcal{F}) \longrightarrow \coprod_{x \in X^{(1)}} \iota_{x*}\,\mathrm{H}^{1+q}_{\text{ét},x}(X, \mathcal{F}) \longrightarrow \cdots \longrightarrow \coprod_{x \in X^{(p)}} \iota_{x*}\,\mathrm{H}^{p+q}_{\text{ét},x}(X, \mathcal{F}) \longrightarrow \cdots$$

By Proposition (1.4.70), this complex is a flasque resolution of $\mathcal{H}^q_{\text{ét}}(\mathcal{F})$ with global sections (I.1). □

4.2.1. The effacement theorem

— In this section, we present an *effacement theorem* due to Gabber, which allows the Bloch-Ogus theorem to hold for étale cohomology with finite coefficients over an arbitrary field (Bloch and Ogus' original proof only works when the base field is infinite). Actually, Gabber's paper [Gab94] not only allows us to improve their result, but even allows Bloch-Ogus theory to work for a large set of usual cohomology theories with support that are defined by «substrata», such as Betti cohomology, de Rham cohomology, algebraic K-theory, de Rham-Witt cohomology or Voevodsky's version of motivic cohomology. Our focus is on étale cohomology with torsion coefficients, so we will state a simple form of Gabber's theorem for convenience :

Theorem 1.4.72 (Gabber's effacement theorem, [Gab94]). *If $\pi : X \to \mathsf{Spec}\, k$ is a smooth morphism, then any étale torsion sheaf on X of the form $\pi^*\mathcal{F}_0$ is effaceable.*

As promised, specialising to torsion sheaves such as Tate twists, we obtain the following result :

Corollary 1.4.73 (Bloch-Ogus). *Let X be smooth and irreducible over k, R and Y as in Proposition (1.4.70) and $n \geq 1$ an integer invertible on k. Then for every $i \in \mathbb{Z}$ and any $q \geq 0$, on has an exact sequence :*

$$0 \longrightarrow \mathrm{H}^q_{\acute{e}t}(Y, \mathbb{Z}/n(i)) \longrightarrow \mathrm{H}^q(\Bbbk(Y), \mathbb{Z}/n(i)) \longrightarrow \coprod_{x \in Y^{(1)}} \mathrm{H}^{q-1}(\kappa(x), \mathbb{Z}/n(i-1)) \longrightarrow \cdots$$

Remark 1.4.74. A slightly modified version of the above result was also proved in the case of a Dedekind ring A (or more generally an integral Dedekind scheme) by Soulé in [Sou79, Prop. I] using a different method involving the Leray spectral sequence $E_2^{p,q} = \mathrm{H}^p_{\text{ét}}(A, \mathcal{R}^q f_*\mathbb{Z}/n(j)) \Rightarrow \mathrm{H}^{p+q}(k, \mathbb{Z}/n(j))$ with $f : \mathsf{Spec}\, k \to \mathsf{Spec}\, A$ the morphism induced by the inclusion $A \subset k = \mathsf{Frac}(A)$.

4.2.1.1. Pushforwards of effaceable sheaves. Before presenting the proof of the effacement theorem, we mention another way to produce effaceable sheaves by pushing forward effaceable sheaves along finite morphisms :

Proposition 1.4.75. *Let $f : Y \to X$ be a finite morphism between schemes of pure dimension d with Y smooth and $\mathcal{F}$ an étale sheaf on Y. If $t_1, \ldots, t_n \in X$ are such that $\mathcal{F}$ is effaceable at $f^{-1}(\{t_1, \ldots, t_n\})$, then $f_*\mathcal{F}$ is effaceable at $t_1, \ldots, t_n$.*

Proof. Let $T := \{t_1, \ldots, t_n\}$, Z be as in Definition (1.4.69), $T' := f^{-1}(T)$ and $Z' := f^{-1}(Z)$. If we apply the effacement theorem to $(Y, T', Z', \mathcal{F})$, we obtain a pair (U', Z'') such that $T' \subset U'$ and the composite map

$$\mathrm{H}^n_{\text{ét},Z'}(Y, \mathcal{F}) \longrightarrow \mathrm{H}^n_{\text{ét},Z'\cap U'}(U', \mathcal{F}) \longrightarrow \mathrm{H}^n_{\text{ét},Z''\cap U'}(U', \mathcal{F})$$

is zero for every $n \geq 0$. If we now let $U := X \setminus f(Y \setminus U')$ and $V := f(Z'')$, then $T \subset U$, $Z \subset V$, $\mathsf{codim}_X(V) \geq p$ and $f^{-1}(U) \subseteq U'$, $Z'' \subseteq f^{-1}(V)$. We therefore obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{H}^n_{\text{ét},Z}(X, f_*\mathcal{F}) \arrow[r] \arrow[dd, "\sim"'] & \mathrm{H}^n_{\text{ét},Z\cap U}(U, f_*\mathcal{F}) \arrow[rr] & & \mathrm{H}^n_{\text{ét},V\cap U}(U, f_*\mathcal{F}) \arrow[d, "\sim"] \\
& & \circlearrowleft & \mathrm{H}^n_{\text{ét},f^{-1}(V)\cap f^{-1}(U)}(f^{-1}(U), \mathcal{F}) \\
\mathrm{H}^n_{\text{ét},Z'}(Y, \mathcal{F}) \arrow[r] & \mathrm{H}^n_{\text{ét},Z'\cap U'}(U', \mathcal{F}) \arrow[r, "0"'] & \mathrm{H}^n_{\text{ét},Z''\cap U'}(U', \mathcal{F}) \arrow[r] & \mathrm{H}^n_{\text{ét},f^{-1}(V)\cap U'}(U', \mathcal{F}) \arrow[u]
\end{tikzcd}$$

where the left vertical map and the top right vertical map are isomorphisms (the direct image functor $f_* : \mathrm{Sh}(Y_{\text{ét}}) \to \mathrm{Sh}(X_{\text{ét}})$ associated to a finite morphism being exact, see *e.g.* [Mil80, Chap. II, Cor. 3.6]), hence the desired result. □

A natural consequence of this, together with the Bloch-Ogus theorem, is a version of Shapiro's lemma for Zariski cohomology :

Corollary 1.4.76. *Let $f : Y \to X$ be a finite flat morphism of smooth varieties over a field k and $\mathcal{F}$ an effacable étale sheaf on Y. Then the natural map*

$$\mathrm{H}^n_{Zar}(X, f_*\mathcal{H}^q_{\acute{e}t}(\mathcal{F})) \longrightarrow \mathrm{H}^n_{Zar}(Y, \mathcal{H}^q_{\acute{e}t}(\mathcal{F}))$$

is an isomorphism.

Proof. Since f is finite, then $f_* : \mathrm{Sh}(Y_{\text{ét}}) \to \mathrm{Sh}(X_{\text{ét}})$ is exact. Specialising to cohomology with support, we obtain that for $q \geq 0$, for any closed subset $Z \subset X$ and for $Z' := f^{-1}(Z)$, we have an isomorphism $\mathrm{H}^q_{\text{ét},Z}(X, f_*\mathcal{F}) \xrightarrow{\sim} \mathrm{H}^q_{\text{ét},Z'}(Y, \mathcal{F})$. By construction, these isomorphisms induce an isomorphism (taking direct limits) on the E_1-terms of the coniveau spectral sequences respectively attached to $\mathcal{F}$ on Y and $f_*\mathcal{F}$ on X :

$$\coprod_{x \in X^{(p)}} \mathrm{H}^{p+q}_{\text{ét},x}(X, f_*\mathcal{F}) \xrightarrow{\sim} \coprod_{y \in Y^{(p)}} \mathrm{H}^{p+q}_{\text{ét},y}(Y, \mathcal{F}).$$

In particular, we get an isomorphism between the associated Cousin complexes (the diagrams commute naturally). This implies that their homology groups are isomorphic. On the other hand, $f_*\mathcal{F}$ is effaceable by the previous proposition, so the Bloch-Ogus theorem for semi-local rings applies here. Thus we recover an isomorphism

$$\mathrm{H}^p_{\mathrm{Zar}}(X, \mathcal{H}^q_{\text{ét}}(f_*\mathcal{F})) \xrightarrow{\sim} \mathrm{H}^p_{\mathrm{Zar}}(Y, \mathcal{H}^q_{\text{ét}}(\mathcal{F})).$$

Finally, we have an isomorphism of sheaves $\mathcal{H}^q_{\text{ét}}(f_*\mathcal{F}) \xrightarrow{\sim} f_*\mathcal{H}^q_{\text{ét}}(\mathcal{F})$ since they are isomorphic at each stalk (once again by the finiteness of f). The claim follows. □

4.2.2. Sketch of proof

— In this paragraph we present a slightly modified version of Gabber's proof for the effacement theorem, due to Colliot-Thélène, Hoobler and Kahn. Quite a few intermediate steps are required to reach its full statement, so we refer to their paper [CTHK97, §§2–4] for a full exposition (and a vast generalisation to other cohomology theories in *loc. cit.*, §§5–8). Let us first restate the effacement theorem in a stronger but simpler manner :

Theorem 1.4.77 (Effacement theorem, compare Theorem (1.3.59)). *Let X be a smooth and affine variety over a field k, $t_1, \dots, t_n$ finitely many points in X, $p \geq 0$ an integer and a closed subvariety Z such that $\operatorname{codim}_X(Z) \geq p+1$. Let $\pi : X \to \operatorname{Spec} k$ be the structural morphism and $\mathscr{F}$ a torsion sheaf of abelian groups on $X_{ét}$ that is a pullback $\mathscr{F} = \pi^*\mathscr{F}_0$ of a Γ_k-module $\mathscr{F}_0$. If k is infinite, then there exists an open subset $U \subseteq X$ that contains $t_1, \dots, t_n$ and a closed subvariety $Z' \subseteq X$ such that*

(i) $\operatorname{codim}_X(Z') \geq p$;

(ii) *the map* $\mathrm{H}^n_{ét,Z\cap U}(U,\mathscr{F}) \to \mathrm{H}^n_{ét,Z'\cap U}(U,\mathscr{F})$ *is zero for each* $n \geq 0$.

Otherwise if k is finite, then there exist U and Z' as above such that the composite morphism

$$\mathrm{H}^n_{ét,Z}(X,\mathscr{F}) \to \mathrm{H}^n_{ét,Z\cap U}(U,\mathscr{F}) \to \mathrm{H}^n_{ét,Z'\cap U}(U,\mathscr{F})$$

is zero for each $n \geq 0$.

Remarks 1.4.78.

- When k is finite, the map $\mathrm{H}^n_{ét,Z\cap U}(U,\mathscr{F}) \to \mathrm{H}^n_{ét,Z'\cap U}(U,\mathscr{F})$ can also be zero for each $n \geq 0$. The theorem states that at least the composite $\mathrm{H}^n_{ét,Z}(X,\mathscr{F}) \to \mathrm{H}^n_{ét,Z\cap U}(U,\mathscr{F}) \to \mathrm{H}^n_{ét,Z'\cap U}(U,\mathscr{F})$ is always zero.
- Note that despite what its presentation suggests, the above statement is not local. We will indeed see in the proof that we can't simply replace U by a smaller open to obtain the same result (the crucial point being (ii) here). See [CTHK97, Rmk. 2.2.8] for a more detailed remark on this phenomenon.

4.2.2.1. A result of Gabber. The proof of this version of the effacement theorem requires a geometric presentation result in the same vein as Quillen's presentation Lemma (1.3.60), however the latter needs to be slightly stronger. We only state it below, as its proof is not so difficult (the only non-strictly elementary algebraic geometry involved being Chevalley's theorem on constructible schemes) but rather long and technical, and not so relevant for the purposes of this text. We refer to [CTHK97, §3] for a complete exposition.

Theorem 1.4.79 (Geometric presentation theorem [CTHK97, Thm. 3.1.1], compare Lemma (1.3.60)). *Let X be a smooth, affine and irreducible variety over an infinite field k, $t_1, \dots, t_r \in X$ finitely many points and Z a closed subvariety of positive codimension. Then there exists a morphism $\varphi = (\psi, v) : X \to \mathbb{A}^{d-1}_k \times_k \mathbb{A}^1_k$, an open set $V \subset \mathbb{A}^{d-1}_k$ and an open set $U \subset \psi^{-1}(V)$ containing $t_1, \dots, t_r$ such that :*

(i) $Z \cap U = Z \cap \psi^{-1}(V)$;

(ii) $\psi|_Z$ *is finite ;*

(iii) $\varphi|_U$ *is étale and yields a closed immersion* $Z \cap U \hookrightarrow \mathbb{A}^1_V$;

(iv) $\varphi(t_i) \notin \varphi(Z)$ *if* $t_i \notin Z$, $i = 1, \dots, r$;

(v) $\varphi^{-1}(\varphi(Z \cap U)) \cap U = Z \cap U$.

Corollary 1.4.80. *With notations as above, $\psi|_{Z\cap U} : Z\cap U \to V$ is a finite morphism, and one has a cartesian square*

$$\begin{tikzcd} Z\cap U \arrow[r, hook] \arrow[d, "\sim"'] & U \arrow[d, "\phi|_U"] \\ \varphi(Z\cap U) \arrow[r, hook] & \mathbb{A}^1_V \end{tikzcd}$$

where the horizontal arrows are closed immersions, the left vertical one is an isomorphism and the right vertical one is étale.

Remark 1.4.81. The assumption that k is infinite is crucial in the proof, as it allows to choose a section on the base, hence a finite morphism ψ satisfying the desired conditions (once one has reduced the theorem to the case where the t_i's are closed points and Z is a principal divisor). See [CTHK97, Lem. 3.3.1].

Lemma 1.4.82 (Key lemma)**.** *Let V be a k-scheme and $\mathcal{F}$ as in the Effacement theorem. Let $\pi : \mathbb{A}^1_V \to V$ and $\bar{\pi} : \mathbb{P}^1_V \to V$ be the natural projections, $j : \mathbb{A}^1_V \hookrightarrow \mathbb{P}^1_k$ the open inclusion and $s_\infty : V \to \mathbb{P}^1_V$ the section of $\bar{\pi}$ at infinity. Let F be a closed subset of V and assume that both V and $V \setminus F$ are quasi-compact and quasi-separated. Then the natural diagram*

$$\begin{tikzcd} \mathrm{H}^n_{ét,\mathbb{A}^1_F}(\mathbb{A}^1_V, \mathcal{F}) & \\ & \mathrm{H}^n_{ét,F}(V, \mathcal{F}) \arrow[ul, "\pi^*"'] \\ \mathrm{H}^n_{ét,\mathbb{P}^1_F}(\mathbb{P}^1_V, \mathcal{F}) \arrow[uu, "j^*"] \arrow[ur, "s^*_\infty"'] \arrow[u, phantom, "\circlearrowleft"] & \end{tikzcd}$$

commutes.

Proof. Without loss of generality we can assume that either

(1) $\mathcal{F}$ has torsion coprime with $\mathrm{char}(k)$, or

(2) $\mathcal{F}$ is a p-primary torsion sheaf, where $p = \mathrm{char}(k)$.

Let us consider the first case. Recall that in the case of divisors, the first étale Chern class agrees with the étale cycle class, so that the étale first Chern class $c^{(m)} := c_1(\mathcal{O}_{\mathbb{P}^1_V}(1)) \in \mathrm{H}^2_{\text{ét}}(\mathbb{P}^1_V, \mathbb{Z}/m(j))$ of $\mathcal{O}_{\mathbb{P}^1_V}(1)$ modulo $m \geq 1$ is the image of the class $[\mathcal{O}_{\mathbb{P}^1_V}(1)] \in \mathsf{Pic}(\mathbb{P}^1_V) \simeq \mathrm{H}^1_{\text{ét}}(\mathbb{P}^1_V, \mathbb{G}_m)$ under the boundary map in the long exact sequence associated to the Kummer sequence $1 \to \mu_m \to \mathbb{G}_m \to \mathbb{G}_m \to 1$ on the étale site of $\mathbb{P}^1_V$. The quasi-compactness and quasi-separatedness of V show together that the cup-products

$$\mathrm{H}^i_{\text{ét}}(V, \mathcal{F}[m]) \xrightarrow{\bar{\pi}^*} \mathrm{H}^i_{\text{ét}}(\mathbb{P}^1_V, \mathcal{F}[m]) \xrightarrow{-\smile c^{(m)}} \mathrm{H}^{i+2}_{\text{ét}}(V, \mathcal{F}[m](1))$$

for $m \geq 1$ yield a limit morphism (this is nothing more than commutativity of étale cohomology with filtered colimits in this case, see *e.g.* [DJ+22, Tag 0EZT]) :

$$\mathrm{H}^i_{\text{ét}}(V, \mathcal{F}) \xrightarrow{c_1(\mathcal{O}_{\mathbb{P}^1_V}(1))} \mathrm{H}^{i+2}_{\text{ét}}(\mathbb{P}^1_V, \mathcal{F}(1)) \ .$$

Now in case (1), we have for each $i \geq 0$ an isomorphism

$$\mathrm{H}^i_{\text{ét}}(V, \mathcal{F}) \oplus \mathrm{H}^{i-2}_{\text{ét}}(V, \mathcal{F}(-1)) \xrightarrow{(\bar{\pi}^*, c_1(\mathcal{O}_{\mathbb{P}^1_V}(1)))} \mathrm{H}^i_{\text{ét}}(\mathbb{P}^1_V, \mathcal{F}) \ ,$$

see [Ill77, Exposé VII, Cor. 2.2.4]. If we are now in the second case, then we already have an isomorphism

$$\mathrm{H}^i_{\text{ét}}(V, \mathscr{F}) \xrightarrow{\sim} \mathrm{H}^i_{\text{ét}}(\mathbb{P}^1_V, \mathscr{F})$$

for every $i \geq 0$ by [Gab93, Lem. 3]. Note that the corresponding isomorphisms in both cases hold for $V \setminus F$ as well. Now applying the long exact sequence with support to the pair $(\mathbb{P}^1_V, \mathbb{P}^1_F)$ and to $\mathscr{F}$, we get a canonical isomorphism :

$$\mathrm{H}^i_{\text{ét},\mathbb{P}^1_F}(\mathbb{P}^1_V, \mathscr{F}) \cong \begin{cases} \mathrm{H}^i_{\text{ét},F}(V, \mathscr{F}) \oplus \mathrm{H}^{i-2}_{\text{ét},F}(V, \mathscr{F}(-1)) & \text{in case (1)} \\ \mathrm{H}^i_{\text{ét},F}(V, \mathscr{F}) & \text{in case (2)} \end{cases}.$$

This already shows the lemma for the case (2). On the other hand, remark that $s_\infty^*(\mathscr{O}_{\mathbb{P}^1_V}(1)) = j^*(\mathscr{O}_{\mathbb{P}^1_V}(1)) = 0$. Taking the cup-product with $c_1(\mathscr{O}_{P^1_V})$ yields a map $\mathrm{H}^{n-2}_{\text{ét},F}(V, \mathscr{F}(-1)) \to \mathrm{H}^n_{\text{ét},\mathbb{P}^1_F}(\mathbb{P}^1_V, \mathscr{F})$, and therefore if we are in case (1), then in the natural diagram of the lemma the restrictions of s_∞^* and j^* to the factor $\mathrm{H}^{n-2}_{\text{ét},F}(V, \mathscr{F}(-1))$ (modulo the above isomorphism) must be zero. The desired commutativity result follows. □

Theorem 1.4.83. *Let V be a k-scheme, F a closed subset of V and F' a closed subset of $\mathbb{A}^1_F$ such that the projection $f : F' \to F$ is a finite morphism. Then, for any torsion sheaf of abelian groups on $V_{ét}$, the map*

$$\mathrm{H}^n_{ét,F'}(\mathbb{A}^1_V, \pi^*\mathscr{F}) \longrightarrow \mathrm{H}^n_{ét,\mathbb{A}^1_F}(\mathbb{A}^1_V, \mathscr{F})$$

is zero, where $\pi : \mathbb{A}^1_V \to V$ is the projection onto the second factor.

Proof. First note that we must have $s_\infty(V) \cap F' = \varnothing$, so that we have a factorisation $s_\infty = \iota \circ s'$ where $\iota : \mathbb{P}^1_V \setminus F' \hookrightarrow \mathbb{P}^1_V$ is the open immersion. We thus obtain a commutative diagram :

$$\begin{tikzcd}
\mathrm{H}^n_{\text{ét},F'}(\mathbb{A}^1_V, \pi^*\mathscr{F}) \arrow[r, "\alpha"] & \mathrm{H}^n_{\text{ét},\mathbb{A}^1_F}(\mathbb{A}^1_V, \pi^*\mathscr{F}) & & \\
\arrow[r, phantom, "\circlearrowleft"] & & \mathrm{H}^n_{\text{ét},F}(V, \mathscr{F}) \arrow[ul, "\pi^*"'] & \\
\mathrm{H}^n_{\text{ét},F'}(\mathbb{P}^1_V, \pi'^*\mathscr{F}) \arrow[uu, "\sim"] \arrow[r, "\beta"'] & \mathrm{H}^n_{\text{ét},\mathbb{P}^1_F}(\mathbb{P}^1_V, \pi'^*\mathscr{F}) \arrow[uu, "j^*"'] \arrow[ur, "s_\infty^*"'] \arrow[rr, "\iota^*"'] & & \mathrm{H}^n_{\text{ét},\mathbb{P}^1_F \setminus F'}(\mathbb{P}^1_V \setminus F', \pi'^*\mathscr{F}) \arrow[ul, "s^*"']
\end{tikzcd}$$

where the left vertical arrow is an isomorphism by excision for étale cohomology and the bottom part is exact as a portion of the long exact sequence with support. Now since $\iota^* \circ \beta = 0$, we deduce that $\alpha = 0$ as well, as wanted. □

4.2.2.2. Proof the effacement theorem. We now have all the tools we need to prove Gabber's effacement theorem.

Proof of Theorem (1.4.77). We can assume that X is irreducible. If k is infinite then let us pick U, V, φ, ψ as in the Geometric Presentation Theorem. Let $Z' := \psi^{-1}(\psi(Z))$. We can thus apply the previous theorem to $V := V$, $F := \psi(Z)$ and $F' := \varphi(Z \cap U)$. Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{H}^n_{\text{ét},Z\cap U}(U, \mathscr{F}) \arrow[r] & \mathrm{H}^n_{\text{ét},Z'\cap U}(U, \mathscr{F}) \\
\mathrm{H}^n_{\text{ét},F'}(\mathbb{A}^1_V, \pi^*\mathscr{F}) \arrow[u, "\varphi^*"] \arrow[r] \arrow[ur, phantom, "\circlearrowleft"] & \mathrm{H}^n_{\text{ét},\mathbb{A}^1_F}(\mathbb{A}^1_V, \pi^*\mathscr{F}) \arrow[u, "\varphi^*"']
\end{tikzcd}$$

where the left vertical map is an isomorphism by Corollary (1.4.80) followed by excision for étale cohomology, and the bottom one is zero by the above theorem. By commutativity, we thus obtain that $\mathrm{H}^n_{\text{ét},Z\cap U}(U,\mathscr{F}) \to \mathrm{H}^n_{\text{ét},Z'\cap U}(U,\mathscr{F})$ is the zero map, hence the desired result in this case.

Now assume that k is finite. Let p, ℓ be two distinct prime numbers and let L_1, L_2 denote the $\mathbb{Z}_p$ and $\mathbb{Z}_\ell$-extensions of k respectively[14]. Since these fields are infinite, let $(\psi_1, \varphi_1, V_1, U_1)$ and Z'_1, and $(\psi_2, \varphi_2, V_2, U_2)$ and Z'_2 be as above given by the Geometric Presentation Theorem for (X_{L_1}, Z_{L_1}) and (X_{L_2}, Z_{L_2}) respectively. Consider two finite subextensions $k \subseteq k_1 \subset L_1$ and $k \subseteq k_2 \subset L_2$ on which $(\psi_1, \varphi_1, V_1, U_1, Z'_1)$ and $(\psi_2, \varphi_2, V_2, U_2, Z'_2)$ are respectively defined.

Since the projections $\varphi_i(Z_i \cap U_i) \to \psi_i(Z_i)$ are finite for $i = 1, 2$, then the previous theorem applies, and therefore by the same procedure as above we obtain that the effacement theorem holds over k_1 and k_2 with these choices. Now, let $U := X \setminus (\mathsf{pr}_1(X_{k_1} \setminus U_1) \cup \mathsf{pr}_2(X_{k_2} \setminus U_2))$ and $Z' := \mathsf{pr}_1(Z'_1) \cup \mathsf{pr}_2(Z'_2)$ where $\mathsf{pr}_1 : X \times_k k_1 \to X$ and $\mathsf{pr}_2 : X \times_k k_2 \to X$ are *both* the projection onto the first factor. First note that since we reduced to the case where the t_i's are closed points, we obtain that U contains all of them, and we have $U_{k_i} \subseteq U_i$ and $Z_i \subseteq Z_{k_i}$ for $i = 1, 2$. Moreover for $i = 1, 2$ we have a commutative diagram :

$$\begin{tikzcd}
\mathrm{H}^n_{\text{ét},(Z\cap U)_{k_i}}(U_{k_i},\mathscr{F}) \arrow[r] & \mathrm{H}^n_{\text{ét},(Z'\cap U)_{k_i}}(U_{k_i},\mathscr{F}) \\
\mathrm{H}^n_{\text{ét},Z\cap U_i}(U_i,F) \arrow[u] \arrow[r, "0"'] \arrow[ur, phantom, "\circlearrowleft"] & \mathrm{H}^n_{\text{ét},Z'\cap U_i}(U_i,F) \arrow[u] \\
\mathrm{H}^n_{\text{ét},Z}(X,F) \arrow[u] &
\end{tikzcd} .$$

This shows that the composition

$$\mathrm{H}^n_{\text{ét},Z}(X,\mathscr{F}) \longrightarrow \mathrm{H}^n_{\text{ét},Z_{k_i}}(X_{k_i},\mathscr{F}) \longrightarrow \mathrm{H}^n_{\text{ét},(Z\cap U)_{k_i}}(U_{k_i},\mathscr{F}) \longrightarrow \mathrm{H}^n_{\text{ét},(Z'\cap U)_{k_i}}(U_{k_i},\mathscr{F})$$

must be zero for $i = 1, 2$, or equivalently, the composite

$$\mathrm{H}^n_{\text{ét},Z}(X,\mathscr{F}) \longrightarrow \mathrm{H}^n_{\text{ét},Z\cap U}(U,\mathscr{F}) \longrightarrow \mathrm{H}^n_{\text{ét},Z'\cap U}(U,\mathscr{F}) \longrightarrow \mathrm{H}^n_{\text{ét},(Z'\cap U)_{k_i}}(U_{k_i},\mathscr{F})$$

is zero. But by restriction-corestriction, the composition

$$\mathrm{H}^n_{\text{ét},Z'\cap U}(U,\mathscr{F}) \longrightarrow \mathrm{H}^n_{\text{ét},(Z'\cap U)_{k_i}}(U_{k_i},\mathscr{F}) \longrightarrow \mathrm{H}^n_{\text{ét},Z'\cap U}(U,\mathscr{F})$$

is the multiplication by $[k_i : k]$ (where the first map is induced by the projection and the second is the usual transfer map). Since $[k_1 : k]$ and $[k_2 : k]$ are coprime, we obtain that any element in the image of $\mathrm{H}^n_{\text{ét},Z}(X,\mathscr{F}) \longrightarrow \mathrm{H}^n_{\text{ét},Z\cap U}(U,\mathscr{F}) \longrightarrow \mathrm{H}^n_{\text{ét},Z'\cap U}(U,\mathscr{F})$ has order dividing their gcd, which is 1, therefore this map is zero. $\square$

[14] Such extensions always exist, since the absolute Galois group of a finite field is $\widehat{\mathbb{Z}} \simeq \prod_{\ell \text{ prime}} \mathbb{Z}_\ell$.

4.3. Bloch's Method

— Let X be an integral k-variety and $n \geq 1$ an integer invertible on k. Consider the following commutative diagram of complexes :

$$\begin{array}{lcccccc}
\text{(degrees)} & 2 & & 1 & & 0 & \\
\mathcal{C} & \bigoplus_{x\in X^{(i-2)}} K_2(\kappa(x)) & \xrightarrow{\oplus T_{\kappa(x)}} & \bigoplus_{x\in X^{(i-1)}} \kappa(x)^\times & \xrightarrow{\oplus \mathrm{div}} & \bigoplus_{x\in X^{(i)}} \mathbb{Z} & \longrightarrow 0 \\
& \downarrow{\cdot n} & & \downarrow{\cdot n} & & \downarrow{\cdot n} & \\
\mathcal{C} & \bigoplus_{x\in X^{(i-2)}} K_2(\kappa(x)) & \xrightarrow{\oplus T_{\kappa(x)}} & \bigoplus_{x\in X^{(i-1)}} \kappa(x)^\times & \xrightarrow{\oplus \mathrm{div}_x} & \bigoplus_{x\in X^{(i)}} \mathbb{Z} & \longrightarrow 0 \\
& \downarrow{\oplus R_{n,\kappa(x)}} & & \downarrow{\oplus K_{n,\kappa(x)}} & & \downarrow & \\
\mathcal{D} & \bigoplus_{x\in X^{(i-2)}} \mathrm{H}^2(\kappa(x), \mu_n^{\otimes 2}) & \xrightarrow{u_n} & \bigoplus_{x\in X^{(i-1)}} \mathrm{H}^1(\kappa(x), \mu_n) & \xrightarrow{\partial_n} & \bigoplus_{x\in X^{(i)}} \mathrm{H}^0(\kappa(x), \mathbb{Z}/n) & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
& 0 & & 0 & & 0 &
\end{array}$$

Here the maps $T_{\kappa(x)}$ are the tame symbols in Milnor's K-theory defined in §3.1.2.1, the maps div_x are the divisor maps (after normalisation), the $R_{n,\kappa(x)}$ are provided by the Merkurjev-Suslin theorem [GS17, Thm. 8.6.1] and the $K_{n,\kappa(x)}$ are the boundary maps in the Kummer sequence. The maps u_n and ∂_n are the (sums of the) residues in Galois cohomology.

Note that the vertical complexes are exact. Indeed, for the middle one, this comes from the Kummer sequence and the fact that the Picard group of a field is trivial thanks to Hilbert 90 ; for the left one, this is ensured once again by the Merkurjev-Suslin theorem.

A quick diagram chasing shows that the following sequence is exact :

$$0 \longrightarrow \mathrm{H}_1(\mathcal{C})/n \longrightarrow \mathrm{H}_1(\mathcal{D}) \longrightarrow \mathrm{H}_0(\mathcal{C})[n] \longrightarrow 0.$$

Indeed, if some function $f_0 \in \bigoplus_{x\in X^{(i-1)}} \kappa(x)^\times$ represents an element h_0 of the kernel of ∂_n, then its image is the class in $\mathrm{H}_0(\mathcal{C})$ of the cycle $z_0 \in \bigoplus_{x\in X^{(i)}} \mathbb{Z}$ defined by $\mathrm{div}(f_0) = nz_0$. By Merkurjev-Suslin, the arrows $R_{n,\kappa(x)}$ are surjective, therefore the above map factors through the image of u_n. This defines a surjection $\mathrm{H}_1(\mathcal{D}) \twoheadrightarrow \mathrm{H}_0(\mathcal{C})[n]$ whose kernel is $\mathrm{H}_1(\mathcal{C})/n$. Now remark that the elements of $\mathrm{H}_0(\mathcal{C})$ are precisely the cycles of codimension i in X modulo rational equivalence, *i.e.* $\mathrm{H}_0(\mathcal{C}) = \mathcal{Z}^i(X)/\mathcal{Z}^i(X)_{\mathrm{rat}} = \mathrm{CH}^i(X)$.

The subsidiary question is : how can we control the left and middle groups? By Quillen's formalism of algebraic K-theory, we can extend the complex $\mathcal{C}$ to a larger Gersten complex as in §3.2.2.1 :

$$\bigoplus_{x\in X^{(0)}} K_i(\kappa(x)) \longrightarrow \cdots \longrightarrow \bigoplus_{x\in X^{(j)}} K_{i-j}(\kappa(x)) \longrightarrow \cdots \longrightarrow \bigoplus_{x\in X^{(i-1)}} \kappa(x)^\times \longrightarrow \bigoplus_{x\in X^{(i)}} \mathbb{Z} \longrightarrow 0.$$

If we further assume X to be smooth over k, then Quillen's proof of Gersten's conjecture (see Theorem (1.3.59)) shows that this complex arises as the complex of global sections of a flasque resolution of the Zariski sheaf $\mathcal{K}_i$ given by the sheafification of the preseheaf $U \mapsto K_i(\Gamma(U, \mathcal{O}_X))$ on X_{Zar}. This implies (after re-indexing in a cohomological setting) that

$$\mathrm{H}_1(\mathcal{C}) \simeq \mathrm{H}^{i-1}_{\mathrm{Zar}}(X, \mathcal{K}_i).$$

Similarly, the complex $\mathcal{D}$ fits into a Cousin complex :

$$\bigoplus_{x\in X^{(0)}} \mathrm{H}^i(\kappa(x), \mathbb{Z}/n(i)) \longrightarrow \cdots \longrightarrow \bigoplus_{x\in X^{(i-j)}} \mathrm{H}^{i-j}(\kappa(x), \mathbb{Z}/n(i-j)) \longrightarrow \cdots \longrightarrow \bigoplus_{x\in X^{(i)}} \mathrm{H}^0(\kappa(x), \mathbb{Z}/n) \longrightarrow 0.$$

If we keep the assumption that X is smooth over k, then as discussed before, Bloch-Ogus theory shows that this complex coincides with the complex given by global sections of a flasque resolution of the Zariski sheaf $\mathcal{H}^i_{\text{ét}}(\mathbb{Z}/n(i))$. We thus obtain that $\mathrm{H}_1(\mathcal{D}) \simeq \mathrm{H}^{i-1}_{\mathrm{Zar}}(X, \mathcal{H}^i_{\text{ét}}(\mathbb{Z}/n(i)))$. Putting everything together, we have the following result :

Proposition 1.4.84. *If X is smooth over k, then for each $i \geq 1$ and each integer $n \geq 1$ that is invertible on k, there exists a short exact sequence*

$$0 \longrightarrow \mathrm{H}^{i-1}_{\mathrm{Zar}}(X, \mathcal{K}_i)/n \longrightarrow \mathrm{H}^{i-1}_{\mathrm{Zar}}(X, \mathcal{H}^i_{\acute{e}t}(\mathbb{Z}/n(i))) \longrightarrow \mathrm{CH}^i(X)[n] \longrightarrow 0.$$

I.5. Unramified cohomology

5.1. Definition and main properties

— Here we give the original and standard definition of unramified cohomology. The ideas behind this notion stem from a paper of Artin and Mumford in 1972, where they used the Brauer-Grothendieck group in order to disprove the rationality of certain varieties. In [CTO89], Colliot-Thélène and Ojanguren extended the definition of the unramified Brauer group to higher cohomological degrees and managed to produce examples of unirational varieties that are not (stably) rational. Let X be a scheme and $n \geq 1$ an integer invertible on X. The starting result is the following localisation long exact sequence in étale cohomology :

Theorem 1.5.85 ([Sou79, Prop. 1]). *Let R be a Dedekind ring with fraction field k, $n \geq 1$ an integer that is invertible on k and $j \in \mathbb{Z}$. Then we have a long exact sequence :*

$$0 \longrightarrow \mathrm{H}^1_{\acute{e}t}(R, \mathbb{Z}/n(j)) \longrightarrow \mathrm{H}^1(k, \mathbb{Z}/n(j)) \longrightarrow \bigoplus_{\mathfrak{p}} \mathrm{H}^0(\kappa(\mathfrak{p}), \mathbb{Z}/n(j-1)) \longrightarrow \mathrm{H}^2_{\acute{e}t}(R, \mathbb{Z}/n(j)) \longrightarrow \cdots$$

that is functorial in R, where $\mathfrak{p}$ runs among the prime ideals of R of height 1.

An interesting case is when $X = \mathsf{Spec}\, R$ is the spectrum of a discrete valuation ring. Suppose indeed that $R \subset A$ is an extension of discrete valuation rings, $k \subset L$ is the corresponding extension of fraction fields, and κ_R and κ_A are the respective residue fields. Let π_R be a uniformiser for R and $e_{A/R} := \nu_A(\pi_R)$ be the ramification index of A over R. By the above exact sequence, for each integer $n \geq 1$ invertible on R and for each $j \geq 1$, we have a commutative diagram :

$$\begin{tikzcd}
\mathrm{H}^j(L, \mu_n^{\otimes i}) \arrow[r, "\partial_{j,A}"] & \mathrm{H}^{j-1}(\kappa_A, \mu_n^{\otimes(i-1)}) \\
\mathrm{H}^j(k, \mu_n^{\otimes i}) \arrow[r, "\partial_{j,R}"'] \arrow[u, "\mathrm{Res}_{L/k}"] \arrow[ur, phantom, "\circlearrowleft"] & \mathrm{H}^{j-1}(\kappa_R, \mu_n^{\otimes(i-1)}) \arrow[u, "e_{A/R}\cdot\mathrm{Res}_{\kappa_A/\kappa_R}"']
\end{tikzcd}$$

were the horizontal maps are the residues in Galois cohomology and $\mathrm{Res}_{\kappa_A/\kappa_R}$ is the usual restriction. It therefore makes sense to define the following Galois cohomology groups :

Definition 1.5.86. Let k be a field, $n, i \geq 1$ two integers with n invertible on k and j an arbitrary integer. Given any function field L over k, we define the i^{th} *unramified cohomology group* of L/k as

$$\mathrm{H}^i_{\mathrm{nr}}(L/k, \mathbb{Z}/n(j)) := \ker\left[\mathrm{H}^i(L, \mathbb{Z}/n(j)) \overset{\oplus\partial_{i,R}}{\longrightarrow} \bigoplus_{R \in \mathcal{P}(L/k)} \mathrm{H}^{i-1}(\kappa_R, \mathbb{Z}/n(j-1))\right] \subset \mathrm{H}^i(L, \mathbb{Z}/n(j)),$$

where $\mathcal{P}(L/k)$ denotes the set of rank one discrete valuation rings containing k with fraction field L. An element of $\mathrm{H}^i_{\mathrm{nr}}(L/k, \mathbb{Z}/n(j))$ is called an *unramified class over k*.

5.1.1. Comparison with residues in codimension one

— Since unramified cohomology is defined in terms of function fields, it may seem natural that such a notion only appears in a purely birational context. However, Bloch-Ogus theory allows us to reconcile this point of view with the scheme-theoretic one. The main result in this regard is codimension 1 purity property for étale cohomology which, roughly stated, allows us to compute unramified cohomology by only looking at residues along codimension 1 points on a smooth and proper model.

Lemma I.5.87 (Injectivity property [CT95, Thm. 3.8.1]). *Let k be a field, i, j two integers with $i \geq 0$ and $n \geq 1$ an integer that is invertible over k. If X is an integral k-variety and A is a semi-local ring of X, then the natural map*

$$\mathrm{H}^i_{\acute{e}t}(A, \mathbb{Z}/n(j)) \longrightarrow \mathrm{H}^i(\Bbbk(X), \mathbb{Z}/n(j))$$

is injective.

Proof. First assume that k is infinite. As in the proof of [Ker09, Thm. 6.1], we may use Néron-Popescu desingularisation [Swa95, Thm. 1.1] in order to assume X to be smooth and affine. Let $S = \{t_1, \dots, t_r\}$ be the finite set of points defining A and $\alpha \in \ker[\mathrm{H}^i_{\text{ét}}(A, \mathbb{Z}/n(j)) \to \mathrm{H}^i(\Bbbk(X), \mathbb{Z}/n(j))]$. Since étale cohomology commutes with direct systems with flat and affine transition maps [DJ+22, Tag 03Q5], up to shrinking X we may assume that α comes from a class $\beta \in \mathrm{H}^i_{\text{ét}}(X, \mathbb{Z}/n(j))$ vanishing in $\mathrm{H}^i_{\text{ét}}(U, \mathbb{Z}/n(j))$ for some open subset $U \subset X$. Taking cohomology with supports yields an exact sequence

$$\mathrm{H}^i_{\text{ét},Z}(X, \mathbb{Z}/n(j)) \longrightarrow \mathrm{H}^i_{\text{ét}}(X, \mathbb{Z}/n(j)) \longrightarrow \mathrm{H}^i_{\text{ét}}(U, \mathbb{Z}/n(j))$$

where $Z := X \setminus U$. By exactness and the assumption on β, it must come from a class $\gamma \in \mathrm{H}^i_{\text{ét},Z}(X, \mathbb{Z}/n(j))$. Now, the effacement theorem (I.4.77) applied to A provides a closed subset Z' of X containing Z and a nonempty open subset $V \subset X$ which contains S such that the composite map

$$\mathrm{H}^i_{\text{ét},Z}(X, \mathbb{Z}/n(j)) \longrightarrow \mathrm{H}^i_{\text{ét},Z'}(X, \mathbb{Z}/n(j)) \longrightarrow \mathrm{H}^i_{\text{ét},Z'\cap V}(V, \mathbb{Z}/n(j))$$

is zero. By functoriality the images of γ and β in $\mathrm{H}^i_{\text{ét}}(V, \mathbb{Z}/n(j))$ must be zero, so $\alpha = 0$ in this case.

Now if k is finite, then as in the proof of Theorem (I.4.77) we can find two finite extensions k_1/k and k_2/k with coprime degrees such that α vanishes in both $\mathrm{H}^i_{\text{ét}}(A \otimes_k k_1, \mathbb{Z}/n(j))$ and $\mathrm{H}^i_{\text{ét}}(A \otimes_k k_2, \mathbb{Z}/n(j))$. Again by using transfer maps and restriction-corestriction, we obtain that $\alpha = 0$. ☐

Lemma I.5.88 (Codimension one purity [CT95, Thm. 3.8.2]). *Let k be a field, i, j two integers with $i \geq 0$ and $n \geq 1$ an integer that is invertible over k. Let X a smooth and integral k-variety and A a semi-local ring of X. If a class $\alpha \in \mathrm{H}^i(\Bbbk(X), \mathbb{Z}/n(j))$ lies in the the image of $\mathrm{H}^i_{\acute{e}t}(A_{\mathfrak{p}}, \mathbb{Z}/n(j))$ for each prime $\mathfrak{p}$ of height 1 of A, then it must come from a unique class in $\mathrm{H}^i_{\acute{e}t}(A, \mathbb{Z}/n(j))$.*

Proof. By the above lemma, the class α comes from a unique class in $\mathrm{H}^i_{\text{ét}}(A, \mathbb{Z}/n(j))$. Once again we distinguish the cases where k is infinite or not.

If k is infinite, let S be the set of points of X defining A. We claim that there exists an open subset $U \subset X$ and a lift $\beta \in \mathrm{H}^i_{\text{ét}}(U, \mathbb{Z}/n(j))$ of α such that U contains all the irreducible closed subvarieties of codimension 1 in X passing through at least one point of X. Indeed, the universal property of the generic point provides an open subset U such that α comes from a class $\beta \in \mathrm{H}^i_{\text{ét}}(U, \mathbb{Z}/n(j))$. Suppose that there exists a closed irreducible variety that contains a point $t \in S$ and whose generic point x is not contained in U. By assumption we know that α comes from a class $\alpha_x \in \mathrm{H}^i_{\acute{e}t}(A_t, \mathbb{Z}/n(j))$; let V be an open subset containing x such that α_x extends to a class $\gamma\, \mathrm{H}^i_{\text{ét}}(V, \mathbb{Z}/n(j))$. Since $\mathsf{Spec}\, \Bbbk(X) = \varprojlim_{x \in V} (U \cap V)$ and β and γ agree at the generic point, then they have to agree on some intersection $U \cap V$ with V adequate. Applying the Mayer-Vietoris sequence to U and V (this holds for any abelian sheaf, see [Mil80, Chap. III, Prop. 2.24]) gives an exact sequence

$$\mathrm{H}^i_{\text{ét}}(U \cup V, \mathbb{Z}/n(j)) \longrightarrow \mathrm{H}^i_{\text{ét}}(U, \mathbb{Z}/n(j)) \oplus \mathrm{H}^i_{\text{ét}}(V, \mathbb{Z}/n(j)) \longrightarrow \mathrm{H}^i_{\text{ét}}(U \cap V, \mathbb{Z}/n(j))$$

and since β and γ agree on $U \cap V$ then they must lift to a class in $\mathrm{H}^i_{\text{ét}}(U \cup V, \mathbb{Z}/n(j))$, *a fortiori* α comes from this class. Hence, we can inductively construct an open subset U satisfying our condition.

Up to shrinking X around the points of S, we can assume that $Z := X \setminus U$ has codimension ≥ 2 in X. If $i \leq 2$, then Gabber's purity theorem ensures that $\mathrm{H}^i_{\text{ét},Z}(X, \mathbb{Z}/n(j)) = 0$ and thus the long exact sequence of the pair (X, Z) shows that the restriction $\mathrm{H}^i_{\text{ét}}(X, \mathbb{Z}/n(j)) \to \mathrm{H}^i_{\text{ét}}(U, \mathbb{Z}/n(j))$ is surjective, hence the desired result. Otherwise if $i \geq 3$, then by the effacement theorem (1.4.77) there exists a closed subset F of codimension ≥ 1 containing Z and an open subset $V \subset X$ containing S such that in the following commutative diagram given by long exact sequences with support

$$\begin{tikzcd}
\mathrm{H}^i_{\text{ét}}(X, \mathbb{Z}/n(j)) \arrow[r] \arrow[d] & \mathrm{H}^i_{\text{ét}}(U, \mathbb{Z}/n(j)) \arrow[r] \arrow[d] & \mathrm{H}^i_{\text{ét},Z}(X, \mathbb{Z}/n(j)) \arrow[d] \\
\mathrm{H}^i_{\text{ét}}(X, \mathbb{Z}/n(j)) \arrow[r] \arrow[d] & \mathrm{H}^i_{\text{ét}}(X \setminus F, \mathbb{Z}/n(j)) \arrow[r] \arrow[d] & \mathrm{H}^i_{\text{ét},F}(X, \mathbb{Z}/n(j)) \arrow[d] \\
\mathrm{H}^i_{\text{ét}}(V, \mathbb{Z}/n(j)) \arrow[r] & \mathrm{H}^i_{\text{ét}}(V \setminus F, \mathbb{Z}/n(j)) \arrow[r] & \mathrm{H}^i_{\text{ét},F\cap V}(V, \mathbb{Z}/n(j))
\end{tikzcd}$$

the right vertical composite map is zero. Therefore the image of β in $\mathrm{H}^i_{\text{ét}}(X \setminus F, \mathbb{Z}/n(j))$ must come from a class in $\mathrm{H}^i_{\text{ét}}(V, \mathbb{Z}/n(j))$, which proves the lemma in this case. If k is finite, we may apply the same arguments as in the proof of the previous lemma. □

Proposition 1.5.89 (Summing up). *Let X be an integral variety over a field k and $n \geq 1$ an integer that is invertible on k. Let x be a point in the smooth locus of X. Then,*

(i) *The natural morphism $\mathrm{H}^i_{\acute{e}t}(\mathcal{O}_{X,x}, \mathbb{Z}/n(j)) \to \mathrm{H}^i(\Bbbk(X), \mathbb{Z}/n(j))$ is injective ;*

(ii) *A class $\alpha \in \mathrm{H}^i(\Bbbk(X), \mathbb{Z}/n(j))$ lies in the image of the above morphism if and only if it has trivial residue along each prime divisor of X passing through it.*

Corollary 1.5.90. *Let X be a smooth and proper integral variety over a field k and $n \geq 1$ an integer that is invertible on k. Then a class $\alpha \in \mathrm{H}^i(\Bbbk(X), \mathbb{Z}/n(j))$ is unramified if and only if it has trivial residue along any prime divisor of X.*

Proof. The direct implication is obvious since the local ring at the generic point of any irreducible prime divisor defines a rank one discrete valuation ring containing k with fraction field $\Bbbk(X)$. Conversely, assume that $\alpha \in \mathrm{H}^i(\Bbbk(X), \mathbb{Z}/n(j))$ vanishes along the residue at each prime divisor. Let Y be a normal k-variety together with a birational map $\phi : Y \dashrightarrow X$. We want to show that for any $y \in Y^{(1)}$ we have $\partial_y(\alpha) = 0$, or equivalently, that α lies in the image of the natural map

$$\mathrm{H}^i(\mathcal{O}_{Y,y}, \mathbb{Z}/n(j)) \longrightarrow \mathrm{H}^i(\Bbbk(X), \mathbb{Z}/n(j)).$$

Fixing a $y \in Y^{(1)}$ and up to shrinking Y around y, we can assume that ϕ is a morphism (indeed Y is normal and X is proper so ϕ is defined at all codimension 1 points by the valuative criterion for properness). Let $x := \phi(y) \in X$. Since ϕ is a morphism, we obtain a commutative diagram :

$$\begin{tikzcd}
\mathrm{H}^i_{\text{ét}}(\mathcal{O}_{X,x}, \mathbb{Z}/n(j)) \arrow[dd] \arrow[dr] & \\
\arrow[r, phantom, "\circlearrowleft"] & \mathrm{H}^i(\Bbbk(X), \mathbb{Z}/n(j)) \\
\mathrm{H}^i_{\text{ét}}(\mathcal{O}_{Y,y}, \mathbb{Z}/n(j)) \arrow[ur] &
\end{tikzcd}$$

and by (i) from Proposition (1.5.89), the claim follows. □

In particular, with the same notations as above, we obtain that the natural morphism

$$\mathrm{H}^i_{\text{ét}}(X, \mathbb{Z}/n(j)) \to \mathrm{H}^i(\Bbbk(X), \mathbb{Z}/n(j))$$

given by the inclusion of the generic point factors through a morphism $\mathrm{H}^i_{\text{ét}}(X, \mathbb{Z}/n(j)) \to \mathrm{H}^i_{\text{nr}}(\Bbbk(X)/k, \mathbb{Z}/n(j))$.

Remarks 1.5.91. Let X be an integral k-variety that admits a smooth and proper model $\widetilde{X}$ (*e.g.* by the Hironaka-Nagata compactification theorem if X is smooth).

- The unramified cohomology group $\mathrm{H}^i_{\mathrm{nr}}(\Bbbk(X)/k, \mathbb{Z}/n(j))$ is precisely
$$\mathrm{H}^i_{\mathrm{nr}}(\Bbbk(X)/k, \mathbb{Z}/n(j)) = \mathrm{H}^i_{\mathrm{nr}}(\Bbbk(\widetilde{X})/k, \mathbb{Z}/n(j)) = \ker\left[\mathrm{H}^i(\Bbbk(\widetilde{X}), \mathbb{Z}/n(j)) \xrightarrow{\oplus\partial_x} \bigoplus_{x\in\widetilde{X}^{(1)}} \mathrm{H}^{i-1}(\kappa(x), \mathbb{Z}/n(j-1))\right].$$
- If in particular $j = 1$ and $i = 2$, then $\mathrm{H}^2_{\mathrm{nr}}(\Bbbk(\widetilde{X})/k, \mu_n) = \mathsf{Br}(\widetilde{X})[n] = \mathsf{Br}_{\mathrm{nr}}(X)[n]$ is the n-torsion of the unramified Brauer group of X, see [CTS21, Thm. 3.7.2] (this follows from Gabber's absolute purity theorem). More generally,
$$\mathrm{H}^2_{\mathrm{nr}}(\Bbbk(\widetilde{X})/k, \mathbb{Q}/\mathbb{Z}(1)) = \varinjlim_{n\geq 1} \mathrm{H}^2_{\mathrm{nr}}(\Bbbk(\widetilde{X})/k, \mu_n) \simeq \varinjlim_{n\geq 1} \mathsf{Br}_{\mathrm{nr}}(X)[n] = \mathsf{Br}_{\mathrm{nr}}(X).$$

5.1.2. Functorial properties

Proposition 1.5.92. *Let X be a smooth and integral variety over a field k, $n \geq 1$ an integer that is invertible on k and $\alpha \in \mathrm{H}^i_{nr}(\Bbbk(X)/k, \mathbb{Z}/n(j))$. Then,*

(i) *For any $x \in X$, there is a well-defined restriction*
$$\alpha|_{\{x\}} \in \mathrm{H}^i(\kappa(x), \mathbb{Z}/n(j)).$$

(ii) *If furthermore X is proper over k, then the class $\alpha|_{\{x\}} \in \mathrm{H}^i_{nr}(\kappa(x)/k, \mathbb{Z}/n(j))$ is unramified over k.*

Proof. Lemma (1.5.89) shows that the class $\alpha \in \mathrm{H}^i_{\mathrm{nr}}(\Bbbk(X)/k, \mathbb{Z}/n(j))$ admits a unique lift $\tilde{\alpha} \in \mathrm{H}^i_{\text{ét}}(\mathcal{O}_{X,x}, \mathbb{Z}/n(j))$. One can therefore defined $\alpha|_{\{x\}}$ as the image of $\tilde{\alpha}$ under the natural morphism
$$\mathrm{H}^i_{\text{ét}}(\mathcal{O}_{X,x}, \mathbb{Z}/n(j)) \longrightarrow \mathrm{H}^i(\kappa(x), \mathbb{Z}/n(j)).$$
In order to prove (ii), consider a normal k-variety Z such that $\Bbbk(Z) \simeq \kappa(x)$ and let $z \in Z^{(1)}$. We have to prove that the residue $\partial_z(\alpha|_{\{x\}})$ at z is zero, *i.e.* that $\alpha|_{\{x\}}$ lies in the image of $\mathrm{H}^i_{\text{ét}}(\mathcal{O}_{Z,z}, \mathbb{Z}/n(j)) \to \mathrm{H}^i(\kappa(x), \mathbb{Z}/n(j))$. Since X is proper, we can without loss of generality shrink the variety Z so that the isomorphism $\Bbbk(Z) \simeq \kappa(x)$ is induced by a morphism of schemes $f : Z \to X$ which maps the generic point of Z to x. Since α is unramified over k, the lemma implies that it lies in the image $\mathrm{H}^i_{\text{ét}}(\mathcal{O}_{X,f(z)}, \mathbb{Z}/n(j))$ of $\mathrm{H}^i_{\text{ét}}(\mathcal{O}_{Z,z}, \mathbb{Z}/n(j))$ in $\mathrm{H}^i(\Bbbk(X), \mathbb{Z}/n(j))$. By the universal property of the generic point, there must exist an open neighborhood $U \subset X$ of $f(z)$ and some $\tilde{\alpha} \in \mathrm{H}^i(U, \mathbb{Z}/n(j))$ that restricts to α at the generic point of X. But $f(z)$ lies in the closure of $\{x\}$ in X, so U contains x and $\tilde{\alpha}$ has an image in $\mathrm{H}^i_{\text{ét}}(\mathcal{O}_{X,x}, \mathbb{Z}/n(j))$ which must coincide with α. Hence, the restriction $\alpha|_{\{x\}} \in \mathrm{H}^i(\kappa(x), \mathbb{Z}/n(j))$ must coincide with the image of $\tilde{\alpha}$ under the natural map
$$\mathrm{H}^i_{\text{ét}}(U, \mathbb{Z}/n(j)) \longrightarrow \mathrm{H}^i(\kappa(x), \mathbb{Z}/n(j)).$$
Finally, since $f(z) \in U$, we see that $\alpha|_{\{x\}}$ must lie in the image of $\mathrm{H}^i_{\text{ét}}(\mathcal{O}_{Z,z}, \mathbb{Z}/n(j)) \to \mathrm{H}^i(\kappa(x), \mathbb{Z}/n(j))$, as desired. □

Proposition 1.5.93. *Let k be a field, $f : X \to Y$ a morphism between integral, smooth and proper k-varieties and $n \geq 1$ an integer that is invertible on k. Then there is well-defined pullback map*
$$\mathrm{H}^i_{nr}(\Bbbk(Y)/k, \mathbb{Z}/n(j)) \longrightarrow \mathrm{H}^i_{nr}(\Bbbk(X)/k, \mathbb{Z}/n(j))$$
given by restricting a given unramified class $\alpha \in \mathrm{H}^i_{nr}(\Bbbk(Y)/k, \mathbb{Z}/n(j))$ to the generic point of the image of f and pulling the resulting class to $\Bbbk(X)$.

Proof. The above proposition shows that the restriction of such a class α to the generic point of $f(X)$ is necessarily unramified over k, so we easily obtain that it pulls back to a class in $\mathrm{H}^i_{\mathrm{nr}}(\Bbbk(X), \mathbb{Z}/n(j))$ (see [Sch21, Prop. 4.7] for details). □

5.1.3. Birational and stable birational invariance

— It is not very difficult to see that unramified cohomology is (almost by definition) a birational invariant of smooth and projective varieties over a field. Actually, it is also a stable birational invariant. In order to prove the first claim, let us recall a criterion for birational invariance :

Lemma 1.5.94 ([Voi19, Lem. 2.3])**.** *Let* $\mathcal{F} : \underline{\mathbf{Sm}}_k \to \underline{\mathbf{Ab}}$ *be a contravariant functor such that for any smooth variety* X *over* k*, the following conditions hold :*

(i) *For any dense open subset* $U \subset X$*, the restriction morphism* $\mathcal{F}(X) \to \mathcal{F}(U)$ *is injective ;*

(ii) *If furthermore* $\mathsf{codim}_X(X \setminus U) \geq 2$*, then the restriction morphism* $\mathcal{F}(X) \to \mathcal{F}(U)$ *is an isomorphism.*

Then $\mathcal{F}$ *is a birational invariant of smooth and projective varieties over* k*.*

5.1.3.1. Birational invariance. The invariance property of unramified cohomology under birational maps now becomes immediate :

Proposition 1.5.95. *For any pair of integers* $n, i \geq 1$ *with* n *invertible on* k *and any integer* j*, the functor that sends a smooth variety* X *to* $\mathrm{H}^i_{nr}(\Bbbk(X)/k, \mathbb{Z}/n(j))$ *is a birational invariant of smooth and projective varieties over* k*.*

Proof. Let us check the two conditions in the above lemma. Suppose $U \subset X$ is a dense open subset. Then we have a field isomorphism $\Bbbk(U) \simeq \Bbbk(X)$, hence an isomorphism $\mathrm{H}^i(\Bbbk(U), \mathbb{Z}/n(j)) \simeq \mathrm{H}^i(\Bbbk(X), \mathbb{Z}/n(j))$. We have a well defined pullback

$$\mathrm{H}^i_{\mathrm{nr}}(\Bbbk(X)/k, \mathbb{Z}/n(j)) \longrightarrow \mathrm{H}^i_{\mathrm{nr}}(\Bbbk(U)/k, \mathbb{Z}/n(j)).$$

Moreover since $U^{(1)} \subset X^{(1)}$, then comparing the exact sequences defining the corresponding unramified cohomology groups shows that the former maps injectively into the latter, as desired (the right hand side in the exact sequence for the former defines a smaller kernel). Now if $\mathsf{codim}_X(X \setminus U) \geq 2$, then $U^{(1)} = X^{(1)}$, hence the exact sequences define the same kernel, and we obtain an isomorphism $\mathrm{H}^i_{\mathrm{nr}}(\Bbbk(X)/k, \mathbb{Z}/n(j)) \simeq \mathrm{H}^i_{\mathrm{nr}}(\Bbbk(U)/k, \mathbb{Z}/n(j))$. □

5.1.3.2. Stable birational invariance. In [CTO89], the authors actually exhibited a stronger invariance property, namely that unramified cohomology is a *stable* birational invariant of smooth and projective varieties over a field k (recall that a k-variety X is *stably rational* if there exist two integers $m, n \geq 1$ such that $\mathbb{P}^m_k \times_k X$ is birational to $\mathbb{P}^n_k$).

Proposition 1.5.96. *Let* $n, i \geq 1$ *be two integers with* n *invertible on* k *and* j *an arbitrary integer. Let* $K := k(t)$ *where* t *is an indeterminate. Then the natural morphism* $\mathrm{H}^i(k, \mathbb{Z}/n(j)) \to \mathrm{H}^i(k(t), \mathbb{Z}/n(j))$ *induces an isomorphism*

$$\mathrm{H}^i(k, \mathbb{Z}/n(j)) \xrightarrow{\sim} \mathrm{H}^i_{nr}(k(t)/k, \mathbb{Z}/n(j)).$$

Proof. Without loss of generality, one can assume that k is perfect (indeed, étale cohomology is invariant under purely inseparable extensions). Let $Z \subset \mathbb{A}^1_k$ be a proper closed subset. Taking the long exact cohomology sequence with support in Z and using cohomological purity, we obtain an exact sequence

$$\cdots \longrightarrow \mathrm{H}^i_{\text{ét}}(\mathbb{A}^1_k, \mathbb{Z}/n(j)) \longrightarrow \mathrm{H}^i_{\text{ét}}(\mathbb{A}^1_k \setminus Z, \mathbb{Z}/n(j)) \longrightarrow \bigoplus_{x \in Z} \mathrm{H}^{i-1}(\kappa(x), \mathbb{Z}/n(j)) \longrightarrow \cdots$$

Now since $\mathbb{Z}/n(j)$ is torsion and constructible, then $\mathrm{H}^i(k, \mathbb{Z}/n(j)) \simeq \mathrm{H}^i(\mathbb{A}^1_k, \mathbb{Z}/n(j))$ for each $i \geq 0$ by the homotopy invariance of étale cohomology (see *e.g.* [Mil80, Chap. VI, Cor. 4.20]). Moreover, the maps $\mathrm{H}^i_{\text{ét}}(\mathbb{A}^1_k, \mathbb{Z}/n(j)) \to \mathrm{H}^i_{\text{ét}}(\mathbb{A}^1_k \setminus Z, \mathbb{Z}/n(j))$ are all injective (one can see this by specializing to a k-point or by using a 0-cycle of degree one with a norm argument, depending on whether k is infinite or finite[15]). The long exact sequence thus yields short exact sequences

$$0 \longrightarrow \mathrm{H}^i(k, \mathbb{Z}/n(j)) \longrightarrow \mathrm{H}^i_{\text{ét}}(\mathbb{A}^1_k \setminus Z, \mathbb{Z}/n(j)) \longrightarrow \bigoplus_{x \in Z} \mathrm{H}^{i-1}(\kappa(x), \mathbb{Z}/n(j)) \longrightarrow 0,$$

[15] If k is finite, then it has cohomological dimension equal to 1, so the case $i \geq 2$ follows immediately.

and taking the direct limit over the proper closed subsets of $\mathbb{A}^1_k$, we obtain short exact sequences

$$0 \longrightarrow \mathrm{H}^i(k, \mathbb{Z}/n(j)) \longrightarrow \mathrm{H}^i(\Bbbk(\mathbb{A}^1_k), \mathbb{Z}/n(j)) \longrightarrow \bigoplus_{x\in \mathbb{A}^{1\,(1)}_k} \mathrm{H}^{i-1}(\kappa(x), \mathbb{Z}/n(j)) \longrightarrow 0,$$

which identify $\mathrm{H}^i(k, \mathbb{Z}/n(j))$ with $\mathrm{H}^i_{\mathrm{nr}}(k(t)/k, \mathbb{Z}/n(j))$. □

Proposition 1.5.97. *Let $n, i \geq 1$ be two integers with n invertible on k and $j \in \mathbb{Z}$. Let L be a function field over k and $K := L(t_1, \ldots, t_m)$ a rational function field over L. Then the natural morphism $\mathrm{H}^i(L, \mathbb{Z}/n(j)) \to \mathrm{H}^i(K, \mathbb{Z}/n(j))$ induces an isomorphism*

$$\mathrm{H}^i_{nr}(L/k, \mathbb{Z}/n(k)) \xrightarrow{\sim} \mathrm{H}^i_{nr}(K/k, \mathbb{Z}/n(j)).$$

In particular, the natural morphism $\mathrm{H}^i(k, \mathbb{Z}/n(j)) \to \mathrm{H}^i(k(t_1, \ldots, t_m), \mathbb{Z}/n(j))$ induces an isomorphism

$$\mathrm{H}^i(k, \mathbb{Z}/n(j)) \xrightarrow{\sim} \mathrm{H}^i_{nr}(k(t_1, \ldots, t_m)/k, \mathbb{Z}/n(j)).$$

Proof. By an immediate induction it is sufficient to show the result when $m = 1$. The morphism $\mathrm{H}^i(L, \mathbb{Z}/n(k)) \to \mathrm{H}^i(L(t), \mathbb{Z}/n(j))$ is injective : indeed, if $\alpha \in \mathrm{H}^i(L, \mathbb{Z}/n(j))$ vanishes in $\mathrm{H}^i(L(t), \mathbb{Z}/n(j))$, then the universal property of the generic point shows that α vanishes in $\mathrm{H}^i(U, \mathbb{Z}/n(j))$ for some open subset $U \subset \mathbb{A}^1_L$. The previous proposition thus proves the claim. The functoriality of unramified cohomology yields an embedding

$$\mathrm{H}^i_{\mathrm{nr}}(L/k, \mathbb{Z}/n(j)) \hookrightarrow \mathrm{H}^i_{\mathrm{nr}}(L(t)/k, \mathbb{Z}/n(j)).$$

Now fix a class $\beta \in \mathrm{H}^i_{\mathrm{nr}}(L(t)/k, \mathbb{Z}/n(j))$. We want to show that it comes from a unique class in $\mathrm{H}^i_{\mathrm{nr}}(L/k, \mathbb{Z}/n(j))$. By the previous proposition, β comes from a unique class $\gamma \in \mathrm{H}^i(L, \mathbb{Z}/n(j))$. Let $R \subset L$ be a discrete valuation ring containing k such that $\mathsf{Frac}(R) = L$, $\pi_R \in R$ a uniformiser and $A \subset L(t)$ the discrete valuation ring given by the localisation of $R[t]$ at the ideal generated by π_R. It is easy to check that $\kappa_A = \kappa_R(t)$, so the extension κ_A/κ_R is separable and A/R is unramified, *i.e.* $e_{A/R} = 1$. By functoriality of residues, we obtain a commutative diagram

$$\begin{tikzcd} \mathrm{H}^i(L(t), \mathbb{Z}/n(j)) \arrow[r, "\partial_{i,A}"] & \mathrm{H}^{i-1}(\kappa_R(t), \mathbb{Z}/n(j-1)) \\ \mathrm{H}^i(L, \mathbb{Z}/n(j)) \arrow[u, "\mathrm{Res}_{L(t)/L}"] \arrow[r, "\partial_{i,R}"'] \arrow[ur, phantom, "\circlearrowleft"] & \mathrm{H}^{i-1}(\kappa_R, \mathbb{Z}/n(j-1)) \arrow[u, "\mathrm{Res}_{\kappa_A/\kappa_R}"'] \end{tikzcd}$$

By the same argument as before (0-cycle and norm arugment), the right hand side vertical map is injective, so the commutativity of the diagram shows that if $\partial_{i,A}(\beta) = 0$, then $\partial_{i,R}(\gamma) = 0$ as well. Since this holds for any $R \in \mathcal{P}(L/k)$, we obtain that if β is an unramified class, then so is γ, as desired. □

An immediate consequence of this proposition is the following useful criterion for stable rationality :

Corollary 1.5.98. *Let X be an integral variety over an algebraically closed field k and $n \geq 1$ be an integer that is invertible on k. If X is stably rational, then for all pair of integers i, j with $i \geq 1$, we have $\mathrm{H}^i_{nr}(\Bbbk(X)/k, \mathbb{Z}/n(j)) = 0$.*

5.2. Some refinements

5.2.1. Pairings on zero-cycles and correspondences

5.2.1.1. Merkurjev's pairing. Let X be a smooth and proper integral variety over a field k. In [Mer08, §2.4], Merkurjev defined a pairing

$$\mathcal{Z}_0(X) \times \mathrm{H}^i_{\mathrm{nr}}(\Bbbk(X)/k, \mathbb{Z}/n(j)) \longrightarrow \mathrm{H}^i(k, \mathbb{Z}/n(j))$$

by setting for any closed point $z \in X$ with structure morphism $\pi_z : \mathsf{Spec}\,\kappa(x) \to \mathsf{Spec}\,k$ and any $\alpha \in \mathrm{H}^i_{\mathrm{nr}}(\Bbbk(X)/k, \mathbb{Z}/n(j))$:

$$\langle z, \alpha\rangle := \pi_{z*}(\alpha|_{\{z\}}),$$

and extending bilinearly to $\mathcal{Z}_0(X)$. The key fact about this pairing is that it is compatible with rational equivalence, that is, we have the following statement :

Proposition 1.5.99 (**[Mer08, §2.4]**). *Let k be a field and $n \geq 1$ an integer that is invertible on k. Let $g : C \to X$ be a non-constant morphism of smooth and proper varieties where C is a curve. Then for any $\alpha \in \mathrm{H}^i_{nr}(\Bbbk(X)/k, \mathbb{Z}/n(j))$ and any $\phi \in \Bbbk(C)^\times$, we have*

$$\langle g_*\mathrm{div}(\phi), \alpha\rangle = 0.$$

In particular, the pairing defined above descends to a bilinear pairing

$$\mathrm{CH}_0(X) \times \mathrm{H}^i_{nr}(\Bbbk(X)/k, \mathbb{Z}/n(j)) \longrightarrow \mathrm{H}^i(k, \mathbb{Z}/n(j)).$$

Actually, one can obtain a similar pairing on correspondences :

Corollary 1.5.100. *Let X and Y be two smooth and proper integral varieties over a field k and $n \geq 1$ an integer that is invertible on k. Then there exists a bilinear pairing :*

$$\begin{aligned}\mathrm{CH}_{\dim X}(X \times_k Y) \times \mathrm{H}^i_{nr}(\Bbbk(Y)/k, \mathbb{Z}/n(j)) &\longrightarrow \mathrm{H}^i_{nr}(\Bbbk(X)/k, \mathbb{Z}/n(j))\\ (\Gamma, \alpha) &\longmapsto \Gamma^*\alpha\end{aligned},$$

which is defined as follows : if $\Gamma \subset X \times_k Y$ is integral and does not dominate the first factor, then $\Gamma^*\alpha = 0$; otherwise, the first projection induces a finite morphism $f : \mathsf{Spec}\,\Bbbk(\Gamma) \to \mathsf{Spec}\,\Bbbk(X)$ and we put $\Gamma^* := f_*(\mathsf{pr}_{2*}(\alpha|_{\{\eta_\Gamma\}})$ where η_Γ denotes the generic point of Γ.

Proof. Suppose that Γ is an integral subvariety of $X \times_k Y$ of dimension $\dim X$ which dominates X through the first projection. By Proposition (1.5.92), the class $\mathsf{pr}_{2*}(\alpha)|_{\{\eta_\Gamma\}} \in \mathrm{H}^i(\Bbbk(\Gamma), \mathbb{Z}/n(j))$ is unramified over k, so that $f_*\,\mathsf{pr}_{2*}(\alpha)|_{\{\eta_\Gamma\}} \in \mathrm{H}^i_{\mathrm{nr}}(\Bbbk(X)/k, \mathbb{Z}/n(j))$ is unramified over k as well. This shows that the mapping $(\Gamma, \alpha) \mapsto \Gamma^*\alpha$ is well-defined and induces a bilinear pairing

$$\mathcal{Z}_{\dim X}(X \times_k Y) \times \mathrm{H}^i_{\mathrm{nr}}(\Bbbk(Y)/k, \mathbb{Z}/n(j)) \longrightarrow \mathrm{H}^i_{\mathrm{nr}}(\Bbbk(X)/k, \mathbb{Z}/n(j)).$$

Now, to see that this pairing sends rationally trivial correspondences to 0, we proceed as the following. We have two natural group homomorphisms

$$\mathcal{Z}_{\dim X}(X \times_k Y) \longrightarrow \mathcal{Z}_0(Y_{\Bbbk(X)})$$

and

$$\mathrm{H}^i_{\mathrm{nr}}(\Bbbk(Y)/k, \mathbb{Z}/n(j)) \longrightarrow \mathrm{H}^i_{\mathrm{nr}}(\Bbbk(Y_{\Bbbk(X)})/\Bbbk(X), \mathbb{Z}/n(j)).$$

Since by definition $\mathrm{H}^i_{\mathrm{nr}}(\Bbbk(X)/k, \mathbb{Z}/n(j)) \subset \mathrm{H}^i(\Bbbk(X), \mathbb{Z}/n(j))$, we obtain a commutative diagram

$$\begin{tikzcd} \mathcal{Z}_{\dim X}(X \times_k Y) \times \mathrm{H}^i_{\mathrm{nr}}(Y/k, \mathbb{Z}/n(j)) \arrow[d] \arrow[dr] & \\ \arrow[r, phantom, "\circlearrowleft"] & \mathrm{H}^i(\Bbbk(X), \mathbb{Z}/n(j)) \\ \mathcal{Z}_0(Y_{\Bbbk(X)}) \times \mathrm{H}^i_{\mathrm{nr}}((Y_{\Bbbk(X)})/\Bbbk(X), \mathbb{Z}/n(j)) \arrow[ur] & \end{tikzcd}$$

On the other hand, the map $\mathcal{Z}_{\dim X}(X \times_k Y) \to \mathcal{Z}_0(Y_{\Bbbk(X)})$ descends naturally to a map $\mathrm{CH}_{\dim X}(X \times_k Y) \to \mathrm{CH}_0(Y_{\Bbbk(X)})$, so the previous proposition shows that any correspondence Γ that is rationally equivalent to 0 verifies $\Gamma^*\alpha = 0$ for any $\alpha \in \mathrm{H}^i_{\mathrm{nr}}(\Bbbk(Y)/k, \mathbb{Z}/n(j))$, hence the well-definedness of the pairing. □

5.2.1.2. Unramified cohomology with simple normal crossings. If we deal with sufficiently «nice» varieties, that is, varieties with simple normal crossing singularities, then one can make sense of a more intrinsic definition of unramified cohomology (the idea one should have in mind lies in the smooth and proper case, where one just has to look at residues along prime divisors).

Definition I.5.101. A variety X of pure dimension over a field k with irreducible components $\{X_l\}_{l\in I}$ is said to have *simple normal crossings* if for each non-empty subset $J \subset I$, the subscheme

$$X_J := \bigcap_{l\in J} X_l$$

is smooth over k of pure codimension $\#J$.

Definition I.5.102. Let X be a proper variety of pure dimension over a field k with simple normal crossings and $n \geq 1$ be an integer that is invertible on k. For any integer $i \geq 1$, we define the i^{th} *unramified cohomology groups of* X with coefficients in $\mathbb{Z}/n(j)$ (j an arbitrary integer) as the subgroup

$$\mathrm{H}^i_{\mathrm{nr}}(X, \mathbb{Z}/n(j)) \subset \bigoplus_{l\in I} \mathrm{H}^i_{\mathrm{nr}}(\Bbbk(X_l)/k, \mathbb{Z}/n(j))$$

consisting of the collections $\alpha = (\alpha_l)_{l\in I}$ of unramified classes $\alpha_l \in \mathrm{H}^i_{\mathrm{nr}}(\Bbbk(X_l)/k, \mathbb{Z}/n(j))$ which agree on intersections of components, that is,

$$\alpha_l|_{X_l\cap X_{l'}} = \alpha_{l'}|_{X_l\cap X_{l'}}$$

for any $l, l' \in I$.

Remark I.5.103. If X is a smooth and proper integral variety, then Corollary (I.5.90) shows that this notion coincides with the initial definition of unramified cohomology :

$$\mathrm{H}^i_{\mathrm{nr}}(X/k, \mathbb{Z}/n(j)) \simeq \mathrm{H}^i_{\mathrm{nr}}(\Bbbk(X)/k, \mathbb{Z}/n(j)).$$

5.2.1.3. Merkurjev's pairing revisited. Let now X be a proper variety of pure dimension with simple normal crossings over a field k with irreducible components $\{X_l\}_{l\in I}$. We have a natural bilinear pairing

$$\mathcal{Z}_0(X) \times \mathrm{H}^i_{\mathrm{nr}}(X/k, \mathbb{Z}/n(j)) \longrightarrow \mathrm{H}^i(k, \mathbb{Z}/n(j) \tag{I.2}$$

given by the mapping

$$(z, \alpha) \longmapsto \langle z, \alpha\rangle_{\mathrm{nr}} := \sum_{\emptyset\neq J\subset I} (-1)^{\#J-1}\langle z|_{X_J}, \alpha|_{X_J}\rangle,$$

where $z|_{X_J}$ denotes the «naive» intersection of z with X_J, that is, we remove the prime cycles in z that do not have support in X_J.

Lemma I.5.104 ([Sch21, Lem. 6.3])**.** *Let X be a proper variety of pure dimension over a field k with simple normal crossings and $n \geq 1$ an integer that is invertible on k. If $z \in X$ is a closed point and $\pi_z : \mathsf{Spec}\,\kappa(z) \longrightarrow \mathsf{Spec}\,k$ is the structure morphism, and if $\alpha \in \mathrm{H}^i_{nr}(X/k, \mathbb{Z}/n(j))$, then for any component X_l of X containing z, we have*

$$\langle z, \alpha\rangle_{nr} = \pi_{z*}(\alpha_l|_{\{z\}}).$$

Proposition I.5.105 ([Sch21, Prop. 6.4]). *Let X be a proper variety of pure dimension over a field k with simple normal crossings and $n \geq 1$ an integer that is invertible on k. Let $g : C \longrightarrow X$ be a morphism where C is a smooth and proper curve. Then for any $\alpha \in \mathrm{H}^i_{nr}(X/k, \mathbb{Z}/n(j))$ and any $\phi \in \Bbbk(C)^\times$, we have*

$$\langle g_*\mathrm{div}(\phi), \alpha\rangle = 0.$$

Proof. This is a direct consequence of the above lemma and Proposition (I.5.99). □

We can immediately derive this result to obtain that the pairing defined earlier is compatible with rational equivalence in the following sense :

Corollary 1.5.106. *The pairing* (I.2) *descends to a bilinear pairing*

$$\mathrm{CH}_0(X) \times \mathrm{H}^i_{nr}(X/k, \mathbb{Z}/n(j)) \longrightarrow \mathrm{H}^i(k, \mathbb{Z}/n(j)).$$

We would now like to extend this pairing to correspondences. Let X and Y be two proper and reduced varieties over a field k, and assume that Y is pure dimensional with simple normal crossings. Let $n \geq 1$ be an integer that is invertible on k and let $\{X_l\}_{l\in I}$ be the set of irreducible components of X. The goal is to define a bilinear pairing :

$$\begin{aligned} \mathcal{Z}_{\dim X}(X \times_k Y) \times \mathrm{H}^i_{\mathrm{nr}}(Y/k, \mathbb{Z}/n(j)) &\longrightarrow \bigoplus_{l\in I} \mathrm{H}^i(\Bbbk(X_l), \mathbb{Z}/n(j)) \\ (\Gamma, \alpha) &\longmapsto ((\Gamma^*\alpha)_l)_{l\in I} \end{aligned}.$$

Fix some $l \in I$. By flat pullback, we have a natural map

$$\mathcal{Z}_{\dim X}(X \times_k Y) \longrightarrow \mathcal{Z}_0(Y_{\Bbbk(X_l)})$$

which sends cycles rationally equivalent to 0 on $X \times_k Y$ to cycles rationally equivalent to 0 on $Y \times_k \Bbbk(X_l)$. Moreover by functoriality of unramified cohomology we have a natural morphism

$$\mathrm{H}^i_{\mathrm{nr}}(Y/k, \mathbb{Z}/n(j)) \longrightarrow \mathrm{H}^i_{\mathrm{nr}}((Y_{\Bbbk(X_l)})/\Bbbk(X_l), \mathbb{Z}/n(j))).$$

We can therefore define the mapping $(\Gamma, \alpha) \mapsto (\Gamma^*\alpha)_l$ to be the only one that makes the following diagram commute :

$$\begin{tikzcd} \mathcal{Z}_{\dim X}(X \times_k Y) \times \mathrm{H}^i_{\mathrm{nr}}(Y/k, \mathbb{Z}/n(j)) \arrow[d] \arrow[dr] & \\ & \bigoplus_{l'\in I} \mathrm{H}^i(\Bbbk(X_{l'}), \mathbb{Z}/n(j)) \\ \mathcal{Z}_0(Y_{\Bbbk(X_l)}) \times \bigoplus_{l'\in I} \mathrm{H}^i_{\mathrm{nr}}((Y_{\Bbbk(X_{l'})})/\Bbbk(X_{l'}), \mathbb{Z}/n(j)) \arrow[ur] & \end{tikzcd}, \tag{I.3}$$

where the lower horizontal arrow is induced by the pairing on 0-cycles defined earlier. By asking this for any $l \in I$, we define the global mapping $(\Gamma, \alpha) \mapsto ((\Gamma^*\alpha)_l)_{l\in I}$.

Corollary 1.5.107. *let X and Y be two proper and reduced algebraic varieties over a field k and assume that Y is pure dimensional with simple normal crossings. Let $n \geq 1$ be an integer that is invertible on k and $\{X_l\}_{l\in I}$ be the set of irreducible components of X. Then the pairing defined above descends to a well-defined bilinear pairing :*

$$\mathrm{CH}_{\dim X}(X \times_k Y) \times \mathrm{H}^i_{nr}(Y/k, \mathbb{Z}/n(j)) \longrightarrow \bigoplus_{l\in I} \mathrm{H}^i(\Bbbk(X_l), \mathbb{Z}/n(j)).$$

Proof. Since this pairing makes the diagram (I.3) commute for any $l \in I$, then Corollary (1.5.106) implies that $\Gamma^*\alpha = 0$ for any correspondence $\Gamma \in \mathcal{Z}_{\dim X}(X \times_k Y)$ that is rationally equivalent to 0, hence the well-definedness. □

5.2.2. A consequence of the decomposition of the diagonal

— The following theorem is due to Merkurjev and relates the decomposition of the diagonal with unramified cohomology in the particular case of smooth and proper varieties ; more precisely, it states that under these assumptions, a decomposition of the diagonal on a variety highly constrains its unramified cohomology (*a fortiori* its birational equivalence class) :

Theorem 1.5.108 (Merkurjev, [Mer08, Thm. 2.11]). *Let X be a proper scheme over a field k with simple normal crossings (e.g. a smooth and proper variety) and $n \geq 1$ an integer invertible on k. If X admits a decomposition of the diagonal then for each pair of integers i, j with $i \geq 1$, the natural morphism*

$$\mathrm{H}^i(k, \mathbb{Z}/n(j)) \longrightarrow \mathrm{H}^i_{nr}(X, \mathbb{Z}/n(j))$$

is surjective. In particular, if k is algebraically closed, then $\mathrm{H}^i_{nr}(X, \mathbb{Z}/n(j)) = 0$ for each $i \geq 1$.

Proof. Suppose that X admits a decomposition of the diagonal and let $\{X_l\}_{l \in I}$ be the set of irreducible components of X. By Corollary (1.5.107), we have a well-defined pairing

$$\mathrm{CH}_{\dim X}(X \times_k X) \times \mathrm{H}^i_{\mathrm{nr}}(X/k, \mathbb{Z}/n(j)) \longrightarrow \bigoplus_{l \in I} \mathrm{H}^i(\Bbbk(X_l), \mathbb{Z}/n(j)).$$

By definition of this pairing, we have $[\Delta_X]^*\alpha = \alpha$ for each $\alpha \in \mathrm{H}^i_{\mathrm{nr}}(X/k, \mathbb{Z}/n(j))$, and $[Z_X]^*\alpha = 0$ since the cycle Z_X on $X \times_k X$ does not dominate any component of the first factor. The decomposition of the diagonal thus gives :

$$\alpha = [\Delta_X]^*\alpha = [X \times z]^*\alpha.$$

Let us write $z = \sum_s a_s[z_s]$ where the a_s are integers and the z_s are closed points of X. For each such x_s, let π_{x_s} : $\operatorname{Spec} \kappa(x_s) \to \operatorname{Spec} k$ be the structure morphism. If we write $\varphi : \mathrm{H}^i(k, \mathbb{Z}/n(j)) \to \mathrm{H}^i_{\mathrm{nr}}(X/k, \mathbb{Z}/n(j))$ for the natural morphism, we have that

$$[X \times z]^*\alpha = \varphi\left(\sum_s a_s \pi_{x_s*}(\alpha|_{\{x_s\}})\right),$$

hence the desired result. ◻

Chapter II

Unramified cohomology of degree three and Noether's Problem

II.1. Setting and statement of the main theorem

— Let G be a finite group and W a faithful representation of G over a field k. The action of G induces an action on the function field $k(W)$. A natural question raised by Noether and which is known today as *Noether's Problem*, is to determine whether the invariant subfield $k(W)^G$ is a purely transcendental extension of k. In 1984, Saltman provided an example of a group G such that the field $\mathbb{C}(W)^G$ is not stably rational over $\mathbb{C}$, by considering the unramified Brauer group $\mathrm{H}^2_{\mathrm{nr}}(\mathbb{C}(W)^G, \mathbb{Q}/\mathbb{Z})$. Bogomolov later managed to give a general formula for this group in terms of the group G, namely :

$$\mathsf{Br}_{\mathrm{nr}}(\mathbb{C}(W)^G/\mathbb{C}) = \mathrm{H}^2_{\mathrm{nr}}(\mathbb{C}(W)^G, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\sim} \bigcap_{B \in \mathscr{B}_G} \ker[\mathrm{H}^2(G, \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^2(B, \mathbb{Q}/\mathbb{Z})],$$

where $\mathscr{B}_G$ denotes the set of bicyclic subgroups of G (that is, the subgroups of G that are isomorphic to a quotient of $\mathbb{Z}^2$). More recently, Peyre provided in [Pey07, Thm. 3.1] a similar presentation for the unramified cohomology group $\mathrm{H}^3_{\mathrm{nr}}(\mathbb{C}(W)^G/\mathbb{C}, \mathbb{Q}/\mathbb{Z}(2))$. Using the precise description of this group, he managed to construct [Pey07, Thm. 6.1], for any odd prime number p, a central extension

$$0 \longrightarrow (\mathbb{Z}/p)^6 \longrightarrow G \longrightarrow (\mathbb{Z}/p)^6 \longrightarrow 0$$

such that $\mathrm{H}^3_{\mathrm{nr}}(\mathbb{C}(W)^G/\mathbb{C}, \mathbb{Q}/\mathbb{Z}) \neq 0$. *A fortiori*, the stable birational invariance of unramified cohomology shows that the invariant subfield $\mathbb{C}(W)^G \subset \mathbb{C}(W)$ of any adequate faithful representation W of G cannot be purely transcendental over $\mathbb{C}$. The aim of this chapter is to explain the different steps and to emphasise the key ideas of the proof of Peyre's main result.

1.1. Negligible classes

Definition 2.1.1. If G is a finite group, M is a G-module and k is a field, then a class $\lambda \in \mathrm{H}^i(G, M)$ is said to be *totally k-negligible* if and only if for any field extension L/k and any morphism $\rho : \Gamma_L \to G$, the image of λ under

$$\rho^* : \mathrm{H}^i(G, M) \to \mathrm{H}^i(L, M)$$

is zero. If $k = \mathbb{C}$, then we call such a λ a *geometrically negligible class*. When there is no confusion on the choice of k, we denote by $\mathrm{H}^i_{\mathrm{n}}(G, M)$ the subgroup of totally k-negligible classes in $\mathrm{H}^i(G, M)$.

Lemma 2.1.2 ([Sal95, Prop. 4.5])**.** *The group of geometrically negligible classes in* $\mathrm{H}^i(G, M)$ *is precisely*

$$\mathrm{H}^i_n(G, M) = \ker[\mathrm{H}^i(G, M) \to \mathrm{H}^i(\mathbb{C}(W)^G, M)]$$

where W is any faithful complex representation of G.

Consider the tautological exact sequence of trivial G-modules

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

Taking cohomology gives a long exact sequence

$$\cdots \longrightarrow \mathrm{H}^i(G, \mathbb{Q}) \longrightarrow \mathrm{H}^i(G, \mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{H}^{i+1}(G, \mathbb{Z}) \longrightarrow \mathrm{H}^{i+1}(G, \mathbb{Q}) \longrightarrow \cdots$$

and since $\mathbb{Q}$ is uniquely divisible (divisible and torsion-free) then both side terms above are zero, so this yields isomorphisms

$$\mathrm{H}^i(G, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^{i+1}(G, \mathbb{Z}) \quad \forall i \geq 1.$$

We can therefore define a cup-product $\mathrm{H}^*(G, \mathbb{Q}/\mathbb{Z}) \otimes_{\mathbb{Z}} \mathrm{H}^*(G, \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^*(G, \mathbb{Q}/\mathbb{Z})$ as given by the following commutative diagrams :

$$\begin{tikzcd}
\mathrm{H}^i(G, \mathbb{Q}/\mathbb{Z}) \times \mathrm{H}^j(G, \mathbb{Q}/\mathbb{Z}) \arrow[r, "\sim"] \arrow[d, dashed, "-\smile-"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{H}^{i+1}(G, \mathbb{Z}) \times \mathrm{H}^{j+1}(G, \mathbb{Z}) \arrow[d, "-\smile-"] \\
\mathrm{H}^{i+j+1}(G, \mathbb{Q}/\mathbb{Z}) \arrow[r, "\sim"'] & \mathrm{H}^{i+j+2}(G, \mathbb{Z})
\end{tikzcd}$$

for $i, j \geq 1$.

Remark 2.1.3. Note that for simplicity we do not keep track of the twists in the coefficients here : indeed, the map $\mathbb{Q} \to \mathbb{C},\ z \mapsto \exp(2i\pi z)$ yields an isomorphism $\mathbb{Q}/\mathbb{Z} \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}(1)$ of trivial G-modules. However when we deal with the Galois cohomology of a field k it is obviously important to specify these twists (unless k is separably closed) since they yield pairwise non-necessarily isomorphic Galois modules.

Definition 2.1.4. The subgroup of *permutation negligible classes* in $\mathrm{H}^3(G, \mathbb{Q}/\mathbb{Z})$ is the group

$$\mathrm{H}^3_{\mathrm{p}}(G, \mathbb{Q}/\mathbb{Z}) := \sum_{H \subset G} \mathrm{Cores}_H^G \left(\mathrm{Im}[\mathrm{H}^1(H, \mathbb{Q}/\mathbb{Z})^{\otimes 2} \xrightarrow{-\smile-} \mathrm{H}^3(H, \mathbb{Q}/\mathbb{Z})]\right)$$

where H runs among the subgroups of G.

1.2. Unramified classes

— Let $H \subset G$ be a subgroup, $\mathcal{Z}_G(H)$ the centraliser of H in G, and $g \in \mathcal{Z}_G(H)$. Let $I := \langle g \rangle$ and $m : H \times I \to G,\ (h, i) \mapsto hi$. Pulling back on cohomology gives a morphism $m^* : \mathrm{H}^3(G, \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^3(H \times I, \mathbb{Q}/\mathbb{Z})$. On the other hand, the projection $\mathrm{pr}_2 : H \times I \to I$ induces a splitting of the map $\mathrm{H}^3(H \times I, \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^3(I, \mathbb{Q}/\mathbb{Z})$ defined by $i_2 : I \to H \times I,\ i \mapsto (1, i)$. Therefore, we obtain a natural map

$$\begin{aligned} \mathrm{H}^3(H \times I, \mathbb{Q}/\mathbb{Z}) &\xrightarrow{\partial_{H,I}} \ker\left[\mathrm{H}^3(H \times I, \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^3(I, \mathbb{Q}/\mathbb{Z})\right]. \\ \xi &\longmapsto \xi - (\mathrm{pr}_2^* \circ i_2^*)(\xi) \end{aligned}$$

The Hochschild-Serre spectral sequence

$$E_2^{p,q} = \mathrm{H}^p(H, \mathrm{H}^q(I, \mathbb{Q}/\mathbb{Z})) \Rightarrow \mathrm{H}^{p+q}(H \times I, \mathbb{Q}/\mathbb{Z})$$

and the fact that $\mathrm{H}^2(I, \mathbb{Q}/\mathbb{Z}) = 0$ (I is cyclic so it admits a cyclic resolution of order 2) provide a map

$$\mathrm{H}^3(H \times I, \mathbb{Q}/\mathbb{Z}) \longrightarrow \ker\left[\mathrm{H}^3(H \times I, \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^3(I, \mathbb{Q}/\mathbb{Z})\right] \longrightarrow \mathrm{H}^2(H, \mathrm{H}^1(I, \mathbb{Q}/\mathbb{Z})).$$

The evaluation at g gives an injection $\mathsf{Hom}(I, \mathbb{Q}/\mathbb{Z}) \hookrightarrow \mathbb{Q}/\mathbb{Z}$. Hence we obtain a composite morphism :

$$\mathrm{H}^3(H \times I, \mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{H}^2(H, \mathrm{H}^1(I, \mathbb{Q}/\mathbb{Z})) \xrightarrow{\sim} \mathrm{H}^2(H, \mathsf{Hom}(I, \mathbb{Q}/\mathbb{Z})) \longrightarrow \mathrm{H}^2(H, \mathbb{Q}/\mathbb{Z}),$$

let us denote it by ∂. Precomposing with m^* therefore gives a map

$$\partial_{H,g} := \partial \circ m^* : \mathrm{H}^3(G, \mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{H}^3(H, \mathbb{Q}/\mathbb{Z}).$$

Definition 2.1.5. We define the *third unramified cohomology group of G with $\mathbb{Q}/\mathbb{Z}$-coefficients* to be the group

$$\mathrm{H}^3_{\mathrm{nr}}(G, \mathbb{Q}/\mathbb{Z}) := \bigcap_{\substack{H \subset G \\ g \in \mathcal{Z}_G(H)}} \ker \partial_{H,g}.$$

We are now ready to state the main theorem proved in [Pey07] :

Theorem 2.1.6 ([Pey07, Thm. 3.1]). *Let G be a finite group and W a faithful complex representation of G. Then the inflation map induces a surjection*

$$\mathrm{H}^3_{nr}(G, \mathbb{Q}/\mathbb{Z})/\mathrm{H}^3_{p}(G, \mathbb{Q}/\mathbb{Z}) \twoheadrightarrow \mathrm{H}^3_{nr}(\mathbb{C}(W)^G/\mathbb{C}, \mathbb{Q}/\mathbb{Z})$$

whose kernel is killed by a power of 2.

II.2. Proof of the main theorem

2.1. Preliminary results

— Peyre's stategy for the proof of the main theorem stated in the previous section boils down to relating residues at the level of the cohomology of the function field $\mathbb{C}(W)^G$ to the abstract residues at the level of the cohomology of G that we defined in §1.2. To do so, we want to apply Bloch's method (see Chapter I, §4.3) in this specific context and describe the $\mathcal{K}_2$-cohomology and étale cohomology of the geometric quotient of W by G quite precisely in terms of the latter. We will make extensive use of the following auxiliary results.

2.1.1. Excision for $\mathcal{K}$-cohomology, equivariant étale cohomology

Lemma 2.2.7. *Let X be a smooth variety over a field k and Y a subvariety of X of codimension at least $c \geq 0$. Then for any $j \geq 0$ and any $i \leq c-2$, we have*

$$\mathrm{H}^i_{Zar}(X, \mathcal{K}_j) \xrightarrow{\sim} \mathrm{H}^i_{Zar}(X \setminus Y, \mathcal{K}_j).$$

Proof. By Theorem (1.3.59), the groups $\mathrm{H}^i_{\mathrm{Zar}}(X, \mathcal{K}_j)$ coincide with the homology groups of the Cousin complex

$$\cdots \longrightarrow \bigoplus_{x \in X^{(i-1)}} K_{j-i+1}(\kappa(x)) \xrightarrow{d_1^{i-1,-j}} \bigoplus_{x \in X^{(i)}} K_{j-i}(\kappa(x)) \xrightarrow{d_1^{i,-j}} \bigoplus_{x \in X^{(i+1)}} K_{j-i-1}(\kappa(x)) \longrightarrow \cdots$$

Since $\mathsf{codim}_X(Y) \geq c$, then for $j \leq c-1$ we have $(X \setminus Y)^{(j)} = X^{(j)}$. By the same argument, for $i \leq c-1$ the residue maps $d_1^{i,-j}$ are the same for X and for $X \setminus Y$. This proves the claim. □

Proposition 2.2.8. *Let G be a finite group and k be a separably closed field of exponential characteristic p and let $n \geq 1$ be an integer coprime to p. Then for any integers i, j with $0 \leq j < i$, for any faithful complex representation W of G such that there exists an open subset U of W on which G acts freely and such that $\mathsf{codim}_W(W \setminus U) \geq i$, we have*

$$\mathrm{H}^j(G, \mathbb{Z}/n) \xrightarrow{\sim} \mathrm{H}^j_{\acute{e}t}(U/G, \mathbb{Z}/n).$$

Proof. Consider the coniveau spectral sequence

$$E_1^{p,q} = \bigoplus_{x \in U^{(p)}} \mathrm{H}^{q-p}(\kappa(x), \mathbb{Z}/n) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(U, \mathbb{Z}/n)$$

and the similar one for W. Since $\mathsf{codim}(W \setminus U) \geq i$, then for $j \leq i-1$ we have $U^{(j)} = W^{(j)}$, so the coniveau filtration in the respective spectral sequences identifies :

$$\mathrm{H}^j_{\text{ét}}(U, \mathbb{Z}/n) \xleftarrow{\sim} \mathrm{H}^j_{\text{ét}}(W, \mathbb{Z}/n).$$

The latter being an affine space, we have $\mathrm{H}^j_{\text{ét}}(W, \mathbb{Z}/n) \simeq \mathrm{H}^j(k, \mathbb{Z}/n)$ by the homotopy invariance of étale cohomology[16]. Moreover, we have

$$\mathrm{H}^j(k, \mathbb{Z}/n) \simeq \begin{cases} 0 & \text{if } j \neq 0 \\ \mathbb{Z}/n(k) & \text{if } j = 0 \end{cases},$$

so in the associated Hochschild-Serre spectral sequence

$$E_2^{p,q} = \mathrm{H}^p(G, \mathrm{H}^q_{\text{ét}}(U, \mathbb{Z}/n) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(U/G, \mathbb{Z}/n),$$

all of the $E_2^{p,q}$-terms are zero for $q \leq i-1$. For $j \leq i-1$, if we thus look at $E_2^{0,j} = \mathrm{H}^0(G, \mathrm{H}^j(k, \mathbb{Z}/n) = 0$, then the differentials coming to and from this term are both zero, so that $E_2^{0,j} = E_\infty^{0,j} = H^j_{\text{ét}}(U/G, \mathbb{Z}/n)/F^1\, \mathrm{H}^j_{\text{ét}}(U/G, \mathbb{Z}/n)$, and thus $H^j_{\text{ét}}(U/G, \mathbb{Z}/n) = F^1\, \mathrm{H}^j_{\text{ét}}(U/G, \mathbb{Z}/n)$. Applying the same argument inductively, we obtain that

$$\mathrm{H}^j_{\text{ét}}(U/G, \mathbb{Z}/n) \xrightarrow{\sim} F^j\, \mathrm{H}^j_{\text{ét}}(U/G, \mathbb{Z}/n) \xrightarrow{\sim} E_\infty^{j,0} = E_2^{j,0} = \mathrm{H}^j(G, \mathbb{Z}/n),$$

which is precisely the desired isomorphism. □

2.1.2. No-name lemma, equivariant cycles and Chern classes

Proposition 2.2.9. *If k is an algebraically closed field and G is a finite group, then $\mathrm{CH}^2_G(k)$ is canonically isomorphic to the group $\mathrm{H}^3_n(G, \mathbb{Q}/\mathbb{Z}(2))$ of totally k-negligible classes in $\mathrm{H}^3(G, \mathbb{Q}/\mathbb{Z}(2))$.*

Lemma 2.2.10 (No-name lemma)**.** *Let G be a finite étale group scheme over a field k of characteristic 0, let V, W be two faithful representations of G over k, and let $X := \mathbf{SL}_{m,k}/G$ and $Y := \mathbf{SL}_{n,k}/G$ be the two corresponding geometric quotients. Then $\Bbbk(X)$ and $\Bbbk(Y)$ are stably birational.*

Proof. Let us choose two open subsets U and U' of $\mathbf{SL}_{m,k}$ and $\mathbf{SL}_{n,k}$ respectively, on which G acts freely. By making G act diagonally on $U \times_k \mathbf{SL}_{n,k}$, we obtain a cartesian diagram:

$$\begin{tikzcd} U \times_k \mathbf{SL}_{n,k} \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & (U \times_k \mathbf{SL}_{n,k})/G \arrow[d] \\ U \arrow[r] & U/G \end{tikzcd}.$$

As in the proof of Proposition (1.1.21), we use the fact that $U \to U/G$ is faithfully flat of finite presentation, so that $U \times_k \mathbf{SL}_{n,k} \to U$ induces a vector bundle $(U \times_k \mathbf{SL}_{n,k})/G \to U/G$ (since it is trivial for the flat topology, then it is also Zariski-trivial by Grothendieck's Hilbert 90). Therefore, U/G is stably birational to $(U \times_k \mathbf{SL}_{n,k})/G$, and $(U \times_k U')/G$ is an open subset of the latter. Replacing U by U' and $\mathbf{SL}_{n,k}$ by $\mathbf{SL}_{m,k}$ and mimicking the arguments, we obtain the

[16] See *e.g.* [Mil80, Chap. VI, Cor. 4.20] ; one can alternatively consider the Kummer sequence on the étale site of W and the similar one for $\mathsf{Spec}\, k$ and compare the long exact cohomology sequences.

desired result. □

This result holds in particular when G is an abstract group. An interesting consequence in this case is the following: if L/k is a finite Galois extension such that $\mathsf{Gal}(L/k) = G$ and E is a k-vector space endowed with a semi-linear action of G, then Speiser's lemma [GS17, Lem. 2.3.8] shows that $L \otimes_k E^G \simeq E$. In particular, $L(E)^G/k$ is purely transcendental. Indeed, the previous remark shows that without loss of generality, one can write $L(E) = L(t_1, \cdots, t_n)$ where $n := \dim_k(E^G)$ and G acts trivially on the t_i's; we thus obtain that $L(E)^G = k(t_1, \ldots, t_n)$.

Proof of Proposition (2.2.9). Choose a faithful complex representation W of G together with an open subset U on which G acts freely and such that $\mathsf{codim}_W(W \setminus U) \geq 4$. Fix a field extension L/k and a morphism $\rho : \Gamma_L \to G$, and assume it without loss of generality to be surjective. Let K be the field fixed by $\ker \rho$, so that $K(W)^G$ is purely transcendental (by the No-name lemma and Speiser's lemma) and the natural map

$$\mathrm{H}^3(\Gamma_L, \mathbb{Q}/\mathbb{Z}(2)) \longrightarrow \mathrm{H}^3(\mathsf{Gal}(L(W)_s/K(W)^G), \mathbb{Q}/\mathbb{Z}(2))$$

is injective. We obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{H}^3(G, \mathbb{Q}/\mathbb{Z}(2)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{H}^3(\Gamma_L, \mathbb{Q}/\mathbb{Z}(2)) \arrow[d] \\
\mathrm{H}^3(\mathsf{Gal}(k(W)_s/k(W)^G), \mathbb{Q}/\mathbb{Z}(2)) \arrow[r] & \mathrm{H}^3(\mathsf{Gal}(L(W)_s/K(W)^G), \mathbb{Q}/\mathbb{Z}(2))
\end{tikzcd}.$$

By [Pey99, Thm. 2.3.1], we have an exact sequence

$$(\mathsf{Pic}(X) \otimes k)^G \longrightarrow \ker[\mathrm{H}^3(G, \mathbb{Q}/\mathbb{Z}(2)) \to \mathrm{H}^3(k(W)^G, \mathbb{Q}/\mathbb{Z}(2))] \longrightarrow \mathrm{CH}^2_G(k)_{\mathrm{tors}} \longrightarrow \mathrm{H}^1(G, \mathsf{Pic}(X) \otimes k^\times).$$

On the other hand, both side terms are zero since $X = U/G$, and by restriction-corestriction, we get that $\mathrm{CH}^2_G(k) = \mathrm{CH}^2(X)$ is torsion annihilated by $\#G$ (for more detail about this argument see the proof of Proposition (2.2.13)). As a consequence, we obtain the desired isomorphism. □

Lemma 2.2.11. *Let X be a smooth variety over a field k. Then $\mathrm{CH}^2(X)$ is generated by the second Chern classes of algebraic vector bundles over X of determinant 1. In particular, if G is a finite group, then the group $\mathrm{CH}^2_G(\mathbb{C})$ is generated by the Chern classes of complex representations of G.*

Proof. For $i \geq 1$, consider the composition

$$\mathrm{CH}^i(X) \xrightarrow{\mathrm{Cl}^i} \mathrm{Fil}^i_{\mathrm{cod}} K_0(X) \xrightarrow{c_i} \mathrm{CH}^i(X)$$

where Cl^i is the i^{th} K-theoretic cycle class map, $\mathrm{Fil}^\bullet_{\mathrm{cod}}$ denotes the codimension filtration on $K_0(X)$ in the sense of [BGI71] and c_i is the i^{th} Chern class with values in the Chow group. By Riemann-Roch without denominators (see [Ful98, Thm. 15.3]), we have that

$$c_i \circ \mathrm{Cl}^i = (-1)^{i-1}(i-1)! \cdot \mathrm{Id}_{\mathrm{CH}^i(X)},$$

so that for $i = 2$, this composition is minus the identity. In order to obtain vector bundles of determinant 1, one just has to replace a bundle E by $E \oplus \det(E)$. The claim for an arbitrary X follows. Now if G is a finite group, fix a faithful complex representation W of G and an open subset $U \subset W$ on which G acts freely and whose complement has sufficiently large codimension. As proved by Merkurjev in [Mer99, Cor. 6.5] (the argument is similar to the one provided in the proof of Proposition (1.1.21) from Chapter I), one has a natural surjective morphism of groups

$$\mathcal{R}(G) \twoheadrightarrow K_0(U/G)$$

where $\mathscr{R}(G)$ denotes the representation ring of G[17]. Therefore we get that $\mathrm{CH}^2_G(\mathbb{C}) = \mathrm{CH}^2(U/G)$ is generated by the Chern classes of special complex representations of G, as desired. □

Remark 2.2.12. In 1990, Totaro provided a slightly modified version of this statement, see [Tot14, Thm. 5.1], and even bounded in some cases the number of generators of the full equivariant Chow ring of G as in [Tot14, Chap. 5, §5.2] (note that these results hold for any finite group scheme).

2.2. Proof of the main results

2.2.1. The main exact sequence

— The proof of Peyre's main result revolves around the following short exact sequence (which was already discussed in the proof of [Pey99, Thm. 2.3.1] and was initially discovered by Kahn in the more general context of Lichtenbaum's motivic complexes) :

Proposition 2.2.13. *Let G be a finite group and W a faithful complex representation of G, U an open subset of W on which G acts freely and such that $\mathsf{codim}_W(W \setminus U) \geq 4$. Then there is a canonical exact sequence*

$$0 \longrightarrow \mathrm{CH}^2_G(\mathbb{C}) \longrightarrow \mathrm{H}^3(G, \mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{H}^0(U/G, \mathscr{H}^3_{\acute{e}t}(\mathbb{Q}/\mathbb{Z}(2))) \longrightarrow 0.$$

Proof. Write $X := U/G$. Consider the Leray spectral sequence attached to $\mathrm{Id} : X_{\acute{e}t} \to X_{\mathrm{Zar}}$ and the étale sheaf $\mathbb{Q}/\mathbb{Z}(2)$:

$$E_2^{p,q} = \mathrm{H}^p_{\mathrm{Zar}}(X, \mathscr{H}^q_{\acute{e}t}(\mathbb{Q}/\mathbb{Z}(2))) \Rightarrow \mathrm{H}^{p+q}_{\acute{e}t}(X, \mathbb{Q}/\mathbb{Z}(2)).$$

By Bloch-Ogus theory, and since X is smooth, the coniveau filtration shows that the $E_2^{p,q}$-terms vanish for $p > q$; in particular, any differential coming from a term in position $(p-1, p)$ is zero. Notably, the term in position $(1, 2)$ also has no incoming differential, so that $E_2^{1,2} = E_\infty^{1,2}$ which provides an injection $E_2^{1,2} \hookrightarrow \mathrm{H}^3_{\acute{e}t}(X, \mathbb{Q}/\mathbb{Z}(2))$. The spectral sequence also provides an exact sequence of lower terms

$$0 \longrightarrow E_\infty^{0,3} \longrightarrow E_2^{0,3} \longrightarrow E_2^{2,2} \longrightarrow E_\infty^{2,2} \longrightarrow 0,$$

where $E_2^{2,2} \simeq \mathrm{CH}^2(X) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$ by Bloch's formula. Combining these two facts, we obtain an exact sequence :

$$\begin{aligned} 0 \longrightarrow \mathrm{H}^1_{\mathrm{Zar}}(X, \mathscr{H}^2_{\acute{e}t}(\mathbb{Q}/\mathbb{Z}(2))) \longrightarrow \mathrm{H}^3_{\acute{e}t}(X, \mathbb{Q}/\mathbb{Z}(2)) &\longrightarrow \mathrm{H}^0(X, \mathscr{H}^3_{\acute{e}t}(\mathbb{Q}/\mathbb{Z}(2))) \\ \longrightarrow \mathrm{CH}^2(X) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} &\longrightarrow \mathrm{H}^4_{\acute{e}t}(X, \mathbb{Q}/\mathbb{Z}(2)). \end{aligned}$$

On the other hand, Proposition (1.4.84) shows that Bloch's complex gives the short exact sequence (after taking direct limits)

$$0 \longrightarrow \mathrm{H}^1_{\mathrm{Zar}}(X, \mathscr{K}_2) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} \longrightarrow \mathrm{H}^1_{\mathrm{Zar}}(X, \mathscr{H}^2_{\acute{e}t}(\mathbb{Q}/\mathbb{Z}(2))) \longrightarrow \mathrm{CH}^2(X)_{\mathrm{tors}} \longrightarrow 0.$$

Note that since $X = U/G$, then $\mathrm{CH}^2(X) = \mathrm{CH}^2_G(\mathbb{C})$. Now the localisation exact sequence for Chow groups yields

$$\mathrm{CH}^2(W \setminus U) \longrightarrow \mathrm{CH}^2(W) \longrightarrow \mathrm{CH}^2(U) \longrightarrow 0,$$

and since we assumed $\mathsf{codim}_W(W \setminus U) \geq 4$, we obtain that $\mathrm{CH}^2(U) = \mathrm{CH}^2(W) = 0$ (W is an affine space, so we have $\mathrm{CH}^i(W) = 0$ for $i \neq 0$, see *e.g.* [Ful98, Chap. 1, §1.9]). Moreover, Lemma (2.2.7) shows that

$$\mathrm{H}^1_{\mathrm{Zar}}(U, \mathscr{K}_2) = \mathrm{H}^1_{\mathrm{Zar}}(W, \mathscr{K}_2) = 0$$

[17]Indeed, the category $\underline{\mathbf{Rep}}_{\mathbb{C}}(G)$ of complex representations of G is abelian and monoidal, so it admits a Grothendieck ring $\mathscr{R}(G) = K_0(\underline{\mathbf{Rep}}_{\mathbb{C}}(G))$.

(the vanishing follows from the fact that W is affine of dimension ≥ 4 and a direct calculation with the BGQ spectral sequence $E_1^{p,q} = \bigoplus_{x\in W^{(p)}} K_{-p-q}(\kappa(x)) \Rightarrow K_{-p-q}(W)$). Now let $\pi : U \to X$ be the projection map and π^* and π_* be the induced restriction and the corestriction maps on cohomology and Chow groups respectively. We have that the compositions

$$\mathrm{H}^1_{\mathrm{Zar}}(X, \mathcal{K}_2) \xrightarrow{\pi^*} \mathrm{H}^1_{\mathrm{Zar}}(U, \mathcal{K}_2) \xrightarrow{\pi_*} \mathrm{H}^1_{\mathrm{Zar}}(X, \mathcal{K}_2)$$

and

$$\mathrm{CH}^2(X) \xrightarrow{\pi^*} \mathrm{CH}^2(U) \xrightarrow{\pi_*} \mathrm{CH}^2(X)$$

coincide with the multiplication by $\#G$, see *e.g.* [Ros96, p. 330] (indeed, the projection map $U \to X$ is finite étale, so this is more or less a projection formula). On the other hand, the vanishing of $\mathrm{H}^1_{\mathrm{Zar}}(U, \mathcal{K}_2)$ and $\mathrm{CH}^2(U)$ show that the above composite maps are zero. This implies that $\mathrm{CH}^2(X)$ and $\mathrm{H}^1_{\mathrm{Zar}}(X, \mathcal{K}_2)$ are torsion groups annihilated by $\#G$. In particular,

$$\mathrm{CH}^2(X) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = 0 \quad \text{and} \quad \mathrm{H}^1_{\mathrm{Zar}}(X, \mathcal{K}_2) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = 0.$$

As a consequence, we respectively obtain a short exact sequence

$$0 \longrightarrow \mathrm{H}^1_{\mathrm{Zar}}(X, \mathcal{H}^2_{\text{ét}}(\mathbb{Q}/\mathbb{Z}(2))) \longrightarrow \mathrm{H}^3_{\text{ét}}(X, \mathbb{Q}/\mathbb{Z}(2)) \longrightarrow \mathrm{H}^0(X, \mathcal{H}^3_{\text{ét}}(\mathbb{Q}/\mathbb{Z}(2))) \longrightarrow 0$$

and an isomorphism

$$\mathrm{H}^1_{\mathrm{Zar}}(X, \mathcal{H}^2_{\text{ét}}(\mathbb{Q}/\mathbb{Z}(2))) \xrightarrow{\sim} \mathrm{CH}^2_G(\mathbb{C}).$$

Finally, Proposition (2.2.8) shows that $\mathrm{H}^3_{\text{ét}}(X, \mathbb{Q}/\mathbb{Z}(2)) \simeq \mathrm{H}^3(G, \mathbb{Q}/\mathbb{Z})$, hence the desired short exact sequence. □

2.2.2. Analysing image and inverse image of neglibible and unramified classes

— The goal from now on is to describe, in the exact sequence of Proposition (2.2.13), the image of the injective map $\mathrm{CH}^2_G(\mathbb{C}) \to \mathrm{H}^3(G, \mathbb{Q}/\mathbb{Z})$ and the inverse image of $\mathrm{H}^3_{\mathrm{nr}}(\mathbb{C}(W)^G/\mathbb{C}, \mathbb{Q}/\mathbb{Z}(2)) \subset \mathrm{H}^0(U/G, \mathcal{H}^3_{\text{ét}}(\mathbb{Q}/\mathbb{Z}(2)))$ in $\mathrm{H}^3(G, \mathbb{Q}/\mathbb{Z})$.

2.2.2.1. Almost every permutation abstract unramified class is permutation neglibible. We begin by determining the image of $\mathrm{CH}^2_G(\mathbb{C}) \to \mathrm{H}^3(G, \mathbb{Q}/\mathbb{Z})$: the main idea is to compare this map to the cycle class from a purely scheme theoretic point of view, so that we can relate it to the Chern class and use some intersection theory.

Proposition 2.2.14. *If G is a finite group, then the prime-to-2 part of* $\mathrm{H}^3_{\mathrm{n}}(G, \mathbb{Q}/\mathbb{Z})$ *is contained in* $\mathrm{H}^3_{\mathrm{p}}(G, \mathbb{Q}/\mathbb{Z})$.

Proof. Let p be a prime factor of $\#G$ and G_p a p-Sylow subgroup of G. By definition, we have

$$\mathrm{Cores}^G_{G_p}(\mathrm{H}^3_{\mathrm{p}}(G_p, \mathbb{Q}/\mathbb{Z})) \subset \mathrm{H}^3_{\mathrm{p}}(G, \mathbb{Q}/\mathbb{Z}).$$

Moreover, we have a commutative diagram :

$$\begin{tikzcd}
 & \mathrm{H}^3(G, \mathbb{Q}/\mathbb{Z}) \arrow[r] \arrow[d, "\mathrm{Res}^G_{G_p}"] & \mathrm{H}^3(\mathbb{C}(W)^G, \mathbb{Q}/\mathbb{Z}) \arrow[d, "\mathrm{Res}^G_{G_p}"] \\
\mathrm{H}^3_{\mathrm{p}}(G_p, \mathbb{Q}/\mathbb{Z}) \arrow[r] \arrow[d, "\mathrm{Cores}^G_{G_p}"] & \mathrm{H}^3(G_p, \mathbb{Q}/\mathbb{Z}) \arrow[r] \arrow[d, "\mathrm{Cores}^G_{G_p}"] & \mathrm{H}^3(\mathbb{C}(W)^{G_p}, \mathbb{Q}/\mathbb{Z}) \arrow[d, "\mathrm{Cores}^G_{G_p}"] \\
\mathrm{H}^3_{\mathrm{p}}(G, \mathbb{Q}/\mathbb{Z}) \arrow[r] & \mathrm{H}^3(G, \mathbb{Q}/\mathbb{Z}) \arrow[r] & \mathrm{H}^3(\mathbb{C}(W)^G, \mathbb{Q}/\mathbb{Z})
\end{tikzcd} .$$

By [Pey07, Rmk. 6], the group $\mathrm{H}^3_{\mathrm{p}}(G, \mathbb{Q}/\mathbb{Z})$ is contained in $\mathrm{H}^3_{\mathrm{n}}(G, \mathbb{Q}/\mathbb{Z})$, so the middle and bottom rows are complexes. Suppose that the middle row is exact. Since $\mathrm{Cores}^G_{G_p} \circ \mathrm{Res}^G_{G_p} = (G : G_p) \cdot \mathrm{Id}$, then a quick diagram chasing shows that the bottom row is exact on the p-primary torsion subgroups. Therefore, we only need to prove the claim when G is a p-group.

We now claim that the map $\partial : \mathrm{CH}^2_G(\mathbb{C}) \to \mathrm{H}^3(G, \mathbb{Q}/\mathbb{Z})$ appearing in the exact sequence of the previous proposition coincides with the cycle class $c\ell_2 : \mathrm{CH}^2(X) \to \mathrm{H}^3_{\text{ét}}(X, \mathbb{Q}/\mathbb{Z}(2))$. Indeed, it is proved in [CTSS83, Cor. 1, p.772][18] that we have a diagram that commutes up to sign

$$\begin{tikzcd}
\mathrm{CH}^2(X)_{\mathrm{tors}} \arrow[rr, "c\ell_2"] & & \mathrm{H}^4_{\text{ét}}(X, \widehat{\mathbb{Z}}(2)) \\
& \circlearrowleft & \\
\mathrm{H}^1_{\mathrm{Zar}}(X, \mathcal{H}^2_{\text{ét}}(\mathbb{Q}/\mathbb{Z}(2))) \arrow[uu] \arrow[rr] & & \mathrm{H}^3_{\text{ét}}(X, \mathbb{Q}/\mathbb{Z}(2)) \arrow[uu]
\end{tikzcd}$$

where the left vertical map is constructed using Bloch's complex and the Merkurjev-Suslin theorem as in Chapter I, §4.3, the bottom map comes from Bloch-Ogus theory applied to the Leray spectral sequence

$$E_2^{p,q} = \mathrm{H}^p_{\mathrm{Zar}}(X, \mathcal{H}^q_{\text{ét}}(\mathbb{Q}/\mathbb{Z}(2))) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(X, \mathbb{Q}/\mathbb{Z}(2))$$

(as in the proof of the previous proposition), and the right vertical map is the limit of the Bockstein morphisms in the long exact sequences attached to the exact sequences of étale sheaves

$$0 \to \mathbb{Z}/m(i) \to \mathbb{Z}/mn(i) \to \mathbb{Z}/n(i) \to 0$$

for $m, n \geq 1$ (direct limit on n and inverse limit on m). But it is a well known fact that the cycle class commutes with the étale Chern class, *i.e.* in our case $c\ell_2 \circ c_2 = c_2$, see [Ful98, Prop. 19.1.2]. In particular, applying Proposition (2.2.9) we obtain that the group

$$\mathrm{H}^3_{\mathrm{n}}(G, \mathbb{Q}/\mathbb{Z}) = \ker[\mathrm{H}^3(G, \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^3(\mathbb{C}(W)^G, \mathbb{Q}/\mathbb{Z})] \simeq \mathrm{CH}^2_G(\mathbb{C})$$

must be generated by the second Chern classes of representations of G. For convenience we may now identify (isomorphism classes of) representations of G with the vector bundles that they induce in $K_0(X)$. If $x, y \in \mathcal{R}(G)$, let us write $x + y$ for the class of the direct sum representation. The Whitney formula [Ful98, Thm. 3.2, (e)] for the induced exact sequence of vector bundles yields

$$c_2(x+y) = c_2(x) + c_1(x) \smile c_1(y) + c_2(y) \in \mathrm{H}^3(G, \mathbb{Q}/\mathbb{Z}).$$

But by definition, $c_1(x) \smile c_1(y)$ is a permutation negligible class, so the second Chern class factors as a morphism of groups

$$\mathcal{R}(G) \xrightarrow{\bar{c}_2} \mathrm{H}^3(G, \mathbb{Q}/\mathbb{Z})/\mathrm{H}^3_{\mathrm{p}}(G, \mathbb{Q}/\mathbb{Z}).$$

The aim of the proof from now on is to show that this map is zero. We know by Brauer's main theorem that each character of G is an integral combination of characters induced by subgroups, see *e.g.* [Ser71, §10.5, Thm. 20]. On the other hand the representation category $\underline{\mathbf{Rep}}_{\mathbb{C}}(G)$ is semisimple and $\mathcal{R}(G)$ is generated as a group by the characters of G ; in particular, it is therefore generated by induced characters of subgroups of G, so we just need to show that for any subgroup H of G and any character χ on H of dimension 1, we have $c_2(\mathrm{Ind}_H^G(\chi)) \in \mathrm{H}^3_{\mathrm{p}}(G, \mathbb{Q}/\mathbb{Z})$.

In [FM87, §I.3], Fulton and MacPherson constructed for any finite étale covering of k-schemes $f : X \to Y$ (here we suppose for convenience that $\mathrm{char}(k) = 0$) some *higher transfer morphisms* $f_*^{(n)} : \mathrm{H}^i_{\text{ét}}(X, \mathbb{Z}) \to \mathrm{H}^{in}_{\text{ét}}(Y, \mathbb{Z})$ for $n \geq 1$, satisfying *Newton formulæ*, that is, for any $z \in \mathrm{H}^i_{\text{ét}}(X, \mathbb{Z})$, we have

$$f_*(z^n) - f_*^{(1)}(z) \smile f_*(z^{n-1}) + \ldots + (-1)^n n f_*^{(n)}(z) = 0.$$

[18] Altough this result is only stated here, we would like to point out that its verification is quite technical and far for being trivial.

In our case, any subgroup H of G induces a finite étale covering $f : U/H \to U/G$, and hence it yields transfer morphisms

$$\mathrm{Cores}^{(n)} : \mathrm{H}^i(H, \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^{n(i+1)-1}(G, \mathbb{Q}/\mathbb{Z})^{[19]}.$$

But by [FM87, Cor. 5.3], for any e-dimensional representation class $x \in \mathscr{R}(G)$, the first and second Chern classes satisfy the following identity :

$$\begin{aligned} c_2(\mathrm{Ind}_H^G(x)) = c_2(f_*x) &= f_*c_2(x) + f_*^{(2)}c_1(x) + ec_1(V) \smile f_*c_1(x) + c_2(V^{\oplus e}) \\ &= \mathrm{Cores}(c_2(x)) + \mathrm{Cores}^{(2)}(c_1(x)) + e \cdot c_1(\mathrm{Ind}_H^G(1)) \smile \mathrm{Cores}(c_1(x)) + c_2(\mathrm{Ind}_H^G(1)^{\oplus e}) \end{aligned}$$

where V is the trivial representation. In the particular case where $x = \chi$ is (the class of) a character, we obtain that

$$c_2(\mathrm{Ind}_H^G(\chi)) = \mathrm{Cores}(c_2(\chi)) + \mathrm{Cores}^{(2)}(c_1(\chi)) + c_1(\mathrm{Ind}_H^G(1)) \smile \mathrm{Cores}(c_1(\chi)) + c_2(\mathrm{Ind}_H^G(1)).$$

First note that since χ is a representation of dimension 1, then $c_2(\chi) = 0$, *a fortiori* $\mathrm{Cores}(c_2(\chi)) = 0$. On the other hand if $p \neq 2$, then from the identity

$$\mathrm{Cores}(c_2(\chi)^2) - \mathrm{Cores}(c_2(\chi))^2 + 2\mathrm{Cores}^{(2)}(c_2(\chi)) = 0$$

we get that $\mathrm{Cores}^{(2)}(c_2(\chi)) = (\mathrm{Cores}(c_2(\chi))^2 - \mathrm{Cores}(c_2(\chi)^2))/2$. Hence,

$$c_2(\mathrm{Ind}_H^G(\chi)) = \frac{1}{2}(\mathrm{Cores}(c_1(\chi))^2 - \mathrm{Cores}(c_1((\chi^2))) + c_1(\mathrm{Ind}_H^G(1)) \smile \mathrm{Cores}(c_1(\chi)) + c_2(\mathrm{Ind}_H^G(1))$$

and it is clear that the first three cup-products in the right hand side are permutation negligible classes. Therefore we only need to prove that $c_2(\mathrm{Ind}_H^G(1))$ is permutation negligible as well.

Now since G is a p-group, we shall proceed by induction on $(G : H)$. The case $(G : H) = 1$ is direct since $c_2(1) = 0$. If $(G : H) \geq 2$, write $(G : H) = p^m$ for some $m \geq 1$ and assume that the claim is true for any subgroup of G of index stricly smaller than p^m. One can always find a subgroup H_1 of G such that H is normal in H_1 of index p. Indeed by [Suz82, Chap. 2, Thm. 1.6], we know that any proper subgroup of G is a proper normal subgroup of its normaliser ; if we take H_1 to be the normaliser of H in G, then either H is maximal in H_1 and thus H_1/H must be cyclic of order p since it contains non non-trivial subgroup, or there exists a subgroup of H_1 in which H is maximal for the inclusion relation. This gives the desired H_1. On the other hand, we have

$$c_2(\mathrm{Ind}_H^G(1)) = c_2(\mathrm{Ind}_{H_1}^G(\mathrm{Ind}_H^{H_1}(1))).$$

Since H is normal in the finite group H_1, we may choose a character $\chi : H_1 \to \mathbb{C}^\times$ such that $H = \ker\chi$. In particular we have an isomorphism of H_1-modules $\mathrm{Ind}_H^{H_1}(1) \simeq \mathbb{C}[H_1/H]$; moreover the class of $\mathbb{C}[H_1/H]$ in $\mathscr{R}(H_1)$ is given by

$$\mathbb{C}[H_1/H] = 1 + \chi + \chi^2 + \ldots + \chi^{p-1}.$$

Applying the Whitney formula to the induced chain of sequences of vector bundles on X, we get that

$$\begin{aligned} c_2(\mathrm{Ind}_H^G(1)) &= c_2(\mathrm{Ind}_{H_1}^G(1)) + \ldots + \mathrm{Ind}_{H_1}^G(\chi^{p-1})) \\ &\equiv c_2(\mathrm{Ind}_{H_1}^G(1)) + \ldots + c_2(\mathrm{Ind}_{H_1}^G(\chi^{p-1})) \mod \mathrm{H}^3_{\mathrm{p}}(G, \mathbb{Q}/\mathbb{Z}). \end{aligned}$$

But by the induction hypothesis, we know that all of these classes are permutation negligible, which concludes the proof. □

[19] Where we identified $\mathrm{H}^i(H, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^{i+1}(H, \mathbb{Z}) \xrightarrow{f_*^{(n)}} \mathrm{H}^{n(i+1)}(G, \mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^{n(i+1)-1}(G, \mathbb{Q}/\mathbb{Z})$. For $n = 1$, the map $\mathrm{Cores}^{(1)} = \mathrm{Cores}$ is just the usual corestriction.

2.2.2.2. Every class in $\mathrm{H}^3_{\mathrm{nr}}(\mathbb{C}(W)^G/\mathbb{C}, \mathbb{Q}/\mathbb{Z}(2))$ arises from a class in $\mathrm{H}^3_{\mathrm{nr}}(G, \mathbb{Q}/\mathbb{Z})$. It now remains to determine the inverse image of $\mathrm{H}^3_{\mathrm{nr}}(\mathbb{C}(W)^G/\mathbb{C}, \mathbb{Q}/\mathbb{Z}(2))$ in $\mathrm{H}^3(G, \mathbb{Q}/\mathbb{Z})$ along the exact sequence of Proposition (2.2.13).

For convenience let us fix some notations by writing $k := \mathbb{C}(W)^G$ and $L := \mathbb{C}(W)$. Let $\rho : \mathsf{Gal}(\overline{L}/k) \to G$ be the natural surjection and $\gamma \in \mathrm{H}^3_{\mathrm{nr}}(G, \mathbb{Q}/\mathbb{Z})$. We would like to determine whether $\rho^*(\gamma) \in \mathrm{H}^3(\mathbb{C}(W)^G, \mathbb{Q}/\mathbb{Z}(2))$ is unramified. Let A and B be rank one discrete valuation rings containing $\mathbb{C}$ with respective fraction fields $\mathbb{C}(W)^G$ and $\mathbb{C}(W)$, and such that $B \cap \mathbb{C}(W)^G = A$.

Let $\widehat{k}_A$ be the completion of k at A and $\widehat{L}_B$ the completion of L at B, $\overline{L}_B$ an algebraic closure of $\widehat{L}_B$ and $\widehat{k}_A^{\mathrm{nr}}$ and $\widehat{L}_B^{\mathrm{nr}}$ the maximal unramified extensions of $\widehat{k}_A$ and $\widehat{L}_B$ in $\overline{L}_B$ respectively. We denote by $\mathscr{D}$ the decomposition group of B in G and $\mathscr{I}$ its inertia group. We also put $\mathscr{G}_A := \mathsf{Gal}(\overline{L}_B/\widehat{k}_A)$ and $\mathscr{G}_B := \mathsf{Gal}(\overline{L}_B/\widehat{L}_B)$, and $\mathscr{I}_A$ and $\mathscr{I}_B$ the corresponding inertia groups. We obtain a commutative diagram with (tautological) exact rows :

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{I}_A \arrow[r, "j_A"] \arrow[d, two heads, "f_{\mathscr{I}}"] & \mathscr{G}_A \arrow[r, "\pi_A"] \arrow[d, two heads, "f_{\mathscr{G}}"] & \mathscr{G}_A/\mathscr{I}_A \arrow[r] \arrow[d, two heads] & 0 \\ 0 \arrow[r] & \mathscr{I} \arrow[r, "j"'] & \mathscr{D} \arrow[r, "\pi"'] & \mathscr{D}/\mathscr{I} \arrow[r] & 0 \end{tikzcd} \quad . \tag{II.1}$$

Since $\mathscr{I}_A \simeq \widehat{\mathbb{Z}}(1)$ has cohomological dimension equal to 1 (*cf.* [GS17, Prop. 6.1.3]), then the Hochschild-Serre spectral sequence

$$E_2^{p,q} = \mathrm{H}^p(\mathscr{G}_A/\mathscr{I}_A, \mathrm{H}^q(\mathscr{I}_A, \mathbb{Q}/\mathbb{Z}(2)) \Rightarrow \mathrm{H}^{p+q}(\mathscr{G}_A, \mathbb{Q}/\mathbb{Z}(2))$$

has nonzero terms only in the rectangle $\{E_2^{p,q} \mid 0 \le q \le 1\}$. Moreover, by Kummer theory then for any $j \ge 1$ we have an isomorphism of Galois modules $\mathrm{H}^1(\mathscr{I}_A, \mathbb{Q}/\mathbb{Z}(j)) \simeq \mathsf{Hom}(\mathscr{I}_A, \mathbb{Q}/\mathbb{Z}(j)) \simeq \mathbb{Q}/\mathbb{Z}(j-1)$. We obtain a map

$$\mathrm{H}^3(\widehat{k}_A, \mathbb{Q}/\mathbb{Z}(2)) \longrightarrow \mathrm{H}^2(\mathscr{G}_A/\mathscr{I}_A, \mathrm{H}^1(\mathscr{I}_A, \mathbb{Q}/\mathbb{Z}(2)),$$

hence a well-defined residue map

$$\partial_A : \mathrm{H}^3(k, \mathbb{Q}/\mathbb{Z}(2)) \longrightarrow \mathrm{H}^2(\kappa_A, \mathbb{Q}/\mathbb{Z}(1))$$

as the composite

$$\mathrm{H}^3(k, \mathbb{Q}/\mathbb{Z}(2)) \longrightarrow \mathrm{H}^3(\widehat{k}_A, \mathbb{Q}/\mathbb{Z}(2)) \longrightarrow \mathrm{H}^2(\mathscr{G}_A/\mathscr{I}_A, \mathrm{H}^1(\mathscr{I}_A, \mathbb{Q}/\mathbb{Z}(2)) \xrightarrow{\sim} \mathrm{H}^2(\kappa_A, \mathbb{Q}/\mathbb{Z}(1))$$

(where the first map is the usual corestriction $\mathrm{Cores}_k^{\widehat{k}_A}$).

Proposition 2.2.15. *The group $\mathrm{H}^3_{nr}(G, \mathbb{Q}/\mathbb{Z})$ is precisely the inverse image of $\mathrm{H}^3_{nr}(\mathbb{C}(W)^G/\mathbb{C}, \mathbb{Q}/\mathbb{Z}(2))$ in $\mathrm{H}^3(G, \mathbb{Q}/\mathbb{Z})$.*

Proof. Since $\mathbb{C}$ contains all the roots of unity, then the extension

$$0 \longrightarrow \mathscr{I}_A \longrightarrow \mathscr{G}_A \longrightarrow \mathscr{G}_A/\mathscr{I}_A \longrightarrow 0 \tag{II.2}$$

is central. On the other hand $\widehat{k}_A$ is isomorphic to the field of Laurent series $\kappa_A(\!(t)\!)$ and we have an isomorphism of field extensions

$$\widehat{k}_A^{\mathrm{nr}}/\widehat{k}_A \simeq \varinjlim_{\ell/\kappa_A} \ell(\!(t)\!)/\kappa_A(\!(t)\!),$$

where ℓ runs among the finite separable extensions of κ_A[20] (see *e.g.* [Ser80, Chap. II, §4, Thm. 2]). Similarly, since the residue field of $\widehat{k}_A$ is algebraically closed of characteristic zero, then Puiseux's theorem [Ser80, Chap. IV, §2, Prop. 8] shows that

$$\overline{k_A} \simeq \varinjlim_{n \ge 1} \widehat{k}_A^{\mathrm{nr}}(t^{1/n}).$$

[20] One should be careful here because the extension $\widehat{k}_A^{\mathrm{nr}}/\widehat{k}_A$ is not complete in general.

Therefore, the central extension (II.2) splits[21]. Now by the commutativity of the diagram (II.1) and the surjectivity of its vertical arrows, we get that $\mathscr{I}$ is central in $\mathscr{D}$ as well. If we denote by r a retraction for j_A, then we get a commutative diagram

$$\begin{tikzcd}
\mathscr{G}_A \arrow[r, "(\mathrm{Id}-r)\times r"] \arrow[d, "f_{\mathscr{G}}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathscr{G}_A \times \mathscr{I}_A \arrow[d, "f_{\mathscr{G}}\times f_{\mathscr{I}}"] \\
\mathscr{D} & \mathscr{D}\times\mathscr{I} \arrow[l, "di \leftarrow\!\shortmid (d,i) : m"]
\end{tikzcd}$$

Let $\psi := (f_{\mathscr{G}} \times f_{\mathscr{I}}) \circ (\mathrm{Id} - r) \times r$. The factorisation $f_{\mathscr{G}} = m \circ \psi$ then fits into a bigger commutative diagram with exact rows :

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{I}_A \arrow[r, "j_A"] \arrow[d, "f_{\mathscr{I}}"] & \mathscr{G}_A \arrow[r, "\pi_1"] \arrow[d, "\psi"] & \mathscr{G}_A/\mathscr{I}_A \arrow[r] \arrow[d, dashed, "\tau"] & 0 \\
0 \arrow[r] & \mathscr{I} \arrow[r, "i_2"] \arrow[d, equal] & \mathscr{D}\times\mathscr{I} \arrow[r, "\mathsf{pr}_1"] \arrow[d, "m"] & \mathscr{D} \arrow[r] \arrow[d, "\pi"] & 0 \\
0 \arrow[r] & \mathscr{I} \arrow[r, "j"] & \mathscr{D} \arrow[r, "\pi"] & \mathscr{D}/\mathscr{I} \arrow[r] & 0
\end{tikzcd}, \tag{II.3}$$

where τ is the unique map that makes this diagram commute. Taking cohomology, we deduce a big commutative diagram :

$$\begin{tikzcd}
\mathrm{H}^3(\mathscr{D}\times\mathscr{I}, \mathbb{Q}/\mathbb{Z}) \arrow[r, "\psi^*"] \arrow[d, "s_{\mathscr{D},\mathscr{I}}"'] & \mathrm{H}^3(\widehat{k}_A, \mathbb{Q}/\mathbb{Z}(2)) \arrow[d, equal] \\
\ker[\mathrm{H}^3(\mathscr{D}\times\mathscr{I}, \mathbb{Q}/\mathbb{Z}) \xrightarrow{i_2^*} \mathrm{H}^3(\mathscr{I}, \mathbb{Q}/\mathbb{Z})] \arrow[r, "\psi^*"] \arrow[d] & \mathrm{H}^3(\widehat{k}_A, \mathbb{Q}/\mathbb{Z}(2)) \arrow[d] \\
\mathrm{H}^2(\mathscr{D}, \mathrm{H}^1(\mathscr{I}, \mathbb{Q}/\mathbb{Z})) \arrow[r] & \mathrm{H}^2(\kappa_A, \mathbb{Q}/\mathbb{Z}(1))
\end{tikzcd}.$$

Here, the map $s_{\mathscr{D},\mathscr{I}}$ is defined as in §I.2 and the left and right bottom vertical maps are given by the Hoschchild-Serre spectral sequences as discussed in §I.2 and earlier in the proof respectively. Let us give a brief explanation of the commutativity of this diagram. The top square commutes since $\psi^* \circ \mathsf{pr}_2^* = r^* \circ f_{\mathscr{I}}^* = 0$ (by commutativity of the previous diagram and the fact that $\mathrm{H}^3(\mathscr{I}_A, \mathbb{Q}/\mathbb{Z}) = 0$ since $\mathrm{cd}(\mathscr{I}_A) = 1$) ; for the bottom one, it comes from the definition of ψ and the functoriality of the Hochschild-Serre spectral sequence.

Now remark that the diagram (II.3) also provides a factorisation :

$$\psi^* \circ m^* \circ \mathrm{Res}^G_{\mathscr{D}} = \mathrm{Cores}^{\widehat{k}_A}_k \circ \rho^* : \mathrm{H}^3(G, \mathbb{Q}/Z) \longrightarrow \mathrm{H}^3(\mathscr{G}_A, \mathbb{Q}/\mathbb{Z}(2)).$$

Together with the above diagram and fixing a suitable generator g of $\mathscr{I}$, this yields a commutative diagram :

$$\begin{tikzcd}
\mathrm{H}^3(G, \mathbb{Q}/\mathbb{Z}) \arrow[r, "\partial_{\mathscr{D},g}"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{H}^2(\mathscr{D}, \mathbb{Q}/\mathbb{Z}) \arrow[d] \\
\mathrm{H}^3(k, \mathbb{Q}/\mathbb{Z}(2) \arrow[r, "\partial_A"'] & \mathrm{H}^2(\kappa_A, \mathbb{Q}/\mathbb{Z}(1))
\end{tikzcd}, \tag{II.4}$$

[21] If κ_A had positive characteristic p, we would always get a splitting since $\mathscr{G}_A/\mathscr{I}_A$ has p-cohomological dimension ≤ 1 by [Ser97, Chap. 2, §2, Prop. 3], so that $\mathsf{Ext}^1(\mathscr{G}_A/\mathscr{I}_A, \mathscr{I}_A) \simeq \mathrm{H}^2(\mathscr{G}_A/\mathscr{I}_A, \mathscr{I}_A) = 0$.

so that if a class $\gamma \in \mathrm{H}^3(G, \mathbb{Q}/\mathbb{Z})$ vanishes along the residue $\partial_{\mathcal{D},g}$, then its image in $\mathrm{H}^3(k, \mathbb{Q}/\mathbb{Z}(2)) = \mathrm{H}^3(\mathbb{C}(W)^G, \mathbb{Q}/\mathbb{Z}(2))$ must vanish along ∂_A. Since this holds for any choice of residue, we obtain that

$$\mathrm{H}^3_{\mathrm{nr}}(G, \mathbb{Q}/\mathbb{Z}) \subseteq (\rho^*)^{-1}(\mathrm{H}^3_{\mathrm{nr}}(\mathbb{C}(W)^G/\mathbb{C}, \mathbb{Q}/\mathbb{Z}(2))).$$

It therefore remains to show that the reverse inclusion also holds. Let $H \subset G$ be a subgroup. If $i \geq 0$ is an integer, we let

$$\mathrm{H}^i_{\mathrm{gnr}}(G, \mathbb{Q}/\mathbb{Z}) := (\rho^*)^{-1}(\mathrm{H}^i_{\mathrm{nr}}(\mathbb{C}(W)^G/\mathbb{C}, \mathbb{Q}/\mathbb{Z}(2)))$$

(the classes in this group are called *geometrically unramified*). We claim that for any morphism of groups $\varpi : H \to G$, the induced restrictions preserve geometrically unramified classes, that is,

$$\varpi^*(\mathrm{H}^i_{\mathrm{gnr}}(G, \mathbb{Q}/\mathbb{Z})) \subseteq \mathrm{H}^i_{\mathrm{gnr}}(H, \mathbb{Q}/\mathbb{Z}) \quad \forall i \geq 0.$$

Indeed, if we fix a faithful complex representation V of H, then W also yields a representation of H *via* ϖ, and $V \oplus W$ provides a faithful representation ; moreover the chain of field inclusions

$$\mathbb{C}(W)^G \subseteq \mathbb{C}(W)^H \subseteq \mathbb{C}(V \oplus W)^G$$

and the functoriality of unramified cohomology (see Proposition (I.5.93)) show that the image of $\mathrm{H}^i_{\mathrm{nr}}(\mathbb{C}(W)^G/\mathbb{C}, \mathbb{Q}/\mathbb{Z}(2))$ under the induced map $\mathrm{H}^i(\mathbb{C}(W)^G, \mathbb{Q}/\mathbb{Z}(2)) \to \mathrm{H}^i(\mathbb{C}(V \oplus W)^H, \mathbb{Q}/\mathbb{Z}(2))$ is contained in $\mathrm{H}^i_{\mathrm{nr}}(\mathbb{C}(V \oplus W)^H/\mathbb{C}, \mathbb{Q}/\mathbb{Z}(2))$, as desired. Now if we fix an element $g \in \mathcal{Z}_G(H)$ and $I := \langle g \rangle$, then by definition of the residue $\partial_{H,g}$ and the previous remark, we can restrict ourselves to the case where $G = H \times I$. As before, choose a faithful representation V of H and let $\chi : I \to \mathbb{C}^\times$ be the character given by sending the generator g to $\exp(2i\pi/\#I)$. We obtain that $V \oplus \chi$ is a faithful representation of G, and we can identify the field of functions $\mathbb{C}(V \oplus \chi)$ with $\mathbb{C}(V)(t)$ where t is an indeterminate. Taking $G = H \times I$-invariants thus yields $\mathbb{C}(V \oplus \chi)^G = \mathbb{C}(V)^H(t^{\#I})$. Now, consider the rank one discrete valuation ring $B = \mathbb{C}(V)(t)_{(t)}$ and let A be the induced valuation ring inside $\mathbb{C}(V \oplus \chi)^G$. The diagram (II.4) reads in this situation :

$$\begin{tikzcd}
\mathrm{H}^3(G, \mathbb{Q}/\mathbb{Z}) \arrow[r, "\partial_{H,g}"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{H}^2(H, \mathbb{Q}/\mathbb{Z}) \arrow[d] \\
\mathrm{H}^3(\mathbb{C}(V \oplus \chi)^G, \mathbb{Q}/\mathbb{Z}(2) \arrow[r, "\partial_A"'] & \mathrm{H}^2(\mathbb{C}(V)^H, \mathbb{Q}/\mathbb{Z}(1))
\end{tikzcd}.$$

We claim that the right vertical map is injective, as this will immediately imply the desired inclusion. In order to see this, we only need to show that the subgroup of geometrically negligible classes in $\mathrm{H}^2(G, \mathbb{Q}/\mathbb{Z})$ is trivial. Recall that by Serre's lemma we have :

$$\mathrm{H}^2(G, \mathbb{Q}/\mathbb{Z}) = \ker[\mathrm{H}^2(G, \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^2(\mathbb{C}(V)^G, \mathbb{Q}/\mathbb{Z})].$$

On the other hand, remark that we have an isomorphism of trivial G-modules $\mathbb{Q}/\mathbb{Z} \simeq \mu_\infty$; moreover $\mathbb{C}^\times/\mu_\infty$ is a $\mathbb{Q}$-vector space, hence an injective G-module. Therefore, the long exact sequence attached to the exact sequence of trivial G-modules

$$0 \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow \mathbb{C}^\times \longrightarrow \mathbb{C}^\times/\mu_\infty \longrightarrow 0$$

provides an isomorphism $\mathrm{H}^2(G, \mathbb{Q}/\mathbb{Z}) \simeq \mathrm{H}^2(G, \mathbb{C}^\times)$. Now, since V is an affine space we have the tautological exact sequence

$$0 \longrightarrow \mathbb{C}^\times \longrightarrow \mathbb{C}(V)^\times \longrightarrow \mathrm{Div}(V) \longrightarrow 0,$$

and by assumption on V we know that $\mathrm{Div}(V)$ is a permutation G-module. Applying Shapiro's lemma thus yields $\mathrm{H}^1(G, \mathrm{Div}(V)) = 0$, so in the long exact sequence attached to the above exact sequence, we obtain an injection

$$0 \longrightarrow \mathrm{H}^2(G, \mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{H}^2(G, \mathbb{C}(V)^\times).$$

But by [Ser80, Chap. X, §4, Prop. 6], we have an injection $\mathrm{H}^2(G, \mathbb{C}(V)^\times) \hookrightarrow \mathrm{H}^2(\mathbb{C}(V)^G, \mathbb{Q}/\mathbb{Z})$. Therefore, we must also have an inclusion $\mathrm{H}^2(G, \mathbb{Q}/\mathbb{Z}) \hookrightarrow \mathrm{H}^2(\mathbb{C}(V)^G, \mathbb{Q}/\mathbb{Z})$, which concludes the proof. ◻

Chapter III

Unramified cohomology and the integral Hodge conjecture

III.1. Setting and notations

1.1. Integral Hodge classes

— Let X be a smooth, projective and connected complex variety of dimension d. Let $\mathbb{Z}(1) := 2\pi i\mathbb{Z} \subset \mathbb{C}$ and $\mathbb{Z}(j) := \mathbb{Z}(1)^{\otimes j}$ for $j \geq 1$. As in Chapter I, §2.2, for each such j, there is a Betti cycle class

$$\mathrm{CH}^j(X) \longrightarrow \mathrm{H}^{2j}_B(X, \mathbb{Z}(j)),$$

whose image is denoted by $\mathrm{H}^{2j}_{\mathrm{alg}}(X, \mathbb{Z}(j))$. Among the the rational classes in $\mathrm{H}^{2j}_B(X, \mathbb{Q}(i)) = \mathrm{H}^{2j}_B(X, \mathbb{Z}(i)) \otimes_{\mathbb{Z}} \mathbb{Q}$, we have the subgroup of Hodge classes, denoted $\mathrm{Hdg}^{2j}(X, \mathbb{Q}(j))$, which consists of the classes that have type (j, j) for the usual Hodge decomposition of $\mathrm{H}^{2j}_B(X, \mathbb{Q}(i)) \otimes_{\mathbb{Q}} \mathbb{C}$ (see *e.g.* [Voi03, §7.1.1]). We have a natural inclusion $\mathrm{H}^{2j}_{\mathrm{alg}}(X, \mathbb{Z}(j)) \otimes_{\mathbb{Z}} \mathbb{Q} \subset \mathrm{Hdg}^{2j}(X, \mathbb{Q}(j))$, and the Hodge conjecture asks whether this inclusion is actually an equality. This question has a positive answer in general for $j = 1$ (Lefschetz theorem on $(1, 1)$-classes [Voi02, Thm. 7.2]) or $j = d - 1$ (hard Lefschetz theorem [Voi03, Thm. 1.23]).

Definition 3.1.1. The group of *integral Hodge classes* of degree $2j$ is defined as the subgroup $\mathrm{Hdg}^{2j}(X, \mathbb{Z}(j)) \subset \mathrm{H}^{2j}_B(X, \mathbb{Z}(j))$ given by the inverse image of $\mathrm{Hdg}^{2j}(X, \mathbb{Q}(j))$ in $\mathrm{H}^{2j}_B(X, \mathbb{Z}(j))$ *via* the natural map from integral Betti cohomology to rational Betti cohomology.

From this observation, we have a natural inclusion

$$\mathrm{H}^{2j}_{\mathrm{alg}}(X, \mathbb{Z}(j)) \subset \mathrm{Hdg}^{2j}(X, \mathbb{Z}(j)).$$

We write $Z^{2j}(X) := \mathrm{Hdg}^{2j}(X, \mathbb{Z}(j))/\mathrm{H}^{2j}_{\mathrm{alg}}(X, \mathbb{Z}(j))$ for the group that measures the potential failure of the integral Hodge conjecture for codimension j cycles on X. We naturally have the equality

$$Z^{2j}(X)_{\mathrm{tors}} = (\mathrm{H}^{2j}_B(X, \mathbb{Z}(j))/\mathrm{H}^{2j}_{\mathrm{alg}}(X, \mathbb{Z}(j)))_{\mathrm{tors}},$$

and the rational Hodge conjecture in degree $2j$ precisely asks whether these two groups coincide with $Z^{2j}(X)$. The *integral Hodge conjecture* (also called *integral Hodge question*) aks whether this group is zero in general. There are some counterexamples to this question (for instance thanks to the work of Atiyah-Hirzebruch), but they are in general quite difficult to construct. In [CTV12], a link between the unramified cohomology group $\mathrm{H}^3_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}(2))$ and $Z^4(X)$ has been established by Colliot-Thélène and Voisin, which provides a new point of view on the integral Hodge conjecture for cycles of codimension 2, and the techniques that can be used to measure its failure. In particular, the main result of Colliot-Thélène and Voisin allowed them to construct unirational varieties for the which the integral Hodge conjecture is not satisfied in codimension (or dimension) 2.

1.2. Bloch-Ogus for Betti cohomology

— As explained in Chapter I, §2.2, for any complex algebraic variety X, there is a commutative diagram of sites :

$$\begin{tikzcd} X_{\mathrm{cl}} \arrow[r, "\partial"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & X(\mathbb{C}) \arrow[d, "h"] \\ X_{\text{ét}} \arrow[r, "g"'] & X_{\mathrm{Zar}} \end{tikzcd},$$

and we put $\pi : X_{\mathrm{cl}} \to X_{\mathrm{Zar}}$ for the composite morphism of sites. For a given abelian group A (seen alternatively as a constant sheaf on X_{cl}) and an arbitrary integer j, we let $A(j) := A \otimes_{\mathbb{Z}} \mathbb{Z}(j)$, where $\mathbb{Z}(j) := (2\pi i\mathbb{Z})^{\otimes j}$ if $j \geq 0$ and $\mathbb{Z}(j) := \mathsf{Hom}_{\mathbb{Z}}(\mathbb{Z}(-j), \mathbb{Z})$ otherwise. We have seen in Chapter I, §4.2 that for any $i \geq 0$ and any torsion sheaf $\mathscr{F}$ on $X_{\text{ét}}$, Bloch-Ogus theory provides an acyclic resolution for the Zariski sheaf $\mathscr{H}^i_{\text{ét}}(\mathscr{F})$, which is given by the filtration by codimension of support. As mentioned in the same paragraph, their effacement argument for étale cohomology with torsion coefficients also applies to a wide variety of other cohomology theories. Betti cohomology is one of them. Indeed, if we write $\mathscr{H}^i_B(A)$ for the Zariski sheaf associated to the presheaf $U \mapsto \mathrm{H}^i_B(X, A)$, then a result symmetric to the Bloch-Ogus theorem (1.4.71) applied to this sheaf is proved in their original paper. For any closed integral subvariety $\iota_D : D \hookrightarrow X$, write :

$$\mathrm{H}^i_B(\Bbbk(D), A) := \varinjlim_{D \subset U} \mathrm{H}^i_B(U, A),$$

where U ranges among the Zariski open subset of X that meet D. We then have the following acyclic resolution :

Theorem 3.1.2 ([BO74, Thm. 4.2]). *For any abelian group A, any smooth and connected variety X over $\mathbb{C}$ and any integer $i \geq 1$, we have the following exact sequence of Zariski sheaves on X :*

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{H}^i_B(A) \arrow[r] & \iota_{\eta_X *}\, \mathrm{H}^i_B(\Bbbk(X), A) \arrow[r] & \bigoplus_{D \in X^{(1)}} \iota_{D*}\, \mathrm{H}^{i-1}_B(\Bbbk(D), A(-1)) \arrow[dll, rounded corners] \\ & \cdots \arrow[r] & \bigoplus_{D \in X^{(i)}} \iota_{D*}\, \mathrm{H}^0_B(\Bbbk(D), A(-i)) \arrow[r] & 0, \end{tikzcd}$$

where the differentials are induced by the topological residues $\mathrm{Res}_{D,D'}$ when $D' \subset D$ (and zero otherwise), $\iota_{\eta_X} : \mathsf{Spec}\,\Bbbk(X) \hookrightarrow X$ is the inclusion of the generic point, and $\iota_D : \mathsf{Spec}\,\Bbbk(D) \hookrightarrow X$ is the inclusion of the generic point of D.

III.2. Failure of the integral Hodge conjecture

2.1. Preliminary results

2.1.1. A generalisation of an argument of Bloch-Srinivas

— The following argument was initially discussed by Bloch and Srinivas in [BS83, Thm. 1] ; it states that the sheaves $\mathscr{H}^p_B(\mathbb{Z})$ have no torsion for each $p \geq 1$ on a general connected complex variety. However the original statement of Bloch-Srinivas could only revolve around the 2-torsion of $\mathscr{H}^3_B(\mathbb{Z})$, since by that time only the Merkurjev-Suslin theorem was known to be true. The proof of Voisin and Colliot-Thélène is roughly the same but relies on the general Bloch-Kato conjecture proved by Voevodsky and Rost, and the Gersten conjecture for Milnor's K-theory proved by Kerz (see Chapter I, §3.2.2.4).

Theorem 3.2.3. *Let X be a smooth connected complex algebraic variety. For any integer i, the multiplication by an integer $n \geq 1$ on the sheaves $\mathcal{H}^p_B(\mathbb{Z}(i))$ ($p \geq 0$) induces short exact sequences of Zariski sheaves :*

$$0 \longrightarrow \mathcal{H}^p_B(\mathbb{Z}(i)) \xrightarrow{\cdot n} \mathcal{H}^p_B(\mathbb{Z}(i)) \longrightarrow \mathcal{H}^p_B(\mathbb{Z}/n(i)) \longrightarrow 0.$$

In particular, the sheaves $\mathcal{H}^p_B(\mathbb{Z}(i))$ are torsion-free.

Proof. As twists don't really matter in Betti cohomology, we can just treat the case where $i = p$. The exact sequence of abelian groups

$$0 \longrightarrow \mathbb{Z}(p) \xrightarrow{\cdot n} \mathbb{Z}(p) \longrightarrow \mathbb{Z}/n(p) \longrightarrow 0$$

gives rise to a long exact sequence of Zariski sheaves :

$$\cdots \longrightarrow \mathcal{R}^{p-1}\pi_*\mathbb{Z}/n(p) \longrightarrow \mathcal{R}^p\pi_*\mathbb{Z}(p) \xrightarrow{\cdot n} \mathcal{R}^p\pi_*\mathbb{Z}(p) \longrightarrow \mathcal{R}^p\pi_*\mathbb{Z}/n(p) \longrightarrow \cdots$$

so we only have to show that the arrows $\mathcal{R}^p\pi_*\mathbb{Z}(p) \longrightarrow \mathcal{R}^p\pi_*\mathbb{Z}/n(p)$ are surjective. We have a commutative diagram with exact rows :

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}(1) \arrow[r] \arrow[d] & \mathcal{O}_{X,\mathrm{cl}} \arrow[r, "\exp"] \arrow[d] & \mathcal{O}^\times_{X,\mathrm{cl}} \arrow[r] \arrow[d, equal] & 1 \\
0 \arrow[r] & \mathbb{Z}/n(1) \arrow[r] & \mathcal{O}^\times_{X,\mathrm{cl}} \arrow[r, "z\mapsto z^n"'] & \mathcal{O}^\times_{X,\mathrm{cl}} \arrow[r] & 1
\end{tikzcd}$$

(where $\mathcal{O}_{X,\mathrm{cl}}$ denotes the sheaf of continuous fonctions on $X(\mathbb{C})$ and the two left vertical arrows are induced by $z \mapsto \exp(z/n)$). Taking cohomology, we obtain a commutative diagram of sheaves

$$\begin{tikzcd}
\pi_*\mathcal{O}^\times_{X,\mathrm{cl}} \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{R}^1\pi_*\mathbb{Z}(1) \arrow[d] \\
\pi_*\mathcal{O}^\times_{X,\mathrm{cl}} \arrow[r] & \mathcal{R}^1\pi_*\mathbb{Z}/n(1)
\end{tikzcd}.$$

On the other hand, the comparison theorems between étale and Betti cohomology with torsion coefficients (see [DA73, Exposé XI, Thm. 4.4]) show that $\mathcal{R}^if_*\mathbb{Z}/n(1) = 0$ for $i \geq 1$, so using the Kummer sequence

$$1 \longrightarrow \mathbb{Z}/n(1) \longrightarrow \mathcal{O}^\times_{X,\mathrm{cl}} \xrightarrow{z\mapsto z^n} \mathcal{O}^\times_{X,\mathrm{cl}} \longrightarrow 1$$

on X_{cl}, we obtain a commutative with exact rows :

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{Z}/n(1) \arrow[r] \arrow[d, "\sim"'] & \mathcal{O}^\times_{X,\mathrm{cl}} \arrow[r, "z\mapsto z^n"] \arrow[d] & \mathcal{O}^\times_{X,\mathrm{cl}} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & f_*\mathbb{Z}/n(1) \arrow[r] & f_*\mathcal{O}^\times_{X,\mathrm{cl}} \arrow[r] & f_*\mathcal{O}^\times_{X,\mathrm{cl}} \arrow[r] & 1
\end{tikzcd},$$

hence a commutative diagram of Zariski sheaves :

$$\begin{tikzcd}
g_*f_*\mathcal{O}^\times_{X,\mathrm{cl}} \arrow[r] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{R}^1g_*f_*\mathbb{Z}/n(1) \\
g_*\mathbb{G}_{m,X} \arrow[r] \arrow[u] & \mathcal{R}^1g_*\mathbb{Z}/n(1) \arrow[u, "\sim"']
\end{tikzcd}.$$

But since $\mathcal{R}^1 f_* \mathbb{Z}/n(1) = 0$, the Grothendieck spectral sequence $\mathcal{R}^p g_* \mathcal{R}^q f_* \mathbb{Z}/n(1) \Rightarrow \mathcal{R}^{p+q}\pi_* \mathbb{Z}/n(1)$ shows that the natural map $\mathcal{R}^1 g_* f_* \mathbb{Z}/n(1) \to \mathcal{R}^1 \pi_* \mathbb{Z}/n(1)$ is an isomorphism. Therefore, up to rearranging the above diagram, we can extend it to a bigger one :

$$\begin{tikzcd} \pi_* \mathcal{O}_{X,\mathrm{cl}}^\times \arrow[r] \arrow[d, equal] & \mathcal{R}^1 \pi_* \mathbb{Z}(1) \arrow[d] \\ \pi_* \mathcal{O}_{X,\mathrm{cl}}^\times \arrow[r] & \mathcal{R}^1 \pi_* \mathbb{Z}/n(1) \\ g_* \mathbb{G}_{m,X} \arrow[r] \arrow[u] & \mathcal{R}^1 g_* \mathbb{Z}/n(1) \arrow[u, "\sim"'] \end{tikzcd} \cdot$$

Since $\mathcal{R}^1 g_* \mathbb{G}_{m,X} = 0$ by Grothendieck's Hilbert 90, we get that the connecting map $g_* \mathbb{G}_{m,X} \to \mathcal{R}^1 g_* \mathbb{Z}/n(1)$ is surjective, hence in the above diagram the map $\mathcal{R}^1 \pi_* \mathbb{Z}(1) \longrightarrow \mathcal{R}^1 \pi_* \mathbb{Z}/n(1)$ is surjective. More, we know that $\mathcal{O}_X^\times \to g_* \mathbb{G}_{m,X}$ is an isomorphism. We thus get that the composite map

$$\mathcal{O}_X^\times \xrightarrow{\sim} g_* \mathbb{G}_{m,X} \longrightarrow \pi_* \mathcal{O}_{X,\mathrm{cl}}^\times \longrightarrow \mathcal{R}^1 \pi_* \mathbb{Z}(1) \longrightarrow \mathcal{R}^1 \pi_* \mathbb{Z}/n(1) \longrightarrow \mathcal{R}^1 g_* \mathbb{Z}/n(1)$$

coincides with the natural surjection $\mathcal{O}_X^\times \to \mathcal{R}^1 g_* \mathbb{Z}/n(1)$ deduced by the Kummer sequence and the direct image functor $g_* : \mathrm{Sh}(X_{\text{ét}}) \to \mathrm{Sh}(X_{\mathrm{Zar}})$. Taking cup-products, we obtain a sequence of morphisms :

$$(\mathcal{O}_X^\times)^{\otimes p} \longrightarrow \mathcal{R}^p \pi_* \mathbb{Z}(p) \longrightarrow \mathcal{R}^p \pi_* \mathbb{Z}/n(p) \longrightarrow \mathcal{R}^p g_* \mathbb{Z}/n(p),$$

and the composite map $(\mathcal{O}_X^\times)^{\otimes p} \to \mathcal{R}^p g_* \mathbb{Z}/n(p)$ is nothing more than the composition

$$(\mathcal{O}_X^\times)^{\otimes p} \longrightarrow (\mathcal{R}^1 g_* \mathbb{Z}/n(1))^{\otimes p} \longrightarrow \mathcal{R}^p g_* \mathbb{Z}/n(p)$$

where the first map is given by the Kummer sequence. But again, the vanishing of $\mathcal{R}^p f_* \mathbb{Z}/n(p)$ and the Grothendieck spectral sequence associated to f and g provide an isomorphism $\mathcal{R}^p g_* \mathbb{Z}/n(p) \xrightarrow{\sim} \mathcal{R}^p \pi_* \mathbb{Z}/n(p)$.

We now want to show that the induced map $(\mathcal{O}_X^\times)^{\otimes p} \to \mathcal{R}^p g_* \mathbb{Z}/n(p)$ is surjective. By Rost-Voevodsky's proof of the Bloch-Kato conjecture, we know that at for any field k together with an integer $n \geq 1$ that is coprime to $\mathrm{char}(k)$, we have isomorphisms $K_p^M(k)/n \simeq \mathrm{H}^p(k, \mathbb{Z}/n(p))$ for $p \geq 1$. Applying Kerz's proof of the Gersten conjecture and the Bloch-Ogus theorem, we have an isomorphism of acyclic resolutions of Zariski sheaves :

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{K}_p^M/n \arrow[r] \arrow[d, dashed] & \iota_{\eta_X *} K_p^M(\mathbb{k}(X))/n \arrow[r] \arrow[d, "\sim"'] & \cdots \arrow[r] & \bigoplus_{x \in X^{(p)}} \iota_{x*} K_0^M(\kappa(x))/n \arrow[r] \arrow[d, "\sim"'] & 0 \\ 0 \arrow[r] & \mathcal{H}_{\text{ét}}^p(\mathbb{Z}/n(p)) \arrow[r] & \iota_{\eta_X *} \mathrm{H}^p(\mathbb{k}(X), \mathbb{Z}/n(p)) \arrow[r] & \cdots \arrow[r] & \bigoplus_{x \in X^{(p)}} \iota_{x*} \mathrm{H}^0(\kappa(x), \mathbb{Z}/n) \arrow[r] & 0 \end{tikzcd}$$

hence an induced isomorphism of sheaves $\mathcal{K}_p^M/n \xrightarrow{\sim} \mathcal{H}_{\text{ét}}^p(\mathbb{Z}/n(p))$. This yields a surjective morphism :

$$(\mathcal{O}_X^\times)^{\otimes p} \twoheadrightarrow (\mathcal{O}_X^\times)^{\otimes p}/n \twoheadrightarrow \mathcal{K}_p^M/n \xrightarrow{\sim} \mathcal{H}_{\text{ét}}^p(\mathbb{Z}/n(p)).$$

We thus showed that the composite map

$$(\mathcal{O}_X^\times)^{\otimes p} \longrightarrow \mathcal{R}^p \pi_* \mathbb{Z}(p) \longrightarrow \mathcal{R}^p \pi_* \mathbb{Z}/n(p) \longrightarrow \mathcal{R}^p g_* \mathbb{Z}/n(p)$$

is surjective, which forces $\mathcal{R}^p \pi_* \mathbb{Z}(p) \to \mathcal{R}^p \pi_* \mathbb{Z}/n(p)$ to be surjective as well, hence the theorem. □

Remark 3.2.4. We could also drop the smoothness assumption on the variety X (this is implicitely done in the paper of Voisin and Colliot-Thélène). Indeed, as explained in Chapter I, §3.2.2.4, we still have an isomorphism $K_p^M(\mathcal{O}_{X,x})/n \simeq \mathrm{H}_{\text{ét}}^p(\mathcal{O}_{X,x}, \mathbb{Z}/n(p))$ for any $x \in X$, because X is defined over an infinite field ; we then obtain a surjection of sheaves $\mathcal{K}_p^M/n \twoheadrightarrow \mathcal{H}_{\text{ét}}^p(\mathbb{Z}/n(p))$ which induces a surjective morphism

$$(\mathcal{O}_X^\times)^{\otimes p} \longrightarrow \mathcal{R}^p \pi_* \mathbb{Z}(p) \longrightarrow \mathcal{R}^p \pi_* \mathbb{Z}/n(p) \longrightarrow \mathcal{R}^p g_* \mathbb{Z}/n(p)$$

(one should however carefully check that the morphisms considered in this composite agree with the morphism of sheaves $\mathcal{K}_p^M/n \twoheadrightarrow \mathcal{H}_{\text{ét}}^p(\mathbb{Z}/n(p))$ constructed *via* the stalks).

2.1.2. Decomposition of the diagonal and action of correspondences on $\mathcal{H}_B^\bullet(-)$

Proposition 3.2.5. *Let X be a projective, smooth and connected complex variety of dimension d. Then :*

(i) *If there exists a closed subvariety $j : Y \hookrightarrow X$ of dimension r such that the pushforward $j_* \operatorname{CH}_0(Y) \to \operatorname{CH}_0(X)$ is surjective, then there exists an integer $N \geq 1$ such that $\mathrm{H}^0(X, \mathcal{H}_B^p(A))$ is annihilated by N for $p > r$;*

(ii) *If there exists a closed subvariety $j : Y \hookrightarrow X$ of codimension r such that the pushforward $j_* \operatorname{CH}_0(Y) \to \operatorname{CH}_0(X)$ is surjective, then there exists an integer $N \geq 1$ such that $\mathrm{H}^p(X, \mathcal{H}_B^d(A))$ is annihilated by N for $p < r$;*

(iii) *If $X = \mathbb{P}^d_{\mathbb{C}}$, then $\mathrm{H}^p_{Zar}(X, \mathcal{H}_B^q(A)) = 0$ for $p \neq q$ and $\mathrm{H}^p(X, \mathcal{H}_B^p(A)) = A$ for every $p \leq d$.*

Remark 3.2.6. If we specialise (i) to $A = \mathbb{Z}$, then Theorem (3.2.3) shows that $\mathcal{H}_B^p(\mathbb{Z})$ is torsion-free, *a fortiori* the group $\mathrm{H}^0(X, \mathcal{H}_B^p(\mathbb{Z}))_{\mathrm{tors}}$ is zero, hence actually $\mathrm{H}^0(X, \mathcal{H}_B^p(\mathbb{Z})) = 0$.

Proof. As proved in Appendix A, the correspondences act naturally on the cohomology of the sheaves $\mathcal{H}^q(A)$, in such a way that composition of correspondences is compatible with this action.

Let us first prove (i). By the Bloch-Srinivas decomposition of the diagonal (1.1.18) discussed in Chapter I, §1.4.2, we know that there exists an integer $N \geq 1$ and two correspondences Γ_1, Γ_1 such that

$$N[\Delta_X] = \Gamma_1 + \Gamma_2 \in \operatorname{CH}^d(X \times_{\mathbb{C}} X),$$

where $\operatorname{Supp}(\Gamma_1) \subset Y \times_{\mathbb{C}} X$ and Y can be supposed of pure dimension r, and $\operatorname{Supp}(\Gamma_2) \subset X \times_{\mathbb{C}} D$ where D is a proper subvariety of X. The action of these correspondences on the cohomology of $\mathcal{H}_B^p(A)$ yields the identity :

$$N \cdot \mathrm{Id} = \Gamma_{1*} + \Gamma_{2*} : \mathrm{H}^0(X, \mathcal{H}_B^p(A)) \longrightarrow \mathrm{H}^0(X, \mathcal{H}_B^p(A)).$$

Invoking Hironaka's theorem, we can consider resolutions of singularities $\widetilde{D} \to D$ and $\widetilde{Y} \to Y$ and corresponding lifts of Γ_1, Γ_2 such that Γ_{1*} factors through the restriction map

$$\mathrm{H}^0(X, \mathcal{H}_B^p(A)) \longrightarrow \mathrm{H}^0(\widetilde{Y}, \mathcal{H}_B^p(A)).$$

On the other hand, the sheaf $\mathcal{H}_B^p(A)$ is identically zero on $\widetilde{Y}$. Indeed, since $\dim \widetilde{Y} < p$, then for any affine open subset $U \subset \widetilde{Y}$, we have $\mathrm{H}_B^p(U, A) = 0$ by the usual vanishing of Betti cohomology in the affine case (see *e.g.* [Voi03, Thm. 1.22][22]) ; for a general open subset $U \subset X$, we can always fix an affine open cover, and the claim follows from the sheaf property. Moreover, since Γ_2 is supported on $D \times_{\mathbb{C}} X$, then the image of Γ_{2*} consists of classes with support in D which, by the Gersten resolution of $\mathcal{H}_B^p(A)$, implies that they are zero. Thus $N\mathrm{Id} : \mathrm{H}^0(X, \mathcal{H}_B^p(A)) \longrightarrow \mathrm{H}^0(X, \mathcal{H}_B^p(A))$ is the zero map.

By permuting the factors in the Bloch-Srinivas decomposition of $[\Delta_X] \in \operatorname{CH}^d(X \times_{\mathbb{C}} X)$, we can now assume that Γ_1 is supported on $D \times_{\mathbb{C}} X$ with D a proper closed subvariety, and Γ_2 is supported on $X \times_{\mathbb{C}} Y$ with Y of pure codimension r. As before, by looking at the action of correspondences on the cohomology of $\mathcal{H}_B^d(A)$, we obtain that

$$N \cdot \mathrm{Id} = \Gamma_{1*} + \Gamma_{2*} : \mathrm{H}^p_{\mathrm{Zar}}(X, \mathcal{H}_B^d(A)) \longrightarrow \mathrm{H}^p_{\mathrm{Zar}}(X, \mathcal{H}_B^d(A)).$$

Once again, by Hironaka's theorem, we have desingularisations $\widetilde{D} \to D$ and $\widetilde{Y} \to Y$ and corresponding lifts of Γ_1 and Γ_2 so that Γ_{1*} factors through

$$\mathrm{H}^p_{\mathrm{Zar}}(X, \mathcal{H}_B^d(A)) \longrightarrow \mathrm{H}^p_{\mathrm{Zar}}(\widetilde{D}, \mathcal{H}_B^d(A)).$$

But this map is zero by the same argument as before, since $\dim D < d$. On the other hand, Γ_{2*} factors through

$$\tilde{j}_* : \mathrm{H}^{p-r}_{\mathrm{Zar}}(\widetilde{Y}, \mathcal{H}_B^{d-r}(A)) \longrightarrow \mathrm{H}^p_{\mathrm{Zar}}(X, \mathcal{H}_B^d(A)),$$

[22] This is more or less given by Artin's version of the weak Lefschetz theorem, combined with Poincaré duality.

where $\tilde{j}$ denotes the composite map $\widetilde{Y} \longrightarrow Y \hookrightarrow X$. As $p < r$, we have that $\mathrm{H}^p_{\mathrm{Zar}}(X, \mathscr{H}^d_B(A)) = 0$, so Γ_{2*} is zero as well. This proves (ii).

If $X = \mathbb{P}^d_{\mathbb{C}}$, as discussed in Chapter I, §1.4, Examples (1.1.10), we have a decomposition

$$[\Delta_X] = \sum_{i=0}^{d} h_1^i h_2^{d-i}$$

where $h_j := \mathrm{pr}_j^* c_1(\mathscr{O}_{\mathbb{P}^d_{\mathbb{C}}}(1))$ for $j = 1, 2$. As a consequence, if we pick a class $\alpha \in \mathrm{H}^p_{\mathrm{Zar}}(X, \mathscr{H}^q_B(A))$, then we have $[\Delta_X]_*\alpha = \sum_{i=0}^d (h_1^i h_2^{d-i})_*\alpha$. If we write $j_i : X_i := \mathbb{P}^i_{\mathbb{C}} \hookrightarrow \mathbb{P}^d_{\mathbb{C}}$ for the canonical inclusion and $\pi_i : \mathbb{P}^i_{\mathbb{C}} \to \{\star\}$ for the constant map, then $h_1^i h_2^{d-i}$ is the class of $X_{d-i} \times_{\mathbb{C}} X_i$ in $\mathbb{P}^d_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{P}^d_{\mathbb{C}}$, so that we have the identity :

$$(h_1^i h_2^{d-i})_*\alpha = j_{i*}(\pi_i^*(\pi_{d-i*}(j_{d-i}^*\alpha))).$$

As $\pi_{d-i*} \circ j_{d-i}^*$ is identically zero on $\mathrm{H}^p_{\mathrm{Zar}}(X, \mathscr{H}^q_B(A))$ for $p \neq q$, we thus get the vanishing of $\mathrm{H}^p_{\mathrm{Zar}}(X, \mathscr{H}^q_B(A))$ for these choices of p and q. The assertion $\mathrm{H}^p_{\mathrm{Zar}}(X, \mathscr{H}^p_B(A)) = A$ for $p \leq d$ results from the same arguments, so (iii) is proved. □

2.2. Proof of the main results

2.2.1. Unramified cohomology of degree 3 with torsion coefficients and failure of the integral Hodge conjecture in degree 4

— We first discuss the integral Hodge question for codimension 2 cycles on X. Indeed, the Leray spectral sequence with $\mathbb{Z}(2)$-coefficients associated to the morphism of sites $X_{\mathrm{cl}} \to X_{\mathrm{Zar}}$ has no nonzero term under the diagonal thanks to Bloch-Ogus theory. Therefore, by considering the lower terms in the first quadrant, we can derive an interesting short exact sequence :

Theorem 3.2.7 (Colliot-Thélène-Voisin [CTV12, Thm. 3.7]). *Let X be a smooth, projective and connected complex variety. For any integer $n \geq 1$, we have a short exact sequence :*

$$0 \longrightarrow \mathrm{H}^0(X, \mathscr{H}^3_B(\mathbb{Z}(2)))/n \longrightarrow \mathrm{H}^0(X, \mathscr{H}^3_B(\mathbb{Z}/n(2))) \longrightarrow Z^4(X)[n] \longrightarrow 0.$$

Proof. Consider the Leray spectral sequence :

$$E_2^{p,q} = \mathrm{H}^p_{\mathrm{Zar}}(X, \mathscr{H}^q_B(\mathbb{Z}(2))) \Rightarrow \mathrm{H}^{p+q}_B(X, \mathbb{Z}(2)).$$

As usual, since X is smooth, we know by Bloch-Ogus theory for Betti cohomology that the $E_2^{p,q}$-terms vanish for $p > q$, so that among the E_2-terms of degree 4, no nonzero differential starts from $E_2^{2,2}$ or $E_2^{1,3}$, and no nonzero differential arrives at $E_2^{1,3}$ or $E_2^{0,4}$. This implies that

$$E_2^{2,2} \twoheadrightarrow \mathrm{coker}[E_2^{2,2} \to E_2^{4,1}] = E_\infty^{2,2}, \quad E_2^{1,3} = E_\infty^{1,3} \quad \text{and} \quad E_\infty^{0,4} = \ker[E_2^{0,4} \to E_2^{2,3}] \subset E_2^{0,4},$$

so that there is a natural composite map

$$\mathrm{CH}^2(X)/\mathrm{alg} \longrightarrow E_\infty^{2,2} \subset \mathrm{H}^4_B(X, \mathbb{Z}(2))$$

whose image is the deepest level $F^2\,\mathrm{H}^4_B(X, \mathbb{Z}(2))$ in the Leray filtration which, by Bloch-Ogus theory, coincides with the coniveau filtration. This identification on the respective filtration also allows us to identify this map with the cycle class $c\ell_2$, so that $\mathrm{coker}[E_2^{2,2} \to \mathrm{H}^4_B(X, \mathbb{Z}(2))] = \mathrm{H}^4_B(X, \mathbb{Z}(2))/\mathrm{H}^4_{\mathrm{alg}}(X, \mathbb{Z}(2))$. But this group has a filtration coming from the Leray spectral sequence, whose graded pieces are

$$E_\infty^{1,3} = \mathrm{H}^1_{\mathrm{Zar}}(X, \mathscr{H}^3_B(\mathbb{Z}(2))) \quad \text{and} \quad E_\infty^{0,4} \subset \mathrm{H}^0(X, \mathscr{H}^4_B(\mathbb{Z}(2))).$$

As $\mathrm{H}^0(X, \mathscr{H}_B^4(\mathbb{Z}(2)))$ is torsion-free, we obtain an isomorphism

$$Z^4(X)[n] = (\mathrm{H}_B^4(X, \mathbb{Z}(2))/\mathrm{H}_{\mathrm{alg}}^4(X, \mathbb{Z}(2)))[n] = \mathrm{H}_{\mathrm{Zar}}^1(X, \mathscr{H}_B^3(\mathbb{Z}(2)))[n].$$

Finally, the short exact sequence of sheaves $0 \to \mathscr{H}_B^3(\mathbb{Z}(2)) \to \mathscr{H}_B^3(\mathbb{Z}(2)) \to \mathscr{H}_B^3(\mathbb{Z}/n(2)) \to 0$ provides an isomorphism

$$\mathrm{H}_{\mathrm{Zar}}^1(X, \mathscr{H}_B^3(\mathbb{Z}(2)))[n] \simeq \frac{\mathrm{H}^0(X, \mathscr{H}_B^3(\mathbb{Z}/n(2)))}{\mathrm{coker}[\mathrm{H}^0(X, \mathscr{H}_B^3(\mathbb{Z}(2))) \xrightarrow{\cdot n} \mathrm{H}^0(X, \mathscr{H}_B^3(\mathbb{Z}(2)))]},$$

hence the desired exact sequence. □

2.2.1.1. Relating integral and rational Hodge classes. The exact sequence obtained earlier is not completely satisfying, because we cannot control the group $\mathrm{H}^0(X, \mathscr{H}_B^3(\mathbb{Z}(2)))/n$ in general. Moreover, it only provides information on the torsion classes in $Z^4(X)$. Under the assumption that $\mathrm{CH}_0(X)$ is supported on a surface, we can however make use of the following celebrated classical result, due to Bloch and Srinivas :

Theorem 3.2.8 (**Bloch-Srinivas [BS83, Thm. 1, (iv)]**). *Let X be a smooth and projective complex variety such that there exists a closed subvariety $j : Y \hookrightarrow X$ of dimension ≤ 3, such that the proper pushforward $j_* : \mathrm{CH}_0(Y) \to \mathrm{CH}_0(X)$ is surjective. Then the (rational) Hodge conjecture holds for cycles of codimension 2 on X.*

Remark 3.2.9. As codimension 2 cycles on a variety of dimension 3 are precisely 1-dimensional cycles, then the rational Hodge conjecture on Y as above is a direct consequence of the Lefschetz theorem on $(1,1)$-classes and the hard Lefschetz theorem. Bloch and Srinivas then considered a suitable decomposition of the diagonal $[\Delta_X] = \Gamma_1 + \Gamma_2 \in \mathrm{CH}^{\dim X}(X \times_{\mathbb{C}} X)$ and observed that the action of the correspondences Γ_1 and Γ_2 on $\mathrm{H}_B^4(X, \mathbb{Q})$ sends $\mathrm{Hdg}^4(X, \mathbb{Q})$ to $\mathrm{H}_{\mathrm{alg}}^4(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Theorem 3.2.10 (**Colliot-Thélène-Voisin [CTV12, Thm. 3.8]**). *Let X be a smooth, projective and connected complex variety such that there exists a closed subvariety $j : Y \hookrightarrow X$ that is smooth, projective and connected of dimension ≤ 3 and such that $j_* : \mathrm{CH}_0(Y) \to \mathrm{CH}_0(X)$ is surjective. Then $Z^4(X)$ is a finite group, and there is a short exact sequence :*

$$0 \longrightarrow \mathrm{H}^0(X, \mathscr{H}_B^3(\mathbb{Z}(2))) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} \longrightarrow \mathrm{H}^0(X, \mathscr{H}_B^3(\mathbb{Q}/\mathbb{Z}(2))) \longrightarrow Z^4(X) \longrightarrow 0.$$

Proof. The previous theorem shows that the rational Hodge conjecture for codimension 2 cycles holds for X, so that $Z^4(X) = \mathrm{H}_B^4(X, \mathbb{Z}(2))/\mathrm{H}_{\mathrm{alg}}^4(X, \mathbb{Z}(2))$ is a torsion group (every Hodge class in $\mathrm{Hdg}^4(X, \mathbb{Z}(2))$ is equal to an algebraic class up to a rational factor). The statement is thus directly obtained by taking the direct limit of the short exact sequences displayed in Theorem (3.2.7). □

Corollary 3.2.11. *Let X be a smooth, projective and connected complex variety such that there exists a closed subvariety $j : Y \hookrightarrow X$ that is smooth, projective and connected of dimension 2 and such that $j_* : \mathrm{CH}_0(Y) \to \mathrm{CH}_0(X)$ is surjective. Then $Z^4(X)$ is finite and there is an isomorphism*

$$Z^4(X) \xrightarrow{\sim} \mathrm{H}_{nr}^3(X, \mathbb{Q}/\mathbb{Z}(2)).$$

Proof. By Proposition (3.2.5), we know that $\mathrm{H}^0(X, \mathscr{H}_B^3(\mathbb{Z}(2)))$ is annihilated by some integer $N \geq 1$, so that in the exact sequence

$$0 \longrightarrow \mathrm{H}^0(X, \mathscr{H}_B^3(\mathbb{Z}(2))) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} \longrightarrow \mathrm{H}^0(X, \mathscr{H}_B^3(\mathbb{Q}/\mathbb{Z}(2))) \longrightarrow Z^4(X) \longrightarrow 0$$

provided by the above theorem, the group $\mathrm{H}^0(X, \mathscr{H}_B^3(\mathbb{Z}(2))) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$ is zero. □

Remark 3.2.12. Note that the proof of Colliot-Thélène and Voisin heavily relies on Rost-Voevodsky's proof of the Bloch-Kato conjecture ; Kahn later showed in [Kah12] that Bloch-Kato in degree 2, *i.e.* the Merkurjev-Suslin theorem, suffices to prove the result, using Jannsen's formalism of continuous ℓ-adic cohomology. However both approaches use the Gersten conjecture proven by Bloch-Ogus. In a very recent paper [Sch23], Schreieder introduced the notion of *refined unramified homology* of varieties and proved comparison theorems that identify some of these groups with groups of algebraic cycles. In particular, he provided a simpler argument that does not need Bloch-Kato in any degree and which does not make use of the Gersten conjecture. Actually, the proof he presented generalizes quite easily to give a similar result in arbitrary codimension and in fact on possibly singular schemes (see [Sch23, Thm. 7.7]).

2.2.2. Failure of the integral Hodge conjecture in degree $2d-2$

— Similarly to codimension 2 cycles, one can identify the group $Z^{2d-2}(X)$ (parametrising the obstruction to the integral Hodge conjecture for cycles of dimension 2 on X) with a suitable Zariski cohomology group of the sheaf $\mathcal{H}_B^{\dim X}(\mathbb{Q}/\mathbb{Z})$. The arguments are essentially symmetric to the ones we discussed earlier :

Theorem 3.2.13. *Let X be a smooth, projective and connected complex variety of dimension d. Then there is an exact sequence*

$$0 \longrightarrow \mathrm{H}^{d-3}_{Zar}(X, \mathcal{H}^d_B(\mathbb{Z}(d-1))) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} \longrightarrow \mathrm{H}^{d-3}_{Zar}(X, \mathcal{H}^d_B(\mathbb{Q}/\mathbb{Z}(d-1))) \longrightarrow Z^{2d-2}(X) \longrightarrow 0,$$

where $Z^{2d-2}(X)$ is finite.

Proof. By Theorem (3.2.3), we have a short exact sequence of Zariski sheaves

$$0 \longrightarrow \mathcal{H}^d_B(\mathbb{Z}(d-1)) \xrightarrow{\cdot n} \mathcal{H}^d_B(\mathbb{Z}(d-1)) \longrightarrow \mathcal{H}^d_B(\mathbb{Z}/n(d-1)) \longrightarrow 0$$

which, by taking cohomology, yields a short exact sequence

$$0 \longrightarrow \mathrm{H}^{d-3}_{\mathrm{Zar}}(X, \mathcal{H}^d_B(\mathbb{Z}(d-1)))/n \longrightarrow \mathrm{H}^{d-3}_{\mathrm{Zar}}(X, \mathcal{H}^d_B(\mathbb{Z}/n(d-1))) \longrightarrow \mathrm{H}^{d-2}_{\mathrm{Zar}}(X, \mathcal{H}^d_B(\mathbb{Z}(d-1)))[n] \longrightarrow 0.$$

Passing to the limit over all $n \geq 1$, we obtain a short exact sequence :

$$0 \longrightarrow \mathrm{H}^{d-3}_{\mathrm{Zar}}(X, \mathcal{H}^d_B(\mathbb{Z}(d-1))) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} \longrightarrow \mathrm{H}^{d-3}_{\mathrm{Zar}}(X, \mathcal{H}^d_B(\mathbb{Q}/\mathbb{Z}(d-1))) \longrightarrow \mathrm{H}^{d-2}_{\mathrm{Zar}}(X, \mathcal{H}^d_B(\mathbb{Z}(d-1)))_{\mathrm{tors}} \longrightarrow 0. \tag{III.1}$$

As in the proof of Theorem (3.2.7), The Leray spectral sequence

$$E_2^{p,q} = \mathrm{H}^p_{\mathrm{Zar}}(X, \mathcal{H}^q_B(\mathbb{Z}(d-1)) \Rightarrow \mathrm{H}^{p+q}_B(X, \mathbb{Z}(d-1))$$

has nonzero terms only in the cone $\{E_2^{p,q} \mid 0 \leq p \leq q \leq d\}$, so that we obtain an exact sequence

$$E_2^{d-3,d} \longrightarrow E_2^{d-1,d-1} \longrightarrow E_\infty^{d-1,d-1} \longrightarrow 0.$$

On the other hand, the image of

$$E_2^{d-1,d-1} = \mathrm{CH}^{d-1}(X)/\mathrm{alg} \longrightarrow E_\infty^{d-1,d-1} = F^{d-1}\,\mathrm{H}^{2d-2}_B(X, \mathbb{Z}(d-1))$$

is the deepest level in the Leray filtration which coincides with the coniveau filtration, so this map is precisely the cycle class $c\ell_{d-1}$. The next term in the filtration of $\mathrm{H}^{2d-2}_B(X, \mathbb{Z}(d-1))$ is given by $E_\infty^{d-2,d}$ which has a surjection by $E_2^{d-2,d}$, so we obtain an exact sequence :

$$\mathrm{H}^{d-3}_{\mathrm{Zar}}(X, \mathcal{H}^d_B(\mathbb{Z}(d-1))) \longrightarrow \mathrm{CH}^{d-1}(X)/\mathrm{alg} \xrightarrow{c\ell_{d-1}} \mathrm{H}^{2d-2}_B(X, \mathbb{Z}(d-1)) \longrightarrow \mathrm{H}^{d-2}_{\mathrm{Zar}}(X, \mathcal{H}^d_B(\mathbb{Z}(d-1))) \longrightarrow 0.$$

By the hard Lefschetz theorem and the Lefschetz theorem on $(1,1)$-classes, see *e.g.* [Voi03, Thm. 1.23] and [Voi02, Thm. 7.2] respectively, we know that $Z^{2d-2}(X)$ is finite and coincides with the torsion of the cokernel of $\mathrm{CH}^{d-1}(X)/\mathrm{alg} \to \mathrm{H}^{2d-2}_B(X, \mathbb{Z}(d-1))$ which is, by the exact sequence (III.1), equal to $\mathrm{H}^{d-2}_{\mathrm{Zar}}(X, \mathcal{H}^d_B(\mathbb{Z}(d-1)))_{\mathrm{tors}}$. □

Corollary 3.2.14. *Let X be a smooth, projective and connected complex variety of dimension d such that there exists a closed subvariety $j : Y \hookrightarrow X$ that is smooth, projective and connected of dimension 2 and such that $j_* : \mathrm{CH}_0(Y) \to \mathrm{CH}_0(X)$ is surjective. Then we have an isomorphism of finite groupd*

$$\mathrm{H}^{d-3}_{Zar}(X, \mathscr{H}^d_B(\mathbb{Q}/\mathbb{Z}(d-1)) \xrightarrow{\sim} Z^{2d-2}(X).$$

Proof. Under the above hypotheses and by Proposition (3.2.5), the group $\mathrm{H}^{d-3}_{\mathrm{Zar}}(X, \mathscr{H}^d_B(\mathbb{Z}(d-1))$ is torsion, so this is a direct consequence of the previous theorem. □

2.3. Some examples obtained through K-theory

— One can apply the results of Colliot-Thélène and Voisin discussed earlier in this chapter in various cases, especially for studying the geometry and the arithmetic of «good» varieties ; notably, one can discuss the vanishing of the group $\mathrm{H}^3_{\mathrm{nr}}(\mathscr{X}, \mathbb{Q}/\mathbb{Z}(2))$ when one considers families of varieties $\mathscr{X} \to B$ over a base of low dimension (the example one should have in mind is a fibration over a smooth, projective and connected curve). Indeed, by controlling the unramified cohomology at the level of the generic fibre, it is possible to deduce some interesting results on the family itself.

2.3.1. Unramified cohomology under fibrations

— Let $\mathscr{X}$ and B be two smooth, projective and connected complex varieties and $f : \mathscr{X} \to B$ a dominant morphism whose generic fibre is smooth and connected. Let $F := \Bbbk(B)$ and write $X := \mathscr{X} \times_B F$. Comparing the Bloch-Ogus resolutions for the sheaf $\mathscr{H}^3_{\text{ét}}(\mathbb{Z}/n(2))$ on X and $\mathscr{X}$ respectively, we obtain a natural injection $\mathrm{H}^3_{\mathrm{nr}}(\mathscr{X}, \mathbb{Z}/n(2)) \hookrightarrow \mathrm{H}^3_{\mathrm{nr}}(X, \mathbb{Z}/n(2))$ for each integer $n \geq 1$. If we thus manage to determine the vanishing of $\mathrm{H}^3_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}(2))$, then Theorem (3.2.7) implies that $Z^4(\mathscr{X})_{\mathrm{tors}} = 0$.

Let F be a field of characteristic zero, X a smooth, projective and geometrically connected variety over F. A first interesting case is the one where X is a quadric :

Theorem 3.2.15 (Kahn, Rost, Sujatha [KRS98, Thm. 5, Cor. 10]). *Let Q be a smooth quadric of dimension at least 1 over F, that is not an Albert form (that is, given by a quadratic form of rank 6 of the form $\langle a, b, ab, -c, -d, -cd\rangle$). Then the restriction map*

$$\mathrm{H}^3_{nr}(F, \mathbb{Q}/\mathbb{Z}(2)) \longrightarrow \mathrm{H}^3_{nr}(Q, \mathbb{Q}/\mathbb{Z}(2))$$

is surjective[23]

When $\mathrm{cd}(F) \leq 2$, we obtain in particular that $\mathrm{H}^3_{\mathrm{nr}}(Q, \mathbb{Q}/\mathbb{Z}(2)) = 0$ unless Q is an Albert quadric. Actually, if Q arises as the generic fibre of a dominant morphism to a surface, then $\mathrm{H}^3_{\mathrm{nr}}(Q, \mathbb{Q}/\mathbb{Z}(2))$ always vanishes :

Corollary 3.2.16. *Let $\mathscr{X} \to B$ be a dominant morphism of smooth, projective and connected complex varieties where $\dim B = 2$, and whose generic fibre is a quadric X of dimension at least 1. Then $\mathrm{H}^3_{nr}(\mathscr{X}, \mathbb{Q}/\mathbb{Z}(2)) = 0$, a fortiori $Z^4(\mathscr{X}) = 0$.*

Proof. If $\dim X \in \{1, 2\}$, then X cannot be an Albert quadric, so the previous theorem provides the vanishing $\mathrm{H}^3_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}(2)) = 0$. If X has dimension at least 3, then it admits a $\Bbbk(B)$-point ; since B is a surface, then $\Bbbk(B)$ is a C_2-field, so this implies that X is birational to a projective space $\mathbb{P}^d_{\Bbbk(B)}$ over $\Bbbk(B)$, and by the stable birational invariance of unramified cohomology, we have an isomorphism

$$\mathrm{H}^3_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}(2)) \xrightarrow{\sim} \mathrm{H}^3_{\mathrm{nr}}(\mathbb{P}^d_{\Bbbk(B)}, \mathbb{Q}/\mathbb{Z}(2)) \xrightarrow{\sim} \mathrm{H}^3(\Bbbk(B), \mathbb{Q}/\mathbb{Z}(2)),$$

and the latter vanishes since $\mathrm{cd}(\Bbbk(B)) = 2$. Hence in any case, we have an injection $\mathrm{H}^3_{\mathrm{nr}}(\mathscr{X}, \mathbb{Q}/\mathbb{Z}(2)) \hookrightarrow \mathrm{H}^3_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}(2))$ and the latter is zero. □

[23] This theorem actually holds in any characteristic different from 2, one just has to replace the coefficients $\mathbb{Q}/\mathbb{Z}(2)$ by $\varinjlim_{\gcd(n, \mathrm{char}\, F)=1} \mathbb{Z}/n(2)$.

2.3.2. Varieties over a field of cohomological dimension one

2.3.2.1. Galois descent for cycles of codimension 2. It is well-known that for a proper, geometrically reduced and geometrically connected variety over a field F, one has an exact sequence :

$$0 \longrightarrow \mathsf{Pic}(X) \longrightarrow \mathsf{Pic}(\overline{X})^{\Gamma_F} \longrightarrow \mathsf{Br}(F) \longrightarrow \ker[\mathsf{Br}(X) \to \mathsf{Br}(\overline{X})] \longrightarrow \mathrm{H}^1(F, \mathsf{Pic}(\overline{X})) \longrightarrow \mathrm{H}^3(F, F_s^\times)$$

(this is nothing more than the exact sequence of lower terms of the Leray spectral sequence $E_2^{p,q} = \mathrm{H}^p(F, \mathrm{H}^q_{\text{ét}}(\overline{X}, \mathbb{G}_m)) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(X, \mathbb{G}_m)$, together with the fact that $\mathrm{H}^0(\overline{X}, \mathbb{G}_m) = F_s^\times$ under the above hypotheses on X). In particular, the natural map $\mathsf{Pic}(X) \to \mathsf{Pic}(\overline{X})^{\Gamma_F}$ is always injective, and its cokernel is controlled by the constant classes in $\mathsf{Br}(X)$. However, if we instead consider codimension 2 cycles on X, then the corresponding map is in general neither injective nor surjective. In this paragraph, under the assumption that $\mathrm{cd}(F) \leq 1$ (and supposing that $\mathrm{char}(F) = 0$ for convenience about torsion problems), we explain how to derive a similar exact sequence for codimension 2 cycles on X by replacing the Brauer group with unramified cohomology of degree 3.

Proposition 3.2.17. *Let X be a geometrically integral variety over F.*

(i) *There exists a natural isomorphism :*

$$\ker[\mathrm{H}^3(\Bbbk(X), \mathbb{Q}/\mathbb{Z}(2)) \to \mathrm{H}^3(\Bbbk(\overline{X}), \mathbb{Q}/\mathbb{Z}(2))] \xrightarrow{\sim} \mathrm{H}^2(F, K_2(\Bbbk(\overline{X}))).$$

(ii) *If X is furthermore smooth, then this isomorphism induces an isomorphism :*

$$\ker[\mathrm{H}^0(X, \mathscr{H}^3_{\acute{e}t}(\mathbb{Q}/\mathbb{Z}(2))) \to \mathrm{H}^0(\overline{X}, \mathscr{H}^3_{\acute{e}t}(\mathbb{Q}/\mathbb{Z}(2)))] \xrightarrow{\sim} \ker\left[\mathrm{H}^2(F, K_2(\Bbbk(\overline{X}))) \to \mathrm{H}^2(F, \oplus_{x \in \overline{X}^{(1)}} K_1(\kappa(x)))\right].$$

Proof. By the Merkurjev-Suslin theorem (Bloch-Kato in degree 2), the Galois symbol provides a Γ_F-equivariant isomorphism $K_2(\Bbbk(\overline{X})) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} \xrightarrow{\sim} \mathrm{H}^2(\Bbbk(\overline{X}), \mathbb{Q}/\mathbb{Z}(2))$. On the other hand, we have the tautological exact sequence defining torsion :

$$0 \longrightarrow K_2(\Bbbk(\overline{X}))_{\text{tors}} \longrightarrow K_2(\Bbbk(\overline{X})) \longrightarrow K_2(\Bbbk(\overline{X})) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow K_2(\Bbbk(\overline{X})) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} \longrightarrow 0,$$

which we can split into two short exact sequences of Γ_F-modules

$$0 \longrightarrow K_2(\Bbbk(\overline{X}))_{\text{tors}} \longrightarrow K_2(\Bbbk(\overline{X})) \longrightarrow L \longrightarrow 0$$

and

$$0 \longrightarrow L \longrightarrow K_2(\Bbbk(\overline{X})) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow K_2(\Bbbk(\overline{X})) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

Taking the long exact cohomology sequence associated to the second one, and using the fact that $K_2(\Bbbk(\overline{X})) \otimes_{\mathbb{Z}} \mathbb{Q}$ is acyclic, we obtain an isomorphism $\mathrm{H}^1(F, K_2(\Bbbk(\overline{X})) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^2(F, L)$. On the other hand, the first sequence provides an exact portion

$$\mathrm{H}^2(F, K_2(\Bbbk(\overline{X}))_{\text{tors}}) \longrightarrow \mathrm{H}^2(F, K_2(\Bbbk(\overline{X}))) \longrightarrow \mathrm{H}^2(F, L) \longrightarrow \mathrm{H}^3(F, K_2(\Bbbk(\overline{X}))_{\text{tors}}),$$

and the side terms are zero since $\mathrm{cd}(F) \leq 1$. Therefore, we obtain an isomorphism

$$\mathrm{H}^2(F, K_2(\Bbbk(\overline{X}))) \xrightarrow{\sim} \mathrm{H}^1(F, K_2(\Bbbk(\overline{X})) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}).$$

Taking these identifications in account, the Hochschild-Serre spectral sequence

$$E_2^{p,q} = \mathrm{H}^p(F, \mathrm{H}^q(\Bbbk(\overline{X}, \mathbb{Q}/\mathbb{Z}(2)))) \Rightarrow \mathrm{H}^{p+q}(\Bbbk(X), \mathbb{Q}/\mathbb{Z}(2))$$

provides a homomorphism

$$\ker[\mathrm{H}^3(\Bbbk(X), \mathbb{Q}/\mathbb{Z}(2)) \to \mathrm{H}^3(\Bbbk(\overline{X}), \mathbb{Q}/\mathbb{Z}(2))] \longrightarrow \mathrm{H}^1(F, \mathrm{H}^2(\Bbbk(\overline{X}), \mathbb{Q}/\mathbb{Z}(2)))).$$

Indeed, we have an exact sequence

$$0 \longrightarrow \frac{F^1\,\mathrm{H}^3(\Bbbk(X),\mathbb{Q}/\mathbb{Z}(2))}{F^2\,\mathrm{H}^3(\Bbbk(X),\mathbb{Q}/\mathbb{Z}(2))} \longrightarrow E_2^{1,2} \longrightarrow E_2^{3,1},$$

and the right term is zero since $\mathrm{cd}(F) = 1$; moreover we have an exact sequence

$$0 \longrightarrow F^1\,\mathrm{H}^3(\Bbbk(X),\mathbb{Q}/\mathbb{Z}(2)) \longrightarrow \mathrm{H}^3(\Bbbk(X),\mathbb{Q}/\mathbb{Z}(2)) \longrightarrow E_\infty^{0,3} \longrightarrow 0$$

where $E_\infty^{0,3} = \ker[E_2^{0,3} \to E_2^{2,2}] = E_2^{0,3}$. From this we also see that the desired morphism is bijective, hence the first claim of the proposition.

Now let us assume that X is smooth over F. If we fix a point $x \in X^{(1)}$, one can look at the semi-local ring $\mathcal{O}_{\overline{X},x}$ at the points of $\overline{X}$ above x. By Hoobler's trick (see Chapter I, §3.2.2.4), we have a Γ_F-equivariant isomorphism

$$K_2(\mathcal{O}_{\overline{X},x}) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} \xrightarrow{\sim} \mathrm{H}^2_{\text{ét}}(\mathcal{O}_{\overline{X},x}, \mathbb{Q}/\mathbb{Z}(2)).$$

By the same arguments as before, we have an isomorphism :

$$\ker[\mathrm{H}^3_{\text{ét}}(\mathcal{O}_{X,x},\mathbb{Q}/\mathbb{Z}(2)) \to \mathrm{H}^3_{\text{ét}}(\mathcal{O}_{\overline{X},x},\mathbb{Q}/\mathbb{Z}(2))] \xrightarrow{\sim} \mathrm{H}^2(F, K_2(\mathcal{O}_{\overline{X},x})).$$

On the other hand, as x has codimension 1 in X, we know that Quillen's proof of the Gersten conjecture (Chapter I, §3.2.2.2, Theorem (I.3.59)) applied to $\mathcal{O}_{\overline{X},x}$ provides a short exact sequence of Γ_F-modules

$$0 \longrightarrow K_2(\mathcal{O}_{\overline{X},x}) \longrightarrow K_2(\Bbbk(\overline{X})) \longrightarrow \overline{F} \otimes_F \kappa(x) \longrightarrow 0.$$

Therefore, the kernel of

$$\mathrm{H}^2(F, K_2, (\Bbbk(\overline{X}))) \longrightarrow \mathrm{H}^2(F, \oplus_{x\in X^{(1)}} \overline{F} \otimes_F \kappa(x))$$

can be identified with the subgroup of $\mathrm{H}^2(F, K_2(\Bbbk(\overline{X})))$ consisting of elements that lie in the image of $\mathrm{H}^2(F, K_2(\mathcal{O}_{\overline{X},x}))$ for every $x \in X^{(1)}$. By Bloch-Ogus theory (Chapter I, §4.2.1, Corollary (I.4.73)), we know that $\mathrm{H}^0(X, \mathcal{H}^3_{\text{ét}}(\mathbb{Q}/\mathbb{Z}(2)))$ consists of the classes in $\mathrm{H}^3(\Bbbk(X),\mathbb{Q}/\mathbb{Z}(2))$ that lie in the image of $\mathrm{H}^3_{\text{ét}}(\mathcal{O}_{X,x},\mathbb{Q}/\mathbb{Z}(2))$ for every $x \in X^{(1)}$. This shows that the two kernels $\ker[\mathrm{H}^0(X, \mathcal{H}^3_{\text{ét}}(\mathbb{Q}/\mathbb{Z}(2))) \to \mathrm{H}^0(\overline{X}, \mathcal{H}^3_{\text{ét}}(\mathbb{Q}/\mathbb{Z}(2)))]$ and $\ker\big[\mathrm{H}^2(F, K_2(\Bbbk(\overline{X}))) \to \mathrm{H}^2(F, \oplus_{x\in\overline{X}^{(1)}} K_1(\kappa(x)))\big]$ are equal. □

Theorem 3.2.18 (Compare [CTS21, §5.4.1, (5.20)]). *Let X be a smooth, projective and geometrically integral variety over F. There exists an exact sequence :*

$$\begin{tikzcd}
0 \arrow[r] & \ker[\mathrm{CH}^2(X) \to \mathrm{CH}^2(\overline{X})^{\Gamma_F}] \arrow[r] & \mathrm{H}^1(F, \mathrm{H}^1_{Zar}(\overline{X}, \mathcal{K}_2)) \arrow[dll, rounded corners] \\
\ker[\mathrm{H}^3_{nr}(X,\mathbb{Q}/\mathbb{Z}(2)) \to \mathrm{H}^3_{nr}(\overline{X},\mathbb{Q}/\mathbb{Z}(2))] \arrow[r] & \operatorname{coker}[\mathrm{CH}^2(X) \to \mathrm{CH}^2(\overline{X})^{\Gamma_F}] \arrow[r] & 0.
\end{tikzcd}$$

Proof. If E/F is a Galois extension with group G and V is a general smooth and geometrically integral variety over F, then by Theorem (I.3.59) the homology of the complex

$$0 \longrightarrow K_2(\Bbbk(V_E)) \longrightarrow \bigoplus_{x\in V_E^{(1)}} K_1(\kappa(x)) \longrightarrow \bigoplus_{x\in V_E^{(2)}} K_0(\kappa(x)) \longrightarrow 0$$

in degree i is precisely $\mathrm{H}^i_{\mathrm{Zar}}(X_E, \mathcal{K}_2)$. If we denote by Z the kernel of the second arrow and I its image, then we obtain three short exact sequences of G-modules

$$0 \longrightarrow Z \longrightarrow \bigoplus_{x\in V_E^{(1)}} \kappa(x)^\times \longrightarrow I \longrightarrow 0,$$

and

$$0 \longrightarrow \frac{K_2(\Bbbk(V_E))}{\mathrm{H}^0(V_E, \mathcal{K}_2)} \longrightarrow Z \longrightarrow \mathrm{H}^1_{\mathrm{Zar}}(V_E, \mathcal{K}_2) \longrightarrow 0$$

and

$$0 \longrightarrow I \longrightarrow \bigoplus_{x \in V_E^{(2)}} \mathbb{Z} \longrightarrow \mathrm{CH}^2(V_E) \longrightarrow 0.$$

On the other hand,

$$\mathrm{H}^1(G, \oplus_{x \in V^{(2)}} \mathbb{Z}) = \mathrm{H}^1(G, (\oplus_{x \in V_E^{(2)}} \mathbb{Z})^G) = 0$$

by Shapiro's lemma (see [GS17, Cor. 3.3.2]). Similarly, by Hilbert 90 and Shapiro's lemma we have that $\mathrm{H}^1(G, \oplus_{x \in V_E^{(2)}} \kappa(x)^\times) = 0$. Taking the long exact sequences associated to the three previous exact sequences (and using Bloch's formula $\mathrm{H}^2(V, \mathcal{K}_2) \simeq \mathrm{CH}^2(V)$), we obtain two exact portions :

$$0 \longrightarrow \mathrm{H}^1(G, Z) \longrightarrow \mathrm{CH}^2(V) \longrightarrow \mathrm{CH}^2(V_E)^G \longrightarrow \mathrm{H}^1(G, I) \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{H}^1(G, I) \longrightarrow \mathrm{H}^2(G, Z) \longrightarrow \mathrm{H}^2(G, \oplus_{x \in V_E^{(1)}} \kappa(x)^\times).$$

Passing to the limit over all Galois subextensions of $\overline{F}/F$, we deduce the same result for $E = \overline{F}$. Now let us specialise this to a variety X as in the theorem. By [CTR85, Thm. 1.8], we know that $\mathrm{H}^0(\overline{X}, \mathcal{K}_2)$ is the extension of a finite group by a divisible group, and similarly for $\mathrm{H}^1_{\mathrm{Zar}}(\overline{X}, \mathcal{K}_2)$ (see [CTR85, Thm. 2.2]), we have an extension

$$0 \longrightarrow D \longrightarrow \mathrm{H}^1_{\mathrm{Zar}}(\overline{X}, \mathcal{K}_2) \longrightarrow \bigoplus_{\ell \text{ prime}} \mathrm{H}^3_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(2))\{\ell\} \longrightarrow 0,$$

where D is divisible and the right hand side is finite (as X is smooth, proper and connected over a separably closed field, this follows by smooth base change [Mil80, Chap. VI, Cor. 4.2] and comparison with Betti cohomology). In particular, as $\mathrm{cd}(F) \leq 1$, then we obtain the vanishing of $\mathrm{H}^r(F, \mathrm{H}^i_{\mathrm{Zar}}(\overline{X}, \mathcal{K}_2))$ for $i = 0, 1$ and $r \geq 2$. On the other hand, a version of Hilbert 90 for K_2 (see [GS17, Thm. 8.4.1]) shows that $\mathrm{H}^1(F, K_2(\Bbbk(\overline{X}))) = 0$, so that the map $\mathrm{H}^2(F, K_2(\Bbbk(\overline{X}))) \to \mathrm{H}^2(F, K_2(\Bbbk(\overline{X})))/\mathrm{H}^0(\overline{X}, \mathcal{K}_2)$ is an isomorphism. Putting everything together, we derive an exact sequence :

$$0 \longrightarrow \mathrm{H}^1(F, Z) \longrightarrow \mathrm{H}^1(F, \mathrm{H}^1_{\mathrm{Zar}}(\overline{X}, \mathcal{K}_2)) \longrightarrow \mathrm{H}^2(F, K_2(\Bbbk(\overline{X}))) \longrightarrow \mathrm{H}^2(F, Z) \longrightarrow 0,$$

where the left term is identified with $\ker[\mathrm{CH}^2(X) \to \mathrm{CH}^2(\overline{X})]$. We have a map $\mathrm{H}^2(F, K_2(\Bbbk(\overline{X}))) \to \mathrm{H}^2(F, \oplus_{x \in \overline{X}^{(1)}} \kappa(x)^\times)$ induced by residues in codimension 1, which therefore induces a map $\mathrm{H}^2(F, Z) \to \mathrm{H}^2(F, \oplus_{x \in \overline{X}^{(1)}} \kappa(x)^\times)$ and provides the exact sequence :

$$\begin{tikzcd} 0 \arrow[r] & \ker[\mathrm{CH}^2(X) \to \mathrm{CH}^2(\overline{X})^{\Gamma_F}] \arrow[r] & \mathrm{H}^1(F, \mathrm{H}^1_{\mathrm{Zar}}(\overline{X}, \mathcal{K}_2)) \arrow[dll, rounded corners] \\ \ker[\mathrm{H}^2(F, K_2(\Bbbk(\overline{X}))) \to \mathrm{H}^2(F, \oplus_{x \in \overline{X}^{(1)}} \kappa(x)^\times)] \arrow[r] & \ker[\mathrm{H}^2(F, Z) \to \mathrm{H}^2(F, \oplus_{x \in \overline{X}^{(1)}} \kappa(x)^\times)] \arrow[r] & 0, \end{tikzcd}$$

which is precisely the exact sequence we are looking for (indeed, the previous proposition provides the desired indentifications). □

2.3.2.2. Controlling $\mathcal{K}_2$-cohomology. In order to understand the map $\mathrm{CH}^2(X) \to \mathrm{CH}^2(\overline{X})^{\Gamma_F}$ in the above theorem, the subsidiary problem is to control the group $\mathrm{H}^1_{\mathrm{Zar}}(\overline{X}, \mathcal{K}_2)$. Thanks to the work of Colliot-Thélène and Raskind in [CTR85], we have the following result (note that the first point below was used in the proof of the previous theorem) :

Theorem 3.2.19 (Colliot-Thélène-Raskind [CTR85, Thm. 2.1, Thm. 2.2, Thm. 2.12]). *Let X be a smooth, projective and geometrically integral variety over F, and $M := \bigoplus_{\ell\ prime} \mathrm{H}^3_{ét}(\overline{X}, \mathbb{Z}_\ell(2))\{\ell\}$. Then :*

(i) *There exists a natural exact sequence*

$$0 \longrightarrow D \longrightarrow \mathrm{H}^1_{Zar}(\overline{X}, \mathcal{K}_2) \longrightarrow M \longrightarrow 0,$$

where M is finite and D is divisible.

(ii) *For every prime number ℓ, there exists a natural isomorphism :*

$$\mathrm{H}^2_{ét}(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) \xrightarrow{\sim} \mathrm{H}^1_{Zar}(\overline{X}, \mathcal{K}_2)\{\ell\}.$$

(iii) *If K (resp. C) denotes the kernel (resp. cokernel) of the natural map $\mathsf{Pic}(\overline{X}) \otimes \overline{F}^\times \to \mathrm{H}^1(\overline{X}, \mathcal{K}_2)$, then under the hypothesis $\mathrm{H}^2(X, \mathcal{O}_X) = 0$, the group K is uniquely divisible (resp. C is the direct sum of M and a uniquely divisible group).*

Remark 3.2.20. In particular, if there exists an integer $i \geq 1$ such that $\mathrm{cd}(F) \leq i$, then the groups $\mathrm{H}^r(F, \mathrm{H}^0(\overline{X}, \mathcal{K}_2))$ and $\mathrm{H}^r(F, \mathrm{H}^1_{\mathrm{Zar}}(\overline{X}, \mathcal{K}_2))$ are zero for $r \geq i+1$.

If we thus assume that $\mathrm{H}^2(X, \mathcal{O}_X) = 0$ (for instance if X is rationally connected, see *e.g.* [Deb01, Cor. 4.18] for a proof of this claim), then we can precisely describe the groups arising in the long exact sequence of Theorem (3.2.18) as follows :

Corollary 3.2.21 (Colliot-Thélène-Voisin). *Let X be a smooth, projective and geometrically integral variety over F such that $\mathrm{H}^2(X, \mathcal{O}_X) = 0$. There is an exact sequence :*

$$\begin{aligned} 0 \longrightarrow \ker[\mathrm{CH}^2(X) \to \mathrm{CH}^2(\overline{X})^{\Gamma_F}] \longrightarrow \mathrm{H}^1(F, \textstyle\bigoplus_{\ell\ prime} \mathrm{H}^3_{ét}(\overline{X}, \mathbb{Z}_\ell(2))\{\ell\}) \\ \longrightarrow \ker[\mathrm{H}^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2)) \to \mathrm{H}^3_{nr}(\overline{X}, \mathbb{Q}/\mathbb{Z}(2))] \longrightarrow \mathrm{coker}[\mathrm{CH}^2(X) \to \mathrm{CH}^2(\overline{X})^{\Gamma_F}] \longrightarrow 0. \end{aligned}$$

Remark 3.2.22. The same exact sequence arises unconditionally if we replace F by a finite field k. However the proof crucially relies on the finiteness of $\mathrm{CH}^2(X)_{\mathrm{tors}}$, and the latter fact has been proved thanks to the finiteness results of Deligne for ℓ-adic cohomology (Weil conjectures, with twisted coefficients). Indeed, using a Hochschild-Serre spectral sequence, one shows that if $j \neq 2i, 2i+1$, then $\mathrm{H}^j_{\text{ét}}(X, \mathbb{Q}_\ell(i)) = 0$ and $\mathrm{H}^j_{\text{ét}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))$ is finite (the second fact is also proven in [Del77, Th. finitude]). On the other hand $\mathrm{CH}^2(X)\{\ell\}$ is a sub-quotient of $\mathrm{H}^3_{\text{ét}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$ (this is essentially given by Bloch-Ogus theory, see [CT93, Thm. 3.2.2]) so it is finite, and there are finitely many prime numbers $\ell \neq \mathrm{char}(k)$ such that $\mathrm{H}^3_{\text{ét}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) \neq 0$. Finally, if $p = \mathrm{char}(k)$, then Gabber showed that the p-torsion of $\mathrm{CH}^2(X)$ is always finite, but the argument is more delicate (it uses the crystalline comparison theorem of Illusie on the de Rham-Witt complex together with a result of Bloch-Gabber-Kato), see [CTSS83, §1.4] for details.

Proof. This essentially directly follows from the above theorem. Indeed, keeping the corresponding notations, we have an exact sequence of Γ_F-modules

$$0 \longrightarrow K \longrightarrow \mathsf{Pic}(\overline{X}) \otimes \overline{F}^\times \longrightarrow \mathrm{H}^1_{\mathrm{Zar}}(\overline{X}, \mathcal{K}_2) \longrightarrow C \longrightarrow 0$$

(where K is uniquely divisible and $C = M \oplus C'$ where $M = \bigoplus_{\ell\ \mathrm{prime}} \mathrm{H}^3_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(2))\{\ell\}$ and C' is uniquely divisible), which we can break into two exact sequences

$$0 \longrightarrow K \longrightarrow \mathsf{Pic}(\overline{X}) \otimes \overline{F}^\times \longrightarrow R \longrightarrow 0$$

and

$$0 \longrightarrow R \longrightarrow \mathrm{H}^1_{\mathrm{Zar}}(\overline{X}, \mathcal{K}_2) \longrightarrow C \longrightarrow 0$$

where R is divisible. As X is smooth and proper, we have the usual short exact sequence

$$0 \longrightarrow \underline{\mathbf{Pic}}^0_{\overline{X}/\overline{F}}(\overline{F}) \longrightarrow \mathsf{Pic}(\overline{X}) \longrightarrow \mathsf{NS}(\overline{X}) \longrightarrow 0$$

which gives a long exact sequence

$$\cdots \longrightarrow \mathrm{Tor}^1(\mathsf{NS}(\overline{X}), \overline{F}^\times) \longrightarrow \underline{\mathbf{Pic}}^0_{\overline{X}/\overline{F}}(\overline{F}) \otimes \overline{F}^\times \longrightarrow \mathsf{Pic}(\overline{X}) \otimes \overline{F}^\times \longrightarrow \mathsf{NS}(\overline{X}) \otimes \overline{F}^\times \longrightarrow 0.$$

We therefore deduce two short exact sequences

$$0 \longrightarrow P \longrightarrow \underline{\mathbf{Pic}}^0_{\overline{X}/\overline{F}}(\overline{F}) \otimes \overline{F}^\times \longrightarrow Q \longrightarrow 0$$

and

$$0 \longrightarrow Q \longrightarrow \underline{\mathbf{Pic}}^0_{\overline{X}/\overline{F}}(\overline{F}) \otimes \overline{F}^\times \longrightarrow \mathsf{NS}(\overline{X}) \otimes \overline{F}^\times \longrightarrow 0$$

where P is finite and Q is divisible. Now remark that $\underline{\mathbf{Pic}}^0_{\overline{X}/\overline{F}}(\overline{F}) \otimes \overline{F}^\times$ is uniquely divisible as a tensor product of two divisible groups, so it is cohomologically acyclic. Moreover, since $\mathrm{cd}(F) \leq 1$, then for any Γ_F-module W that is finite or divisible, we have $\mathrm{H}^2(F, W) = 0$; we also have that

$$\mathrm{H}^1(F, \mathsf{NS}(\overline{X}) \otimes \overline{F}^\times) = \mathrm{H}^1(F, (\mathsf{NS}(\overline{X})/\mathsf{NS}(\overline{X})_{\mathrm{tors}}) \otimes \overline{F}^\times) = 0$$

because $(\mathsf{NS}(\overline{X})/\mathsf{NS}(\overline{X})_{\mathrm{tors}}) \otimes \overline{F}^\times$ is a torus (since $\mathsf{NS}(\overline{X})$ is finitely generated) and $\mathrm{cd}(F) \leq 1$, see *e.g.* [Ser97, Chap. III, §2.2, Thm. 1]. Therefore, taking the long exact cohomology sequences associated to the two previous exact sequences, we obtain that $\mathrm{H}^1(F, Q) = 0$, thus $\mathrm{H}^1(F, \mathsf{Pic}(\overline{X}) \otimes \overline{F}^\times) = 0$. Similarly, as K is uniquely divisible and given the previous vanishing, we obtain that $\mathrm{H}^1(F, R) = \mathrm{H}^2(F, R) = 0$. Finally, taking the long exact cohomology sequence

$$\cdots \longrightarrow \mathrm{H}^1(F, R) \longrightarrow \mathrm{H}^1(F, \mathrm{H}^1_{\mathrm{Zar}}(\overline{X}, \mathcal{K}_2)) \longrightarrow \mathrm{H}^1(F, C') \oplus \mathrm{H}^1(F, M) \longrightarrow \mathrm{H}^1(F, R) \longrightarrow \cdots,$$

we obtain an isomorphism $\mathrm{H}^1(F, \mathrm{H}^1_{\mathrm{Zar}}(\overline{X}, \mathcal{K}_2)) \xrightarrow{\sim} \mathrm{H}^1(F, M)$, hence the desired result. ◻

2.3.2.3. Some vanishing results. We now discuss the vanishing of $\mathrm{H}^1(F, \bigoplus_\ell \mathrm{H}^3_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(2))\{\ell\})$. It is well-known that the coefficient group $\bigoplus_\ell \mathrm{H}^3_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(2))\{\ell\}$ is closely related to the Brauer group of $\overline{X}$, *a fortiori* to the geometry of X (*e.g.* the vanishing of $\mathrm{H}^2(X, \mathcal{O}_X)$), see [CTS21, Chap. 5, §5.1–§5.5] for a detailed treatment. When $\overline{X}$ is (stably) rational, one can then describe the cokernel of $\mathrm{CH}^2(X) \to \mathrm{CH}^2(\overline{X})^{\Gamma_F}$ in terms of $\mathrm{H}^3_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}(2))$.

Corollary 3.2.23 (Colliot-Thélène-Voisin). *Let X be a smooth, projective and geometrically integral variety over F. Then :*

(i) *If* $\mathsf{Br}(\overline{X}) = 0$*, then we have an exact sequence :*

$$0 \longrightarrow \mathrm{CH}^2(X) \longrightarrow \mathrm{CH}^2(\overline{X})^{\Gamma_F} \longrightarrow \mathrm{H}^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2)) \longrightarrow \mathrm{H}^3_{nr}(\overline{X}, \mathbb{Q}/\mathbb{Z}(2)).$$

(ii) *If (furthermore) $\overline{X}$ is rational, then we have an exact sequence :*

$$0 \longrightarrow \mathrm{CH}^2(X) \longrightarrow \mathrm{CH}^2(\overline{X})^{\Gamma_F} \longrightarrow \mathrm{H}^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2)) \longrightarrow 0.$$

(iii) *If $\overline{X}$ is rationally connected of dimension* 3 *and such that $\mathrm{H}^3_B(\overline{X}, \mathbb{Z})$ is torsion-free, then we have an exact sequence*

$$0 \longrightarrow \mathrm{CH}^2(X) \longrightarrow \mathrm{CH}^2(\overline{X})^{\Gamma_F} \longrightarrow \mathrm{H}^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2)) \longrightarrow 0.$$

Proof. First note that we have an extension

$$0 \longrightarrow (\mathbb{Q}/\mathbb{Z})^{b_2-\rho} \longrightarrow \mathsf{Br}(\overline{X}) \longrightarrow \bigoplus_{\ell \text{ prime}} \mathrm{H}^3_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(1))\{\ell\} \longrightarrow 0$$

where $\rho := \operatorname{rk} \mathsf{NS}(\overline{X})$ and $b_2 := \operatorname{rk} \mathrm{H}^2_B(\overline{X}, \mathbb{Z})$, see *e.g.* [CTS21, Prop. 5.2.9] for a proof of this claim. Therefore, the vanishing of $\mathsf{Br}(\overline{X})$ ensures (in characteristic 0) that $b_2 = \rho$, which is equivalent to the fact that $\mathrm{H}^2(\overline{X}, \mathcal{O}_{\overline{X}}) = 0$ (the exponential sequence provides an exact portion $0 \longrightarrow \mathsf{NS}(\overline{X}) \longrightarrow \mathrm{H}^2_B(\overline{X}, \mathbb{Z}) \longrightarrow \mathrm{H}^2(\overline{X}, \mathcal{O}_{\overline{X}})$; the converse implication follows from Hodge theory, see *e.g.* [Voi02, Thm. 7.2]), so the previous corollary applies and (i) holds. Now, the conditions $\mathsf{Br}(\overline{X}) = 0$ and $\mathrm{H}^3_{\mathrm{nr}}(\overline{X}, \mathbb{Q}/\mathbb{Z}(2)) = 0$ are satisfied when $\overline{X}$ is rational since both groups are (stable) birational invariants of smooth and proper connected varieties (for the Brauer group, see [CTS21, Cor. 6.2.11] ; for unramified cohomology, this has been discussed in Chapter I, §5.1.3.2, Corollary (1.5.98)). Therefore, the exact sequence in (i) arises as soon as $\overline{X}$ is stably rational, and we obtain (ii). Finally, if $\overline{X}$ is rationally connected, then $\mathrm{H}^2(X, \mathcal{O}_X) = 0$ (see [Deb01, Cor. 4.18]). In particular, the invariance of étale cohomology under separably closed extensions (*cf.* [Mil80, Chap. VI, Cor. 4.3]) and the usual comparison with Betti cohomology provide an inclusion

$$\mathsf{Br}(\overline{X}) \simeq \bigoplus_{\ell \text{ prime}} \mathrm{H}^3_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(1))\{\ell\} \hookrightarrow \mathrm{H}^3_B(\overline{X}, \mathbb{Z})_{\text{tors}} = 0,$$

so we have an exact sequence as in (i). Now it is a (highly) non-trivial result that a rationally connected threefold X verifies $\mathrm{H}^3_{\mathrm{nr}}(\overline{X}, \mathbb{Q}/\mathbb{Z}(2)) = 0$ (see [CTV12, Cor. 6.2] for more details[23]). □

2.3.2.4. The case of surfaces. Let X be a variety over an arbitrary field k. The *index* of X is defined as the integer $I(X) := \gcd\{\deg_k z \mid z \in \mathrm{CH}_0(X)\}$ (one can alternatively only consider the gcd of the degrees over k of all the closed points on X). By definition, we thus have that $I(X) = 1$ if and only if X admits a 0-cycle of degree 1 (this is for instance true when X admits a decomposition of the diagonal, see Chapter I, §1.4.1, Lemma (1.1.11)). It is well-known that the index is a birational invariant of smooth varieties, see [GLL13, Prop. 6.8]. In the specific situation where $k = F$ has characteristic 0 and $\mathrm{cd}(F) = 1$, and X is a surface, then under rather mild geometric conditions one can control the index in terms of the unramified cohomology of X.

Proposition 3.2.24 (Colliot-Thélène-Voisin). *Let X be a smooth, projective and geometrically integral surface over F such that $\mathrm{H}^1(X, \mathcal{O}_X) = 0$. Then if $\mathrm{H}^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2)) = 0$, we have that $I(X) = 1$.*

Proof. Consider the tautological exact sequence of Γ_F-modules :

$$0 \longrightarrow A_0(\overline{X}) \longrightarrow \mathrm{CH}_0(\overline{X}) \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0.$$

As $\mathrm{H}^1(X, \mathcal{O}_X) = 0$, we have that $\underline{\mathbf{Alb}}_{\overline{X}/\overline{F}} = 0$ so Roitman's theorem (see Appendix C, §C.1) shows that $A_0(\overline{X})$ is torsion-free (since char $F = 0$). Moreover $A_0(\overline{X})$ is always divisible, see *e.g.* [Blo10, Lem. 1.3], so it is in particular uniquely divisible, hence cohomologically acyclic. Taking the long exact sequence attached to the above exact sequence, we obtain that the degree map $\mathrm{CH}_0(\overline{X})^{\Gamma_F} \to \mathbb{Z}$ is surjective. On the other hand, we may apply Theorem (3.2.18) and use the vanishing of $\mathrm{H}^3_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}(2))$ in order to obtain that $\operatorname{coker}[\mathrm{CH}_0(X) \to \mathrm{CH}_0(\overline{X})^{\Gamma_F}] = 0$ (as X is a surface), so we get a surjection $\mathrm{CH}_0(X) \to \mathrm{CH}_0(\overline{X})^{\Gamma_F}$. This implies the claim of the proposition by definition of the index. □

[23] One first proves the claim for $F = \mathbb{C}$. Indeed, a rationally connected complex variety is uniruled : there is a rational curve passing through every point $x \in X$, so the evaluation map $\mathrm{ev} : \mathbb{P}^1_{\mathbb{C}} \times_{\mathbb{C}} \mathrm{Mor}(\mathbb{P}^1_{\mathbb{C}}, X) \to X$ is dominant. As $\mathrm{Mor}(\mathbb{P}^1_{\mathbb{C}}, X)$ has at most countably many components (see [Deb01, §2.1]) and X is irreducible, then there is a dominant map $\mathbb{P}^1_{\mathbb{C}} \times_{\mathbb{C}} Y \to X$ where $Y \subset \mathrm{Mor}(\mathbb{P}^1_{\mathbb{C}}, X)$ is irreducible of dimension $\leq \dim X - 1$, as desired. Moreover, Voisin proved in [Voi04, Thm. 2] that the integral Hodge conjecture for codimension 2 cycles is true for uniruled threefolds. By Corollary (3.2.11), we thus have $\mathrm{H}^3_{\mathrm{nr}}(\overline{X}, \mathbb{Q}/\mathbb{Z}(2)) = 0$; then one can use the fact that unramified cohomology with torsion coefficients is invariant under extensions of algebraically closed fields, *cf.* [CT95, Thm. 4.4.1].

Proposition 3.2.25 (Colliot-Thélène-Voisin). *Let X be a smooth, projective and geometrically integral surface over F and $n \geq 2$ an integer such that :*

(i) $\mathrm{H}^1(X, \mathcal{O}_X) = 0$;

(ii) $\mathsf{NS}(\overline{X})[n] = 0$;

(iii) $\mathrm{H}^3_{nr}(X, \mathbb{Z}/n(2)) = 0$.

Then $I(V)$ is coprime to n.

Proof. As discussed before, Bloch-Ogus theory applied to the Leray spectral sequence $E_2^{p,q} = \mathrm{H}^p_{\mathrm{Zar}}(X, \mathcal{H}^q_{\text{ét}}(\mathbb{Z}/n(2)) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(X, \mathbb{Z}/n(2))$ provides an exact sequence of lower terms :

$$\mathrm{H}^3_{\text{ét}}(X, \mathbb{Z}/n(2)) \longrightarrow \mathrm{H}^3_{\mathrm{nr}}(X, \mathbb{Z}/n(2)) \longrightarrow \mathrm{CH}^2(X)/n \longrightarrow \mathrm{H}^4_{\text{ét}}(X, \mathbb{Z}/n(2))$$

where the map on the right is the cycle class. The hypothesis (iii) thus shows that $c\ell_2 : \mathrm{CH}^2(X)/n \to \mathrm{H}^4_{\text{ét}}(X, \mathbb{Z}/n(2))$ is injective. On the other hand, since X is a surface, then Poincaré duality provides an isomorphism $\mathrm{CH}^2(\overline{X})/n \xrightarrow{\sim} \mathrm{H}^4_{\text{ét}}(\overline{X}, \mathbb{Z}/n(2)) \simeq \mathbb{Z}/n$. On the other hand, the Leray spectral sequence for $X \to \mathsf{Spec}\, F$ and $\mathbb{Z}/n(2)$ reads

$$E_2^{p,q} = \mathrm{H}^p(F, \mathrm{H}^q_{\text{ét}}(\overline{X}, \mathbb{Z}/n(2))) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(X, \mathbb{Z}/n(2)).$$

By the finiteness of étale cohomology with torsion coefficients over a separably closed field (see [Mil80, Chap. VI, Cor. 2.8]) and using the fact that $\mathrm{cd}(F) \leq 1$, we derive an exact sequence of lower terms

$$0 \longrightarrow \mathrm{H}^1(F, \mathrm{H}^3_{\text{ét}}(\overline{X}, \mathbb{Z}/n(2))) \longrightarrow \mathrm{H}^4_{\text{ét}}(X, \mathbb{Z}/n(2)) \longrightarrow \mathrm{H}^4_{\text{ét}}(\overline{X}, \mathbb{Z}/n(2))^{\Gamma_F} \longrightarrow 0.$$

The hypotheses (i) and (ii) show that $\mathsf{Pic}(\overline{X})[n] \simeq \mathsf{NS}(\overline{X})[n] = 0$ (by invoking the usual exact sequence $0 \longrightarrow \underline{\mathbf{Pic}}^0_{X/F}(\overline{F}) \longrightarrow \mathsf{Pic}(\overline{X}) \longrightarrow \mathsf{NS}(\overline{X}) \longrightarrow 0$), so the Kummer sequence provides the vanishing of $\mathrm{H}^1_{\text{ét}}(\overline{X}, \mathbb{Z}/n(1))$. By Poincaré duality, and as X is a surface, we obtain that $\mathrm{H}^3_{\text{ét}}(\overline{X}, \mathbb{Z}/n(2)) = 0$. Therefore, $\mathrm{H}^4_{\text{ét}}(X, \mathbb{Z}/n(2)) \xrightarrow{\sim} \mathrm{H}^4_{\text{ét}}(\overline{X}, \mathbb{Z}/n(2))^{\Gamma_F}$. As the latter verifies $\mathrm{H}^4_{\text{ét}}(\overline{X}, \mathbb{Z}/n(2))^{\Gamma_F} \simeq \mathbb{Z}/n \simeq \mathrm{H}^4_{\text{ét}}(\overline{X}, \mathbb{Z}/n(2))$, we thus obtain a commutative diagram with injective rows :

$$\begin{tikzcd} \mathrm{CH}^2(X) \arrow[r, hook, "c\ell_2"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{H}^4_{\text{ét}}(X, \mathbb{Z}/n(2)) \arrow[d, "\sim"] \\ \mathrm{CH}^2(\overline{X}) \arrow[r, hook, "c\ell_2"'] & \mathrm{H}^4_{\text{ét}}(\overline{X}, \mathbb{Z}/n(2)) \end{tikzcd}.$$

Therefore, the degree map $\deg : \mathrm{CH}^2(X) = \mathrm{CH}_0(X) \to \mathbb{Z}$ (whose image is precisely $I(V)\mathbb{Z}$) induces an injection $\mathrm{CH}_0(X)/n \hookrightarrow \mathbb{Z}/n$, hence an injection $I(V)\mathbb{Z}/nI(V) \hookrightarrow \mathbb{Z}/n$, so that $I(V)$ is coprime to n. □

Proposition 3.2.26 (Colliot-Thélène-Voisin). *Let X be a smooth, projective and geometrically integral surface over F such that $\mathrm{H}^2(X, \mathcal{O}_X) = 0$ and $\mathsf{NS}(\overline{X})$ is torsion-free. Then there exists an exact sequence :*

$$0 \longrightarrow \mathrm{CH}_0(X) \longrightarrow \mathrm{CH}_0(\overline{X})^{\Gamma_F} \longrightarrow \mathrm{H}^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2)) \longrightarrow 0.$$

If moreover $\mathrm{H}^1(X, \mathcal{O}_X) = 0$, then we have an isomorphism $\mathrm{H}^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2))/\mathrm{H}^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2))_{div} \xrightarrow{\sim} \mathbb{Z}/I(V)$. If furthermore the degree map $\deg : \mathrm{CH}_0(\overline{X}) \to \mathbb{Z}$ is an isomorphism, then $\mathrm{H}^3_{nr}(\overline{X}, \mathbb{Q}/\mathbb{Z}(2))_{div} = 0$.

Proof. Note that for a surface, Poincaré duality provides an isomorphism

$$\bigoplus_{\ell \text{ prime}} \mathrm{H}^3_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(2))\{\ell\} \xrightarrow{\sim} \bigoplus_{\ell \text{ prime}} \mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(1))\{\ell\},$$

and the latter is zero if and only if $\mathsf{NS}(\overline{X})_{\mathrm{tors}} = 0$ (see for instance [CTS21, Chap. 5, (5.13)]). Therefore, as $\mathrm{H}^2(X, \mathcal{O}_X) = 0$, we get that $\mathsf{Br}(\overline{X}) = 0$, and we may apply Corollary (3.2.23) in order to obtain an exact sequence :

$$0 \longrightarrow \mathrm{CH}_0(X) \longrightarrow \mathrm{CH}_0(\overline{X})^{\Gamma_F} \longrightarrow \mathrm{H}^3_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}(2)) \longrightarrow \mathrm{H}^3_{\mathrm{nr}}(\overline{X}, \mathbb{Q}/\mathbb{Z}(2)).$$

Moreover since X is smooth and connected, then Bloch-Ogus theory provides the natural inclusion $\mathrm{H}^3_{\mathrm{nr}}(\overline{X}, \mathbb{Q}/\mathbb{Z}(2)) \hookrightarrow \mathrm{H}^3(\Bbbk(\overline{X}), \mathbb{Q}/\mathbb{Z}(2))$, and the latter is zero because X is a surface (we know that $\Bbbk(\overline{X})$ therefore has cohomological dimension 2). This proves the main statement. If we suppose that $\mathrm{H}^1(X, \mathcal{O}_X) = 0$, then once again $A_0(\overline{X})$ is uniquely divisible by Roitman's theorem, so $\mathrm{H}^1(F, A_0(\overline{X})) = 0$, and the snake lemma provides an exact sequence :

$$A_0(\overline{X})^{\Gamma_F} \longrightarrow \mathrm{H}^3_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}(2)) \longrightarrow \mathbb{Z}/I(X) \longrightarrow 0.$$

But the image of $A_0(\overline{X})^{\Gamma_F} \to \mathrm{H}^3_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}(2))$ is precisely $\mathrm{H}^3_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{div}}$ (by identifying once again 0-cycles with codimension 2 cycles on X), hence the second claim. Finally, if $\deg : \mathrm{CH}_0(\overline{X}) \to \mathbb{Z}$ is an isomorphism, then this precisely means that $A_0(\overline{X}) = 0$, hence in particular $A_0(\overline{X})^{\Gamma_F} = 0$ and we conclude using the above exact sequence. □

Chapter IV

Motivic classes and the integral Hodge question

IV.1. Setting and preliminary notions

— The Grothendieck ring of complex varieties $K_0(\underline{\mathbf{Var}}_{\mathbb{C}})$ first appeared in a letter of Grothendieck to Serre in 1964. The motivation behind Grothendieck's definition was that the class of a variety in this ring contains a lot of geometric information, while the ring itself is not too «harsh» to compute thanks to the scissors relations and the projective bundle formula that is satisfies. This ring has been deeply used for developing theories of motivic integration. For example, the topological Euler characteristic, Hodge polynomials, stably-birational properties, or even number of points (if the variety is defined over a number field) can be read through the motivic class in the Grothendieck ring. Analogously, one can make sense of a Grothendieck ring of varieties over a finite field, especially for counting points, which is a useful heuristical method to obtain information on cohomological statements or ℓ-adic Galois representations for the varieties considered. However, we know very little about this ring.

Moving back to the complex case, the study of the equality of some classes in the Grothendieck ring of complex varieties has recently given some important results in birational geometry. In particular for a given finite group G, although no specific implication is known, the equality $[\mathrm{B}G] = 1$ in $K_0(\underline{\mathbf{Stck}}_{\mathbb{C}})$ (where $\mathrm{B}G = [\operatorname{Spec}\mathbb{C}/G]$ is the *classifying stack* of G, see §1.1.2.2) appears to be closely related to the stable birational class of the field of invariants $\mathbb{C}(V)^G$ of a suitable faithful complex representation of G. An interesting result in this regard is due to Ekedahl, who showed in [Eke09a] that the non-vanishing of the unramified Brauer group $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(V)^G/\mathbb{C})$ implies that $[\mathrm{B}G] \neq 1$ in $K_0(\underline{\mathbf{Stck}}_{\mathbb{C}})$. Scavia recently showed in [Sca21] that the converse in not true, that is, there exist finite groups G such that $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(V)^G/\mathbb{C}) = 0$ but the class $[\mathrm{B}G]$ is non-trivial in the Grothendieck ring of complex stacks. To do so, he considered the examples of groups provided by Peyre in [Pey07] that satisfy $\mathrm{Br}(\mathbb{C}(V)^G/\mathbb{C}) = 0$ but $\mathrm{H}^3_{\mathrm{nr}}(\mathbb{C}(V)^G/\mathbb{C}, \mathbb{Q}/\mathbb{Z}) \neq 0$ together with the comparison between the groups $\mathrm{H}^3_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z})$ and $Z^4(X)$ for suitable complex varieties, due to Colliot-Thélène and Voisin in [CTV12]. This suggests that higher unramified cohomology allows a refinement of the understanding of the motivic class $[\mathrm{B}G] \in K_0(\underline{\mathbf{Stck}}_{\mathbb{C}})$ and supports the conjecture, formulated by Totaro, that the stable rationality of $\mathbb{C}(V)^G$ «controls» the motivic class of $\mathrm{B}G$.

1.1. Recollection on stacks

— The goal of this section is to provide a very short list of key concepts in the general theory of stacks ; it should not be considered as a thorough introduction to stacks and algebraic spaces, but rather as a guideline through the next section which deals with the Grothendieck ring of stacks. It would be of course impossible to provide a detailed yet concise exposition to the general theory, so we will assume some familiarity with the standard definitions and systematically refer the reader to [LMB00], [Ols16] or [DJ+22] (in particular for a complete exposition) when needed. We will follow almost to the letter the excellent short introduction due to Colliot-Thélène and Skorobogatov in [CTS21].

1.1.1. Fibred categories

— Let us begin our discussion with the definition of a fibred category, see [DJ+22, Tag 02XJ], [Ols16, §3.1] and [LMB00, §2] for more details. Let $\mathscr{C}$ be a category, the prototypical example being the category $\underline{\mathbf{Sch}}_S$ of schemes over a fixed base scheme S.

Definition 4.1.1. A *category over* $\mathscr{C}$ is a pair (F, p) where F is a category and $p : F \to \mathscr{C}$ is a functor. For any object $U \in \mathrm{Ob}(\mathscr{C})$, the *fibre* $F(U)$ over U is the category whose objects are the objects u of F over U, *i.e.* such that $p(u) = U$, and whose morphisms are morphisms in F that lift $\mathrm{Id}_U : U \to U$.

Definition 4.1.2. A morphism $\phi : u \to v$ in F is said to be *cartesian* if for any object w in F, a morphism $\psi : w \to v$ and a factorisation

$$p(w) \xrightarrow{h} p(u) \xrightarrow{p(\phi)} p(v)$$

of $p(\psi)$, there exists a unique morphism $\lambda : w \to u$ in F such that $p(\lambda) = h$ and $\phi \circ \lambda = \psi$.

In this case, the morphism u is called a *pullback* of v along $f = p(\phi)$ and it is usually denoted by $u = f^*v$. It is unique up to a unique isomorphism.

Definition 4.1.3. A *fibred category* over $\mathscr{C}$ is a category $p : F \to \mathscr{C}$ over $\mathscr{C}$ such that for every morphism $f : U \to V$ in $\mathscr{C}$ and every $v \in F(V)$, there exists an object $u \in F(U)$ and a cartesian morphism $\phi : u \to v$ which lifts f, that is, $p(\phi) = f$.

A *morphism* of fibred categories $p : F \to \mathscr{C}$ to $q : G \to \mathscr{C}$ is a functor $g : F \to G$ sending cartesian morphisms to cartesian morphisms such that there is an equality of functors $p = q \circ g$.

1.1.1.1. Categories fibred in groupoids. The references for this paragraph are [DJ+22, Tag 003S], [Ols16, §3.4] and [LMB00, §2]. We recall that a *groupoid* is a category in which every morphism is an isomorphism.

Definition 4.1.4. A fibred category $p : F \to \mathscr{C}$ is a *category fibred in groupoids* if the fibre $F(U)$ is a groupoid for any object $U \in \mathrm{Ob}(\mathscr{C})$.

Equivalently, one can remark that $p : F \to \mathscr{C}$ is fibred in groupoids if and only if every morphism in F is cartesian, see [Ols16, Exer. 3.D, p.85]. For a given $x \in \mathrm{Ob}(F)$, the functor p gives rise to an equivalence of categories between the «localised» categories F/x and $\mathscr{C}/p(x)$.

Let $p : F \to \mathscr{C}$ be a category fibred in groupoids. For any object X of $\mathscr{C}$ and any pair of objects x_1, x_2 of $F(X)$, we define the functor

$$\underline{\mathrm{Isom}}(x_1, x_2) : (\mathscr{C}/X)^{\mathrm{op}} \longrightarrow \underline{\mathbf{Sets}}$$

that associates to $f : Y \to X$ the set $\mathrm{Isom}_{F(Y)}(f^*x_1, f^*x_2)$ for some chosen pullbacks f^*x_1 and f^*x_2 along f. By the definition of a category fibred in groupoids, this implies that a given morphism $g : Z \to Y$ gives rise to a canonical map :

$$\underline{\mathrm{Isom}}(x_1, x_2)(f : Y \to X) \longrightarrow \underline{\mathrm{Isom}}(x_1, x_2)(fg : Z \to X),$$

so this is a functor. Moreover up to a canonical isomorphism, it does not depend on the choice of the pullbacks f^*x_1 and f^*x_2.

In particular, for $x \in \mathrm{Ob}(F(X))$, we obtain a functor :

$$\underline{\mathrm{Aut}}_x := \underline{\mathrm{Isom}}(x, x) : (\mathscr{C}/X)^{\mathrm{op}} \longrightarrow \underline{\mathbf{Grps}}.$$

1.1.1.2. Yoneda's lemma. For a given S-scheme X, we are provided with the natural functor of points $h_X : (\underline{\mathbf{Sch}}_S)^{\mathrm{op}} \to \underline{\mathbf{Sets}}$ given by $h_X(Y) := \mathsf{Hom}_S(Y, X)$. By Yoneda's lemma, we know that this functor is fully faithful $\underline{\mathbf{Sch}}_S \to \mathrm{Fun}((\underline{\mathbf{Sch}}_S)^{\mathrm{op}}, \underline{\mathbf{Sets}})$, hence it provides an embedding of $\underline{\mathbf{Sch}}_S$ into the category of contravariant functors from $\underline{\mathbf{Sch}}_S$ to $\underline{\mathbf{Sets}}$. On the other hand, for any functor $F : (\underline{\mathbf{Sch}}_S)^{\mathrm{op}} \to \underline{\mathbf{Sets}}$, there is a natural bijection :

$$\mathsf{Hom}(h_X, F) \xrightarrow{\sim} F(X)$$

given by evaluating the object $\mathrm{Id}_X : X \to X$ of $h_X(X)$. Therefore, we can replace an S-scheme X by its functor of points h_X, which is an object of a larger category (according to Laumon and Moret-Bailly in [LMB00], this is the «functorial point of view», which we can oppose to the «geometric point of view», *i.e.* viewing the category $\underline{\mathbf{Sch}}$ as a full subcategory of the category of ringed spaces). In what follows, we will often not make the distinction between an S-scheme X and its functor of points h_X for a matter of convenience.

Remark that this Yoneda embedding operation can actually be refined. Indeed, for a given S-scheme X, we note that the category $\underline{\mathbf{Sch}}_X$ is naturally fibred over $\underline{\mathbf{Sch}}_S$ *via* the forgetful functor. There is an analogue of Yoneda's lemma for 2-categories (see [DJ+22, Tag 003G] for a detailed account on 2-categories), see [DJ+22, Tag 004B], which provides the following statement : if $p : F \to \underline{\mathbf{Sch}}_S$ is another fibred category, then the functor

$$\xi : \mathsf{Hom}_{\underline{\mathbf{Sch}}_S}(\underline{\mathbf{Sch}}_X, F) \longrightarrow F(X)$$

that sends a morphism of fibred categories to the value of this morphism on the object $\mathrm{Id} : X \to X$ of $\underline{\mathbf{Sch}}_X$, is an equivalence of categories.

1.1.1.3. Sheaves on a category fibred in groupoids over a site.

Let $p : \mathscr{X} \to S_{\text{Ét}}$ be a category fibred in groupoids over the category of schemes over a base scheme S that is equipped with the étale topology (we could alternatively consider the small étale site or the fppf site of S). First note that the site $S_{\text{Ét}}$ induces a site $\mathscr{X}_{\text{Ét}}$ where the coverings are families of morphisms $\{x_i \to x\}_{i\in I}$ in $\mathscr{X}$ such that $\{p(x_i) \to p(x)\}_{i\in I}$ is a covering in $S_{\text{Ét}}$, see [DJ+22, Tag 06NU] for details. Therefore, we have a natural framework for the notion of sheaves on $\mathscr{X}_{\text{Ét}}$. The functor p induces a natural equivalence between the localised categories $\mathscr{X}/x$ and $\underline{\mathbf{Sch}}/p(x)$ for any $x \in \mathrm{Ob}(\mathscr{X})$; more, one can show that the sites $\mathscr{X}_{\text{Ét}}/x$ and $p(x)_{\text{Ét}}$ are actually equivalent, see [DJ+22, Tag 06W0].

One can also define a structure scheaf $\mathscr{O}_{\mathscr{X}}$ as follows : the structure sheaf $\mathscr{O}$ on $S_{\text{Ét}}$ associates to any S-scheme T the ring $\Gamma(T, \mathscr{O}_T)$. We let $\mathscr{O}_{\mathscr{X}}$ be the sheaf of rings on $\mathscr{X}_{\text{Ét}}$ such that $\mathscr{O}_{\mathscr{X}}(x) := \mathscr{O}(p(x))$ for any object $x \in \mathrm{Ob}(\mathscr{X})$. It is therefore possible to talk about sheaves of $\mathscr{O}_{\mathscr{X}}$-modules. One can check that the localised sites $\mathscr{X}_{\text{Ét}}/x$ and $p(x)_{\text{Ét}}$ are equivalent as ringed sites if we take the structure sheaves into account. Then one can define the usual classes of $\mathscr{O}_{\mathscr{X}}$ by analogy with ringed spaces, [DJ+22, Tag 03DL] for details :

Definition 4.1.5. Let $\mathscr{F}$ be a sheaf of $\mathscr{O}_{\mathscr{X}}$-modules.

(i) $\mathscr{F}$ is *locally free* if for every object $x \in \mathrm{Ob}(\mathscr{X})$, there is an étale covering $\{x_i \to x\}_{i\in I}$ such that the restriction of $\mathscr{F}$ to each x_i is a free $\mathscr{O}_{x_i}$-module ;

(ii) $\mathscr{F}$ is *finite locally free* if for every object $x \in \mathrm{Ob}(\mathscr{X})$, there is an étale covering $\{x_i \to x\}_{i\in I}$ such that the restriction of $\mathscr{F}$ to each x_i is isomorphic to $\mathscr{O}_{x_i}^{\oplus n}$ for some $n \geq 1$;

(iii) $\mathscr{F}$ is of *finite type* if for every object $x \in \mathrm{Ob}(\mathscr{X})$, there is an étale covering $\{x_i \to x\}_{i\in I}$ such that the restriction of $\mathscr{F}$ to each x_i is isomorphic to a quotient of $\mathscr{O}_{x_i}^{\oplus n}$ for some $n \geq 1$;

(iv) $\mathscr{F}$ is *quasi-coherent* if for every object $x \in \mathrm{Ob}(\mathscr{X})$, there is an étale covering $\{x_i \to x\}_{i\in I}$ such that the restriction of $\mathscr{F}$ to each x_i is isomorphic to the cokernel of a map of free $\mathscr{O}_{x_i}$-modules ;

(v) $\mathscr{F}$ is *coherent* if it is of finite type and for every object $x \in \mathrm{Ob}(\mathscr{X})$ and any $n \geq 1$, the kernel of any map $\mathscr{O}_x^{\oplus n} \to \mathscr{F}$ is of finite type.

Remark 4.1.6. One can also make sense of what a *vector bundle* on $\mathscr{X}$ is, by considering it as a locally free $\mathscr{O}_{\mathscr{X}}$-module of finite constant rank $n \geq 1$.

1.1.2. Stacks and spaces

— We refer to [Ols16, §4.2, §4.6] and [LMB00, §3]. Let $p : F \to \mathscr{C}$ be a category fibred in groupoids where $\mathscr{C}$ admits finite fibre products. For a given set of morphisms $\{X_i \to X\}_{i\in I}$ in $\mathscr{C}$, one defines $F(\{X_i \to X\}_{i\in I})$ to be the category of *descent data*, that is, the category consisting of objects E_i of $F(X_i)$ for $i \in I$ and isomorphisms $\sigma_{i,j} : \mathsf{pr}_1^*(E_i) \to \mathsf{pr}_2^*(E_j)$ in $F(X_i \times_X X_j)$ for $i, j \in I$ satisfying the usual *cocycle condition*, on triple intersections :

$$\begin{tikzcd} \mathsf{pr}_0^* X_i \arrow[rr, "\mathsf{pr}_{02}^* \sigma_{i,k}"] \arrow[dr, "\mathsf{pr}_{01}^* \sigma_{i,j}"'] & & \mathsf{pr}_2^* X_k \\ & \mathsf{pr}_1^* X_j \arrow[ur, "\mathsf{pr}_{12}^* \sigma_{j,k}"'] & \end{tikzcd} .$$

If the natural functor $F(X) \to F(\{X_i \to X\}_{i\in I})$ is an equivalence of categories, we say that the set of morphisms $\{X_i \to X\}_{i\in I}$ is of *effective descent* for F.

Let now $\mathscr{C}$ be a site, *e.g.* $\mathscr{C} = S_{\text{Ét}}$ where S is a fixed scheme.

Definition 4.1.7. A category fibred in groupoids $p : F \to \mathscr{C}$ is a *stack* if for any object $X \in \mathrm{Ob}(\mathscr{C})$, then every covering family $\{X_i \to X\}_{i\in I}$ is of effective descent for F.

See [LMB00, Def. 3.1] for further details. Equivalently, for any covering of any $X \in \mathrm{Ob}(\mathscr{C})$, then any descent datum with respect to this covering is effective, and $\underline{\mathrm{Isom}}(x_1, x_2)$ is a sheaf on $\mathscr{C}/X$ for any pair of objects x_1, x_2 of $F(X)$, *cf.* [Ols16, Prop. 4.6.2]. (In particular, $\underline{\mathrm{Aut}}_x$ is a sheaf on $\mathscr{C}/X$ for any $x \in \mathrm{Ob}(F(X))$.) We give two absolutely fundamental examples :

Examples 4.1.8.

(i) *(The stack associated to a sheaf on a site.)* A set can be canonically viewed as a groupoid by defining morphisms to be the identity maps on the elements of this set. A functor $f : \mathscr{C}^{\mathrm{op}} \to \underline{\mathbf{Sets}}$ then naturally gives rise to a category fibred in sets over $\mathscr{C}$, whose fibre over an object X is the set $f(X)$, see [CTS21, Ex. 4.1.5]. Hence it can be seen as a category fibred in groupoids. If $\mathscr{C}$ is in particular a site, then this fibred category over $\mathscr{C}$ is a stack if and only if f is a sheaf.

(ii) *(The stack associated to an S-scheme.)* By the 2-Yoneda lemma, one can replace an S-scheme X by the fibred category $\underline{\mathbf{Sch}}_X \to \underline{\mathbf{Sch}}_S$. One then checks that this actually is a category fibred in groupoids, more precisely in sets with the identity maps. More, it is a stack for the usual topologies on $\underline{\mathbf{Sch}}_S$ (fpqc, fppf, étale, Nisnevich, Zariski, *etc.*) since by a theorem on Grothendieck [DJ+22, Tag 0AI2], the functor of points h_X is a sheaf for the fpqc topology, which is finer than all of the topologies mentioned above.

1.1.2.1. Algebraic spaces. We first give the definition of an algebraic space, see [Ols16, Chap. 5] and [LMB00, §1] (actually we don't need the notion of categories fibred in groupoids and descent data in order to define these objects, but they are crucial to define algebraic stacks). See also [Ols16, §3.4] for a complete treatment of the notion of 2-fibred products of categories fibred in groupoids.

Definition 4.1.9. Let S be a scheme. A morphism of sheaves of sets $F \to G$ on $S_{\text{Ét}}$ is said to be *representable by schemes* (or more simply *representable*) if for any S-scheme T and any morphism $T \to G$ the fibre product $F \times_G T$ is a scheme.

If F and G are already representable, say $F = h_X$ and G_Y for some S-schemes X and Y, then by Yoneda's lemma, we know that any morphism $F \to G$ is induced by a morphism of schemes $X \to Y$, which implies that the morphism $F \to G$ is itself representable.

Let now F be a sheaf of sets on $S_{\text{Ét}}$. If the diagonal map $F \to F \times_S F$ is representable, then any S-morphism $T \to F$ where T is an S-scheme is representable as well (this is indeed a consequence of the isomorphism $T \times_F Z \simeq (T \times_S Z) \times_{F\times_S F} F$ for any S-scheme Z and any S-morphism $Z \to F$).

Let (P) be a property of morphisms of schemes that is stable under base change (being proper, flat, separated, *etc.*). More precisely, we ask that for every covering $\{U_i \to U\}_{i\in I}$ of S-schemes, the morphism $U \to S$ has property (P) if and only if each $U_i \to S$ has property (P). If F and G are functors $(\underline{\mathbf{Sch}}_S)^{\mathrm{op}} \to \underline{\mathbf{Sets}}$, then a morphism of functors $F \to G$ has property (P) if it is representable by schemes, *i.e.* for every $T \in \underline{\mathbf{Sch}}_S$ and any morphism $T \to G$, the fibre product functor $F \times_G T$ is isomorphic to h_Y for some S-scheme Y, and the resulting morphism of schemes $Y \to T$ has property (P).

Definition 4.1.10. A sheaf of sets X on $S_{\text{Ét}}$ is an *algebraic space* over S if :

(i) the diagonal $\Delta_X : X \to X \times_S X$ is representable, and

(ii) there is a surjective étale S-morphism $U \to X$ where U is an S-scheme (étale presentation property).

Note that the first condition implies that the morphism $U \to X$ in (ii) is representable, so the condition «surjective étale» makes sense. We define a morphism of S-algebraic spaces in the obvious way (being a morphism of sheaves of sets), so that we can consider the category of S-algebraic spaces $\underline{\mathbf{Spc}}_S$. It is then clear that an S-scheme is an S-algebraic space for any suitable Grothendieck topology (view it as a sheaf of sets for the fpqc topology *via* its functor of points), so that the category of S-schemes is a full subcategory of the category of S-algebraic spaces. Like schemes, algebraic spaces are sheaves for the fpqc topology on $\underline{\mathbf{Sch}}_S$ (this is a recent result of Gabber, see [DJ+22, Tag 03W8]).

1.1.2.2. Algebraic stacks. Let us now consider stacks over $S_{\text{Ét}}$. Since an algebraic space is in particular a sheaf on the big étale site of S, then it naturally gives rise to a scheme by the previous examples.

Definition 4.1.11. A morphism of stacks $\mathscr{X} \to \mathscr{Y}$ is *representable by algebraic spaces* if for every algebraic space V and every morphism $V \to \mathscr{Y}$, the fibre product $\mathscr{X} \times_{\mathscr{Y}} V$ is an algebraic space.

Definition 4.1.12. A stacks $\mathscr{X}$ over $S_{\text{Ét}}$ is said to be *algebraic* (also called an *Artin stack*) if :

(i) the diagonal $\Delta_{\mathscr{X}} : \mathscr{X} \to \mathscr{X} \times_S \mathscr{X}$ is representable by algebraic spaces, and

(ii) there exists a smooth surjective S-morphism $U \to \mathscr{X}$ where U is an S-scheme.

Similarly, one can define a morphism of stacks in the evident way and therefore consider the 2-category of S-algebraic stacks $\underline{\mathbf{AlgStck}}_S$. Furthermore, an algebraic stack $\mathscr{X}$ is said to be *Deligne-Mumford* if there exists a surjective étale S-morphism $U \to \mathscr{X}$ where U is an S-scheme (compare with the étale presentation property for algebraic spaces). Note also that property (i) in the above definition is equivalent to the following property (*cf.* [Ols16, Lem. 8.1.8]) : for every S-scheme U and any pair of objects $u_1, u_2 \in \mathrm{Ob}(\mathscr{X}(U))$, the sheaf $\underline{\mathrm{Isom}}(u_1, u_2)$ is an algebraic space. We finish this section with a fundamental notion that we will use extensively in the rest of this chapter :

Example 4.1.13. *(Quotient stacks.)* An important example of algebraic stack over $S_{\text{Ét}}$ arises in the context of the action of an S-group scheme G over an S-scheme X. Indeed, if G is a smooth S-group scheme that acts on an S-algebraic space X, then we define a *quotient stack* $[X/G]$ as the stack whose objects are triples $(T, \mathcal{P}, \pi)$ where T is an S-scheme, $\mathcal{P}$ is a sheaf of torsors for $G \times_S T$ on $T_{\text{Ét}}$ and $\pi : \mathcal{P} \to X \times_S T$ is a $G \times_S T$-equivariant morphism of sheaves. One can then check that this stack is algebraic, see [Ols16, Ex. 8.1.12]. A smooth covering can be obtained from the natural map $X \to [X/G]$ given by the trivial G_X-torsor over X.

In particular, if $X = S$ and G acts trivially on S, the quotient stack $[S/G]$ is called the *classifying stack* of G and is denoted by $\mathrm{B}_S G$ (when G is a k-group scheme, then we usually adopt the notation $\mathrm{B}G$ instead of $\mathrm{B}_k G$[24]).

Remark 4.1.14. The category of quasi-coherent sheaves on an algebraic stack is abelian. Actually, we can say more : it is a Grothendieck category (*cf.* [DJ+22, Tag 079A] for the definition); so it admits in particular direct sums, tensor products, and direct and inverse limits. Also, the dual of a sheaf of $\mathcal{O}_{\mathscr{X}}$-modules that is locally of finite presentation is quasi-coherent, see [DJ+22, Tag 06WU].

1.2. Grothendieck ring of stacks

— Quite recently, the problem of counting points on varieties extended to the setting of moduli problems or equivariant geometry (for instance geometric invariant theory), where one often has to deal with algebraic spaces, or even algebraic stacks. This led to a generalisation of the Grothendieck ring of varieties to the notion of *Grothendieck ring of stacks*, due to Ekedahl in [Eke09b]. Let us give a definition of this ring, by first defining its underlying group :

Definition 4.1.15. The *Grothendieck group of stacks* over a field k is the group $K_0(\underline{\mathbf{Stck}}_k)$ generated by classes $[\mathscr{X}]$ of algebraic stacks $\mathscr{X}$ of finite type over k and whose automorphism group scheme (also called *stabiliser*) is affine, modulo the relations :

[24] This should not be mistaken with the classifying space $\mathbf{B}G = |N(G)|$ of G (viewed as a category) that was defined in Chapter I, §3.1.1.1.

(i) The class $[\mathcal{X}]$ depends only on the isomorphism class of $\mathcal{X}$ in $\underline{\mathbf{Stck}}_k$;

(ii) If $\mathcal{Y} \hookrightarrow \mathcal{X}$ is closed immersion of stacks with complement $\mathcal{U}$, then $[\mathcal{X}] = [\mathcal{Y}] + [\mathcal{U}]$;

(iii) If $\mathcal{E} \to \mathcal{X}$ is a vector bundle of constant rank $n \geq 1$, then $[\mathcal{E}] = [\mathcal{X} \times_k \mathbb{A}^n_k]$.

Remark 4.1.16. One can also make sense of a Grothendieck ring $K_0(\underline{\mathbf{Spc}}_k)$ of k-algebraic spaces of finite type, by imposing the relations $[X] = [Y] + [U]$ only for closed subschemes $Y \hookrightarrow X$ of a given k-algebraic space X (with complement $U := X \setminus Y$). As pointed out by Ekedahl in [Eke09b, §1], this gives the same group as $K_0(\underline{\mathbf{Var}}_{\mathbb{C}})$ because a quasi-compact and quasi-separated k-algebraic space always admits a finite stratification by closed subschemes (see *e.g.* [DJ+22, Tag 0A4I]), and the projective bundle formula follows naturally from the scissors relations.

One can then enrich the group structure into a ring structure by setting $[\mathcal{X} \times_k \mathcal{Y}] := [\mathcal{X}] \cdot [\mathcal{Y}]$ for $\mathcal{X}$ and $\mathcal{Y}$ two algebraic stacks (the multiplicative unit being $[\mathsf{Spec}\, k]$). We will write $\mathbb{L} := [\mathbb{A}^1_k]$ for the class of the affine line.

If we fix a class $\mathcal{G}$ of connected group schemes of finite type over k (for instance, connected algebraic groups such as $\mathbf{GL}_{n,k}$ or $\mathbf{SL}_{n,k}$ for $n \geq 1$), then we define $K_0^{\mathcal{G}}(\underline{\mathbf{Stck}}_k)$ to be the quotient of $K_0(\underline{\mathbf{Stck}}_k)$ by the relations $[Y] = [G] \cdot [X]$ whenever $G \in \mathcal{G}$ and $Y \to X$ is a G-torsor of algebraic spaces (note here that we impose the condition only for algebraic spaces, see also [Ols16, Chap. 4, §4.5] and [DJ+22, Tag 04TV] for further structural results on torsors in the context of algebraic spaces). There exists a unique obvious ring structure on $K_0^{\mathcal{G}}(\underline{\mathbf{Stck}}_k)$ that makes the quotient map $K_0(\underline{\mathbf{Stck}}_k) \to K_0^{\mathcal{G}}(\underline{\mathbf{Stck}}_k)$ into a ring morphism, and if $\mathcal{G}' \subset \mathcal{G}$ is another class of connected group schemes, then the quotient map $K_0(\underline{\mathbf{Stck}}_k) \to K_0^{\mathcal{G}'}(\underline{\mathbf{Stck}}_k)$ factors through $K_0(\underline{\mathbf{Stck}}_k) \to K_0^{\mathcal{G}}(\underline{\mathbf{Stck}}_k)$. Moreover, the natural inclusion of $\underline{\mathbf{Spc}}_k$ into $\underline{\mathbf{Stck}}_k$ as a full 2-subcategory (see [LMB00, Rmq 4.1.1]) induces a canonical map $K_0(\underline{\mathbf{Spc}}_k) \to K_0(\underline{\mathbf{Stck}}_k)$.

1.2.1. Some general results

— We now state the following proposition due to Ekedahl, which provides a lot of useful basic properties of the ring $K_0(\underline{\mathbf{Stck}}_k)$ and its «equivariant analogues» :

Proposition 4.1.17 ([Eke09b, Prop. 1.1]). *The following statements hold :*

(i) *We have* $[\mathbf{GL}_{n,k}] = (\mathbb{L}^n - 1)(\mathbb{L}^n - \mathbb{L}) \ldots (\mathbb{L}^n - \mathbb{L}^{n-1}) \in K_0(\underline{\mathbf{Stck}}_k)$ *;*

(ii) *If* $\mathcal{X} \to \mathcal{Y}$ *is a* $\mathbf{GL}_{n,k}$*-torsor of* k*-algebraic stacks of finite type, then* $[\mathcal{X}] = [\mathbf{GL}_{n,k}] \cdot [\mathcal{Y}] \in K_0(\underline{\mathbf{Stck}}_k)$ *;*

(iii) *If* $G \in \mathcal{G}$ *and* $\mathcal{X} \to \mathcal{Y}$ *is a* G*-torsor of* k*-algebraic stacks of finite type, then* $[\mathcal{X}] = [G] \cdot [\mathcal{Y}] \in K_0(\underline{\mathbf{Stck}}_k)$ *;*

(vi) *If* $N, H \in \mathcal{G}$ *and* G *is an extension of algebraic groups of* H *by* N*, and if* $\mathcal{X} \to \mathcal{Y}$ *is a* G*-torsor, then* $[\mathcal{X}] = [G] \cdot [\mathcal{Y}] \in K_0(\underline{\mathbf{Stck}}_k)$ *;*

(v) *If* $G \in \mathcal{G}$*, then* $[G] \cdot [\mathrm{B}G] = 1 \in K_0(\underline{\mathbf{Stck}}_k)$ *;*

(vi) *If* $G \in \mathcal{G}$ *and* F *is a* G*-space,* $\mathcal{X} \to \mathcal{Y}$ *is a* G*-torsor of* k*-algebraic stacks of finite type and* $\mathcal{Z} \to \mathcal{Y}$ *is the* F*-fibration attached to* $\mathcal{X} \to \mathcal{Y}$ *and the* G*-action on* F*, then* $[\mathcal{Z}] = [F] \cdot [\mathcal{Y}] \in K_0(\underline{\mathbf{Stck}}_k)$ *;*

(vii) *If* $G \in \mathcal{G}$ *and* $\mathcal{X} \to \mathcal{Y}$ *and* $\mathcal{X}' \to \mathcal{Y}$ *are two* G*-torsors of* k*-algebraic stacks of finite type, and if* $\mathcal{Z} \to \mathcal{Y}$ *is the stack of isomorphisms* $\mathcal{X} \xrightarrow{\sim} \mathcal{X}'$*, then* $[\mathcal{Z}] = [G] \cdot [\mathcal{Y}] \in K_0^{\mathcal{G}}(\underline{\mathbf{Stck}}_k)$ *;*

(viii) *If* $G \in \mathcal{G}$*,* F *is a* G*-space,* H *is a* k*-algebraic group,* $H \to G$ *is a morphism of algebraic groups,* $\mathcal{X} \to \mathcal{Y}$ *is an* H*-torsor of* k*-algebraic stacks of finite type and* $\mathcal{Z} \to \mathcal{Y}$ *is the* F*-fibration associated to* $\mathcal{X} \to \mathcal{Y}$ *and the* H*-action on* F *(given by its* G*-action and* $H \to G$*), then* $[\mathcal{Z}] = [F] \cdot [\mathcal{Y}] \in K_0^{\mathcal{G}}(\underline{\mathbf{Stck}}_k)$ *;*

(ix) *If* $G \in \mathcal{G}$ *and* $H \hookrightarrow G$ *is a subgroup* k*-scheme, then* $[\mathrm{B}H] = [G/H] \cdot [\mathrm{B}G] \in K_0^{\mathcal{G}}(\underline{\mathbf{Stck}}_k)$.

Ekedahl then remarked that one can relate the groups $K_0(\underline{\mathbf{Spc}}_k)$ and $K_0(\underline{\mathbf{Stck}}_k)$ more precisely, or at least «measure the defect» of the natural map $K_0(\underline{\mathbf{Spc}}_k) \to K_0(\underline{\mathbf{Stck}}_k)$. Indeed, the key argument is to localise $K_0(\underline{\mathbf{Spc}}_k)$ with respect to the class $\mathbb{L}$ of the affine line as well as the classes $\mathbb{L}^n - 1$ for $n \geq 1$. We will denote the resulting ring by $K_0(\underline{\mathbf{Spc}}_k)'$.

Theorem 4.1.18 ([Eke09b, Thm. 1.2]). *For any class $\mathscr{G}$ of connected group schemes of finite type over a field k, the natural map*

$$K_0^{\mathscr{G}}(\underline{\mathbf{Spc}}_k) \longrightarrow K_0^{\mathscr{G}}(\underline{\mathbf{Stck}}_k)$$

induces an isomorphism $K_0^{\mathscr{G}}(\underline{\mathbf{Spc}}_k)' \xrightarrow{\sim} K_0^{\mathscr{G}}(\underline{\mathbf{Stck}}_k)$.

Proof. By the above proposition, we know that $[\mathbf{GL}_{n,k}] = (\mathbb{L}^n - 1)\dots(\mathbb{L}^n - \mathbb{L}^{n-1})$ is invertible (consider the $\mathbf{GL}_{n,k}$-torsor $\operatorname{Spec} k \to \mathrm{B}\mathbf{GL}_{n,k}$), so that $\mathbb{L}^i$ and $\mathbb{L}^n - \mathbb{L}^i$ are invertible in $K_0^{\mathscr{G}}(\underline{\mathbf{Stck}}_k)$ for $i \geq 0$ as well. In particular, we get a natural factorisation

$$K_0^{\mathscr{G}}(\underline{\mathbf{Spc}}_k) \longrightarrow K_0^{\mathscr{G}}(\underline{\mathbf{Spc}}_k)' \longrightarrow K_0^{\mathscr{G}}(\underline{\mathbf{Stck}}_k).$$

The goal from now on is to define a map in the other direction. First assume that $\mathscr{X} = [X/\mathbf{GL}_{n,k}]$ is a global quotient of a k-scheme X of finite type. We define the class $[\mathscr{X}] \in K_0^{\mathscr{G}}(\underline{\mathbf{Spc}}_k)'$ to be $[X]/[\mathbf{GL}_{n,k}]$. To check well-definedness, suppose that one can also write $\mathscr{X}$ as another global quotient $\mathscr{X} = [Y/\mathbf{GL}_{m,k}]$. Then we can always construct a 2-cartesian diagram

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & \mathscr{X} \end{tikzcd}$$

where $Z \to X$ is a $\mathbf{GL}_{m,k}$-torsor and $Z \to Y$ is a $\mathbf{GL}_{n,k}$-torsor. By the previous proposition, we thus get that $[\mathbf{GL}_{m,k}] \cdot [X] = [Z] = [\mathbf{GL}_{n,k}] \cdot [Y]$, hence the claim in this case. If we consider a general stack $\mathscr{X}$, then Kresch showed that under the hypothesis that $\mathscr{X}$ is of finite type with affine stabilisers, then one can always stratify this stack by finitely many global quotients by $\mathbf{GL}_{n,k}$ for suitable n's (see [DJ+22, Tag 04UZ]) ; we can therefore define the class $[\mathscr{X}] \in K_0^{\mathscr{G}}(\underline{\mathbf{Spc}}_k)'$ by summing-up the classes of the global quotients arising in the stratification. □

Remark 4.1.19. We would like to point out the fact that localising with respect to the affine line is not a trivial operation at all. For instance, over the complex numbers, Borisov recently showed in [Bor15] that $\mathbb{L} = [\mathbb{A}^1_{\mathbb{C}}]$ is a zero-divisor in $K_0(\underline{\mathbf{Var}}_{\mathbb{C}})$.

1.2.2. Bittner's presentation

— We finish this preliminary section by going back to the complex setting. Indeed, thanks to Hironaka's theorem on resolution of singularities, Bittner managed to provide a useful presentation of $K_0(\underline{\mathbf{Var}}_{\mathbb{C}})$ in terms of generators and relations, which we state below :

Theorem 4.1.20 ([Bit04, Thm. 3.1], [Eke09b, p. 14]). *As an abelian group, $K_0(\underline{\mathbf{Var}}_{\mathbb{C}})[\mathbb{L}^{-1}]$ may be presented by formal fractions of the form $[X]/\mathbb{L}^m$ for $m \geq 0$ where X is a smooth and projective complex variety, modulo the condition that $[\varnothing] = 0$ and the relations :*

(i) *For every smooth and projective complex variety X and every blowup $\widetilde{X} \to X$ at a smooth closed subscheme $Y \hookrightarrow X$ with exceptional divisor $E \to Y$, and for every $m \geq 0$, we have*

$$[\widetilde{X}]/\mathbb{L}^m - [X]/\mathbb{L}^m = [E]/\mathbb{L}^m - [Y]/\mathbb{L}^m;$$

(ii) *For every smooth and projective complex variety X and every $m \geq 0$, we have*

$$[X \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}}]/\mathbb{L}^{m+1} - [X]/\mathbb{L}^{m+1} = [X]/\mathbb{L}.$$

IV.2. Main theorem

2.1. Dimension filtration and statement of the theorem

— There is a quite natural filtration $\mathrm{Fil}^\bullet K_0(\underline{\mathbf{Var}_{\mathbb{C}}})[\mathbb{L}^{-1}]$ that one can define on the localisation $K_0(\underline{\mathbf{Var}_{\mathbb{C}}})[\mathbb{L}^{-1}]$. Indeed, the isomorphism

$$K_0(\underline{\mathbf{Var}_{\mathbb{C}}})' \xrightarrow{\sim} K_0(\underline{\mathbf{Stck}_{\mathbb{C}}})$$

provided by Theorem (4.1.18) together with the presentation of $K_0(\underline{\mathbf{Var}_{\mathbb{C}}})$ in Theorem (4.1.20) suggest that we consider the elements «bounded» by a fixed power of the class of the affine line. More precisely, for $n \in \mathbb{Z}$, we let $\mathrm{Fil}^n K_0(\underline{\mathbf{Var}_{\mathbb{C}}})[\mathbb{L}^{-1}]$ be the subgroup of $K_0(\underline{\mathbf{Var}_{\mathbb{C}}})[\mathbb{L}^{-1}]$ generated by elements of the form $[X]/\mathbb{L}^m$ where X is a smooth and projective complex variety and $\dim X - m \leq n$ (indeed, by Hironaka's theorem, we could alternatively consider the classes of all complex varieties X such that $\dim X - m \leq n$, but this would give the same subgroup, see [Eke09b, Lem. 3.1]). We denote by $\widehat{K_0}(\underline{\mathbf{Var}_{\mathbb{C}}})$ the formal completion of $K_0(\underline{\mathbf{Var}_{\mathbb{C}}})[\mathbb{L}^{-1}]$ with respect to this filtration. It is clear that for $m, n \in \mathbb{Z}$, we have :

$$\mathrm{Fil}^m K_0(\underline{\mathbf{Var}_{\mathbb{C}}})[\mathbb{L}^{-1}] \cdot \mathrm{Fil}^n K_0(\underline{\mathbf{Var}_{\mathbb{C}}})[\mathbb{L}^{-1}] \subseteq \mathrm{Fil}^{m+n} K_0(\underline{\mathbf{Var}_{\mathbb{C}}})[\mathbb{L}^{-1}],$$

so that the completion $\widehat{K_0}(\underline{\mathbf{Var}_{\mathbb{C}}})$ comes naturally equipped with a multiplication that makes it a commutative ring with identity and that extends the ring structure on $K_0(\underline{\mathbf{Var}_{\mathbb{C}}})[\mathbb{L}^{-1}]$. Moreover, in $\widehat{K_0}(\underline{\mathbf{Var}_{\mathbb{C}}})$, we have the identity :

$$(1-\mathbb{L}^n)\sum_{i\geq 0}\mathbb{L}^{ni} = 1$$

for any $n \geq 1$, so that the natural map $K_0(\underline{\mathbf{Var}_{\mathbb{C}}}) \to \widehat{K_0}(\underline{\mathbf{Var}_{\mathbb{C}}})$ factors through $K_0(\underline{\mathbf{Var}_{\mathbb{C}}}) \to K_0(\underline{\mathbf{Stck}_{\mathbb{C}}})$.

We now describe the main theorem proved in [Sca21]. Recall as in Chapter III, §1.1 that for any smooth and projective complex variety X, there is a Betti cycle class map :

$$c\ell_i : \mathrm{CH}^i(X) \longrightarrow \mathrm{H}^{2i}_B(X, \mathbb{Z}(i))$$

for each $0 \leq i \leq \dim X$, which naturally lands in the sugroup of integral Hodge classes $\mathrm{Hdg}^{2i}(X, \mathbb{Z}(i))$ of type (i, i). For any integer i, we defined the finitely generated abelian groups $Z^{2i}(X) := \mathrm{Hdg}^{2i}(X, \mathbb{Z}(i))/\mathrm{H}^{2i}_{\mathrm{alg}}(X, \mathbb{Z}(i))$ which measure the failure of the integral Hodge question on X (by convention, we put $Z^{2i} = 0$ for $i < 0$ or $i > \dim X$). By symmetry, we define the groups $Z_{2i}(X) := Z^{2\dim X - 2i}(X)$ which measure the failure of the integral Hodge question for cycles of dimension i on X. What Scavia remarked is that one can view the assignment Z_{2i} as a map from the Grothendieck group of varieties to the Grothendieck group of the category of abelian groups in such a way that it is «compatible» with the topology induced by the dimension filtration :

Theorem 4.2.21 (Scavia [Sca21, Thm. 1]). *Let i be any integer. Then,*

(i) *There exists a unique group homomorphism :*

$$Z_{2i} : K_0(\underline{\mathbf{Var}_{\mathbb{C}}})[\mathbb{L}^{-1}] \longrightarrow K_0(\underline{\mathbf{Ab}})$$

that sends $[X]/\mathbb{L}^m$ to $[Z_{2i+2m}(X)]$ for every smooth and projective complex variety X and every $m \geq 0$.

(ii) *This homomorphism is continuous with respect to the dimension filtration topology on $K_0(\underline{\mathbf{Var}_{\mathbb{C}}})[\mathbb{L}^{-1}]$ on the one hand and for the discrete topology on $K_0(\underline{\mathbf{Ab}})$ on the other hand. In particular, it extends uniquely to a group morphism :*

$$\widehat{Z}_{2i} : \widehat{K_0}(\underline{\mathbf{Var}_{\mathbb{C}}}) \longrightarrow K_0(\underline{\mathbf{Ab}}).$$

2.2. Proof of the main theorem

— The proof of Scavia revolves around the presentation from Theorem (4.1.20) provided by Bittner for the group $K_0(\underline{\mathbf{Var}_{\mathbb{C}}})$. Indeed, thanks to this result, we are reduced to showing that the assignment Z_{2i} is compatible with the dimension filtration in the case of a blowup on the one hand and a projective bundle on the other hand.

Proof of Theorem (4.2.21). We first need to check that the assignement $[X]/\mathbb{L}^m \mapsto [Z_{2i+2m}(X)]$ respects the relations of Theorem (4.1.20). We begin with the case of a blowup. Let $m \geq 0$, $Y \hookrightarrow X$ a closed immersion of codimension r smooth and projective complex varieties where X has dimension d, and $\widetilde{X} \to X$ the blowup of X along Y ; let $E \to Y$ be the exceptional divisor. We need to show that

$$[Z_{2i+2m}(X)] - [Z_{2i+2m}(Y)] = [Z_{2i+2m}(\widetilde{X})] - [Z_{2i+2m}(E)]$$

in $K_0(\underline{\mathbf{Ab}})$. If we let $j := d - i - m$, then we can rewrite the above equality as :

$$[Z^{2j}(X)] - [Z^{2j-2r}(Y)] = [Z^{2j}(\widetilde{X})] - [Z^{2j-2}(E)]. \tag{IV.1}$$

We now make use of the blowup formula for Chow groups. Indeed, as proven in [Voi03, Thm. 9.27], there is a natural isomorphism of groups :

$$\phi^j : \bigoplus_{h=0}^{r-2} \mathrm{CH}^{j-1-h}(Y) \oplus \mathrm{CH}^j(X) \xrightarrow{\sim} \mathrm{CH}^j(\widetilde{X}).$$

Similarly, there is a blowup formula for the corresponding Hodge structures, see [Voi02, Thm. 7.31] :

$$\psi^j : \bigoplus_{h=0}^{r-2} \mathrm{Hdg}^{2j-2-2h}(Y, \mathbb{Z}) \oplus \mathrm{Hdg}^{2j}(X, \mathbb{Z}) \xrightarrow{\sim} \mathrm{Hdg}^{2j}(\widetilde{X}, \mathbb{Z})$$

(note that we dropped the twists in the notation for convenience here). Moreover one can check in the proofs of the two statements that these isomorphisms are constructed in such a way that they are compatible with the cycle class maps, see [Voi14, Chap. 2, §2.2.2], so that we get a commutative diagram :

$$\begin{tikzcd}
\bigoplus_{h=0}^{r-2} \mathrm{CH}^{j-1-h}(Y) \oplus \mathrm{CH}^j(X) \arrow[r, "\phi^j"] \arrow[d, "(\bigoplus_{h=0}^{r-2} c\ell_{h,Y}) \oplus c\ell_{j,X}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{CH}^j(\widetilde{X}) \arrow[d, "c\ell_{j,\widetilde{X}}"] \\
\bigoplus_{h=0}^{r-2} \mathrm{Hdg}^{2j-2-2h}(Y, \mathbb{Z}) \oplus \mathrm{Hdg}^{2j}(X, \mathbb{Z}) \arrow[r, "\psi^j"'] & \mathrm{Hdg}^{2j}(\widetilde{X}, \mathbb{Z})
\end{tikzcd} .$$

Therefore, we obtain an isomorphism of groups :

$$Z^{2j}(\widetilde{X}) \xrightarrow{\sim} \bigoplus_{h=0}^{r-2} Z^{2j-2-2h}(Y) \oplus Z^{2j}(X).$$

On the other hand, the morphism $E \to Y$ induced by the blowup makes E into a projective bundle of rank $r-1$ given by the projectivisation $E = \mathbb{P}(\mathrm{N}_{Y/X})$ of the normal bundle $\mathrm{N}_{Y/X}$ of Y inside X, see [Voi02, Chap. 3, §3.3.3]. As shown in [Voi03, Thm. 9.25], the pullback along $E \to Y$ induces an isomorphism on Chow groups :

$$\tau^{j-1} : \bigoplus_{h=0}^{r-1} \mathrm{CH}^{j-1-h}(Y) \xrightarrow{\sim} \mathrm{CH}^{j-1}(E),$$

and similarly for Hodge structures it induces an isomorphism :

$$\gamma^{j-1} : \bigoplus_{h=0}^{r-1} \mathrm{Hdg}^{2j-2-2h}(Y, \mathbb{Z}) \xrightarrow{\sim} \mathrm{Hdg}^{2j-2}(E, \mathbb{Z}),$$

cf. [Voi02, Lem. 7.32] Once again, these isomorphisms are constructed in such a way that they are compatible with the cycle class maps, so we deduce an isomorphism of groups :

$$Z^{2j-2}(E) \xrightarrow{\sim} \bigoplus_{h=0}^{r-1} Z^{2j-2-2h}(Y).$$

This shows that the formulæ of the form (IV.1) are respected by Z_{2i}.

We now check the compatibility with projective bundles. Let X be a smooth and projective variety of dimension d and $m \geq 0$ an integer. We want to show that

$$[Z_{2i+2m+2}(X \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}})] - [Z_{2i+2m+2}(X)] = [Z_{2i+2m}(X)].$$

Rearranging once again by putting $j := d - i - m$, we see that we need to show the following equality :

$$[Z^{2j}(X \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}})] - [Z^{2j-2}(X)] = [Z^{2j}(X)]. \tag{IV.2}$$

We again make use of [Voi02, Lem. 7.32] and [Voi03, Thm. 9.25], this time applied to the trivial projective bundle $X \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}} \to X$, so that we obtain a commutative diagram :

$$\begin{tikzcd}
\mathrm{CH}^j(X) \oplus \mathrm{CH}^{j-1}(X) \arrow[r, "\sim"] \arrow[d, "c\ell_{j,X} \oplus c\ell_{j-1,X}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{CH}^j(X \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}}) \arrow[d, "c\ell_{j,X \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}}}"] \\
\mathrm{Hdg}^{2j}(X, \mathbb{Z}) \oplus \mathrm{Hdg}^{2j-2}(X, \mathbb{Z}) \arrow[r, "\sim"'] & \mathrm{Hdg}^{2j}(X \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}}, \mathbb{Z})
\end{tikzcd},$$

hence an isomorphism of groups :

$$Z^{2j}(X \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}}) \xrightarrow{\sim} Z^{2j}(X) \oplus Z^{2j-2}(X).$$

Taking motivic classes, this yields formula (IV.2) as desired, and concludes the proof of (i).

We now prove (ii), which follows easily. Let X be a smooth and projective variety of dimension d and let $m \geq d - i$, so that $2i + 2m \geq 2d$. We thus get that $Z_{2_i}([X]/\mathbb{L}^m) = [Z_{2i+2m}(X)] = 0$, in other words, the assigment Z_{2i} sends the i^{th} piece $\mathrm{Fil}^i K_0(\underline{\mathbf{Var}}_{\mathbb{C}})[\mathbb{L}^{-1}]$ of the dimension filtration to zero. If we therefore endow $K_0(\underline{\mathbf{Var}}_{\mathbb{C}})[\mathbb{L}^{-1}]$ with the topology induced by this filtration and $K_0(\underline{\mathbf{Ab}})$ with the discrete topology, then Z_{2i} is naturally continuous, hence the claim. □

2.3. Some consequences

— The main theorem in Scavia's paper has some very interesting applications, as it somehow allows one to «detect» the non-triviality of certain classes in the Grothendieck ring of stacks. In particular, thanks to the results of Colliot-Thélène and Voisin discussed in Chapter III (notably Corollary (3.2.11)), we know that for a suitable smooth and projective complex variety X (that is, whose Chow group of 0-cycles is supported on a surface, for instance if X is rationally connected), we can identify the group $Z^4(X)$ with the unramified cohomology group $\mathrm{H}^3_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z})$, and both groups are birational invariants of general smooth and projective complex varieties. For convenience, for any integer i, we will still denote by Z_{2i} the composite map :

$$K_0(\underline{\mathbf{Stck}}_{\mathbb{C}}) \longrightarrow \widehat{K_0}(\underline{\mathbf{Var}}_{\mathbb{C}}) \xrightarrow{\widehat{Z}_{2i}} K_0(\underline{\mathbf{Ab}}).$$

Proposition 4.2.22 (Scavia [Sca21, Prop. 7]). *Let G be a finite group and let V be a faithful complex representation of G. Then $Z_{2i}([\mathrm{B}G]) = 0$ for every $i \geq -1$, and we have :*

$$Z_{-4}([\mathrm{B}G]) = [\mathrm{H}^3_{nr}(\mathbb{C}(V)^G/\mathbb{C}, \mathbb{Q}/\mathbb{Z})] \in K_0(\underline{\mathbf{Ab}}).$$

Proof. Let d be the dimension of V. As proven by Ekedahl in [Eke09a, Prop. 3.1, (ii)], we can write $[\mathrm{B}G]$ as :

$$[\mathrm{B}G] = \lim_{m\to+\infty} [V^m/G] \cdot \mathbb{L}^{-md}.$$

If we fix an integer i, then Theorem (4.2.21, (ii)) shows that the natural map $\widehat{Z}_{2i} : \widehat{K_0}(\underline{\mathbf{Var}}_{\mathbb{C}}) \to K_0(\underline{\mathbf{Ab}})$ is continuous. Therefore, in $\widehat{Z}_{2i}(\underline{\mathbf{Var}}_{\mathbb{C}})$, we have for m sufficiently large the equality :

$$Z_{2i}([\mathrm{B}G]) = \widehat{Z}_{2i}([\mathrm{B}G]) = \widehat{Z}_{2i}([V^m/G]\cdot\mathbb{L}^{-md}) = Z_{2i}([V^m/G]\cdot\mathbb{L}^{-md}).$$

Let us fix such an integer m. By Hironaka's theorem we can find a finite sequence of smooth blowups starting from a smooth and projective complex variety X and finishing at the quotient variety V^m/G. Using the scissors formulæ defining $K_0(\underline{\mathbf{Var}}_{\mathbb{C}})$, we can thus write :

$$[V^m/G] = [X] + \sum_{q\geq 0} n_q[X_q] \in K_0(\underline{\mathbf{Var}}_{\mathbb{C}})$$

where the X_q's are smooth and projective complex varieties such that $\dim X_q \leq md-1$ and $n_q \in \mathbb{Z}$ for each $q \geq 0$. Applying Z_{2i} on both sides, we obtain that :

$$Z_{2i}([\mathrm{B}G]) = [Z_{2i+2md}(X)] + \sum_{q\geq 0} n_q[Z_{2i+2md}(X_q)].$$

But by Lefschetz's theorem on (1,1)-classes [Voi02, Thm. 7.2], we have that $Z_{2md-2}(X) = Z^2(X) = 0$. Hence if $i \geq -1$, then every term on the right hand side of the above identity is zero, so that $Z_{2i}([\mathrm{B}G]) = 0$ for all such i's. If now $i=-2$, then we get the equality $[Z_{2md-4}(X)] = [Z^4(X)]$. On the other hand, since X is birationally equivalent to V/G, then we have $\mathrm{H}^3_{\mathrm{nr}}(\Bbbk(X)/\mathbb{C}, \mathbb{Q}/\mathbb{Z}) \simeq \mathrm{H}^3_{\mathrm{nr}}(\mathbb{C}(V)^G/\mathbb{C}, \mathbb{Q}/\mathbb{Z})$. Moreover, remark that X is rationally connected : indeed, V/G is unirational since it is dominated by the affine space V, and X is smooth and projective, so this follows from Chapter I, Remark (1.1.19). This variety thus satisfies the conditions of Chapter III, Corollary (3.2.11), so we naturally have :

$$Z^4(X) \simeq \mathrm{H}^3_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}) \simeq \mathrm{H}^3_{\mathrm{nr}}(\mathbb{C}(V)^G/\mathbb{C}, \mathbb{Q}/\mathbb{Z}).$$

By the previous decomposition of $Z_{2i}([\mathrm{B}G])$ applied to $i=-2$, we obtain the desired identity :

$$Z_{-4}([\mathrm{B}G]) = [Z^4(X)] = [\mathrm{H}^3_{\mathrm{nr}}(\mathbb{C}(V)^G/\mathbb{C}, \mathbb{Q}/\mathbb{Z})].$$

□

We finish this section with the following result, also due to Scavia :

Corollary 4.2.23 (Scavia [Sca21, Thm. 3]). *Let G be a finite group and V be a faithful complex representation of G. Assume that* $\mathrm{H}^3_{nr}(\mathbb{C}(V)^G/\mathbb{C}, \mathbb{Q}/\mathbb{Z}) \neq 0$. *Then* $[\mathrm{B}G] \neq 1$ *in* $K_0(\underline{\mathbf{Stck}}_{\mathbb{C}})$.

Proof. It was shown by Ekedahl in [Eke09b, Prop. 3.3, (i)] that the Grothendieck group $K_0(\underline{\mathbf{Ab}})$ is can be written in terms of generators as :

$$K_0(\underline{\mathbf{Ab}}) = \left\langle [\mathbb{Z}],\ [\mathbb{Z}/p^n] \,\middle|\, p \text{ prime},\ n\geq 1\right\rangle.$$

On the other hand, if $\mathrm{H}^3_{\mathrm{nr}}(\mathbb{C}(V)^G/\mathbb{C}, \mathbb{Q}/Z)$ is non-zero, then it has a non-trivial class in $K_0(\underline{\mathbf{Ab}})$. By the above proposition, this shows that $Z_{-4}([\mathrm{B}G]) \neq 0$ in $K_0(\underline{\mathbf{Ab}})$. But $Z_{-4}([\mathsf{Spec}\,\mathbb{C}]) = 0$ since Z_{-4} is a morphism of groups. We conclude that $[\mathrm{B}G]$ must be nonzero in $K_0(\underline{\mathbf{Stck}}_{\mathbb{C}})$. □

As a consequence, there exist finite groups G such that $\mathsf{Br}_{\mathrm{nr}}(\mathbb{C}(V)^G/\mathbb{C}) = 0$ but $[\mathrm{B}G] \neq 1$ in $K_0(\underline{\mathbf{Stck}}_{\mathbb{C}})$. Indeed, as discussed at the beginning of Chapter II, Peyre constructed in [Pey07, Thm. 3.1] for any odd prime number p a central extension

$$0 \longrightarrow (\mathbb{Z}/p)^6 \longrightarrow G \longrightarrow (\mathbb{Z}/p)^6 \longrightarrow 0$$

such that $\mathsf{Br}_{\mathrm{nr}}(\mathbb{C}(V)^G/\mathbb{C}) = 0$ and $\mathrm{H}^3_{\mathrm{nr}}(\mathbb{C}(V)^G/\mathbb{C}, \mathbb{Q}/\mathbb{Z}) \neq 0$ for any faithful complex representation V of G.

Appendix A

Action of correspondences on the cohomology of $\mathscr{H}_B^\bullet(-)$

A.1. Bloch's formula for Betti cohomology

Lemma 1.1.1. *Let X be a smooth and proper variety over an algebraically closed field k.*

(i) *For any prime ℓ distinct from* char k*, algebraic and homological equivalence relatively to the étale ℓ-adic cycle class coincide for divisors on X.*

(ii) *If moreover $k = \mathbb{C}$, then algebraic and homological equivalence relatively to the Betti cycle class coincide for divisors on X.*

Proof. Let us first prove (ii). The exponential sequence on X provides an exact piece :

$$\mathrm{H}^1(X, \mathscr{O}_X) \longrightarrow \mathsf{Pic}(X) \xrightarrow{c\ell_1} \mathrm{H}^2_B(X, \mathbb{Z}),$$

where the second arrow is the cycle class. Moreover since X is smooth and proper, we have an exact sequence

$$0 \longrightarrow \mathsf{Pic}^0(X) \longrightarrow \mathsf{Pic}(X) \longrightarrow \mathsf{NS}(X) \longrightarrow 0,$$

see *e.g.* [Laz04, Thm. 1.1.16], and $\mathsf{Pic}^0(X) = \ker[\mathsf{Pic}(X) \to \mathrm{H}^2_B(X, \mathbb{Z})]$. Since $\mathsf{Pic}^0(X)$ precisely consists of the invertible sheaves that are algebraically equivalent to 0 (it is represented by a complex abelian variety which, as a complex torus, is the quotient $\mathrm{H}^1(X, \mathscr{O}_X)/\mathrm{H}^1_B(X, \mathbb{Z})$; the claim follows since this torus is connected, because then $\mathrm{H}^0_B(X, \mathbb{Z}) = 0$), we obtain an inclusion $\mathsf{NS}(X) \hookrightarrow \mathrm{H}^2_B(X, \mathbb{Z})$ as the image of the cycle class, which is the desired result.

Now for (i), we know that the Kummer sequence for varying $n \geq 1$ on $X_{\text{ét}}$ provides injections $\mathsf{Pic}(X)/\ell^n \hookrightarrow \mathrm{H}^2_{\text{ét}}(X, \mathbb{Z}/\ell^n(1))$ which, after taking inverse limits, yields an injection (these groups are finite, see [Mil80, Chap. VI, Cor. 2.8]) :

$$\varprojlim_{n \geq 1} \mathsf{Pic}(X)/\ell^n \hookrightarrow \mathrm{H}^2_{\text{ét}}(X, \mathbb{Z}_\ell(1)).$$

On the other hand, using once again the properness and smoothness of X, we have an exact sequence

$$0 \longrightarrow \underline{\mathbf{Pic}}^0_{X/k}(k) \longrightarrow \mathsf{Pic}(X) \longrightarrow \mathsf{NS}(X) \longrightarrow 0,$$

where $\underline{\mathbf{Pic}}^0_{X/k}$ is the Picard variety of X. Since the latter is an abelian variety, then its k-points form a divisible group ; on the other hand, $\mathsf{NS}(X)$ is always finitely generated [BGI71, Exposé XIII, Thm. 5.1], so taking limits we obtain an isomorphism :

$$\varprojlim_{n \geq 1} \mathsf{Pic}(X)/\ell^n \xrightarrow{\sim} \mathsf{NS}(X) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell,$$

hence an injection

$$\mathsf{NS}(X) \hookrightarrow \mathsf{NS}(X) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \xrightarrow{\sim} \varprojlim_{n\geq 1} \mathsf{Pic}(X)/\ell^n \hookrightarrow \mathrm{H}^2_{\text{ét}}(X, \mathbb{Z}_\ell(1)),$$

as we wanted. □

Proposition 1.1.2 (Bloch's formula for Betti cohomology). *For any smooth, projective and connected variety over* $\mathbb{C}$*, we have canonical isomorphisms*

$$\mathrm{CH}^p(X)/\mathrm{alg} \xrightarrow{\sim} \mathrm{H}^p_{Zar}(X, \mathcal{H}^p_B(\mathbb{Z}(p)))$$

for all $p \geq 1$.

Proof. Recall that by the Bloch-Ogus resolution of $\mathcal{H}^p_B(\mathbb{Z}(p))$, we have an isomorphism

$$\mathrm{H}^p_{\mathrm{Zar}}(X, \mathcal{H}^p_B(\mathbb{Z}(p))) \simeq \ker d_1^{p,p}/\mathrm{Im}\, d_1^{p-1,p} = \mathcal{Z}^p(X)/\mathrm{Im}\, d_1^{p-1,p},$$

where $d_1^{p,q}$ denotes the differential in position (p,q) in the E_1-page of the coniveau spectral sequence attached to X and $\mathbb{Z}(p)$. The goal is thus to show that the image of the differential

$$d_1^{p-1,p} : \bigoplus_{D\in X^{(p-1)}} \mathrm{H}^1_B(\Bbbk(D), \mathbb{Z}(1)) \longrightarrow \bigoplus_{D\in X^{(p)}} \mathrm{H}^0_B(\Bbbk(D), \mathbb{Z})$$

is precisely the group $\mathcal{Z}^p(X)_{\mathrm{alg}}$ of cycles of codimension p algebraically equivalent to 0.

If we fix a closed subvariety $D \hookrightarrow X$ of codimension $p-1$, then by Hironaka one can choose a resolution of singularities $\widetilde{D} \to D$ that is projective. In particular, we obtain an isomorphism $\mathrm{H}^1_B(\Bbbk(\widetilde{D}), \mathbb{Z}(1)) \xrightarrow{\sim} \mathrm{H}^1_B(\Bbbk(D), \mathbb{Z}(1))$. As proper morphisms preserve cycles that are algebraically equivalent to 0, see [Ful98, Prop. 10.3], then we are reduced to showing that the image of the differential in the Cousin complex corresponding to $\widetilde{D}$ coincides with the group of cycles of codimension p that are algebraically equivalent to 0, so in this case, the divisors on $\widetilde{D}$ (as the latter has codimension $p-1$ in X). But as $\widetilde{D}$ is smooth and proper, the lemma shows that a divisor D' is algebraically equivalent to 0 if and only if it is homologically equivalent to 0 (for Betti cohomology), and the last condition is satisfied if and only if there exists $\alpha \in \mathrm{H}^1_B(\widetilde{D} \setminus \mathrm{Supp}(D'), \mathbb{Z}(1))$ whose (topological) residue is D', hence the desired result. □

A.2. Main statement

— Let X, Y be two smooth and projective varieties over $\mathbb{C}$. We recall that for a given abelian group A and an integer $j \geq 0$, the sheaf $\mathcal{H}^j_B(A)$ is defined as $\mathcal{R}^j\pi_*A$, where $\pi : X_{\mathrm{cl}} \to X_{\mathrm{Zar}}$ is the usual morphism of sites ; this coincides with the sheafification of the Zariski presheaf that sends an open subset $U \subset X$ to $\mathrm{H}^j_B(U, A)$. In this appendix, we consider correspondences from X to Y with support in codimension $r + \dim X$, where r is a fixed integer.

Proposition 1.2.3. *Any correspondence* $\Gamma \in \mathrm{CH}^{r+\dim X}(X \times_{\mathbb{C}} Y)/alg$ *induces homomorphisms*

$$\Gamma_* : \mathrm{H}^p_{Zar}(X, \mathcal{H}^q_B(A)) \longrightarrow \mathrm{H}^{p+r}_{Zar}(Y, \mathcal{H}^{q+r}_B(A(r)))$$

for $p, q \geq 0$ *that are compatible with the composition of correspondences.*

Proof. Let us write $d := \dim X$. By the above lemma, we know that Γ has a class

$$[\Gamma]_{\mathrm{BO}} \in \mathrm{H}^{r+d}_{\mathrm{Zar}}(X \times_{\mathbb{C}} Y, \mathcal{H}^{r+d}_B(\mathbb{Z}(r+d))).$$

On the other hand, the projection onto the first factor yields a pullback map

$$\mathsf{pr}_1^* : \mathrm{H}^p_{\mathrm{Zar}}(X, \mathcal{H}^q_B(A)) \longrightarrow \mathrm{H}^p_{\mathrm{Zar}}(X \times_{\mathbb{C}} Y, \mathcal{H}^q_B(A)).$$

The cup-products in Betti cohomology $\mathcal{H}^l_B(A) \otimes \mathcal{H}^s_B(\mathbb{Z}(l)) \to \mathcal{H}^{l+s}_B(A(l))$ yield cup-products on cohomology :

$$\mathrm{H}^p_{\mathrm{Zar}}(Z, \mathcal{H}^l_B(A)) \otimes_{\mathbb{Z}} \mathrm{H}^q_{\mathrm{Zar}}(Z, \mathcal{H}^s_B(\mathbb{Z}(l))) \longrightarrow \mathrm{H}^{p+q}_{\mathrm{Zar}}(Z, \mathcal{H}^{l+s}_B(A(l)))$$

for any variety Z. We thus obtain a map

$$(- \smile [\Gamma]_{\mathrm{BO}}) \circ \mathsf{pr}_1^* : \mathrm{H}^p_{\mathrm{Zar}}(X, \mathcal{H}^q_B(A)) \longrightarrow \mathrm{H}^{p+d+r}_{\mathrm{Zar}}(X \times_{\mathbb{C}} Y, \mathcal{H}^{q+d+r}_B(A(d+r))).$$

Note that this is not exactly the map we are looking for ; we want to shift the degree in the right hand side by the dimension of X. By Bloch-Ogus theory, for any pair of integers $i, j \geq 0$, the Gersten resolution for the sheaf $\mathcal{H}^j_B(A(d+r))$ on $X \times_{\mathbb{C}} Y$ gives an isomorphism :

$$\mathrm{H}^i_{\mathrm{Zar}}(X \times_{\mathbb{C}} Y, \mathcal{H}^j_B(A(d+r))) \xrightarrow{\sim} \frac{\ker\left[\bigoplus_{x \in (X\times_{\mathbb{C}}Y)^{(i)}} \mathrm{H}^{j-i}_B(\kappa(x), A(d+r-i)) \longrightarrow \bigoplus_{x \in (X\times_{\mathbb{C}}Y)^{(i+1)}} \mathrm{H}^{j-i-1}_B(\kappa(x), A(d+r-i-1))\right]}{\mathrm{Im}\left[\bigoplus_{x \in (X\times_{\mathbb{C}}Y)^{(i-1)}} \mathrm{H}^{j-i+1}_B(\kappa(x), A(d+r-i+1)) \longrightarrow \bigoplus_{x \in (X\times_{\mathbb{C}}Y)^{(i)}} \mathrm{H}^{j-i}_B(\kappa(x), A(d+r-i))\right]}$$

where the differentials on the right are the obvious residues in the associated Cousin complex. Now, any point $x \in (X \times_{\mathbb{C}} Y)^{(i)}$ has its Zariski closure Z_x that is mapped under pr_2 to a subvariety $Z'_x \subset Y$ of codimension at least $i-d$ (with equality if the projection is generically finite). Let $x' := \mathsf{pr}_2(x)$. For any x such that x' has codimension $> i-d$, we say that

$$\mathsf{pr}_{2*} : \mathrm{H}^{j-i}_B(\kappa(x), A(d+r-i)) \longrightarrow \mathrm{H}^{j-i}_B(\kappa(x'), A(d+r-i))$$

is the zero map. Otherwise, it is induced by the proper morphisms $\mathsf{pr}_2|_{Z_x} : \mathsf{pr}_2^{-1}|_{Z_x}(V) \to V$ for sufficiently small open subsets $V \subset Z'_x$. We now use the following lemma :

Lemma 1.2.4. *The maps* pr_{2*} *commute with the residues in the Cousin complex*

$$0 \longrightarrow \mathrm{H}^j_B(\Bbbk(X), A(d+r)) \longrightarrow \cdots \longrightarrow \bigoplus_{x \in (X\times_{\mathbb{C}}Y)^{(i)}} \mathrm{H}^{j-i}_B(\kappa(x), A(d+r-i)) \longrightarrow \cdots$$

Proof of the lemma. Let $Z \subset X \times_{\mathbb{C}} Y$ be a subvariety of codimension i and $D' \subset Z' = \mathsf{pr}_2(Z)$ a subvariety of Y of codimension $i-d-1$. Suppose without loss of generality that Z and Z' are normal (we will see that the arguments commute with taking normalisations). We want to show that for any integer l, the composite morphism

$$\partial_{Z',D'} \circ \mathsf{pr}_{2*} : \mathrm{H}^k_B(\Bbbk(Z), A(l)) \longrightarrow \mathrm{H}^k_B(\Bbbk(Z'), A(l)) \longrightarrow \mathrm{H}^{k-1}_B(\Bbbk(D'), A(l-1))$$

coincides with the composition

$$\mathsf{pr}_{2*} \circ \left(\sum_{\substack{D \subset Z \\ \mathsf{pr}_2(D) = D'}} \mathsf{pr}_{2*} \right) \circ \partial_{Z,D} : \mathrm{H}^k_B(\Bbbk(Z), A(l)) \longrightarrow \bigoplus_{\substack{D \subset Z \\ \mathsf{pr}_2(D) = D'}} \mathrm{H}^{k-1}_B(\Bbbk(D), A(l-1)) \longrightarrow \mathrm{H}^{k-1}_B(\Bbbk(D'), A(l-1)).$$

First suppose that $\dim Z = \dim Z'$. Since the residues considered here are induced by the long exact cohomology sequences with support for pairs defined by varieties and their divisors, then the two maps must coincide since pr_2 yields proper morphisms between pairs of the form $(Z_0, \cup_{\mathsf{pr}_2(D)=D'} D)$ and (Z'_0, D') where Z'_0 is a smooth open subset of Z' containing a dense open subset of D' and Z_0 is the inverse image of Z'_0 in Z. This morphism remains also proper when taking complements. (If the varieties were not normal, we could always take normalisations and get the same conclusion.)

If now $\dim Z' < \dim Z$, remark that this case occurs only when $\dim Z' = \dim Z - 1$ and $Z' = D'$, so that $\mathsf{pr}_{2*} : \mathrm{H}^k_B(\Bbbk(Z), A(l)) \to \mathrm{H}^k_B(\Bbbk(Z'), A(l))$ is the zero map. As a consequence, the composite $\partial_{Z',D'} \circ \mathsf{pr}_{2*}$ is zero, so we have to show that for any class $\alpha \in \mathrm{H}^k_B(\Bbbk(Z), A(l))$, we have $\mathsf{pr}_{2*}(\partial(\alpha)) = 0$, where $\partial = \sum_{\substack{D \subset Z \\ \mathsf{pr}_2(D) = D'}} \mathsf{pr}_{2*} \circ \partial_{Z,D}$ so that $\partial(\alpha)$ is a finite sum of classes of degree $k-1$ supported on divisors $D \subset Z$ such that $\mathsf{pr}_2 : D \to D'$ is generically finite. The last assumption shows that above a dense open subset Z'_0 of $D' = Z'$, the projection induces a proper and smooth morphism $\mathsf{pr}^0_2 : Z_0 \to Z'_0$ of relative dimension 1, which in turn yields a pushforward

$$\mathsf{pr}_{2*} : \mathrm{H}^{k+1}_B(Z_0, A(l)) \to \mathrm{H}^{k-1}_B(Z'_0, A(l-1)).$$

In particular, for any divisor $j : D \hookrightarrow Z_0$ that is smooth and proper above Z'_0, and for any $\beta \in \mathrm{H}^{k-1}_B(D, A(l-1))$, we have a factorisation $\mathsf{pr}_{2*}(\beta) = (\mathsf{pr}^0_{2*} \circ j_*)(\beta)$ (where pr_{2*} is taken relatively to the induced proper and smooth map from D above Z'_0). But on the other hand,

$$j_* \circ \partial : \mathrm{H}^k_B(Z_0 \setminus D, A(l)) \longrightarrow \mathrm{H}^{k-1}_B(D, A(l-1)) \longrightarrow \mathrm{H}^{k+1}_B(Z_0, A(l))$$

is the zero map. This proves the lemma. □

Let us get back to the proof of the main result. Since the map pr_{2*} in our case commutes with residues, then it descends to a well-defined morphism

$$\begin{aligned}\mathrm{H}^i_{\mathrm{Zar}}(X \times_{\mathbb{C}} Y, \mathscr{H}^j_B(A(d+r))) \longrightarrow{}& \frac{\ker\left[\bigoplus_{x \in (X \times_{\mathbb{C}} Y)^{(i-d)}} \mathrm{H}^{j-i}_B(\kappa(x), A(d+r-i)) \longrightarrow \bigoplus_{x \in (X \times_{\mathbb{C}} Y)^{(i+1-d)}} \mathrm{H}^{j-i-1}_B(\kappa(x), A(d+r-i-1))\right]}{\mathrm{Im}\left[\bigoplus_{x \in (X \times_{\mathbb{C}} Y)^{(i-d-1)}} \mathrm{H}^{j-i+1}_B(\kappa(x), A(d+r-i+1)) \longrightarrow \bigoplus_{x \in (X \times_{\mathbb{C}} Y)^{(i-d)}} \mathrm{H}^{j-i}_B(\kappa(x), A(d+r-i))\right]} \\ &\xrightarrow{\sim} \mathrm{H}^{i-d}_{\mathrm{Zar}}(Y, \mathscr{H}^{j-d}_B(A(r)))\end{aligned}$$

where the second isomorphism comes once again from the Gersten resolution of the sheaf $\mathscr{H}^{i-d}_B(A(r))$ on Y. With these pushforwards now defined, the compositions

$$\mathsf{pr}_{2*} \circ (- \smile [\Gamma]_{\mathrm{BO}}) \circ \mathsf{pr}_1^* : \Gamma_* : \mathrm{H}^p_{\mathrm{Zar}}(X, \mathscr{H}^q_B(A)) \longrightarrow \mathrm{H}^{p+r}_{\mathrm{Zar}}(Y, \mathscr{H}^{q+r}_B(A(r)))$$

are precisely the desired maps in the proposition. The fact that these maps agree with the composition of correspondences is then a natural consequence of the commutativity of Bloch's formula with intersection products, the commutativity of the maps pr_{2*} with cup-products, and a base change formula for the cohomology of $\mathscr{H}^q_B(A(l))$; for complete details, see [CTV12, Lem. 9.3]. □

Appendix B

Quillen's Q-Construction

B.1. Classifying space of a category

— Let us briefly recall some basics about simplicial sets.

Definition 2.1.1. The *category of simpleces* is the category $\underline{\Delta}$ whose objects are finite ordered sets

$$[n] := \{0 < 1 < \ldots < n\}$$

and morphisms $f : [m] \to [n]$ are non-decreasing monotonous maps, that is, $f(i) \leq f(j)$ for $i \leq j$.

When $\mathscr{C}$ is an arbitrary category, we define a *simplicial object* in $\mathscr{C}$ to be a presheaf $F : \underline{\Delta}^{\mathrm{op}} \to \mathscr{C}$. We define a morphism of simplicial objects as a natural transformation of functors, so that we can consider the category of simplicial objects in $\mathscr{C}$ (denoted by $\underline{\mathbf{s}\mathscr{C}}$). In particular, a *simplicial set* is a simplicial object in the category of sets, and a *simplicial space* is a simplicial object in the category of topological spaces.

Definition 2.1.2. Let $\mathscr{C}$ be a small category. The *nerve* of $\mathscr{C}$, denoted $N(\mathscr{C})$, is the simplicial set with n-simplices the set of diagrams

$$x_0 \longrightarrow x_1 \longrightarrow \cdots \longrightarrow x_n \ \text{ with } x_i \in \mathrm{Ob}(\mathscr{C}),\ \text{arrows in } \mathrm{Mor}(\mathscr{C}).$$

Face operators arise by omitting objects and composing arrows, and degeneracies are defined by inserting identity maps, that is :

$$\partial_i(x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} x_n) := x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{i-1}} x_{i-1} \xrightarrow{f_{i+1}\circ f_i} x_{i+1} \xrightarrow{f_{i+2}} \cdots \xrightarrow{f_n} x_n$$

and

$$\sigma_i(x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} x_n) := x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{i-1}} x_{i-1} \xrightarrow{f_i} x_i \xrightarrow{\mathrm{Id}} x_i \xrightarrow{f_{i+1}} \cdots \xrightarrow{f_n} x_n.$$

If X is a general simplical set given by a sequence $\{X_n\}_{n\geq 0}$ of sets together with face operators $\partial_i : X_n \to X_{n-1}$ and degeneracy operators $\sigma_i : X_n \to X_{n+1}$, then the *geometric realisation* of X is the topological space

$$|X| := \left(\coprod_{n\geq 0} X_n \times \Delta^n\right) /\!\sim,$$

where $\Delta^n \subset \mathbb{R}^{n+1}$ denotes the standard n-simplex and $X_n \times \Delta^n$ is the disjoint union of copies of Δ^n indexed by the elements of X_n. The equivalence relation $\sim$ is defined as follows. Any map $f : [m] \to [n]$, induces a map $f^* : X_n \to X_m$ and a continuous map of simpleces $f_* : \Delta^m \to \Delta^n$. Precisely, we define them on vertices $v_0, \ldots, v_m$ by sending $v_i \mapsto v_{f(i)}$ and extending linearly to the faces of Δ^m. For each $x \in X_n$ and $s \in \Delta^m$, we then identify

$$(f^*(x), s) \sim (x, f_*(s)).$$

It follows by construction that the geometric realisation of X is a CW-complex, whose n-cells are given by the elements $x \in X_n$ that are «non-degenerate», that is, which are not of the form $\sigma_i(y)$ for some $y \in X_{n-1}$.

One can then check that any morphism of simplicial sets $f : X \to Y$ induces a continuous map $|X| \to |Y|$: indeed, f is a natural transformation of contravariant functors $X \Rightarrow Y : \underline{\Delta}^{\mathrm{op}} \to \underline{\mathbf{Sets}}$:

$$\begin{array}{ccc} [n] & \quad & \begin{tikzcd} X_n \arrow[r, "f_n"] \arrow[d] & Y_n \arrow[d] \\ X_m \arrow[r, "f_m"'] & Y_m \end{tikzcd} \\ \uparrow & & \\ [m] & & \end{array}$$

so we can define a map

$$X_n \times \Delta^n \longrightarrow Y_n \times \Delta^n \quad (\forall n \geq 0)$$

by sending $(x, s) \mapsto (f_n(x), s)$. One then checks that these maps descend to a well-defined continuous map $|X| \to |Y|$. We thus naturally obtain a functor $|\cdot| : \underline{\mathbf{sSets}} \to \underline{\mathbf{Top}}$ (where $\underline{\mathbf{sSets}}$ denotes the category of simplicial sets).

Definition 2.1.3. Let now X be any topological space. The *singular complex* of X is the simplicial set $\mathcal{S}X : \underline{\Delta}^{\mathrm{op}} \to \underline{\mathbf{Sets}}$ given by

$$[n] \rightsquigarrow \mathrm{Hom}_{\underline{\mathbf{Top}}}(\Delta^n, X).$$

The main point of geometric realisation is encapsulated in the following result :

Proposition 2.1.4 ([May93, Chap. III, §16]). *The geometric realisation functor* $|\cdot| : \underline{\mathbf{sSets}} \to \underline{\mathbf{Top}}$ *is left adjoint to the singular complex functor* $\mathcal{S} : \underline{\mathbf{Top}} \to \underline{\mathbf{sSets}}$ *:*

$$\mathrm{Hom}_{\underline{\mathbf{Top}}}(|X|, Y) \xrightarrow{\sim} \mathrm{Hom}_{\underline{\mathbf{sSets}}}(X, \mathcal{S}Y)$$

for any $Y \in \mathrm{Ob}(\underline{\mathbf{Top}})$ *and* $X \in \mathrm{Ob}(\underline{\mathbf{sSets}})$.

Definition 2.1.5. The *classifying space* $\mathbf{B}\mathscr{C}$ of a small category $\mathscr{C}$ is defined as the geometric realisation

$$\mathbf{B}\mathscr{C} := |N(\mathscr{C})|$$

of the nerve of $\mathscr{C}$.

Proposition 2.1.6 ([Sri96, Lem. 3.6]). *Let* $F, G : \mathscr{C} \to \mathscr{D}$ *be two functors between small categories such that there exists a natural transformation* $\eta : F \Rightarrow G$. *Then the induced maps* $\mathbf{B}F, \mathbf{B}G : \mathbf{B}\mathscr{C} \to \mathbf{B}\mathscr{D}$ *are homotopic.*

An immediate (and often useful) consequence is the following :

Corollary 2.1.7. *Let* $F : \mathscr{C} \to \mathscr{D}$ *be a functor between small categories. If* F *has a left adjoint or a right adjoint, then* $\mathbf{B}F$ *is a homotopy equivalence. In particular, if* $\mathscr{C}$ *and* $\mathscr{D}$ *are equivalent categories, then* $\mathbf{B}\mathscr{C}$ *and* $\mathbf{B}\mathscr{C}$ *are homotopy equivalent.*

Example 2.1.8. Let $\mathscr{C}$ be a small category and $\star$ the category with a single element and identity morphism $\star \to \star$. There exists a unique functor $\mathscr{C} \to \star$.

- If $\mathscr{C}$ has an initial object $I \in \mathrm{Ob}(\mathscr{C})$, then the functor $\star \rightsquigarrow I$ is left adjoint to F :

$$\mathrm{Hom}_{\mathscr{C}}(I, X) \simeq \mathrm{Hom}_{\star}(\star, \star).$$

- If $\mathscr{C}$ has a terminal object $T \in \mathrm{Ob}(\mathscr{C})$, then the functor $\star \rightsquigarrow T$ is right adjoint to F :

$$\mathrm{Hom}_{\star}(\star, \star) \simeq \mathrm{Hom}_{\mathscr{C}}(X, T).$$

Therefore, a small category that admits an initial or a terminal object is *contractible*, that is, its classifying space is contractible.

B.2. The Q-Construction

— Let $\mathscr{C}$ be an exact category (see Chapter I, §3.1.1 for a definition). In this section, we explicit the construction of a category $Q\mathscr{C}$ that satisfies $\pi_1^{\mathrm{top}}(\mathbf{B}Q\mathscr{C}, 0) \simeq K_0(\mathscr{C})$ for a given zero object $0 \in \mathrm{Ob}(\mathscr{C})$. Following [Qui73, §2], we put $\mathrm{Ob}(Q\mathscr{C}) = \mathrm{Ob}(\mathscr{C})$, however we change the morphisms. A morphism in $Q\mathscr{C}$ is a diagram of the form :

$$\begin{tikzcd} & V \arrow[dl, two heads] \arrow[dr, hook] & \\ X & & Y \end{tikzcd} \qquad \begin{array}{c} 0 \longrightarrow V' \longrightarrow V \longrightarrow X \longrightarrow 0 \\[2em] 0 \longrightarrow V \longrightarrow Y \longrightarrow Y' \longrightarrow 0 \end{array}$$

where $V \twoheadrightarrow X$ is an admissible epimorphism in $\mathscr{C}$ and $V \hookrightarrow Y$ is an admissible monomorphism in $\mathscr{C}$. We consider the isomorphism classes of such diagrams, that is, we identify two morphisms given as above if there exists an isomorphism $V \xrightarrow{\sim} V'$ in $\mathscr{C}$ such that the following diagram commutes :

$$\begin{tikzcd} & V \arrow[dl, two heads] \arrow[dr, hook] \arrow[dd, "\sim"] & \\ X & & Y \\ & V' \arrow[ul, two heads] \arrow[ur, hook] & \end{tikzcd}$$

For simplicity, we assume that such isomorphism classes form a set, so that $Q\mathscr{C}$ is small. The composition of two morphisms in $Q\mathscr{C}$ is defined by considering the bi-cartesian square :

$$\begin{tikzcd} & & V \times_Y W \arrow[dl, two heads, "\mathsf{pr}_1"'] \arrow[dr, hook, "\mathsf{pr}_2"] & & \\ & V \arrow[dl, two heads] \arrow[dr, hook] & & W \arrow[dl, two heads] \arrow[dr, hook] & \\ X & & Y & & Z \end{tikzcd}$$

which exists in $\mathscr{C}$, as the latter is closed under extensions by assumption. We obtain a short exact sequence

$$0 \longrightarrow \ker \mathsf{pr}_1 \longrightarrow V \times_Y W \xrightarrow{\mathsf{pr}_1} V \longrightarrow 0$$

and $\ker \mathsf{pr}_1 \simeq \ker[W \to Y]$. By the universal property of pullback squares, we easily see that composition is associative, and it only depends on the considered isomorphism class.

Definition 2.2.9. Let $i : A \hookrightarrow B$ be an admissible monomorphism in $\mathscr{C}$. This provides a morphism $i_! : A \to B$ in $Q\mathscr{C}$ represented by a diagram :

$$\begin{tikzcd} & A \arrow[dl, two heads, "\mathrm{Id}"'] \arrow[dr, hook, "i"] & \\ A & & B \end{tikzcd} .$$

The morphisms of the form $i_!$ are called *injective*. If $p : A \twoheadrightarrow B$ is an admissible epimorphism in $\mathscr{C}$, we define a morphism $p^! : C \to B$ in $Q\mathscr{C}$ represented by a diagram :

$$\begin{tikzcd} & B \arrow[dl, two heads, "p"'] \arrow[dr, hook, "\mathrm{Id}"] & \\ C & & B \end{tikzcd} .$$

The morphisms of the form $p^!$ are called *surjective*.

By definition, any morphism $f : X \to Y$ in $Q\mathscr{C}$ factors as a surjection followed by an injection $i_! \circ p^!$:

$$\begin{tikzcd}[column sep=small]
& & V \arrow[dl, "\mathrm{Id}"'] \arrow[dr, "\mathrm{Id}"] & & \\
& V \arrow[dl, two heads, "p"'] \arrow[dr, "\mathrm{Id}"'] & & V \arrow[dl, "\mathrm{Id}"] \arrow[dr, hook, "i"] & \\
X & & V & & Y
\end{tikzcd}.$$

Similarly, there is a unique factorisation (up to a unique isomorphism) as an injection followed by a surjection $\tilde{p}^! \circ \tilde{i}_!$ given by a bi-cartesian square :

$$\begin{tikzcd}[column sep=small]
& & V \arrow[dl, two heads, "p"'] \arrow[dr, hook, "i"] & & \\
& X \arrow[dl, "\mathrm{Id}"'] \arrow[dr, hook, "\tilde{i}"'] & & Y \arrow[dl, two heads, "\tilde{p}"] \arrow[dr, "\mathrm{Id}"] & \\
X & & X *_V Y & & Y
\end{tikzcd}.$$

The operations $i \mapsfrom \tilde{i}$ and $p \mapsto \tilde{p}$ satisfy the following properties :

(i) If i and j are composable admissible monomorphisms ($A \overset{i}{\hookrightarrow} B \overset{j}{\hookrightarrow} C$), then $(j \circ i)_! = j_! \circ i_!$:

$$\begin{tikzcd}[column sep=small]
& & A \arrow[dl, "\mathrm{Id}"'] \arrow[dr, hook, "i"] & & \\
& A \arrow[dl, "\mathrm{Id}"'] \arrow[dr, hook, "i"'] & & B \arrow[dl, "\mathrm{Id}"] \arrow[dr, hook, "i"] & \\
A & & B & & C
\end{tikzcd}.$$

Dually, if p and q are composable admissible epimorphisms, then $(p \circ q)^! = q^! \circ p^!$. Moreover $(\mathrm{Id}_A)_! = (\mathrm{Id}_A)^! = \mathrm{Id}_A$.

(ii) If one has a bi-cartesian square

$$\begin{tikzcd}
Z & Y \arrow[l, two heads, "\tilde{p}"'] \\
X \arrow[u, hook, "\tilde{i}"] & V \arrow[l, two heads, "p"] \arrow[u, hook, "i"']
\end{tikzcd}$$

where i and $\tilde{i}$ are admissible monomorphisms and p and $\tilde{p}$ are admissible epimorphisms, then $i_! \circ p^! = \tilde{p}^! \circ \tilde{i}_!$.

From these observations, we can actually characterise the category $Q\mathscr{C}$ as follows :

Lemma 2.2.10 ([Sri96, Lem. 6.2])**.** *Let $\mathscr{C}$ be an exact category and $\mathscr{D}$ an arbitrary category. Assume that we are provided with the following data :*

- *For each $A \in \mathrm{Ob}(\mathscr{C})$, there is an object $F(A) \in \mathrm{Ob}(\mathscr{D})$;*
- *for each admissible monomorphism $i : A \hookrightarrow B$ in $\mathscr{C}$, there exists a morphism $i_{!!} : F(A) \to F(B)$ in $\mathscr{D}$;*
- *for each admissible epimorphism $p : B \twoheadrightarrow C$ in $\mathscr{C}$, there exists a morphism $p^{!!} : F(C) \to F(B)$ in $\mathscr{D}$;*

Suppose furthermore that the conditions (i) *and* (ii) *as before hold for the morphisms $i_{!!}$ and $p^{!!}$. Then this data uniquely determines a functor $F : Q\mathscr{C} \to \mathscr{D}$.*

Proposition 2.2.11. *There is an isomorphism of categories*

$$Q\mathscr{C}^{op} \xrightarrow{\sim} Q\mathscr{C}$$

such that injective morphisms in $Q\mathscr{C}$ correspond to surjective morphisms in $Q\mathscr{C}^{op}$, and vice versa.

Proof. If we are provided with a bi-cartesian square in $\mathscr{C}$, then reversing arrows provides a bi-cartesian square in $\mathscr{C}^{\mathrm{op}}$, say of the form :

$$\begin{tikzcd} Z & Y \arrow[l, two heads, "\tilde{p}"'] \\ X \arrow[u, hook, "\tilde{i}"] & V \arrow[l, two heads, "p"] \arrow[u, hook, "i"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Z \arrow[r, hook, "\tilde{p}^{\mathrm{op}}"] \arrow[d, two heads, "\tilde{i}^{\mathrm{op}}"'] & Y \arrow[d, two heads, "i^{\mathrm{op}}"] \\ X \arrow[r, hook, "p^{\mathrm{op}}"'] & V \end{tikzcd} .$$

Now, consider the functor that is the identity on objects and defined on the morphisms by :

$$i_! \circ p^! \rightsquigarrow (\tilde{p}^{\mathrm{op}})_! \circ (\tilde{i}^{\mathrm{op}})^!.$$

One naturally checks that this assignment is fully faithful : $\mathsf{Hom}_{Q\mathscr{C}}(X, Y) \simeq \mathsf{Hom}_{Q\mathscr{C}^{\mathrm{op}}}(X, Y)$. □

B.3. Recovering K_0

— We are now going to explain how the Q-construction allows us to recover, for a given small exact category $\mathscr{C}$, the group $K_0(\mathscr{C})$ as the fundamental group $\pi_1^{\mathrm{top}}(\mathbf{B}Q\mathscr{C}, 0)$ of the classifying space of $\mathscr{C}$ based at a zero object. We first need to define the right notion of covering space in the simplicial setting. For the usual theory of coverings of topological spaces and groupoids, we refer to [May93, Chap. III].

Definition 2.3.12. A morphism of simplicial sets $p : E \to X$ is called a *covering* of X if for any commutative diagram as below in the category of simplicial sets :

$$\begin{tikzcd} \Delta[0] \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ \Delta[n] \arrow[r] \arrow[ur, dashed, "\exists!"] & X \end{tikzcd} ,$$

there is a unique morphism $\Delta[n] \to E$ making the two subtriangles commute.

The coverings of a simplicial set X form a category $\underline{\mathbf{Cov}}_X$, where the morphisms are the following commutative diagrams :

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p"'] & \circlearrowleft & E' \arrow[dl, "p'"] \\ & X & \end{tikzcd} .$$

By construction, the geometric realisation functor is compatible with covering spaces of usual topological spaces :

Proposition 2.3.13 ([GZ67, Appendix I, §3.2]). *The geometric realisation $|p| : |E| \to |X|$ of a simplicial covering $p : E \to X$ is a covering space in the category of topological spaces.*

In his original paper [Qui73], Quillen provided an insightful (and quite helpful) characterisation of the category of coverings for the classifying space of a small category :

Theorem 2.3.14 (**[Qui73, §1, Prop. 1]**). *Let $\mathscr{C}$ be a small category. The category $\underline{\mathbf{Cov}}_{\mathbf{B}\mathscr{C}}$ of coverings of the classifying space of $\mathscr{C}$ is equivalent to the category of morphism-inverting functors $F : \mathscr{C} \to \underline{\mathbf{Sets}}$, i.e. the functors taking each arrow $A \to A'$ to a bijection of sets $F(A) \to F(A')$.*

Now that all the necessary categorical machinery has been introduced and discussed, we are ready to prove the main result of this appendix :

Theorem 2.3.15 (**[Qui73, §2, Thm. 1]**). *Let $\mathscr{C}$ be a skeletally small exact category, and fix 0 a zero-object in $\mathscr{C}$. There is a natural isomorphism :*

$$\pi_1^{top}(\mathbf{B}Q\mathscr{C}, 0) \xrightarrow{\sim} K_0(\mathscr{C}).$$

Proof. By the previous theorem, we know that $\underline{\mathbf{Cov}}_{\mathbf{B}Q\mathscr{C}}$ is equivalent to the category of morphism inverting functors $F : Q\mathscr{C} \to \underline{\mathbf{Sets}}$, which we will simply denote by $\mathscr{I}_{Q\mathscr{C}}$. The point of Quillen's Q-construction is that the morphisms in this category are precisely defined to satisfy compatibility with exact sequences defining $K_0(\mathscr{C})$. Indeed, the category of covering spaces of $\mathbf{B}K_0(\mathscr{C})$ is equivalent to the category of morphism inverting functors $K_0(\mathscr{C}) \to \underline{\mathbf{Sets}}$ (where $K_0(\mathscr{C})$, as a group, is seen as a category), which is precisely the category of $K_0(\mathscr{C})$-sets[25]. As $\pi_1^{\text{top}}(\mathbf{B}Q\mathscr{C})$ is naturally isomorphic to the automorphism group of a universal covering space of $\mathbf{B}Q\mathscr{C}$ (*cf.* [Hat02, Thm. 1.38]), then the theorem amounts to showing an equivalence between $\mathscr{I}_{Q\mathscr{C}}$ and the category of $K_0(\mathscr{C})$-sets.

First, observe that $\mathscr{I}_{Q\mathscr{C}}$ is equivalent to the full subcategory $\mathscr{I}'$ of morphism-inverting functors $F' : Q\mathscr{C} \to \underline{\mathbf{Sets}}$ such that $F'(B) = F'(0)$ and $F'(i_{X!}) = \mathrm{Id}_{F'(0)}$ for every $X \in \mathrm{Ob}(\mathscr{C})$ (where i_X denotes the admissible monomorphism $0 \hookrightarrow X$). Indeed, for any admissible monomorphism $i : A \hookrightarrow B$, we have that $i \circ i_A = i_B$. Therefore, $\mathrm{Id}_{F'(0)} = F'(i_{B!}) = F'(i_!) \circ F'(i_{A!}) = F'(i_!)$. If $F : Q\mathscr{C} \to \underline{\mathbf{Sets}}$ is any morphism-inverting functor, one can define a functor F' in $\mathscr{I}'$ by putting :

$$\begin{tikzcd} & V \arrow[dl, two heads, "p"'] \arrow[dr, hook, "i"] & \\ X & & Y \end{tikzcd}$$

so that we consider $F(i_{V!})^{-1} \circ F(p^!) \circ F(i_{X!})$, with :

$$\begin{tikzcd} & 0 \arrow[dl, two heads] \arrow[dr, hook, "i_X"] & \\ 0 & & X \end{tikzcd} \qquad \begin{tikzcd} & V \arrow[dl, two heads, "p"'] \arrow[dr, hook] & \\ X & & V \end{tikzcd} \qquad \begin{tikzcd} & 0 \arrow[dl, two heads] \arrow[dr, hook, "i_V"] & \\ 0 & & V \end{tikzcd} .$$

$$F(0) \xrightarrow{F(i_{X!})} F(X) \qquad F(X) \xrightarrow{F(p^!)} F(V) \qquad F(V) \xrightarrow{F(i_{V!})^{-1}} F(0)$$

Now consider a natural transformation of functors $F' \Rightarrow F$ given by $X \mapsto F(i_{X!})$. Since $F(i_{X!})$ is a bijection of sets, then $F' \simeq F$, so any object in $\mathscr{I}_{Q\mathscr{C}}$ is isomorphic to an object of $\mathscr{I}'$, as desired.

If S is a $K_0(\mathscr{C})$-set, we can use Lemma (2.2.10) to define a morphism-inverting functor $F_S : Q\mathscr{C} \to \underline{\mathbf{Sets}}$ which belongs to $\mathscr{I}'$, by putting $F_S(A) := S$, $F_S(i_!) := \mathrm{Id}_S$ and by defining $F_S(p^!)$ as the natural action of $[\ker p] \in K_0(\mathscr{C})$ on S.

Conversely, we proceed the following way. If $F : Q\mathscr{C} \to \underline{\mathbf{Sets}}$ is a morphism-inverting functor which belongs to $\mathscr{I}'$, we describe a natural action of $K_0(\mathscr{C})$ on $F(0)$: if $[A] \in K_0(\mathscr{C})$, take $F(p_A^!) \in \mathrm{Aut}(F(0))$ where p_A is the obvious admissible epimorphism $A \twoheadrightarrow 0$. We need to check that this morphism is a homomorphism on $K_0(\mathscr{C})$. If

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is exact in $\mathscr{C}$, we should have

$$F(p_A^!) \circ F(p_C^!) = F(p_C^!) \circ F(p_A^!) \circ F(p_B^!).$$

[25] The idea one should have in mind is that of a Galois category, which is more or less the category of covering spaces of a topological space thanks to the main results of [GR71, §§4–6].

Consider the bi-cartesian square :

$$\begin{tikzcd} C & B \arrow[l, two heads, "p"'] \\ 0 \arrow[u, hook, "i_C"] & A \arrow[l, two heads, "p_A"] \arrow[u, hook, "i"'] \end{tikzcd},$$

from which we deduce $i_! \circ p_A^! = p^! \circ i_{C!}$. Since $F(i_!) = F(i_{C!}) = \mathrm{Id}_{F(0)}$, we conclude that $F(p_A^!) = F(p^!)$. Similarly $p_B = p_C \circ p$ so we have that

$$F(p_B^!) = F((p_C \circ p)^!) = F(p^! \circ p_C^!) = F(p^!) \circ F(p_C^!) = F(p_A^!) \circ F(p_C^!).$$

We also claim that $F(p_A^!) \circ F(p_C^!) = F(p_C^!) \circ F(p_A^!)$. Replacing B by $A \oplus C$ and considering the two split short exact sequences

$$0 \longrightarrow A \longrightarrow A \oplus C \longrightarrow C \longrightarrow 0$$

and

$$0 \longrightarrow C \longrightarrow A \oplus C \longrightarrow A \longrightarrow 0,$$

we readily get the desired result. Finally, since $\ker p_A \simeq A$, we conclude that the assignments $S \mapsto F_S$ and $F \mapsto F(0)$ (as a $K_0(\mathscr{C})$-set) are mutually inverse from each other, hence the equivalence of categories claim at the beginning of the proof. □

Appendix C

Further results on algebraic cycles

C.1. Roitman's Theorem

— We present Bloch's proof of Roitman's theorem [Blo79], which roughly states that the group $A_0(X)$ of degree zero 0-cycles on a smooth and projective connected variety X over a separably closed field k is represented (up to torsion divisible by the characteristic of k) by the Albanese variety of X, that is, the universal initial abelian variety under X (up to the choice of base points). Roitman's original proof uses purely analytic tools, therefore his argument cannot naturally pass to positive characteristic. Bloch's proof bypasses this issue thanks to Bloch-Ogus theory and the Merkurjev-Suslin theorem; Milne later proved that the same statement holds for the p-torsion in characteristic $p > 0$ when the base field is perfect, however his proof is slightly adapted and makes use of de Rham-Witt cohomology together with a duality argument for fppf cohomology, see [Mil82, §§3–5].

Theorem 3.1.1 (Roitman). *Let k be a separably closed field, X a smooth projective and connected k-variety, and ℓ a prime number different from* $\mathrm{char}(k)$*. Then the Albanese map*

$$\mathrm{alb} : A_0(X) \longrightarrow \underline{\mathbf{Alb}}_{X/k}(k)$$

induces an isomorphism $\mathrm{alb} : A_0(X)\{\ell\} \xrightarrow{\sim} \underline{\mathbf{Alb}}_{X/k}(k)\{\ell\}$.

Proof. By Lefschetz's theorem on hyperplane sections, one can find a smooth projective and connected curve $C \subset X$ over k such that there is a morphism of abelian varieties $\underline{\mathbf{Alb}}_{C/k} \to \underline{\mathbf{Alb}}_{X/k}$ that induces a surjection on ℓ-primary torsion groups :

$$\underline{\mathbf{Alb}}_{C/k}(k)\{\ell\} \twoheadrightarrow \underline{\mathbf{Alb}}_{X/k}(k)\{\ell\}.$$

Indeed, as X is smooth and proper, we have the usual exact sequence

$$0 \longrightarrow \underline{\mathbf{Pic}}^0_{X/k}(k) \longrightarrow \mathsf{Pic}(X) \longrightarrow \mathsf{NS}(X) \longrightarrow 0. \tag{C.1}$$

In particular, as $\underline{\mathbf{Alb}}_{X/k} \simeq (\underline{\mathbf{Pic}}^0_{X/k})^\vee$ and $\mathrm{H}^1_{\text{ét}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)) \twoheadrightarrow \mathsf{Pic}(X)\{\ell\}$ (*via* the Kummer sequence), then the claim amounts to showing that $\mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}/\ell^n(1)) \hookrightarrow \mathrm{H}^1_{\text{ét}}(C, \mathbb{Z}/\ell^n(1))$ for all $n \geq 1$. But C is obtained by taking successive hyperplane sections. Consider such a section $H \subset X$, which therefore has affine complement $U \subset X$. By cohomological purity and the long exact cohomology sequence with support, we have an exact portion

$$\mathrm{H}^{2d-3}_{\text{ét}}(H, \mathbb{Z}/\ell^n(d-1)) \longrightarrow \mathrm{H}^{2d-1}_{\text{ét}}(X, \mathbb{Z}/\ell^n(d)) \longrightarrow \mathrm{H}^{2d-1}_{\text{ét}}(U, \mathbb{Z}/\ell^n(d))$$

(where $d := \dim X$) and as U is affine and $2d-1 > d$, the right hand side vanishes. Applying the same process inductively and using Poincaré duality, we obtain the desired injection. Now in the natural commutative diagram

$$\begin{tikzcd} A_0(C) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \underline{\mathbf{Alb}}_{C/k}(k) \arrow[d] \\ A_0(X) \arrow[r] & \underline{\mathbf{Alb}}_{X/k}(k) \end{tikzcd}$$

the top horizontal map is an isomorphism because C is a curve (indeed $A_0(C) \simeq \underline{\mathbf{Pic}}^0_{C/k}(k)$; we then use the duality induced by the Weil pairing). We thus obtain a surjection

$$A_0(X)\{\ell\} \twoheadrightarrow \underline{\mathbf{Alb}}_{X/k}(k)\{\ell\}.$$

It remains to show that this surjection is an isomorphism. Remark that $\mathrm{H}^{2d-1}_{\text{ét}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d)) \simeq \underline{\mathbf{Alb}}_{X/k}(k)\{\ell\}$. Indeed, Poincaré duality provides an isomorphism

$$\mathrm{H}^{2d-1}_{\text{ét}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d)) \xrightarrow{\sim} \mathsf{Hom}\left(\mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}_\ell(1)), \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)\right) \simeq \mathsf{Hom}\left(\varprojlim_{n\geq 1} \mathsf{Pic}(X)[\ell^n], \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)\right),$$

where the second isomorphism comes from the Kummer sequence. Applying the exact sequence (C.1) again and using the fact that $\mathsf{NS}(X)$ is finitely generated (see [BGI71, Exposé XIII, Thm. 5.1]), we obtain an isomorphism

$$\mathsf{Hom}\left(\varprojlim_{n\geq 1} \mathsf{Pic}(X)[\ell^n], \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)\right) \xrightarrow{\sim} \mathsf{Hom}\left(\varprojlim_{n\geq 1} \underline{\mathbf{Pic}}^0_{X/k}(k)[\ell^n], \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)\right).$$

But $\underline{\mathbf{Pic}}^0_{X/k}$ is the dual of $\underline{\mathbf{Alb}}_{X/k}$ under the Weil pairing, that is, $\underline{\mathbf{Pic}}^0_{X/k}(k)[\ell^n] \simeq \mathsf{Hom}\left(\underline{\mathbf{Alb}}_{X/k}(k)[\ell^n], \mathbb{Z}/\ell^n(1)\right)$ for each $n \geq 1$. Combining this fact with the above discussion, we thus obtain a isomorphism :

$$\mathrm{H}^{2d-1}_{\text{ét}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d)) \xrightarrow{\sim} \mathsf{Hom}\left(\varprojlim_{n\geq 1} \underline{\mathbf{Pic}}^0_{X/k}(k)[\ell^n], \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)\right) \xrightarrow{\sim} \varinjlim_{n\geq 1} \underline{\mathbf{Alb}}_{X/k}(k)[\ell^n] = \underline{\mathbf{Alb}}_{X/k}(k)\{\ell\},$$

as desired. Recall from Bloch's method (Chapter I, §4.3, Proposition (I.4.84)) that we have a short exact sequence :

$$0 \longrightarrow \mathrm{H}^{d-1}_{\text{Zar}}(X, \mathcal{K}_d) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell/\mathbb{Z}_\ell \longrightarrow \mathrm{H}^{d-1}_{\text{Zar}}(X, \mathcal{H}^d_{\text{ét}}(\mathbb{Q}_\ell/\mathbb{Z}_\ell(d))) \longrightarrow \mathrm{CH}^d(X)\{\ell\} \longrightarrow 0.$$

Moreover, Bloch-Ogus theory allows us to identify the middle term : indeed, the Leray spectral sequence $E_2^{p,q} = \mathrm{H}^p_{\text{Zar}}(X, \mathcal{H}^q_{\text{ét}}(\mathbb{Q}_\ell/\mathbb{Z}_\ell(d))) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d))$ has nonzero terms only for $0 \leq p \leq q \leq d$, so that $E_2^{d-1,d}$ has no nonzero incoming nor departing differential. Hence $E_2^{d-1,d} = E_\infty^{d-1,d} = F^{d-1}\,\mathrm{H}^{2d-1}_{\text{ét}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d))$. But the other successive quotients in the Leray filtration are also zero since Zariski cohomology vanishes in degree $> d$, so inductively we obtain that $F^{d-1}\,\mathrm{H}^{2d-1}_{\text{ét}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d)) = \mathrm{H}^{2d-1}_{\text{ét}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d))$. Hence in the above exact sequence, we can replace the middle term by $\underline{\mathbf{Alb}}_{X/k}(k)\{\ell\}$, so that we obtain a surjection $\underline{\mathbf{Alb}}_{X/k}(k)\{\ell\} \twoheadrightarrow \mathrm{CH}_0(X)\{\ell\}$. Since $\mathrm{CH}_0(X)_{\text{tors}} = A_0(X)_{\text{tors}}$ (as the kernel of the degree map), then we actually obtain a surjection

$$\underline{\mathbf{Alb}}_{X/k}(k)\{\ell\} \twoheadrightarrow A_0(X)\{\ell\}.$$

On the other hand, the divisibility of $\mathrm{H}^{d-1}_{\text{Zar}}(X, \mathcal{K}_d) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell/\mathbb{Z}_\ell$ shows that the above surjection yields a surjection $\underline{\mathbf{Alb}}_{X/k}(k)[\ell^n] \twoheadrightarrow A_0(X)[\ell^n]$ for each $n \geq 1$. Since the left group is finite (as torsion points of given order on an abelian variety), then both groups are finite and surject onto each other, so we obtain an isomorphism. □

C.2. Specialisation of Chow groups

— As discussed in Chapter I, §1.3, it is possible to define the pullback $f^* : \mathcal{Z}^*(Y) \to \mathcal{Z}^*(X)$ associated to a flat morphism of noetherian schemes $f : X \to Y$, and one can check that such a map descends to Chow groups, that is, yields a well-defined morphism $f^* : \mathrm{CH}^*(Y) \to \mathrm{CH}^*(X)$. When X and Y are schemes over a field k, the projection $\mathsf{pr}_1 : X \times_k Y \to X$ is naturally flat by base change. Under rather mild conditions, the cycles annihilated under the associated pullback on Chow groups are rationally equivalent to 0 up to a suitable integer factor :

Proposition 3.2.2. *Let X and Y be two integral schemes of finite type over a field k and $\mathsf{pr}_1 : X \times_k Y \to X$ the projection onto the first factor. Then the kernel of the pullback*

$$\mathsf{pr}_1^* : \mathrm{CH}^*(X) \longrightarrow \mathrm{CH}^*(X \times_k Y)$$

is a torsion group. Moreover, if $Y(k)$ is dense in Y, then this kernel is actually trivial.

Lemma 3.2.3 (Chevalley). *Let $f : T \to S$ be a dominant morphism of integral noetherian schemes of finite type and let $\eta \in S$ be its generic point. There exists a dense open subset $U \subset S$ such that $\dim T_s = \dim T_\eta$ for every $s \in U$.*

Proof. Note that the set $F := \{t \in T \mid \dim_t T_{f(t)} \geq \dim T_\eta + 1\}$ is closed in T, see *e.g.* [Gro67, (13.1.3)]. By Chevalley's constructibility theorem (*cf.* [Gro67, Thm. 18.4.3]), we know that $f(F)$ is a constructible subset of S that does not contain η. We can thus take $U := S \setminus \overline{f(T)}$. □

Lemma 3.2.4. *Let $\iota : T \hookrightarrow S$ be a closed immersion of schemes where S is integral and noetherian, Z a cycle on S whose support does not contain any irreducible component of T. If Z is rationally equivalent to 0, then so is $\iota^* Z$.*

Proof. Without loss of generality, let us write $Z = \mathrm{div}(f)$ for some rational function $f \in \Bbbk(W)^\times$ with W an integral closed subscheme of X. If we denote by $T_1, \ldots, T_r$ the irreducible components of T, then on each of these the function f restricts to an invertible rational function f_i. Since $\iota^* Z = \sum_{i=1}^r \mathrm{div}(f_i)$, the claim follows. □

Proof of Proposition (3.2.2). If $U \subset Y$ is a dense open subset, then the kernel of $\mathsf{pr}_1^* : \mathrm{CH}^*(X) \to \mathrm{CH}^*(X \times_k Y)$ is contained in the kernel of $\mathrm{CH}^*(X) \longrightarrow \mathrm{CH}^*(X \times_k U)$, and if $Y(k)$ is dense in Y, then $U(k)$ is dense in U. We can therefore (without loss of generality) replace Y by a dense open subset for convenience purposes. If Z is a cycle on X (of whose support can be assumed to be of pure codimension $d \leq \dim X$) such that $\mathsf{pr}_1^* Z$ is rationally equivalent to 0, then by definition one can find integral closed subschemes $W_1, \ldots, W_r$ of $X \times_k Y$ and rational functions $f_i \in \Bbbk(W_i)^\times$ for $i \in [\![1, r]\!]$ such that

$$\mathsf{pr}_1^* Z = \sum_{i=1}^r \mathrm{div}(f_i).$$

The goal of the proof is to pull back this decomposition to a suitable closed fibre of $\mathsf{pr}_2 : X \times_k Y \to Y$ and apply the previous lemma. More precisely, we need to find a closed point $y \in Y$ such that for every $i \in [\![1, r]\!]$, the support of $\mathrm{div}(f_i)$ does not contain any irreducible component of the fibre $(W_i)_y$. First note that $\mathrm{Supp}(\mathsf{pr}_1^* Z) = \mathrm{Supp}(Z) \times_k Y$ does not contain any irreducible component of $(W_i)_y$ for each i.

Up to shrinking Y, one can assume that each component of $\mathrm{Supp}(\mathrm{div}(f_i))$ dominates Y. If η denotes the generic point of Y, then $\mathrm{Supp}(\mathrm{div}(f_i))_\eta$ has pure dimension d, so $\dim(W_i)_\eta = d + 1$ for all i. Therefore, every component of every non-empty fibre of $W_i \to Y$ has dimension $\geq d + 1$ by [Gro67, (13.1.1)]. Now by Chevalley's semicontinuity lemma above, up to shrinking Y again we can suppose that every non-empty fibre of $\mathrm{Supp}(\mathrm{div}(f_i)) \to Y$ has dimension d, so that for any $y \in Y$, the support $\mathrm{Supp}(\mathrm{div}(f_i))$ cannot contain an irreducible component of $(W_i)_y$. By the previous lemma, we know that the pullback of $\mathrm{div}(f_i)$ to $(W_i)_y$ is rationally equivalent to 0.

If we choose a point $y_0 \in Y$, then we have an induced closed immersion $\iota : X_{\kappa(y_0)} \hookrightarrow X \times_k Y$. On the other hand, $\iota^*(\mathsf{pr}_1^* Z) = \sum_{i=1}^r \iota^* \mathrm{div}(f_i)$, so it is rationally equivalent to 0. As the composition

$$\mathcal{Z}(X) \xrightarrow{\mathsf{pr}_1^*} \mathcal{Z}(X \times_k Y) \xrightarrow{\iota^*} \mathcal{Z}(X_{\kappa(y_0)})$$

agrees with the pullback associated to the finite surjective morphism $X_{\kappa(y_0)} \to X$, then Z must be torsion annihilated by the degree of this morphism (by projection formula, see Corollary (1.1.6)), which is equal to $[\kappa(y_0) : k]$, hence the general claim. When $Y(k)$ is dense in Y, then one can choose $y_0 \in Y(k)$ so that $[\kappa(y_0) : k] = 1$ and Z is actually rationally equivalent to 0. ◻

Remark 3.2.5. If we take in particular Y to be affine, then by Noether's normalisation lemma one can choose a finite surjective morphism $Y \to \mathbb{A}^d_k$ for some $d \geq 1$, so that the kernel of

$$\mathrm{CH}^*(X \times_k \mathbb{A}^d_k) \longrightarrow \mathrm{CH}^*(X \times_k Y)$$

is torsion.

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