Schedule : learning seminar on perverse sheaves

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Abstract : Homological theories on smooth algebraic varieties enjoy many nice properties (notably duality results). Intersection cohomology and perverse sheaves have been introduced at the beginning of the 80's by Goresky-MacPherson and Beilinson-Bernstein-Deligne-Gabber in order to provide a good homological framework for singular varieties. The following document is a short programme for a learning seminar on these notions. Except where indicated we aim for each topic to be covered over the course of one week, which translates to one talk of (roughly) two hours. There are plenty of references on the subject, but we choose to focus on [BBDG18], [PS08], [KW01] and [dCM09].

0. Derived categories and t-structures (Drimik(?)) [Possibly 2×2h]

[KW01, Chap. II], [BBDG18, §1], [PS08, Part V, §A.2]

- Quick recollection on triangulated categories: give [KW01, Def. 1.1], discuss localisation, state the 2-out-of-3 property [KW01, Cor. 1.4] and [KW01, Cor. 1.6]. Discuss derived categories as triangulated categories (state [KW01, Rmk. 1.10]), and give some basic examples (*e.g.* [BBDG18, Exemples 1 and 2, p.20]).
- Define t-structures [KW01, Def. 2.1] and state the orthogonality, extension and compatibility lemmas [KW01, Lem. 2.2–2.4]. Give the natural t-structure on the derived category of an abelian category [BBDG18, Exemples 1.3.2, (i)] and explain the dual t-structure of a fixed t-structure [BBDG18, Exemples 1.3.2, (iii)]. Define the core/heart of a t-structure as in [KW01, §II.3] and state [KW01, Thm. 3.1] (the proof is rather long and technical so you may skip it or just provide a sketch). Discuss [BBDG18, Prop. 1.3.3 and Thm. 1.3.6].
- Discuss thick (and in particular, Serre) subcategories of a triangulated category as in [KW01, §II.3.3]. You can provide some examples if you want.
- Discuss the relation between cohomological functors on a derived category and long exact sequences on its heart as in [KW01, §II.4]. In particular, state [KW01, Cor. 4.1-4.2, Thm. 4.3] and deduce [KW01, Thm. 4.4]. Depending on the remaining time, dicsuss the proof of [KW01, Thm. 4.3].
- Discuss gluing of t-structures, in particular [BBDG18, Thm. 1.4.10], and sketch its proof starting from the axiomatic situation in *loc. cit.*, §1.4.3 (beware some typos and inaccuracies, *e.g.* when they say that certain functors (pushforward and extension by zero) are exact, they actually mean these are triangulated functors).

1. Local systems and constructible sheaves (Enrico/Drimik(?))

[BBDG18, §2.1], [PS08, Part V, §B.3 and §C.1]

- Define local systems on a topological space [PS08, Def. B.32] (we will only deal with abelian local systems) and the monodromy representation when the space is path connected. Discuss the case of locally 1-connected spaces. Define locally constant sheaves [PS08, Def. B.33] and discuss the correspondence between locally constant sheaves and local systems on a path connected locally 1-connected space [PS08, Lem. B.34].

- Define the (singular) homology/cohomology of a local system as in [PS08, §B.3.2]. State that this definition coincides with the sheaf cohomology of the associated locally constant sheaf [PS08, Lem. B.35] for «good» topological spaces (see [PS08, Def. B.12]). Discuss Poincaré duality for local systems [PS08, Thm. B.36] (you can say a few words about Verdier duality with a view towards the next talks).

- Define connections for (analytic) vector bundles on smooth manifolds [PS08, Def. B.37] (we could also replace «manifold» by «algebraic variety» by considering Kähler differentials instead) and the covariant derivative associated to a vector field. Define the curvature operator of a connection [PS08, Def. B.39] and the parallel transport map [PS08, Def. B.40]. State [PS08, Cor. B.43].

- Discuss the special case of complex manifolds as in [PS08, §10.1]. Define holomorphic connections [PS08, Def. 10.1] and in particular flat connections [PS08, Def. 10.2] and their associated de Rham complex. Discuss the case where the connection (and underlying vector bundle) comes from a local system, state and sketch the proof of the «baby case» of the Riemann-Hilbert correspondence [PS08, Thm. 10.3 and Cor. 10.4].

2. The intersection homology complex (TBA)

[BBDG18, §5], [PS08, Appendix C and §13.2.1], [KW06]

- Define stratified spaces [PS08, Def. C.1] and give examples of stratifications as in [CL18, §5.1]. Discuss the special case of pseudo-manifolds and give the example of plane curves (Milnor) as 2-dimensional pseudomanifolds. Define Whitney stratifications [PS08, §C.1.2] and state the theorem of Borel-Whitney [PS08, Cor. C.6](quasi-projective complex varieties of dimension n always admit a stratification by closed subvarieties of length $2n$).

- Following https://pub.math.leidenuniv.nl/ vonkjb/other/perversity/pympmyhomology.pdf, define the perversity p for the singular chain complex of a topological space X. Define the notion of a p-allowable singular chain. Give the example of an algebraic curve that is the union of two lines with nodal singularity at the intersection. Define the intersection chain complex and the intersection homology of a space X. Discuss the particular case of middle perversity. Insist on the importance of the choice of stratification for X. Discuss explicit computations as in the example given before.

- State [KW06, Prop. 4.4.1] and give the example of the nodal cubic. State the result of Goresky-MacPherson [KW06, Prop. 4.5.2] that intersection homology doesn't change when taking normalisation for quasi-projective complex varieties, and relate this to the previous example. State the fact that intersection homology admits Poincaré duality, Lefschetz theorems, Hodge decomposition, *etc.* for (non-necessarily smooth) projective varieties.

- Define intersection homology with coefficients in a local system on a stratified pseudomanifold. Recall the definition of Borel-Moore homology and explain how to define intersection homology with locally finite supports. Define the intersection cohomology sheaf complex (briefly explain why restriction maps on open subsets are not obvious and how to define them). State the fact that intersection cohomology sheaves are soft (≠flasque) in all non-positive degrees and for every perversity given a priori, so that their hypercohomology can be computed as the intersection homology with locally finite supports.

3. Verdier duality (TBA)

[KW01, §II.5–II.9], [PS08, §13.1]

- Results on morphisms of algebraic varieties. Pull-backs and push-forwards under morphisms. Constructible sheaves again. Constructible derived category. Six functors preserve constructibility. Results on cohomology of constructible sheaves. Verdier duality. The following notes give a good perspective on what you should talk about precisely: https://www.math.toronto.edu/jkamnitz/seminar/perverse/HyungseopNotes.pdf.

4. Perverse sheaves (TBA)

[BBDG18, §2 and §4.3], [PS08, §13.2], [dCM09, §2]

- Remind the audience about intersection cohomology and perversity as in [PS08, §13.2.1].
- Give the definition of perverse complexes [PS08, Def. 13.14]. Describe the behavior of perversity under six functors [PS08, Prop. 13.15 Rmk. 13.16].
- Give the support and cosupport conditons for perversity [PS08, Lem. 13.17], and use them to provide some examples of perverse complexes [PS08, Ex. 13.18]. Define the perverse extension of an appropriate local system on a pure dimensional complex variety, and state its «universal» property [PS08, Thm. 13.19]. Deduce the commutativity of the perverse extension with Verdier duality [PS08, Cor. 13.20], and the duality on intersection homology [PS08, Prop. 13.21].
- State the fact that perverse complexes form an abelian category [PS08, Lem. 13.22]. The proof of this result essentially relies on the existence of a t-structure on the triangulated category of bounded constructible complexes (the perverse truncation functors are given by support and cosupport conditions), whose heart precisely consists of the perverse complexes. The verification of this fact is done in [BBDG18, §2], so you may sketch it only if you have some spare time. Define the perverse cohomology of a bounded constructible complex, and give some examples [PS08, Ex. 13.23]. Define the perverse intermediate direct image functors, use them to state [PS08, Prop. 13.24] and redefine the perverse extension of a local system [PS08, Def. 13.25].
- Explain how the category of perverse complexes is artinian and describe its simple objects as in [PS08, Lem. 13.26] and [BBDG18, Thm. 4.3.1], and use this to exhibit generators of its Grothendieck group.

5. Vanishing cycles (TBA)

[BBDG18, §4.4], [PS08, §11.2.1–§11.2.3 and §13.2.3], [dCM09, §5.5 and §5.6]

6. Decomposition theorem for perverse sheaves (TBA)

7. D-modules (TBA)

8. Holonomic complexes (TBA)

9. The Riemann-Hilbert correspondence (TBA)

10. Perverse sheaves and quivers (TBA)

11. Mixed Hodge Modules (TBA)

12. Semismall maps, and $S^{[n]}$ (Matteo)

The geometrical approach to proving the Decomposition theorem that first proves the case of so called "semi-small maps" and proceeds by induction: it is interesting especially in some real-life applications of the theory, (i.e. the Hilbert-Chow morphism is semismall! Easy computations quite explicit in terms)

13. Support Theorems

They appear in applications. Just mentioning some examples: Ngô's one was fields medal material.

https://www.math.uchicago.edu/ ngo/support.pdf

References

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