Linear representations TD3

Reminders. Unless explicitely mentioned, a *representation* of a (finite) group G is assumed to be *linear* and *complex* (that is, with values in GL(V) for some finite dimensional vector space V over the field $\mathbf{C}$ of complex numbers). Moreover, we have the following fundamental properties.

(1) Every representation is a direct sum of irreducible representations (Maschke's theorem);
(2) Characters of irreducible representations form an orthonormal basis of the space of central functions on G endowed with the inner product $\langle \varphi, \psi \rangle = \frac{1}{|\mathrm{G}|} \sum_{g \in \mathrm{G}} \varphi(g)\overline{\psi(g)}$;
(3) If we denote by $n_1, \ldots, n_r$ the dimensions of the irreducible representations of G, then they satisfy the relation $\sum_{i=1}^{r} n_i^2 = |\mathrm{G}|$.

Notation. Exercises marked with a  are classical and will be corrected during the exercise class (if time permits it). Those marked with a  are slightly more advanced and facultative.

 **Exercise 1.** Let p be a prime number.

(1) Describe all of the irreducible representations of $\mathbf{Z}/p\mathbf{Z}$. What can one conclude about the orthogonality of characters in this case?

(2) Show that the map $\chi : (\mathbf{Z}/p\mathbf{Z})^* \to \mathbf{C}^*$ given by $\chi(\overline{a}) = 1$ if a is a square mod p, resp. $\chi(\overline{a}) = -1$ otherwise, is a representation. Let now assume that $p \geq 3$; what can one deduce from the fact that this character is orthogonal to the trivial character?

 **Exercise 2.** (1) Let $\rho : \mathfrak{S}_3 \to \mathrm{GL}_3(\mathbf{C})$ be the regular representation, i.e. given by permuting the vectors of the canonical basis $\mathscr{B} = (e_1, e_2, e_3)$ of $\mathbf{C}^3$. Show that the subspace $\mathrm{V} := \mathbf{C}(1,1,1) \subset \mathbf{C}^3$ is a subrepresentation of ρ and that $\mathrm{W} := \{(x,y,z) \in \mathbf{C}^3 \mid x+y+z = 0\}$ is a subrepresentation which is complementary to V.

(2) Show that W is an irreducible representation, and explain how to generalise this construction to $\mathfrak{S}_n$ for $n \geq 4$.

(3) Describe all of the irreducible representations of $\mathfrak{S}_3$ and give its character table.

 **Exercise 3.** Let $\mathrm{G} \subset \mathrm{GL}_n(\mathbf{C})$ $(n \geq 1)$ be a subgroup with finite order. Show that $\sum_{g \in \mathrm{G}} \mathrm{Tr}(g)$ is an integer. Give a geometric interpretation of this fact.

 **Exercise 4.** Let k be a field of characteristic $p > 0$. Show that the following two properties are equivalent:

(i) The group algebra $k[\mathrm{G}]$ is semisimple;
(ii) the order of G is not divisible by p.

[Hint: to prove that (i) $\Rightarrow$ (ii) show that, if p divides $|\mathrm{G}|$, then the augmentation ideal $\mathrm{I} = \{\sum_{g \in \mathrm{G}} a_g[g] \mid \sum_{g \in \mathrm{G}} a_g = 0\} \subset k[\mathrm{G}]$ is not a direct factor (as a module)]

 **Exercise 5.** Consider the dihedral group given by the following presentation:

$$\mathrm{D}_4 := \{r, s \mid r^4 = s^2 = 1,\ srs = r^{-1}\}.$$

(1) Show that D_4 has order 8. Deduce that the dimensions of the irreducible representations of D_4 are $1, 1, 1, 1, 2$.

(2) Show that the map $\mathrm{D}_4 \to \mathrm{GL}_2(\mathbf{R})$ that sends r to the rotation of angle $\pi/2$ with center $(0,0) \in \mathbf{R}^2$ and s to the reflexion of axis $\mathbf{R}(0,1) \subset \mathbf{R}^2$ is an injective group morphism (hence a faithful representation).

(3) Determine all of the irreducible representations of D_4.

 **Exercice 6.** We recall the following notation: given a group G, its *center* is the subgroup $\mathrm{Z}(\mathrm{G}) := \{z \in \mathrm{G} \mid \forall g \in \mathrm{G},\ zg = gz\}$. Fix a representation $\rho : \mathrm{G} \to \mathrm{GL}(\mathrm{V})$ once and for all.

(1) Show that for every $z \in \mathrm{Z}(\mathrm{G})$, one has $\rho(z) = \chi(z) \cdot \mathrm{Id}$ for some $\chi(z) \in \mathbf{C}^*$.

(2) Show that the map $\chi : \mathrm{Z}(\mathrm{G}) \to \mathbf{C}^*$ thus defined is a group morphism (usually called the *central character* of the representation ρ).

(3) Deduce that as soon as G admits a faithful representation (i.e. injective), then its center Z(G) is a cyclic group.

 **Exercise 7.** (0) Let $\zeta_1, \ldots, \zeta_r \in \mathbf{C}^*$ be roots of unity and $\alpha := (\zeta_1 + \ldots + \zeta_r)/r$. Show that if α is an algebraic integer, then either $\alpha = 0$ or $\alpha = \zeta_1 = \ldots = \zeta_r$
[Hint: if one denotes by Π the product of the conjugates of α over $\mathbf{Q}$, show that $|\Pi| \leq 1$].

Let $\rho : \mathrm{G} \to \mathrm{GL(V)}$ be an irreducible representation of degree $n \geq 1$ and character χ. For $g \in \mathrm{G}$, denote by $c(g)$ the cardinal of the conjugacy class of g.

(1) Let $a = \sum_{g \in \mathrm{G}} a_g [g] \in \mathbf{C}[\mathrm{G}]$ be a central element such that each $a_g \in \mathbf{C}$ is an algebraic integer. Show that the element $(1/n) \cdot (\sum_{g \in \mathrm{G}} a_g \cdot \chi(g))$ is an algebraic integer.
[Hint: you may use Question (0) from Exercise 8].

(2) Deduce that $(c(g)/n) \cdot \chi(g)$ is an algebraic integer.

(3) Show that if $c(g)$ and n are relatively prime and $\chi(g) \neq 0$, then $\rho(g)$ is a homothety.
[Hint: $(1/n) \cdot \chi(g)$ is an algebraic integer]

 **Exercise 8.** Let G be a finite group.

(0) Let $a = \sum_{g \in \mathrm{G}} a_g [g] \in \mathbf{C}[\mathrm{G}]$ be a central element such that each $a_g \in \mathbf{C}$ is an algebraic integer. Show that a is integral over $\mathbf{Z}$.
[Hint: let $\mathcal{C}_1, \ldots, \mathcal{C}_r$ denote the conjugacy classes of G, put $e_i := \sum_{g \in \mathcal{C}_i} [g]$ and write $a = \sum_{i=1}^r a_i [e_i]$ for some adequate a_i's. You may then use the fact that the elements of $\mathbf{C}[\mathrm{G}]$ which are integral over $\mathbf{Z}$ form a subring]

(1) Let $\rho : \mathrm{G} \to \mathrm{GL(V)}$ be an irreducible representation. Show that for every $m \geq 0$, the tensor representation $\rho^{\otimes m} : \mathrm{G}^m \to \mathrm{GL}(\mathrm{V}^{\otimes m})$ is irreducible.

(2) Show that the image under $\rho^{\otimes m}$ of any $(z_1, \ldots, z_m) \in \mathrm{Z(G)}^m$ is a homothety.
[Hint: apply Schur's lemma]

The subgroup H of $\mathrm{Z(G)}^m$ consisting of those $(z_1, \ldots, z_m)$ such that $\prod_{i=1}^m z_i = 1$ acts trivially on $\mathrm{V}^{\otimes m}$; taking quotients, we obtain a representation of G^m/H which is irreducible.

(3) Deduce that the degree of this representation divides the order of G^m/H. Varying the choice of $m \geq 0$, infer that the degree of ρ divides the index $(\mathrm{G} : \mathrm{Z(G)})$.