Linear representations TD4

Reminders. Unless explicitely mentioned, a *representation* of a (finite) group G is assumed to be *linear* and *complex* (that is, with values in GL(V) for some finite dimensional vector space V over the field **C** of complex numbers). Moreover, we have the following fundamental properties.

(1) Every representation is a direct sum of irreducible representations (Maschke's theorem);
(2) Characters of irreducible representations form an orthonormal basis of the space of central functions on G endowed with the inner product $\langle \varphi, \psi \rangle = \frac{1}{|\mathrm{G}|} \sum_{g \in \mathrm{G}} \varphi(g)\overline{\psi(g)}$;
(3) If we denote by $n_1, \ldots, n_r$ the dimensions of the irreducible representations of G, then they satisfy the relation $\sum_{i=1}^{r} n_i^2 = |\mathrm{G}|$.

Notation. Exercises marked with a  are classical and will be corrected during the exercise class (if time permits it). Those marked with a  are slightly more advanced and facultative.

 **Exercise 1.** Let G be a finite group, let X be a finite set on which G acts, let $\rho : \mathrm{G} \to \mathrm{GL}(\mathrm{V})$ denote the corresponding permutation representation and let χ be its character.

(1) Show that $\chi(g)$ coincides, for any $g \in \mathrm{G}$, with the number of fixed points of X (under the action of g).

(2) Show that the subspace V^{G} of G-invariant vectors of V identifies with the space of functions on X which are constant on any orbit of the action of G. Deduce that the number of orbits coincides with the multiplicity of the trivial representation in V. Infer that, if the action of G on X is transitive (i.e. if there is only one orbit) then $\rho = \mathbf{1} \oplus \theta$ for some linear representation $\theta : \mathrm{G} \to \mathrm{GL}(\mathrm{W})$ that does not contain the trivial representation. Notably if χ_θ denotes its the character, show that $\chi = \mathbf{1} + \chi_\theta$ and $\langle \chi_\theta, \mathbf{1} \rangle = 0$.

(3) Consider the diagonal action of G on X × X, given by $g \cdot (x, y) := (g \cdot x, g \cdot y)$. Show that the character of the corresponding permutation representation coincides with χ^2.

(4) Assume that the action of G on X is transitive and that X consists of at least two elements. We say that G acts *doubly transitively* on X if, for any $x \neq y$ and $x' \neq y'$ elements in X, there exists a $g \in \mathrm{G}$ such that $g \cdot x = x'$ and $g \cdot y = y'$. Show that the following conditions are equivalent:

(i) G acts doubly transitively on X;
(ii) $\langle \chi^2, \mathbf{1} \rangle = 2$;
(iii) θ is irreducible.

 **Exercise 2.** Consider the group of Hamilton quaternions given by the presentation:

$$\mathrm{Q}_8 := \langle i, j, k \mid i^2 = j^2 = k^2 = ijk = -1 \rangle.$$

(1) Write the multiplication table of Q_8.

(2) Show that Q_8 has 4 irreducible representations of degree 1 and one irreducible representation of degree 2.

(3) Let $\rho : \mathrm{Q}_8 \to \mathrm{GL}_2(\mathbf{C})$ be the unique (up to isomorphism) 2-dimensional representation of Q_8, and let χ_ρ be its character. Show that $\chi_\rho(-1) \in \{\pm 2\}$.

(4) Show that $\chi_\rho(i) = \chi_\rho(j) = \chi_\rho(k) = 0$.

(5) By calculating $\langle \chi_1, \chi_\rho \rangle$, show that $\chi_\rho(-1) = -2$.

(6) Write down the character table of Q_8. Compare it with the character table of D_4 (from TD3). What can you conclude?

 **Exercise 3.** Let G be a finite group and $\rho : \mathrm{G} \to \mathrm{GL}(\mathrm{V})$ be a representation. We denote by **C** the trivial irreducible representation of G. A G-*extension of* **C** *by* V is a representation $\eta : \mathrm{G} \to \mathrm{GL}(\mathrm{W})$ such that there is a short exact sequence of G-representations

$$0 \longrightarrow \mathrm{V} \xrightarrow{i} \mathrm{W} \xrightarrow{p} \mathbf{C} \longrightarrow 0,$$

i.e. we require for all the maps in the above sequence to be G-equivariant. We denote the set of such extensions modulo equivalence as $\mathrm{Ext}^1_{\mathbf{C}[\mathrm{G}]}(\mathbf{C}, \mathrm{V})$. This is a pointed set, that is, pointed at the class of the trivial extension

$$0 \longrightarrow \mathrm{V} \longrightarrow \mathrm{V} \oplus \mathbf{C} \longrightarrow \mathbf{C} \longrightarrow 0.$$

(N.B: this set can also be equipped with a commutative group structure given by the Baer sum of two extensions.)

(1) Let $w \in \mathrm{W}$ be such that $p(w) = 1$ and let $g \in \mathrm{G}$. Show that ${}^g w - w \in \mathrm{V}$. This defines a function:

$$c : \mathrm{G} \longrightarrow \mathrm{V}, \quad g \longmapsto {}^g w - w.$$

(2) Show that $c(gh) = c(g) + {}^g c(h)$ for every $g, h \in \mathrm{G}$. We say that a function $c : \mathrm{G} \to \mathrm{V}$ satisfying this property is a *1-cocycle of* G *with values in* V, and we denote by $\mathrm{Z}^1(\mathrm{G}, \mathrm{V})$ the **C**-vector space of such functions.

(3) Let $c \in \mathrm{Z}^1(\mathrm{G}, \mathrm{V})$ and let W_c be the **C**-vector space $\mathrm{V} \oplus \mathbf{C}$ endowed with the following "twisted" action of G:

$${}^g(v, \lambda) := ({}^g v + \lambda c(g), \lambda)$$

for every $g \in \mathrm{G}$ and $(v, \lambda) \in \mathrm{W}_c$. Check that this formula indeed yields an action, and hence a representation of G.

(4) Let W be an extension of $\mathbf{C}$ by V as in (1) and let $c : \mathrm{G} \to \mathrm{V}$ be the associated 1-cocycle. Show that $\mathrm{W} \simeq \mathrm{W}_c$.

(5) We say that $c \in \mathrm{Z}^1(\mathrm{G}, \mathrm{V})$ is a *1-coboundary* if there exists some $v_0 \in \mathrm{V}$ such that $c(g) = {}^g v_0$ for every $g \in \mathrm{G}$. Show that a 1-coboundary is a 1-cocycle.

We denote by $\mathrm{B}^1(\mathrm{G}, \mathrm{V})$ the space of 1-coboundaries, and we define the first cohomology group $\mathrm{H}^1(\mathrm{G}, \mathrm{V})$ of G with values in V as:

$$\mathrm{H}^1(\mathrm{G}, \mathrm{V}) := \mathrm{Z}^1(\mathrm{G}, \mathrm{V})/\mathrm{B}^1(\mathrm{G}, \mathrm{V}).$$

(6) Show that if $\mathrm{W} = \mathrm{V} \oplus \mathbf{C}$ as a G-representation, then the associated 1-cocycle is a 1-coboundary.

(7) Conversely, show that the extension associated to a 1-coboundary is isomorphic to the direct sum representation $\mathrm{V} \oplus \mathbf{C}$.

(8) Conclude that there is a correspondence of pointed sets:

$$\mathrm{Ext}^1_{\mathbf{C}[\mathrm{G}]}(\mathbf{C}, \mathrm{V}) \overset{1:1}{\longleftrightarrow} \mathrm{H}^1(\mathrm{G}, \mathrm{V}).$$

(9) [Harder] If we now replace the trivial representation by an arbitrary one $\delta : \mathrm{G} \to \mathrm{GL}(\mathrm{U})$, show that there is a correspondence of pointed sets:

$$\mathrm{Ext}^1_{\mathbf{C}[\mathrm{G}]}(\mathrm{U}, \mathrm{V}) \overset{1:1}{\longleftrightarrow} \mathrm{H}^1(\mathrm{G}, \mathrm{U}^* \otimes_{\mathbf{C}} \mathrm{V}).$$

Exercise 4. Let p be an odd prime number. Denote by $\mathrm{Aff}(\mathbf{F}_p)$ the set of matrices of the form:

$$t_{a,b} := \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \quad \text{with} \quad a \in \mathbf{F}_p^*,\ b \in \mathbf{F}_p.$$

(1) Show that matrix multiplication endows $\mathrm{Aff}(\mathbf{F}_p)$ with a group structure, and compute its cardinality.

(2) Show that the subset $\mathrm{H} \subset \mathrm{Aff}(\mathbf{F}_p)$ consisting of those matrices of the form $t_{1,b}$ is a normal subgroup, and that $\mathrm{Aff}(\mathbf{F}_p)/\mathrm{H} \simeq \mathbf{F}_p^*$. Use this to construct $p-1$ irreducible 1-dimensional representations of $\mathrm{Aff}(\mathbf{F}_p)$.

(3) The group $\mathrm{Aff}(\mathbf{F}_p)$ acts on $\mathbf{F}_p$ by the formula $t_{a,b} \cdot z = az + b$. Let V be the $\mathbf{C}$-vector space of functions $f : \mathbf{F}_p \to \mathbf{C}$. Show that the map

$$\mathrm{Aff}(\mathbf{F}_p) \times \mathrm{V} \longrightarrow \mathrm{V},\ (t_{a,b}, f) \longmapsto \big(z \mapsto f(t_{a,b}^{-1} \cdot z)\big)$$

gives rise to a representation $\rho : \mathrm{Aff}(\mathbf{F}_p) \to \mathrm{GL}(\mathrm{V})$.

(4) Let χ_ρ be the character of ρ. Show that:

$$\chi_\rho(t_{a,b}) := \begin{cases} 1 \text{ if } a \neq 1 \\ p \text{ if } a = 1,\ b = 0 \\ 0 \text{ if } a = 1,\ b \neq 0. \end{cases}$$

[Hint: use the fact that a basis of V can be given by the indicator functions of elements of $\mathbf{F}_p$]

(5) Show that the constant functions $f : \mathbf{F}_p \to \mathbf{C}$ (resp. the functions f such that $\sum_{x \in \mathbf{F}_p} f(x) = 0$) yield a subrepresentation $\mathrm{V}_c \subset \mathrm{V}$ (resp. $\mathrm{V}_0 \subset \mathrm{V}$) and that V splits as $\mathrm{V} = \mathrm{V}_c \oplus \mathrm{V}_0$ (as G-representations).

(6) Show that V_0 is irreducible and that any irreducible representation of $\mathrm{Aff}(\mathbf{F}_p)$ is either isomorphic to V_0 or to one of the 1-dimensional representations constructed in (2).