Linear representations TD5

Reminders. Unless explicitely mentioned, a *representation* of a (finite) group G is assumed to be *linear* and *complex* (that is, with values in GL(V) for some finite dimensional vector space V over the field $\mathbf{C}$ of complex numbers). Moreover, we have the following fundamental properties.

(1) Every representation is a direct sum of irreducible representations (Maschke's theorem);
(2) Characters of irreducible representations form an orthonormal basis of the space of central functions on G endowed with the inner product $\langle \varphi, \psi \rangle = \frac{1}{|\mathrm{G}|} \sum_{g \in \mathrm{G}} \varphi(g) \overline{\psi(g)}$;
(3) If we denote by $n_1, \ldots, n_r$ the dimensions of the irreducible representations of G, then they satisfy the relation $\sum_{i=1}^{r} n_i^2 = |\mathrm{G}|$.

Notation. Exercises marked with a  are classical and will be corrected during the exercise class (if time permits it). Those marked with a  are slightly more advanced and/or facultative.

 **Exercise 1.** Let $n \geq 1$. For $\sigma \in \mathfrak{S}_n$, denote by $f_\sigma \geq 1$ the number of fixed points $\{1, \ldots, n\}$ under the action of σ. Show that we have an equality:

$$\sum_{\sigma \in \mathfrak{S}_n} f_\sigma^2 = 2n!.$$

 **Exercise 2.** Let G be a finite abelian group. We aim at describing the structure of G precisely. Denote by $\widehat{\mathrm{G}} := \mathrm{Hom}(\mathrm{G}, \mathbf{C}^*)$ the set of characters of G.

(1) Show that the composition law $f \cdot f' : g \mapsto f(g)f'(g)$ endows $\widehat{\mathrm{G}}$ with a commutative group structure, and that $|\widehat{\mathrm{G}}| = |\mathrm{G}|$.
(2) If H is an abelian group and $\pi : \mathrm{G} \to \mathrm{H}$ is a surjective group morphism, show that the induced map

$$\widehat{\mathrm{H}} \longrightarrow \widehat{\mathrm{G}} \quad f \longmapsto f \circ \pi$$

is an injective group morphism.
(3) Let H be a subgroup of G. Show that restriction of functions induces a group morphism $\widehat{\mathrm{H}} \to \widehat{\mathrm{G}}$ whose kernel is isomorphic to $\widehat{\mathrm{G}/\mathrm{H}}$. Deduce to this morphism is surjective.
(4) Show that the map $\mathrm{G} \to \widehat{(\widehat{\mathrm{G}})}$ that sends an element $g \in \mathrm{G}$ to $\mathrm{ev}_g : f \mapsto f(g)$ is an isomorphism.
(5) Denote by e the exponent of G, that is, the lcm of all the orders of elements of G. Recall why there exists some $g_0 \in \mathrm{G}$ whose order is precisely e. We will fix this g_0 once and for all.
(6) Show that there exists a character $f \in \widehat{\mathrm{G}}$ such that $f(g_0)$ has order e. Deduce that $\mathrm{G} \simeq \langle g_0 \rangle \times \ker(f)$.
(7) Infer that G is a product of cyclic groups, and that $\mathrm{G} \simeq \widehat{\mathrm{G}}$.
(8) Show that, if $g \in \mathrm{G}$ is not the neutral element, then $\sum_{f \in \widehat{\mathrm{G}}} f(g) = 0$.

 **Exercise 3.** Let V be the (left) regular representation of of a finite group G, i.e. V is the vector space of dimension $|\mathrm{G}|$ with a basis consisting of $(e_g)_{g \in \mathrm{G}}$ such that ${}^g e_h = e_{gh}$ for any $g, h \in \mathrm{G}$. Let $\mathrm{H} \subset \mathrm{G}$ be a subgroup and let $\mathrm{W} \subset \mathrm{V}$ be the subspace generated by the vectors $(e_h)_{h \in \mathrm{H}}$.

(1) Show that W is stable under the action of H and that it is isomorphic to the regular representation of H.
(2) Show that V is the induced representation of W from H to G. In particular, when $\mathrm{H} = \{e\}$, this shows that $\mathrm{V} = \mathrm{Ind}_{\{e\}}^{\mathrm{G}}(\mathbf{1})$, where $\mathbf{1}$ denotes the trivial representation of the trivial group.

 **Exercise 4.** Let G be a finite group, let $\mathrm{H} \subset \mathrm{G}$ be a subgroup and consider two representations (W_1, π_1) and (W_2, π_2) of H.

(1) Show that, if $\rho_1 := \mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(\pi_1)$ and $\rho_2 := \mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(\pi_2)$, then $\rho_1 \oplus \rho_2 = \mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(\pi_1 \oplus \pi_2)$.
(2) Generalise this fact to the direct sum of finitely many representations.

 **Exercise 5.** Let G be a finite group, let $\mathrm{H} \subset \mathrm{G}$ be a subgroup, and let (V, ρ) (resp. (W, π)) be a representation of G (resp. of H). Recall that $\mathrm{Hom}_{\mathrm{G}}(\mathrm{V}_1, \mathrm{V}_2)$ (resp. $\mathrm{Hom}_{\mathrm{H}}(\mathrm{W}_1, \mathrm{W}_2)$) denotes the set of morphisms of G-representations from V_1 to V_2 (resp. H-representations from W_1 to W_2).

(1) Show that $\mathrm{Hom}_{\mathrm{H}}(\mathrm{W}, \mathrm{V}) \simeq \mathrm{Hom}_{\mathrm{G}}(\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(\mathrm{W}), \mathrm{V})$.
(2) In the language of categories, we say that induction from H to G yields a functor $\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(-)$ which is left adjoint to the forgetful functor from G-representations to H-representations. Look up the words that you do not understand here and make sense of this sentence.

(3) Show, using (1), that if W is a representation of H and if the induced representation of W from H to G exists, then it must be unique.

(4) The last point yet again translates in the language of categories: namely, the universal property of (2) implies that the induction functor is unique. Try to understand this statement.

Exercise 6. Let G be a finite group and let $H \subset G$ be a subgroup.

(1) Let $\Sigma \subset G$ be a system of coset representatives of G/H. Show that the elements of Σ form a basis of $\mathbf{C}[G]$ viewed as a right $\mathbf{C}[H]$-module, i.e. that there is an isomorphism:

$$\mathbf{C}[\mathrm{G}] \simeq \bigoplus_{g \in \Sigma} [g] \cdot \mathbf{C}[\mathrm{H}]$$

as right $\mathbf{C}[H]$-modules.

(2) Deduce that for any H-representation (W, π), the associated induced representation from H to G satisfies $\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(\mathrm{W}) \simeq \mathbf{C}[\mathrm{G}] \otimes_{\mathbf{C}[\mathrm{H}]} \mathrm{W}$.