Linear representations TD6

Reminders. Unless explicitely mentioned, a *representation* of a (finite) group G is assumed to be *linear* and *complex* (that is, with values in GL(V) for some finite dimensional vector space V over the field **C** of complex numbers). Moreover, we have the following fundamental properties.

(1) Every representation is a direct sum of irreducible representations (Maschke's theorem);
(2) Characters of irreducible representations form an orthonormal basis of the space of central functions on G endowed with the inner product $\langle \varphi, \psi \rangle = \frac{1}{|\mathrm{G}|} \sum_{g \in \mathrm{G}} \varphi(g)\overline{\psi(g)}$;
(3) If we denote by $n_1, \ldots, n_r$ the dimensions of the irreducible representations of G, then they satisfy the relation $\sum_{i=1}^{r} n_i^2 = |\mathrm{G}|$.

Notation. Exercises marked with a  are classical and will be corrected during the exercise class (if time permits it). Those marked with a  are slightly more advanced and/or facultative.

 **Exercise 1.** Let $\mathfrak{A}_5 \subset \mathfrak{S}_5$ be the alternating group on 5 elements.
(1) Show that $\mathfrak{A}_5$ has exactly 5 conjugacy classes, and that it does not admits any nontrivial normal subgroup.
(2) Deduce that $\mathfrak{A}_5$ has no nontrivial 1-dimensional representation.
(3) Construct an irreducible 4-dimensional representation of $\mathfrak{A}_5$.
(4) Infer that the degrees of the irreducible representations of $\mathfrak{A}_5$ are $1, 3, 3, 4, 5$.
(5) Let $\psi : \mathfrak{A}_4 \to \mathbf{C}^*$ be a nontrivial 1-dimensional representation. Show that $\mathrm{Ind}_{\mathfrak{A}_4}^{\mathfrak{A}_5}(\psi)$ is an irreducible 5-dimensional representation of $\mathfrak{A}_5$.
(6) View $\mathfrak{A}_5$ as the group of transformations of a 3-dimensional space which preserve the isocahedron in order to construct an irreducible 3-dimensional representation. As $\mathfrak{A}_5$ is a normal subgroup of $\mathfrak{S}_5$ of index 2, you may use this to construct the other irreducible 3-dimensional representation.
(7) Does every irreducible representation of $\mathfrak{S}_5$ remain irreducible when restricted to $\mathfrak{A}_5$?

 **Exercise 2.** Let G be an abelian group, possibly infinite.
(1) Let (V, ρ) be an irreducible representation of G (possibly infinite dimensional). Under which hypotheses is this representation 1-dimensional? Is this actually always the case?
(2) Let now G be an arbitrary group (possibly infinite), and let (V, ρ) be a representation of G (possibly infinite dimensional). Under which hypotheses does this representation split as a direct sum of irreducible factors (that is, Maschke's Theorem holds)? Is this always the case?

 **Exercise 3.** A finite group G is said to be *supersolvable* if there exists an increasing sequence of subgroups

$$\{1\} = \mathrm{G}_0 \subset \mathrm{G}_1 \subset \ldots \subset \mathrm{G}_{n-1} \subset \mathrm{G}_n = \mathrm{G}$$

such that for any $i \in [\![0, n-1]\!]$, G_i is normal in G_{i+1} and $\mathrm{G}_{i+1}/\mathrm{G}_i$ is cyclic. (In particular, a nilpotent group is supersolvable). We aim at showing that any irreducible representation of such a group is necessarily *monomial*, i.e. induced by a 1-dimensional representation from an adequate subgroup $\mathrm{H} \subset \mathrm{G}$.
(1) Let $\mathrm{N} \subset \mathrm{G}$ be a normal subgroup and $\rho : \mathrm{G} \to \mathrm{GL}(\mathrm{V})$ an irreducible representation. Show that one of the following two statements holds:

(a) there exists a strict subgroup $\mathrm{H} \subset \mathrm{G}$ which contains N and an irreducible representation ρ' of H such that $\rho = \mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(\rho')$, or
(b) the restriction of ρ to N is *isotypic*, i.e. splits as a direct sum of isomorphic irreducible representations.

(2) Suppose that G is non-abelian. Show that there exists a normal abelian subgroup of G which is not contained in Z(G).
(3) Infer that any irreducible representation of G is monomial (one may argue by induction on the order of G and assume without loss of generality that all the representations that are considered are faithful).

 **Exercise 4.** Let G be a finite p-group and let χ be an irreducible character of G. Show that

$$\sum_{\chi'} \chi'(1)^2 \equiv 0 \mod \chi(1)^2,$$

where the sum is taken over the set of irreducible characters χ' such that $\chi'(1) < \chi(1)$.

 **Exercise 5.** Let G be a finite simple group (that is, a group with no nontrivial proper normal subgroup).

(1) Show that every 1-dimensional representation of G is trivial, and infer that any for any irreducible representation (V, ρ) of G, one has $\rho(\mathrm{G}) \subset \mathrm{SL}(\mathrm{V})$.

(2) Show that G admits no irreducible 2-dimensional representation.

 **Exercise 6.** Let G be a finite group and let H be a subgroup of G. Assume that for each $g \notin \mathrm{H}$ we have $\mathrm{H} \cap g\mathrm{H}g^{-1} = \{1\}$; we say that H is a *Frobenius subgroup* of G. Denote by N the set of elements of G which are not conjugate to any element of H.

(1) Show that the number of elements of N is precisely $|\mathrm{G}|/|\mathrm{H}| - 1$.

(2) Let φ be a class function on H. Show that there is exactly one class function $\tilde{\varphi}$ on G which extends φ and takes the value $\varphi(1)$ on N.

(3) Show that $\tilde{\varphi} = \mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(\varphi) - \varphi(1)\psi$, where ψ is the character of $\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(\mathbf{1}) - \mathbf{1}$ (permutation representation minus trivial one).

(4) Show that for any choice of φ_1, φ_2 central functions on H and $\tilde{\varphi}_1, \tilde{\varphi}_2$ lifts of those functions as in (2), one has

$$\langle \varphi_1 | \varphi_2 \rangle_{\mathrm{H}} = \langle \tilde{\varphi}_1 | \tilde{\varphi}_2 \rangle_{\mathrm{G}}.$$

(5) Fix some irreducible character of H. Show, using (3) and (4), that $\langle \tilde{\varphi} | \tilde{\varphi} \rangle_{\mathrm{G}} = 1$, $\tilde{\varphi}(1) \geq 0$ and that $\tilde{\varphi}$ is a linear combination with integer coefficients of irreducible characters of G. Conclude that $\tilde{\varphi}$ is an irreducible character of G. If ρ denotes a corresponding representation, show that $\rho(s) = 1$ for each $s \in \mathrm{N}$.

(6) Show that every representation of H extends to a representation of G whose kernel contains N. Conclude that $\mathrm{N} \cup \{1\}$ is a normal subgroup of G, and that G is the semidirect product of H and $\mathrm{N} \cup \{1\}$ (this is known as *Frobenius's Theorem*).

(7) Suppose conversely that G is the semidirect product of H and a normal subgroup A. Show that H is a Frobenius subgroup of G if and only if, for each $h \in \mathrm{H} \setminus \{1\}$ and each $a \in \mathrm{A} \setminus \{1\}$, we have $hah^{-1} \neq a$ (i.e. H acts freely on $\mathrm{A} \setminus \{1\}$). (If $\mathrm{H} \neq \{1\}$, this property implies that A is nilpotent by combining with a theorem of Thompson.)