Constructions for efficient Private Information Retrieval protocols

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1. The PIR issue
2. Transversal designs for efficient PIR protocols
3. Instances

Informal PIR

Given a file F,

can we retrieve symbol F_i

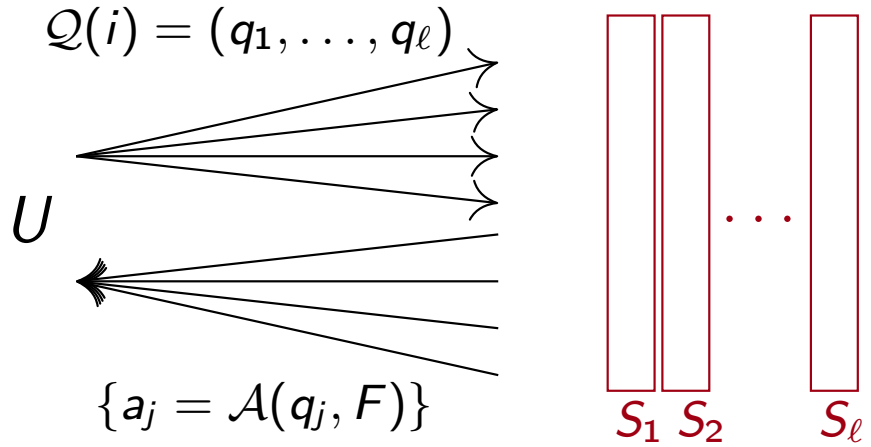
without leaking any information on i?

Formal definition of PIR

Let F be a file (encoded and) stored on ℓ servers $S_1, \ldots, S_\ell$.

Private Information Retrieval (PIR) protocol:
user U wants to recover F_i privately.

1. U generates a query vector $\mathbf{q} = \mathcal{Q}(i)$ and sends q_j to S_j
2. Each server S_j computes $a_j = \mathcal{A}_j(q_j, F_{|S_j})$ and sends it back to U
3. U recovers $F_i = \mathcal{R}(\mathbf{q}, \mathbf{a}, i)$



IT-privacy: we want $\mathbb{P}(i|q_j) = \mathbb{P}(i),\ \forall j = 1, \ldots, \ell$.

Goals

Common goals for PIR:

- Low communication complexity (exchanged bits).
- Low storage overhead (for the servers).
- Low computation complexity for $\mathcal{A}$ (server) and $\mathcal{R}$ (user).

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Our context: file F is frequently queried (*e.g.* a public database.)

- Need very low algorithmic complexity on the server side.

[only (super-)linear complexity for most existing PIR protocols.]

Overview of our solution

Basic ideas:

- Encode the file $F \mapsto c \in \mathcal{C}$, **split** c in ℓ parts and share them among the ℓ servers.
- Use low-weight parity-check equations of $\mathcal{C}$ to retrieve symbols F_i.

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Requirements:

- *privacy:* we need many parity-check equations, with **uniformly distributed supports**,
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Practical solution:

- use codes $\mathcal{C}$ based on transversal designs.

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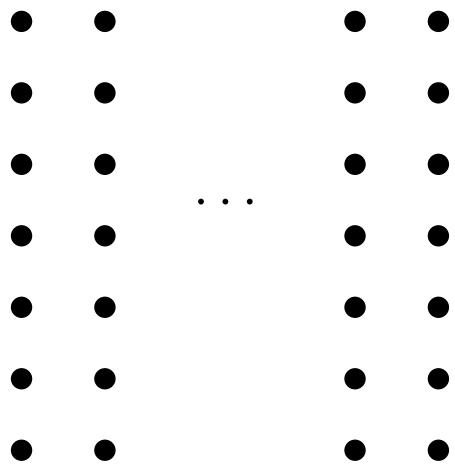
   - First instance: affine transversal designs
   - Second instance: with orthogonal arrays

Transversal designs

A **transversal design** $\mathrm{TD}(\ell, s) = (X, \mathcal{B}, \mathcal{G})$ is given by:

- X a set of *points*, $|X| = n = s\ell$,



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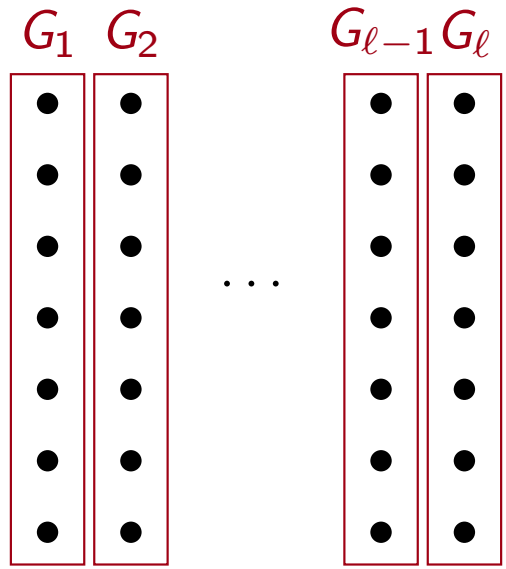

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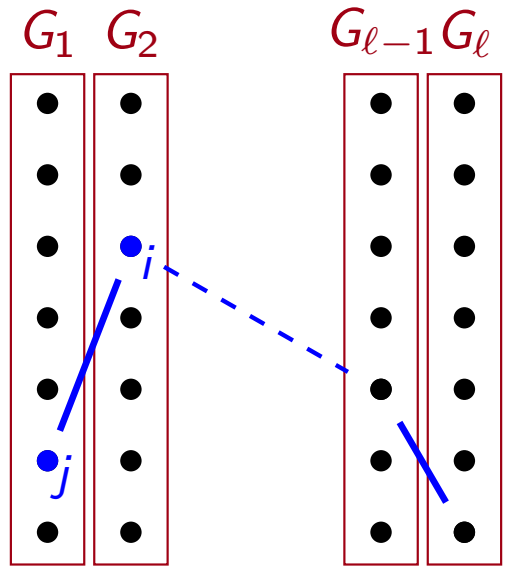
- *blocks* $B \in \mathcal{B}$ satisfying
  - $B \subset X$ and $|B| = \ell$;
  - for all $\{i, j\} \subset X$, $\{i, j\}$ lie:
    either in the same group $G \in \mathcal{G}$,
    or in a unique block $B \in \mathcal{B}$



Codes from designs

Let $\mathcal{T}$ be a transversal design $\mathrm{TD}(\ell, s) = (X, \mathcal{B}, \mathcal{G})$.

Its **incidence matrix** M has size $|\mathcal{B}| \times |X|$ and is defined by:

$$M_{i,j} = \begin{cases} 1 & \text{if } x_j \in B_i \\ 0 & \text{otherwise.} \end{cases}$$

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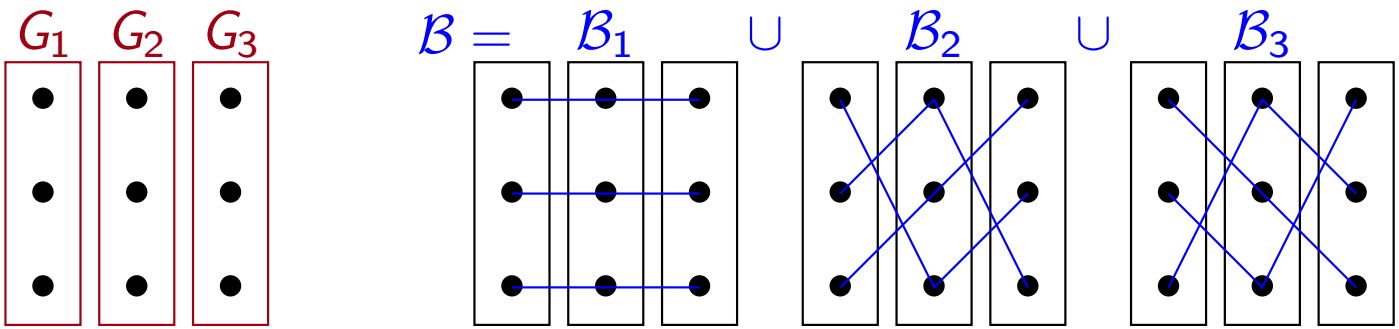
The **code $\mathcal{C}$ based on $\mathcal{T}$ over $\mathbb{F}_q$** is the $\mathbb{F}_q$-linear code having M as parity-check matrix.

- $\mathrm{length}(\mathcal{C}) = |X|$,
- $\dim(\mathcal{C}) = \dim(\ker M)$,
- $B \in \mathcal{B} \longleftrightarrow h \in \mathcal{C}^{\perp}$, such that $\mathrm{wt}(h_{|G_j}) = 1$, $\forall j = 1, \ldots, \ell$.

Example

The transversal design $\mathrm{TD}(3,3)$ represented by:



gives an incidence matrix

$$H = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

whose rank over $\mathbb{F}_3$ is 6. $\quad\Longrightarrow\quad$ $[9,3]_3$ code.

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Let $\mathcal{C} \subseteq \mathbb{F}_q^n$ be a code based on a $\mathrm{TD}(\ell, s)$.

- **Initialisation.** User U encodes $F \mapsto c \in \mathcal{C}$, and gives $c_{|G_j}$ to server S_j for $j = 1, \ldots, \ell$.
- **To recover $F_i = c_i$:**
  1. User U randomly picks a block $B \in \mathcal{B}$ containing i. Then U defines:

$$q_j = \mathcal{Q}(i)_j := \begin{cases} \text{unique } \in B \cap G_j & \text{if } i \notin G_j \\ \text{a random point in } G_j & \text{otherwise.} \end{cases}$$

  2. each server S_j sends back $a_j = \mathcal{A}_j(q_j, c_{|G_j}) := c_{q_j}$
  3. U recovers

$$-\sum_{j:\, i \notin G_j} c_{q_j} = -\sum_{b \in B \setminus \{i\}} c_{q_j} = c_i$$

Privacy and parameters

Theorem.– If the servers do not collude, then our PIR protocol is information-theoretically private.

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Properties. For $|F| = k \log q$ bits, with $k = \dim \mathcal{C} \leq n = s\ell$.

- communication complexity: $\ell(\log s + \log q)$ bits
- computational complexity:
  - $\mathcal{O}(1)$ **for algorithm** $\mathcal{A}$ (somewhat optimal)
  - $\mathcal{O}(\ell)$ $\mathbb{F}_q$-operations for $\mathcal{R}$
- storage overhead: $(n - k) \log q$ bits

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Question: TDs with good k depending on (ℓ, s)?

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A classical TD: points/lines/hyperplanes

Let $\mathcal{T}_A$ be the **classical affine TD**:

- $X = \mathbb{F}_q^m$, $m \geq 2$,
- $\mathcal{G}$ a set of q disjoint hyperplanes partitionning X,
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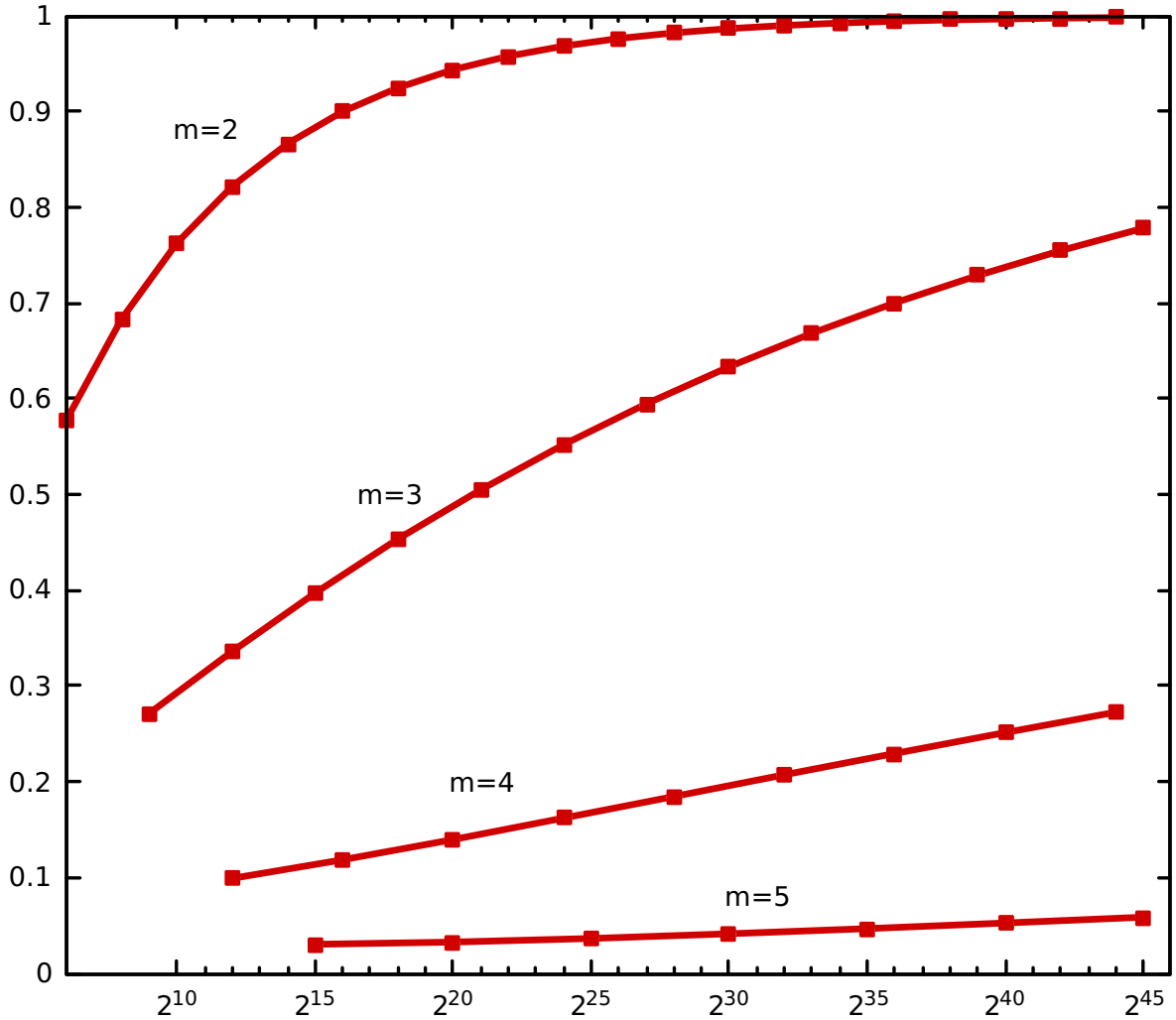
- dimension?
  - its parity-check matrix has q^m columns and q^{2m-2} rows...
  - ... but $\mathcal{C}$ contains $\mathrm{RM}_q(m, q-2)$ which has rate $\simeq 1/m!$,
  - and sometimes it is even larger.

Lower bounds on rates of TD-based codes

rate $R = k/n$



length $n = 2^{em}$

Particular case: $m = 2$

For $m = 2$, $q = p^e$, using Hamada's formula [Ham68] we obtain:

$$n = p^{2e}, \quad k \geq p^{2e} - \binom{p+1}{2}^e, \quad \ell = \sqrt{n}\,.$$

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Asymptotically ($e \to \infty$, fixed p):

$$\begin{cases} R = k/n & = 1 - \Theta(n^{c_p}) \\ \ell & = \Theta(n^{1/2}) \end{cases}$$

where $c_p = \frac{1}{2}(\log_p(\frac{p+1}{2}) - 1) < 0$.

Moreover, $c_p \nearrow$, with $c_2 = -0.208$ and $c_\infty = 0$.

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Open question:

- is this instance rate-optimal?

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TDs and orthogonal arrays

An *orthogonal array* $\mathrm{OA}(t, \ell, s)$ *of strength* t may be seen as a list of codewords over S, with:

- $|S| = s$,
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$$OA(2, 3, 2) = \begin{bmatrix} a & b & b \\ b & b & a \\ b & a & b \\ a & a & a \end{bmatrix}$$

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Construction OA $\rightarrow$ TD :

- $X = S \times [1, \ell]$
- $\mathcal{G} = \{S \times \{i\}, i \in [1, \ell]\}$

$(a,1)$	$(a,2)$	$(a,3)$
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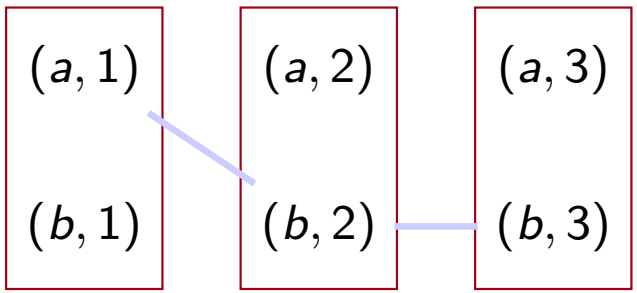

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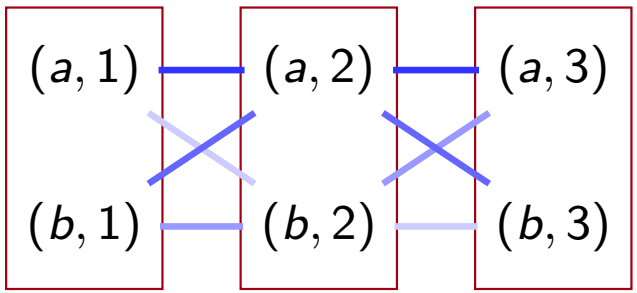

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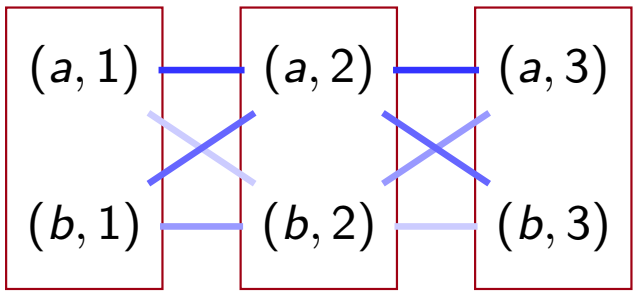

Prop. If $t = 2$, then we obtain a $\mathrm{TD}(\ell, s)$ from an $\mathrm{OA}(t, \ell, s)$.

Resistance to collusions

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What about $\mathrm{OA}(t, \ell, s)$ with $t > 2$?

Resulting TD satisfies: for each t-tuple of points lying in t different groups, there is a block which contains them all.

$\Rightarrow$ Our PIR protocol resists $t - 1$ collusive servers.

The *incidence code* construction

Definition.– We call *incidence code* of $\mathcal{C}_0$ (denoted $I_q(\mathcal{C}_0)$) the $\mathbb{F}_q$-linear code $\mathcal{C}$ coming from the successive constructions:

$$\mathcal{C}_0 = \mathrm{OA}(t, \ell, s) \quad \mapsto \quad \text{generalized } \mathrm{TD}(\ell, s; t) \quad \mapsto \quad \mathcal{C} = I_q(\mathcal{C}_0)$$

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We derive PIR parameters from those of $\mathcal{C}_0$:

- $d^{\perp}(\mathcal{C}_0) - 2$ is the number of collusive servers the protocol resists
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let's use MDS codes for $\mathcal{C}_0$

Incidence codes of Reed-Solomon codes

Example: for $\mathcal{C}_0 = \mathrm{RS}(\mathbb{F}_q, t+1)$,

- $|F| = Rq^2 \log q$ bits, with R the incidence code rate
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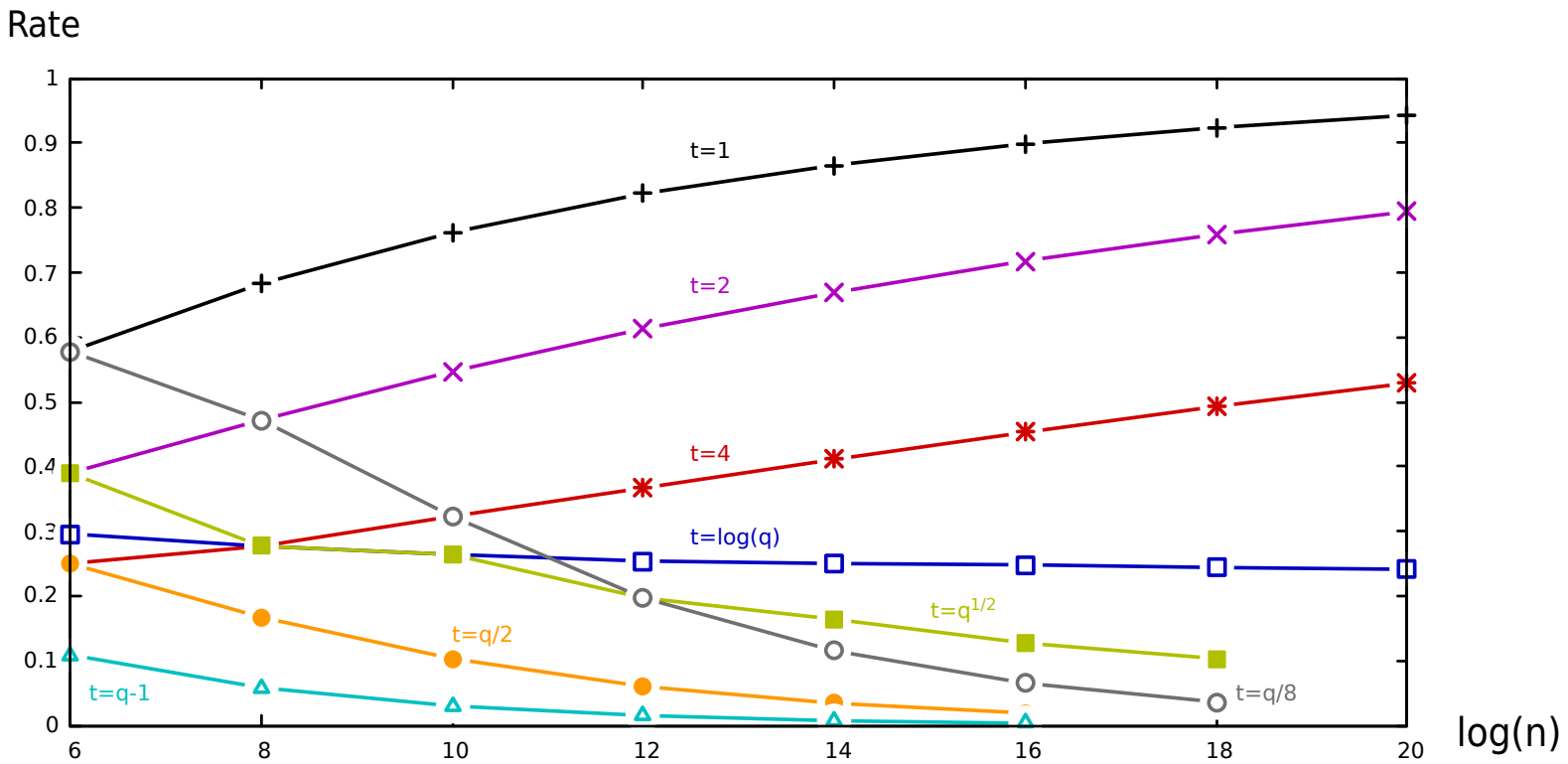

Conclusion

Summary: (server-)efficient PIR protocols can be built with codes based on transversal designs

Current issues:

- find transversal designs leading to largest codes,
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Thank you for your attention.
Questions?