Codes with locality: constructions and applications to cryptographic protocols

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PhD defense

30/11/2018

Outline

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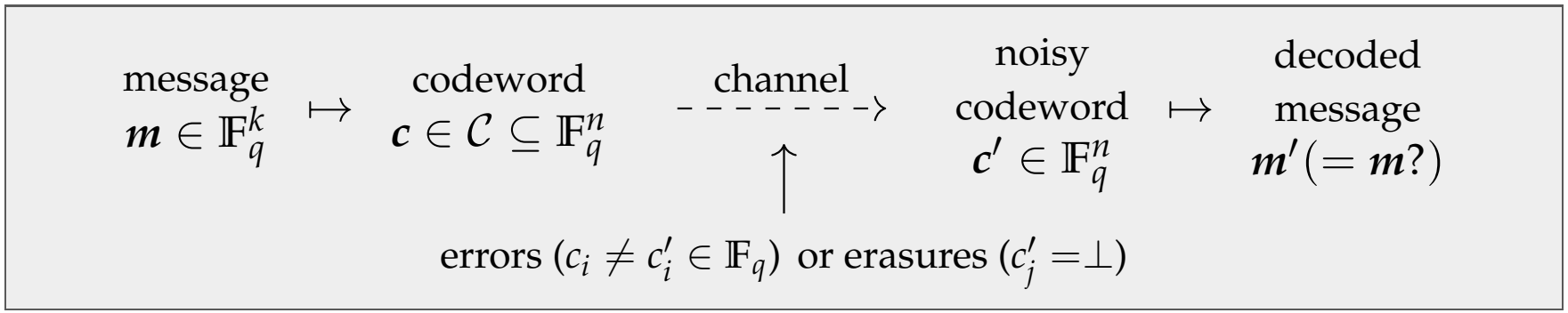
Error-correcting codes

Original goal: transmit information in the presence of noise.



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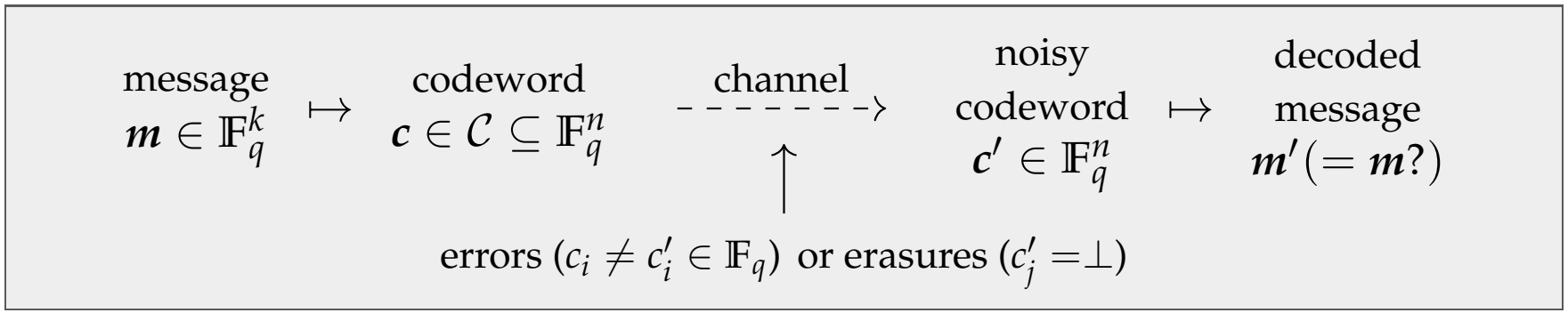

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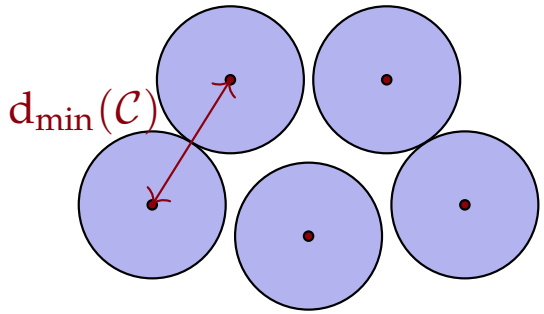
$$\underset{\boldsymbol{m} \in \mathbb{F}_q^k}{\text{message}} \mapsto \underset{\boldsymbol{c} \in \mathcal{C} \subseteq \mathbb{F}_q^n}{\text{codeword}} \xrightarrow{\text{channel}} \underset{\boldsymbol{c}' \in \mathbb{F}_q^n}{\text{noisy codeword}} \mapsto \underset{\boldsymbol{m}'(=\boldsymbol{m}?)}{\text{decoded message}}$$

$$\uparrow$$

errors ($c_i \neq c'_i \in \mathbb{F}_q$) or erasures ($c'_j = \perp$)

Hamming distance $\mathrm{d}(\boldsymbol{u}, \boldsymbol{v}) := |\{i \in [1, n], u_i \neq v_i\}|$.

- $d = \mathrm{d}_{\min}(\mathcal{C}) := \min\{\mathrm{d}(\boldsymbol{c}, \boldsymbol{c}'), \boldsymbol{c} \neq \boldsymbol{c}', (\boldsymbol{c}, \boldsymbol{c}') \in \mathcal{C}^2\}$,
- $\mathcal{C}$ linear over $\mathbb{F}_q$, with $k = \dim(\mathcal{C})$.



Typical example: Reed-Solomon codes

Definition (Reed-Solomon code). Let $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{F}_q^n$, pairwise distinct.

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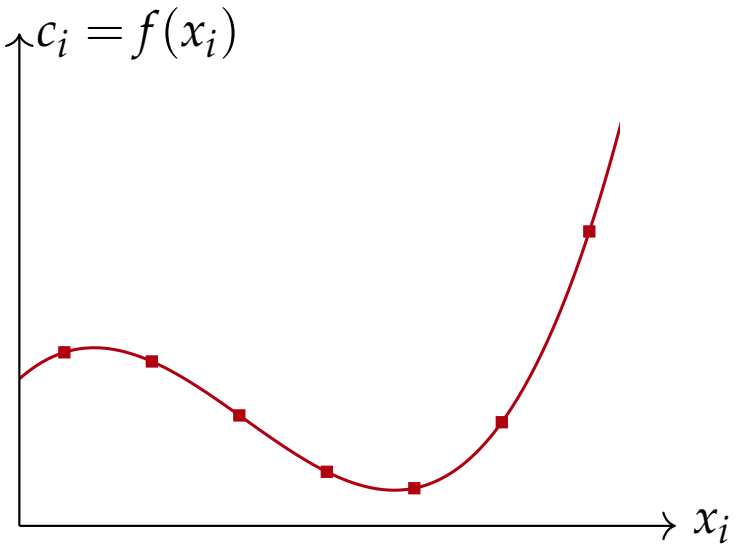

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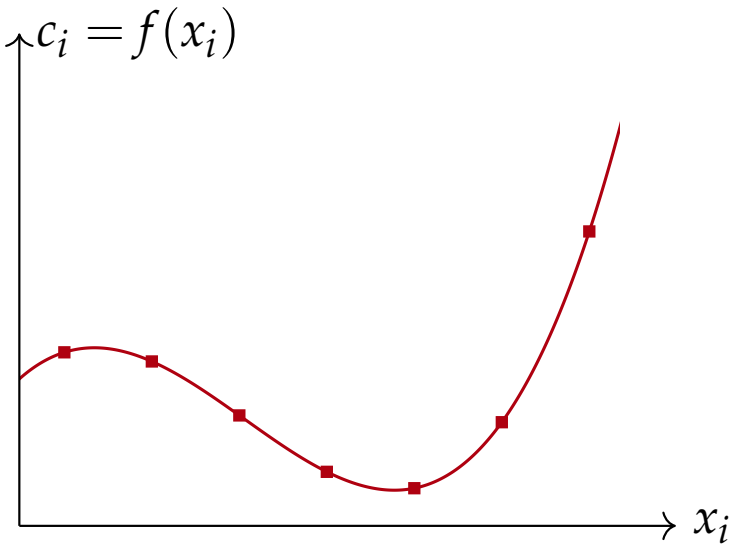

- Dimension $k = r + 1$
- Minimum distance $\mathrm{d}_{\min} = n - r$
- Can decode any b errors and e erasures
  $\rightarrow$ if $e + 2b < \mathrm{d}_{\min}$
  $\rightarrow$ in time $\Theta(n \log^3 n)$.

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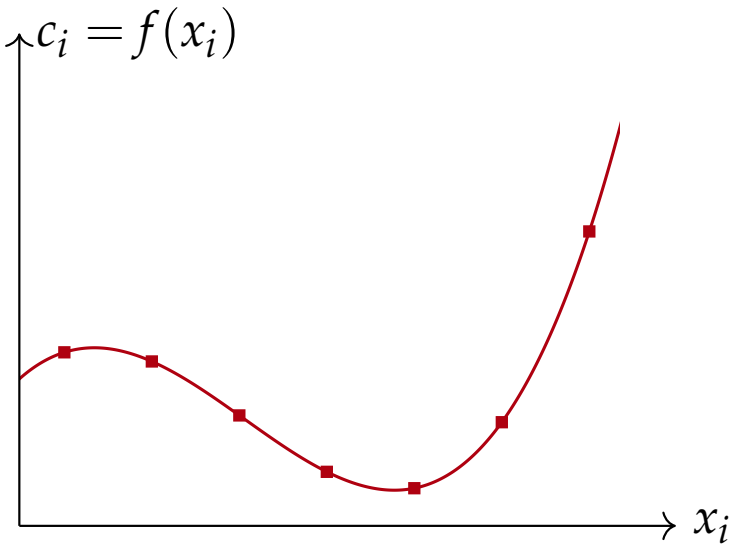

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In this talk,

$$\mathrm{RS}_q(r) := \mathrm{RS}_q(r, \mathbb{F}_q)$$

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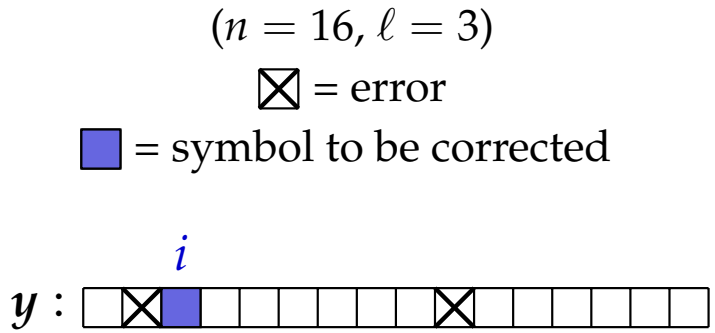
- **locality** $\ell \leq n$,
- failure probability $\varepsilon \in (0,1)$,
- admissible fraction of errors $\delta \in (0,1)$,

if there exists a poly-time **probabilistic algorithm** $\mathcal{D}$ such that, for every $y \in \mathbb{F}_q^n$ and $c \in \mathcal{C}$ satisfying $\mathrm{d}(y,c) \leq \delta n$ and for every $1 \leq i \leq n$:

- $\Pr(\mathcal{D}^{(y)}(i) = c_i) \geq 1 - \varepsilon$;
- $\mathcal{D}^{(y)}(i)$ makes at most ℓ queries to symbols of y.



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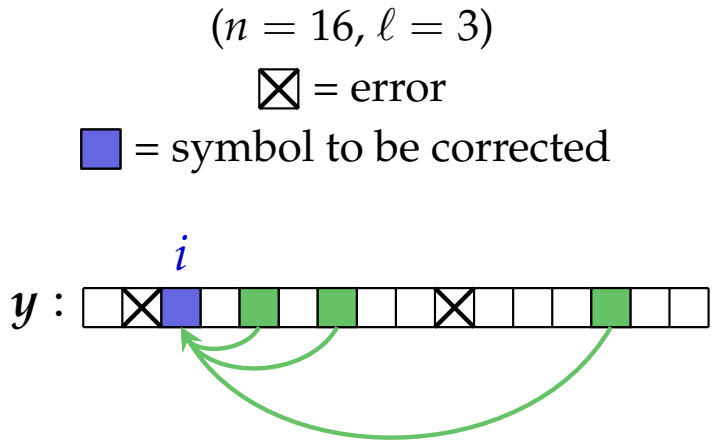

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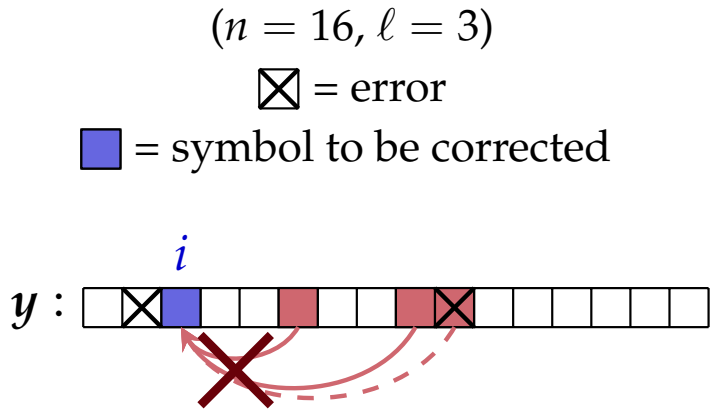

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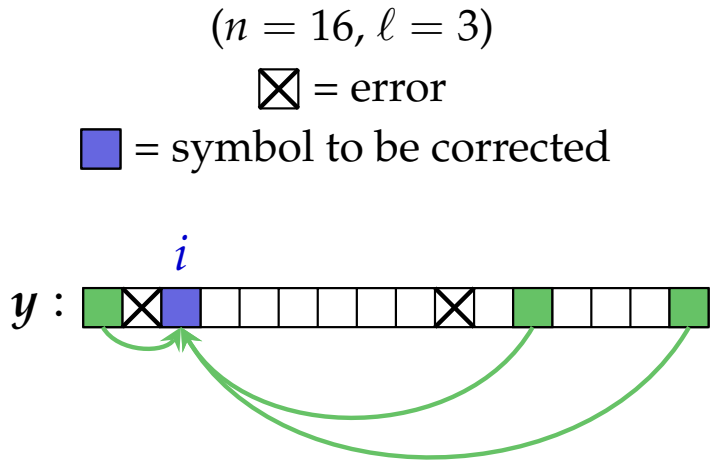

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- failure probability $\varepsilon \leq f(\ell) \cdot \delta$, with $f(\ell) \leq$ cste.
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Some existing constructions:

- constant locality ℓ:
  - Hadamard code (folklore) $\ell = 2$ and $k = \log(n)$
  - Matching vector codes [Yek08] k subexponential in $\log(n)$

- constant rate $R = k/n$:
  - Reed-Muller codes (folklore) $\ell = n^{1/m}$ and $k \leq \frac{1}{m!} \cdot n$
  - Multiplicity codes [KSY14], **lifted codes** [GKS13], expander codes [HOW14]

$$\ell \leq n^{\varepsilon} \quad \text{and} \quad k \geq \alpha \cdot n, \quad \forall \varepsilon, \alpha > 0,\ n \to \infty$$

Example: Reed-Muller codes

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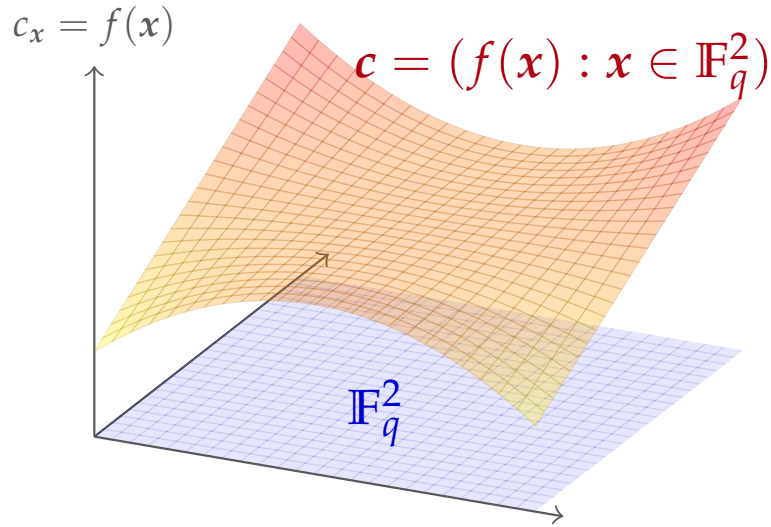

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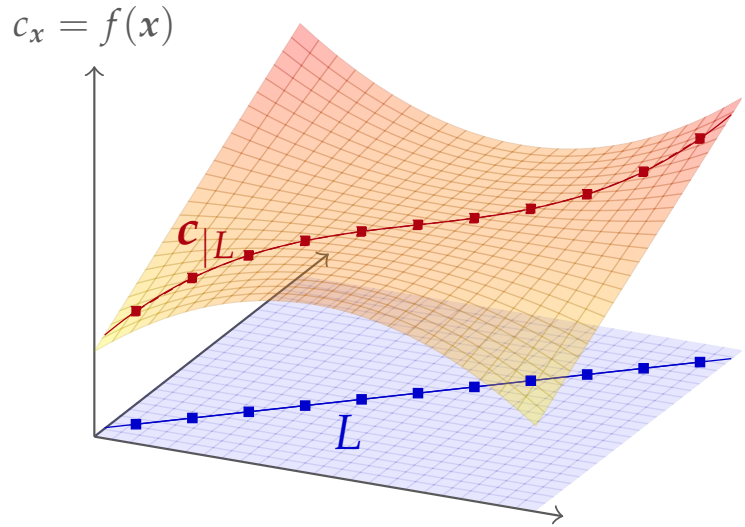
Assume $r \leq q-2$, and let:

- $\boldsymbol{c} = (f(\boldsymbol{x}) : \boldsymbol{x} \in \mathbb{F}_q^m) \in \mathrm{RM}_q(m,r)$
- $\phi : \mathbb{F}_q \to \mathbb{F}_q^m$ affine and injective
  $\Rightarrow$ affine line $L := \phi(\mathbb{F}_q) \subset \mathbb{F}_q^m$

Then, the **restriction** of $\boldsymbol{c}$ to L (*or to* ϕ):

$$\boldsymbol{c}_{|L} := ((f \circ \phi)(t) : t \in \mathbb{F}_q) \in \mathrm{RS}_q(r)$$



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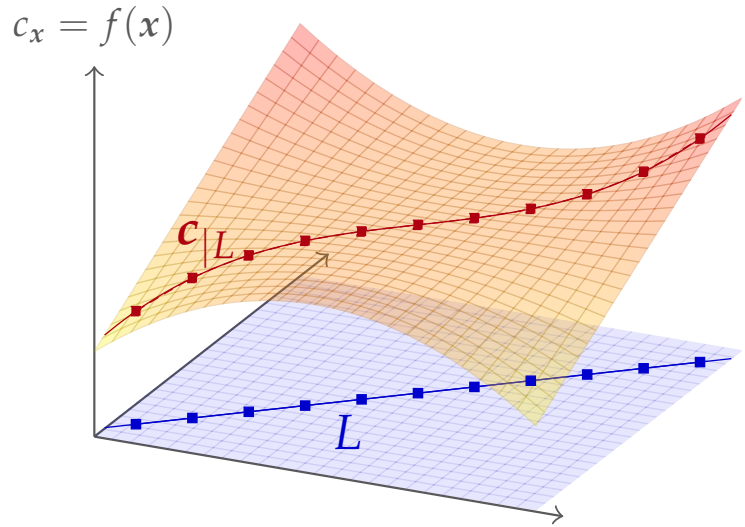

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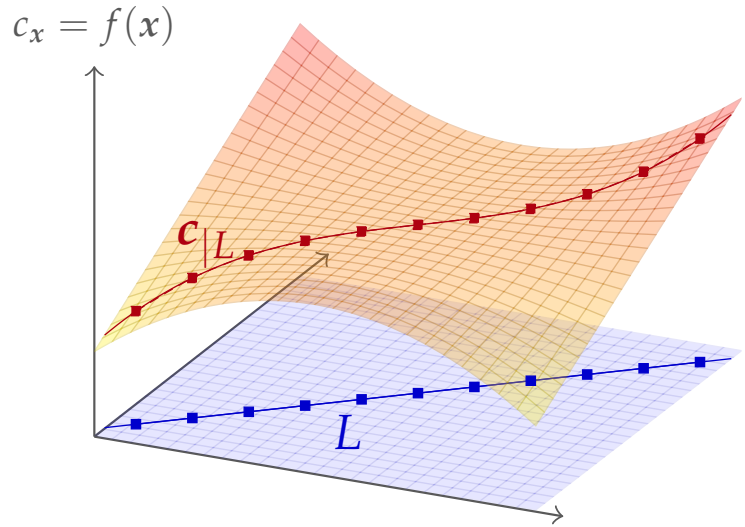

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Example ($q = 4, m = 2, r = 2$).

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$\Rightarrow$ for every ϕ, the "restriction" $(f\circ\phi)(T)$ can be interpolated as a univariate polynomial of degree ≤ 2

High-rate construction: lifted codes (2)

- $\mathbb{A}^m := \mathbb{F}_q^m$ $\qquad\qquad$ $\mathrm{ev}_{\mathbb{A}^m}(f) := (f(\boldsymbol{x}) : \boldsymbol{x} \in \mathbb{F}_q^m) \in \mathbb{F}_q^{\mathbb{A}^m}$
- $\mathrm{Emb}_{\mathbb{A}}(m) := \{\phi : \mathbb{F}_q \to \mathbb{F}_q^m, \text{injective and affine}\}$

Definition (lifted Reed-Solomon code [GKS13] reformulated).

$$\mathrm{Lift}(\mathrm{RS}_q(r), m) := \{\mathrm{ev}_{\mathbb{A}^m}(f), f \in \mathbb{F}_q[\boldsymbol{X}] \mid \forall \phi \in \mathrm{Emb}_{\mathbb{A}}(m), \mathrm{ev}_{\mathbb{A}^1}(f \circ \phi) \in \mathrm{RS}_q(r)\}$$

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What about the dimension/rate?

Theorem (characteristic 2, simplified from [GKS13]).
For every $m \geq 2$ and $0 < R_0 < 1$, there exists $q > 0$ and $r \leq q - 2$ such that

$$\mathrm{Lift}(\mathrm{RS}_q(r), m)$$

is locally correctable with rate $R \geq R_0$.

Rate of lifted codes

Bounds in [GKS13] are **far from being tight**.

- **Ex:** for $m = 2$ and $R_0 = 1/2$, GKS theorem requires $n = q^m \geq 2^{64}$.

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Theorem [characteristic 2, finite length $n = q^2 = 2^{2e}$].
For $m = 2$, $q = 2^e$ and $r = (1 - 2^{-c})q - 1$,

$$R = 1 - \frac{5}{4}\left(\frac{3}{4}\right)^c + \frac{1}{4}\left(\frac{1}{4}\right)^c + \frac{1}{2^e}\left(\frac{3^c - 1}{2^{c+2}}\right).$$

- actually, $n = q^2 \geq 2^6 = 64$ is enough to achieve $R \geq 1/2$.

Degree sets

Lifted codes are **monomial**, *i.e.* generated by evaluations of monomials

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Degree set of a monomial code [GKS13]:

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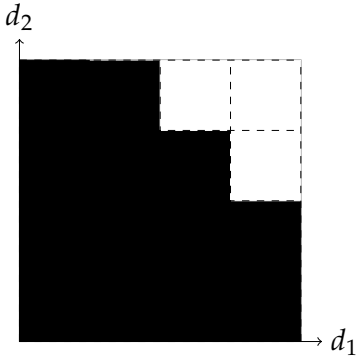
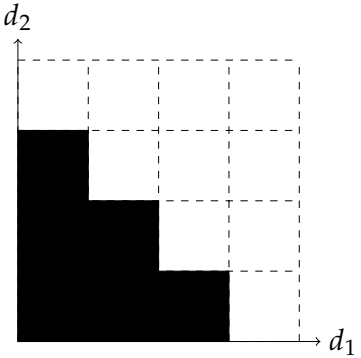
A **representation** for $m = 2$:



$\mathrm{RM}_4(2,4)$



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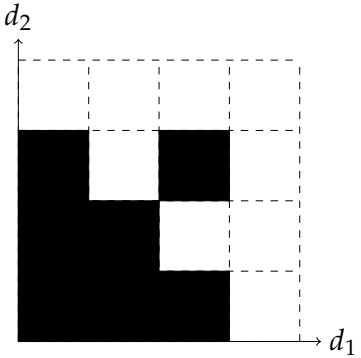
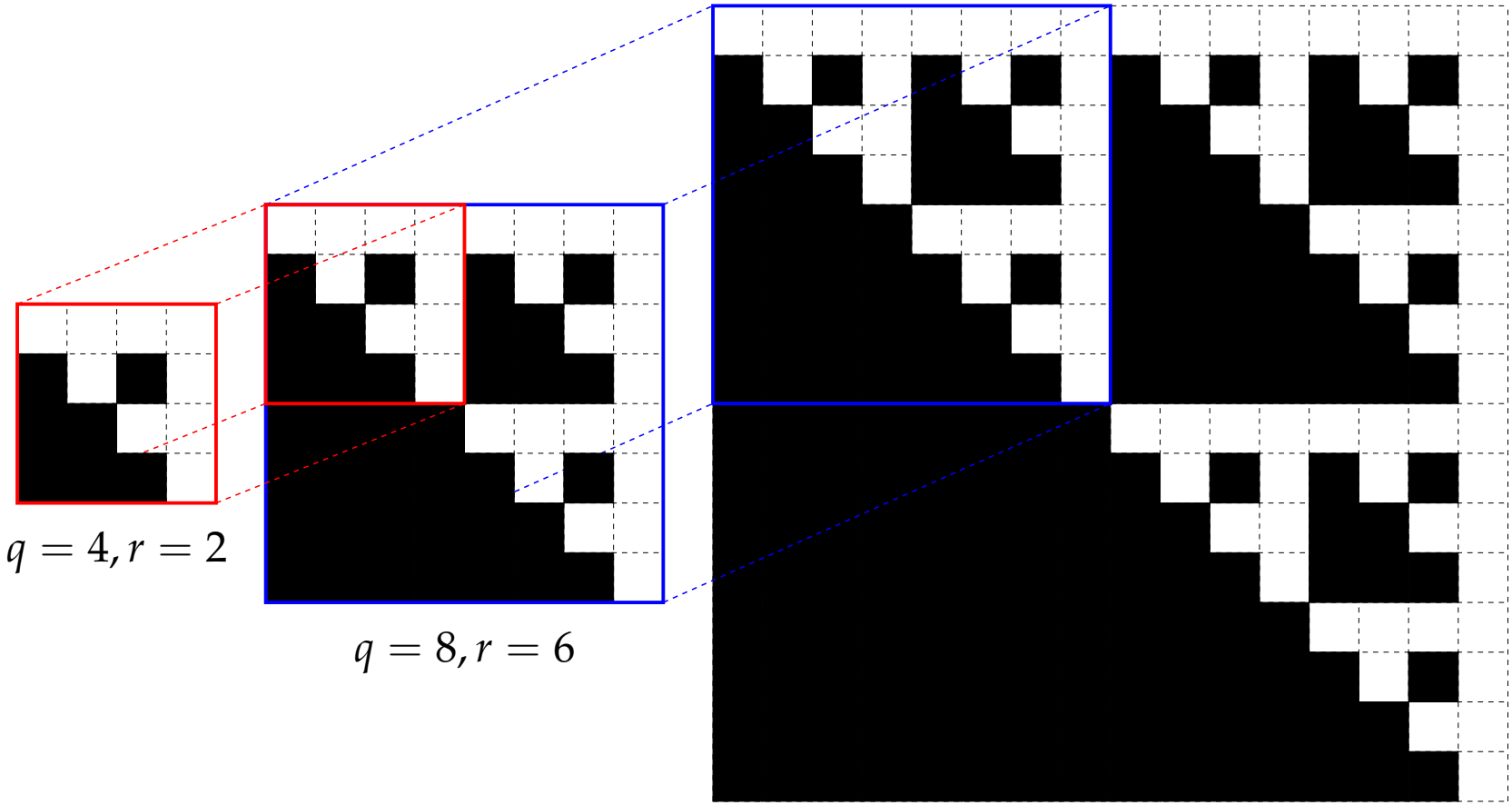


$\mathrm{Lift}(\mathrm{RS}_4(2),2)$

“Fractal” representation of degree sets



Outline

Evaluation on projective spaces

Projective space $\mathbb{P}^m := \left(\mathbb{A}^{m+1} \setminus \{\mathbf{0}\}\right) / \sim$ where $\boldsymbol{a} \sim \boldsymbol{b}$ iff $\exists \lambda \in \mathbb{F}_q^\times, \boldsymbol{a} = \lambda \boldsymbol{b}$

Defining an **evaluation map** over $\mathbb{P}^m$ requires:

- **homogeneous** polynomials $f \in \mathbb{F}_q[\boldsymbol{X}]_v^H$ of fixed degree v,
- to choose a **representative** for every $\boldsymbol{u} \in \mathbb{P}^m$ (see [Lac86]):

$$\boldsymbol{u} = (0 : \cdots : 0 : 1 : * : \cdots : *) \in \mathbb{P}^m$$

We get:

$$f(\boldsymbol{u}) := f(0, \ldots, 0, 1, *, \ldots, *) \in \mathbb{F}_q$$

$$\mathrm{ev}_{\mathbb{P}^m}(f) := (f(\boldsymbol{u}) : \boldsymbol{u} \in \mathbb{P}^m) \in \mathbb{F}_q^{\mathbb{P}^m}$$

Projective lifted codes

Example. Projective Reed-Solomon code:

$$\mathrm{PRS}_q(r) = \{\mathrm{ev}_{\mathbb{P}^1}(f) = (f(\boldsymbol{x}) : \boldsymbol{x} \in \mathbb{P}^1), f \in \mathbb{F}_q[X, Y]_r^H\}$$

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Let $\mathrm{Emb}_{\mathbb{P}}(m) := \{\phi : \mathbb{F}_q^2 \to \mathbb{F}_q^{m+1} \text{ linear and injective}\}$.

Definition (lifted projective RS codes). Let $v = r + (m-1)(q-1)$.

$$\mathrm{Lift}(\mathrm{PRS}_q(r), m) := \{\mathrm{ev}_{\mathbb{P}^m}(f), f \in \mathbb{F}_q[\boldsymbol{X}]_v^H \mid \forall \phi \in \mathrm{Emb}_{\mathbb{P}}(m), \mathrm{ev}_{\mathbb{P}^1}(f \circ \phi) \in \mathrm{PRS}_q(r)\}$$

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- satisfy the **puncturing/shortening** relation

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Details in:
Lifted Projective Reed-Solomon Codes, L., DCC, to appear
10.1007/s10623-018-0552-8

Outline

Lifted codes when $r = q - 2$

Remark. Assume $r = q - 2$. Then, $\mathrm{RS}_q(q-2)$ is the parity-check code.

$$\boldsymbol{a} \in \mathrm{RS}_q(q-2) \iff \sum_{i=1}^{q} a_i = 0$$

$$\boldsymbol{c} \in \mathrm{Lift}(\mathrm{RS}_q(q-2), m) \iff \forall L \subseteq \mathbb{F}_q^m, \sum_{x \in L} c_x = 0$$

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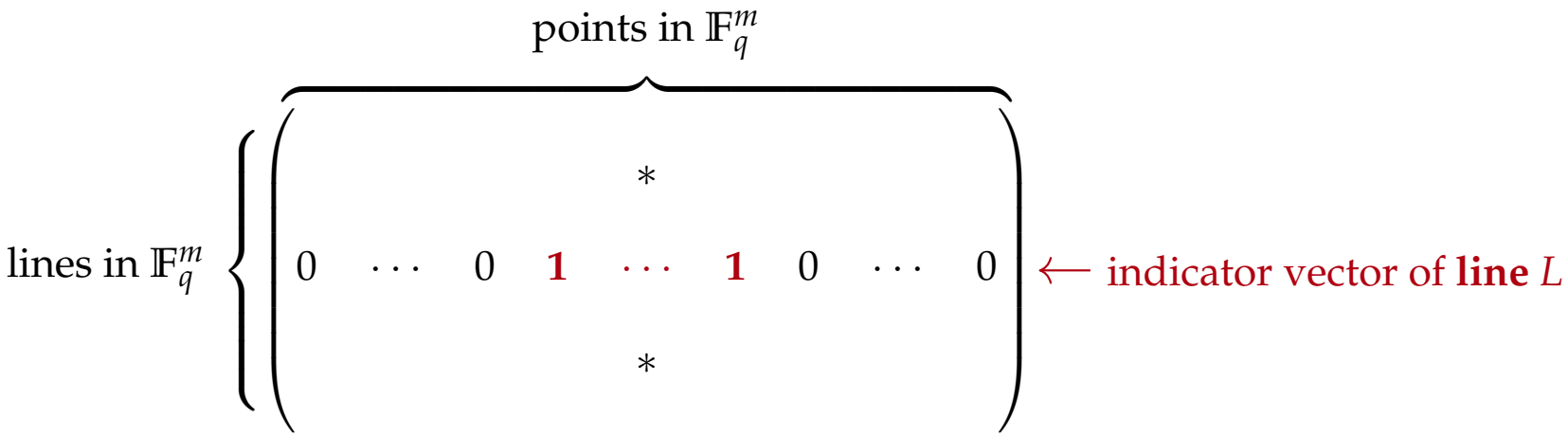
A non-full-rank **parity-check matrix** for $\mathrm{Lift}(\mathrm{RS}_q(q-2), m)$:



Block designs

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Definition. A *t-design* of parameters (n, ℓ, λ) consists in:

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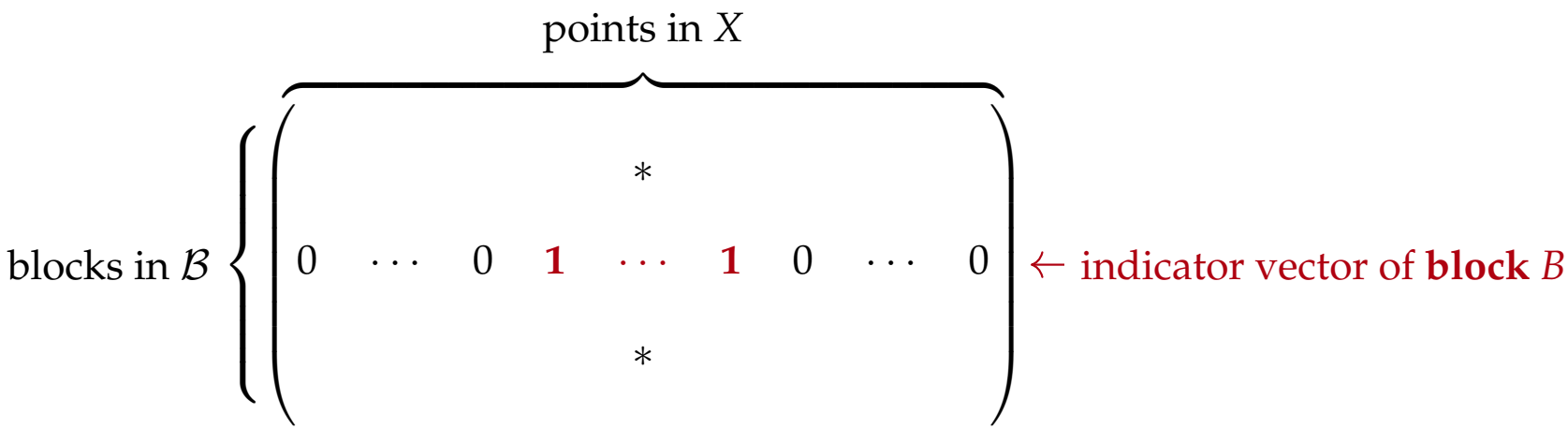
Incidence matrix of a design:



Codes based on designs, and generalisation

The **code based on the design** $\mathcal{D} = (X, \mathcal{B})$ is the code $\mathcal{C} = \mathrm{Code}(\mathcal{D}) \subseteq \mathbb{F}_q^X$ admitting the incidence matrix of $\mathcal{D}$ as a parity-check matrix.

$$\mathrm{Code}(\mathcal{D}) = \{\boldsymbol{c} \in \mathbb{F}_q^X \mid \forall B \in \mathcal{B}, \boldsymbol{c}_{|B} \in \mathrm{Parity}\}$$

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Let $\mathcal{F} = (\mathcal{F}_B \subseteq \mathbb{F}_q^B : B \in \mathcal{B})$ be a family of codes indexed by blocks $B \in \mathcal{B}$. The **generalised design-based code** based on $(\mathcal{D}, \mathcal{F})$ is

$$\mathrm{Code}(\mathcal{D}, \mathcal{F}) := \{\boldsymbol{c} \in \mathbb{F}_q^X \mid \forall B \in \mathcal{B}, \boldsymbol{c}_{|B} \in \mathcal{F}_B\}.$$

Design-based codes and LCCs

Generalised design-based code $\mathcal{C} = \mathrm{Code}(\mathcal{D}, \mathcal{F})$, where

- $\mathcal{D}$ be a t-$(n, \ell+1, \lambda)$-design
- $\tau \in (0, \frac{1}{2})$ is fixed
- $\mathcal{F} = (\mathcal{F}_B : B \in \mathcal{B})$ s.t. every code in $\mathcal{F}$ corrects a fraction τ of errors

Algorithm. Local correction of $y \in \mathbb{F}_q^X$ at $i \in X$

- Pick uniformly at random a block $B \in \mathcal{B}$ such that $i \in B$.
- Correct $y_{|B}$ as a noisy codeword from $\mathcal{F}_B$, and output $\tilde{y}_i$.

Proposition [$t = 2$]. For every $\delta < \tau/2$, $\mathrm{Code}(\mathcal{D}, \mathcal{F})$ is a $(\ell, \delta, \varepsilon)$-LCC, where

$$\varepsilon = \delta/\tau\,.$$

Proposition [$t = 3$]. For every $\delta < \tau - 1/\sqrt{2\ell}$, $\mathrm{Code}(\mathcal{D}, \mathcal{F})$ is a $(\ell, \delta, \varepsilon)$-LCC where

$$\varepsilon = \frac{\delta(1-\delta)}{(\tau-\delta)^2} \cdot \frac{1}{\ell} \leq \frac{1}{\tau^2 \ell} \cdot \delta\,.$$

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Solutions with **better communication complexity:**

- With 1 server, only **computational** privacy is possible [CGKS95, CG97].
- With $\ell \geq 2$ servers, one can achieve **information-theoretic** privacy [CGKS95-98].

Definition of PIR [CGKS95]

Given a file F and ℓ servers $S_1, \ldots, S_\ell$,
user U wants to recover F_i privately.

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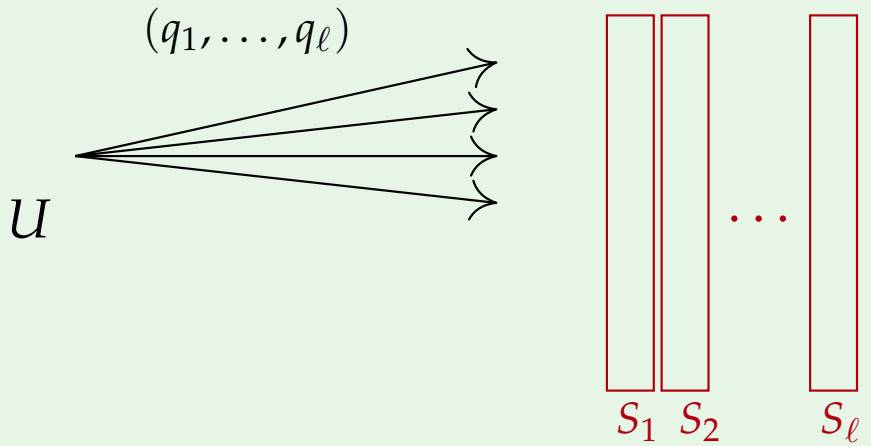

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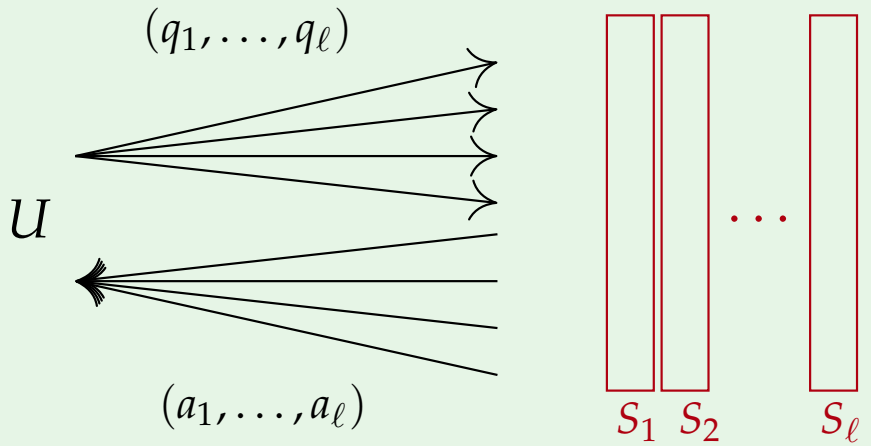

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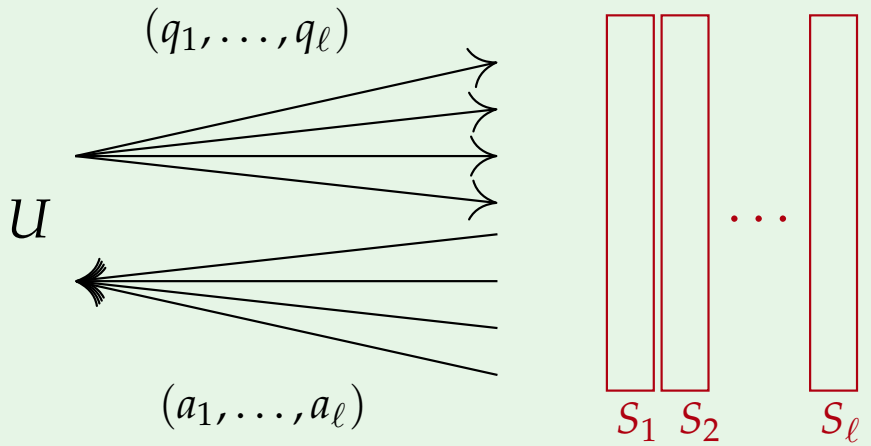

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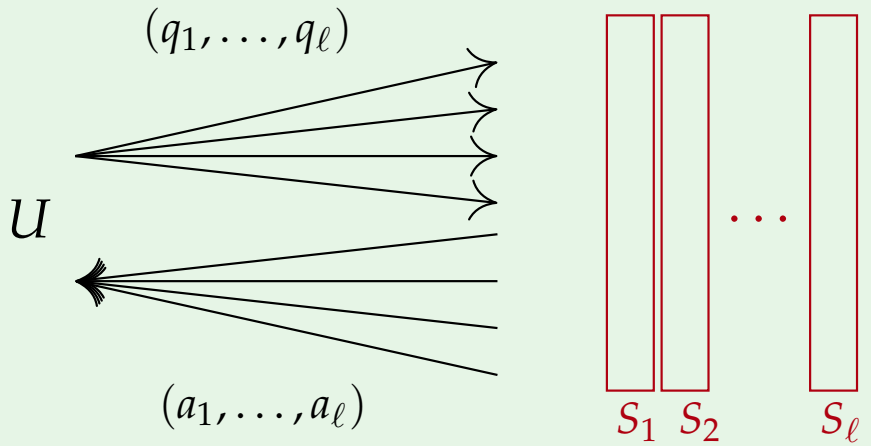

Information-theoretic privacy: $\mathrm{I}(i\,; q_j) = 0,\ \forall j = 1, \ldots, \ell.$

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Usual goals for PIR:

- Low communication complexity
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We here focus on the **computation complexity**, crucial for practicality [OG10].

Outline

Transversal designs

A **transversal design** $\mathrm{TD}(\ell, s) = (X, \mathcal{B}, \mathcal{G})$ is given by:

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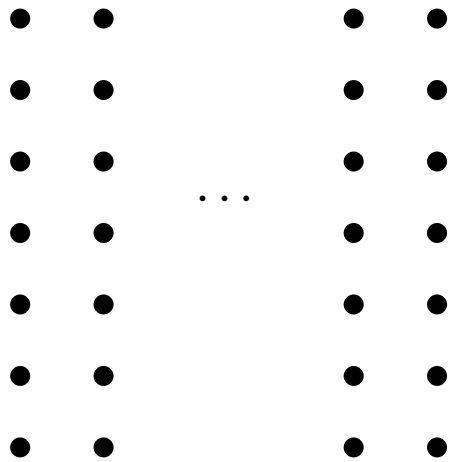

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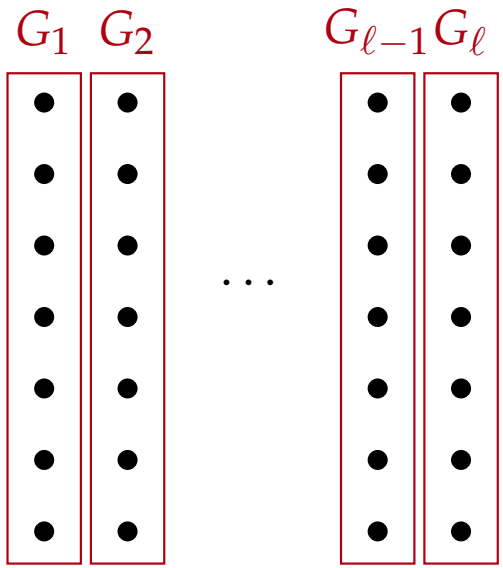

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  - $B \subset X$ and $|B| = \ell$;
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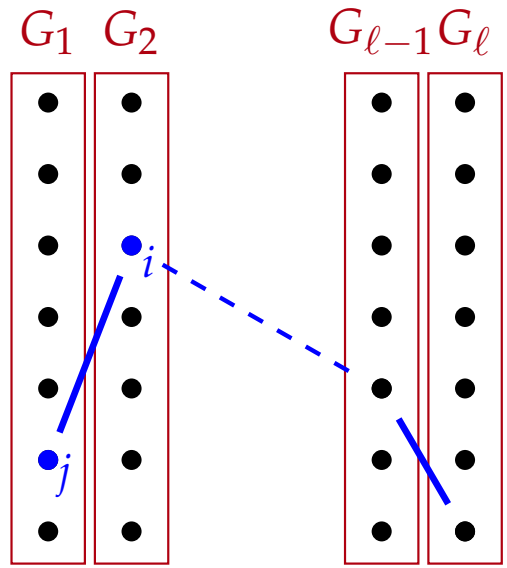

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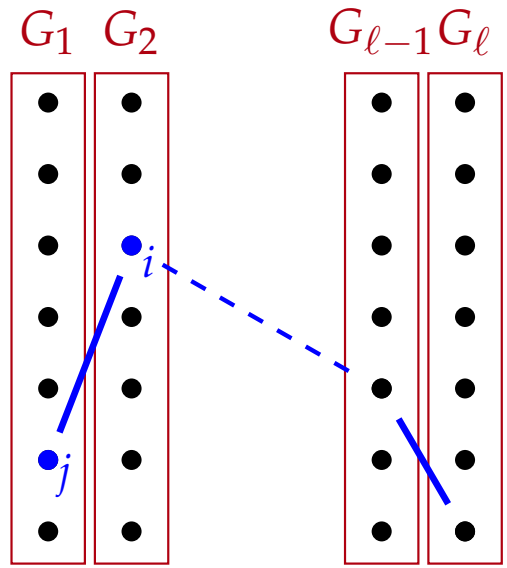
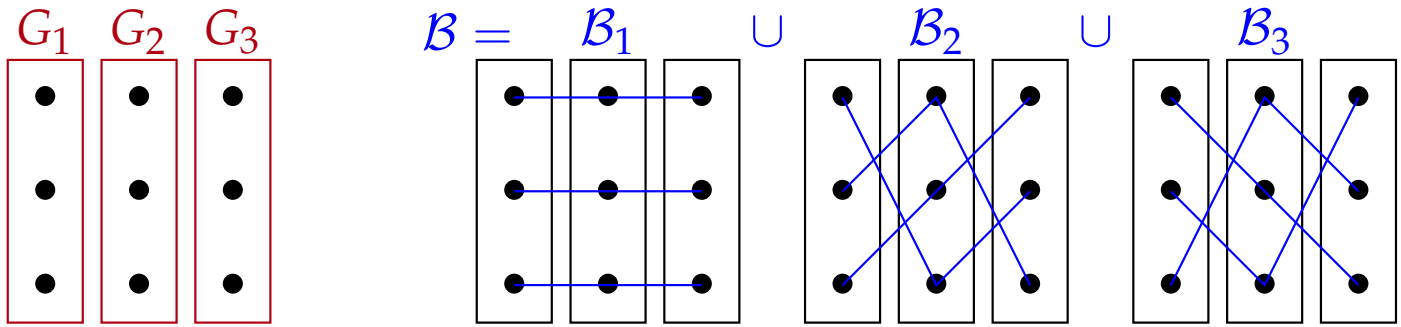


Its incidence matrix (between points and blocks) defines a code.

Example

The transversal design $\mathrm{TD}(3,3)$ represented by:



gives an incidence matrix

$$H = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Its rank over $\mathbb{F}_3$ is 6 $\quad\Longrightarrow\quad$ the associated code $\mathcal{C}$ is a $[9,3]_3$ code.

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- **Initialisation.** User U encodes $F \mapsto \boldsymbol{c} \in \mathcal{C}$, and gives $\boldsymbol{c}_{|G_j}$ to server S_j.
- **To recover** $F_i = c_i$, with $i \in X$:
  1. User U randomly picks a block $B \in \mathcal{B}$ containing i.
     Then U defines:

$$q_j = \mathcal{Q}(i)_j := \begin{cases} \text{unique } \in B \cap G_j & \text{if } i \notin G_j \\ \text{a random point in } G_j & \text{otherwise.} \end{cases}$$

  2. Each server S_j sends back c_{q_j}
  3. U recovers

$$c_i = -\sum_{j:\, i \notin G_j} c_{q_j} = -\sum_{b \in B \setminus \{i\}} c_b$$

Privacy and parameters

Theorem. This PIR protocol is information-theoretically private.

Proof:

- the only server which holds F_i received a random query;
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Features.

- communication complexity: $\ell \log s$ uploaded bits, $\ell \log q$ dowloaded bits
- computational complexity:
  - **only** 1 **read for each server** (somewhat optimal)
  - $\leq \ell$ additions over $\mathbb{F}_q$ for the user
- storage overhead: $(n-k)\log q$ bits

Privacy and parameters

Theorem. This PIR protocol is information-theoretically private.

Proof:

- the only server which holds F_i received a random query;
- for each other server S_j, query q_j gives no information on the block B which has been picked $\Rightarrow$ no information leaks on i.

Features.

- communication complexity: $\ell \log s$ uploaded bits, $\ell \log q$ dowloaded bits
- computational complexity:
  - **only 1 read for each server** (somewhat optimal)
  - $\leq \ell$ additions over $\mathbb{F}_q$ for the user
- storage overhead: $(n-k)\log q$ bits

Question: transversal designs with good k depending on (ℓ, s)?

Outline

Instances with geometric designs

$\mathcal{T}_A$, the **classical affine transversal design**:

- $X = \mathbb{F}_q^m$, $m \geq 2$,
- $\mathcal{G}$ a set of q disjoint hyperplanes partitionning X,
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Question: better instances?

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An **orthogonal array** $\mathrm{OA}(t, \ell, s)$ of strength t is a list A of words
- of length ℓ,
- over a finite set S, $|S| = s$,
- such that, for every $I \subset [1, \ell]$ of size t, $A_{|I} = S^t$.

Equivalently, an $\mathrm{OA}(t, \ell, s)$ is a code $A \subset S^\ell$ with dual distance $t + 1$.

$$S = \{a, b\}$$

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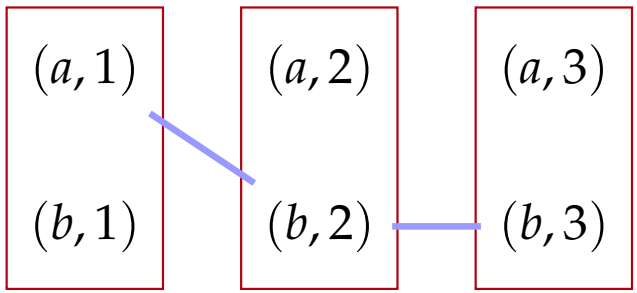

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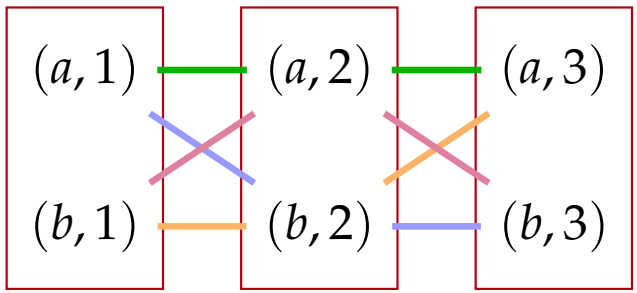

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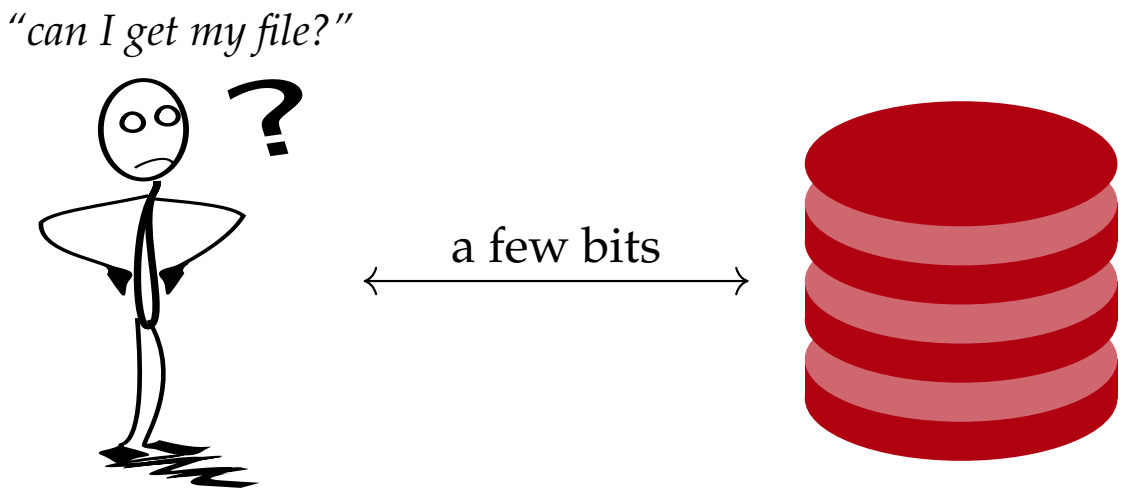
Details in:
Private Information Retrieval from Transversal Designs, L., IEEE TIT, to appear
10.1109/TIT.2018.2861747

Outline

Proofs-of-retrievability [Juels, Kaliski '07]

Issue: a client wants to verify if a file stored on a server is still retrievable, with a low communication challenge-response protocol



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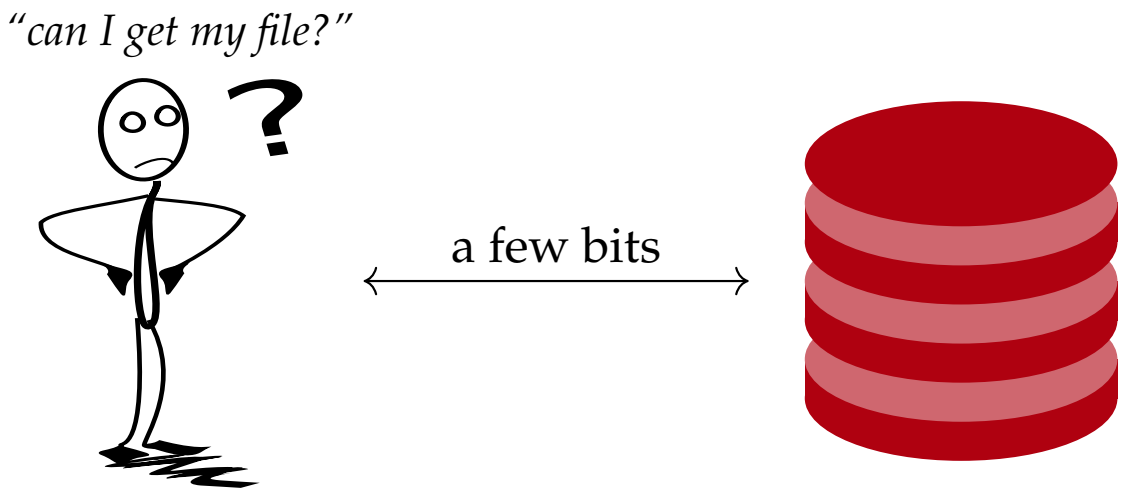

Additional constraints: unbounded-use, low client storage, low computation

PoR with lifted codes

$$\mathcal{C} = \mathrm{Lift}(\mathrm{RS}_q(r), m)$$

Assumption: one can compute independent pseudo-random permutations

$$\sigma_i^{(\kappa)} \in \mathfrak{S}(\mathbb{F}_q), \quad 1 \leq i \leq n, \ \kappa \in \mathcal{K}$$

Initialisation:

- User picks $\kappa \in \mathcal{K}$ at random
- File F is encoded and permuted as follows:

$$F \ \mapsto \ \boldsymbol{c} \in \mathcal{C} \ \mapsto \ \boldsymbol{w} = \boldsymbol{\sigma}(\boldsymbol{c}) = (\sigma_1^{(\kappa)}(c_1), \ldots, \sigma_n^{(\kappa)}(c_n)) \in \mathbb{F}_q^n$$

- User stores κ, server stores $\boldsymbol{w}$

Verification:

- User picks a line $L \subset \mathbb{F}_q^m$ at random and sends it to the server
- Server reads $\boldsymbol{w}_{|L}$ and sends it back to the user
- User accepts iff $\boldsymbol{\sigma}^{-1}(\boldsymbol{w}_{|L}) \in \mathrm{RS}_q(r)$

Results

Informal result (for the lifted code with $m = 2$):

For every $\varepsilon \leq \varepsilon_0 \simeq 1$, we have:
if the server answers correctly to a fraction $\geq 1 - \varepsilon$ of the challenges,
then with probability $\geq 1 - \mathcal{O}\left(\frac{1}{n(\varepsilon_0 - \varepsilon)^2}\right)$ the file is extractable from the server.

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This idea can be **generalised** to other codes such as design-based codes.

Details in:
Generic Constructions of PoRs from Codes and Instantiations, L. & Levy-dit-Vehel
submitted, 2018

Outline

Conclusion

- **Analysis** and **generalisation** of a family of high-rate locally correctable codes, namely **lifted Reed-Solomon codes**
- **Combinatorial formalism** for the construction of locally correctable codes, thanks to **block designs**
- Application to **private information retrieval** (PIR)
- Application to **proofs of retrievability** (PoR)

Future works

- PIR with **low server computation complexity**
  - 1 server read $\longrightarrow$ constant/sublinear number of server reads
- Extend the **lifting** process to other geometric varieties
  - *e.g.* the Hermitian variety
- Design-based codes allow us to **remove probabilistic decoders** from a definition of locally correctable codes
  - "usual" combinatorial coding-theoretic version of LCCs
  - new constructions? new bounds?