Private Information Retrieval Protocols Based on Transversal Designs

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1. The PIR issue
2. Transversal designs for efficient PIR protocols
3. Instances

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2. Transversal designs for efficient PIR protocols

3. Instances

First instance: affine transversal designs

Second instance: with orthogonal arrays

Informal PIR

Given a file F,

can we retrieve the entry F_i

without leaking any information on i?

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Remark:

- PIR $\neq$ anonymity (hidden user)
- PIR $\neq$ encryption (hidden data)

Formal definition of PIR

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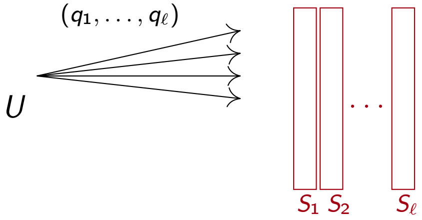

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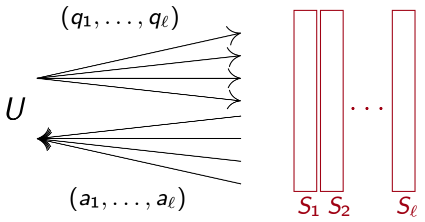

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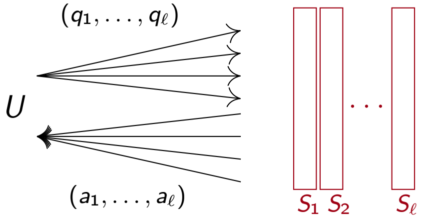

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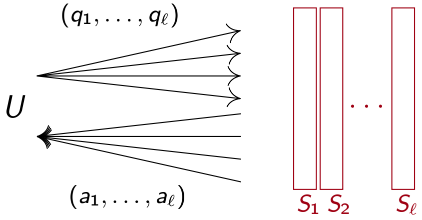


Information-theoretic privacy: $\mathrm{I}(i\,; q_j) = 0,\ \forall j = 1, \dots, \ell.$

Goals

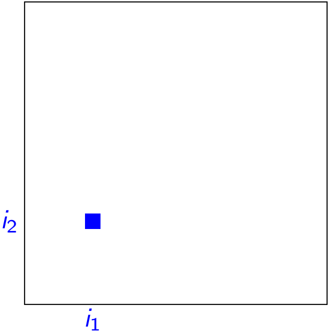
Common goals for PIR:

- Low communication complexity (number of bits exchanged between user and servers).
  $\rightarrow$ number of servers ≥ 2.
- Low storage overhead for the servers (if coded file).
- Low computation complexity for algorithms $\mathcal{A}$ (server) and $\mathcal{R}$ (user).

Seminal work [CGKS'95-98]

Ref: Chor, Goldreich, Kushilevitz, Sudan, *Private Information Retrieval*, FOCS'95, J.ACM'98

- $|F| = n$ bits, with $n = m^2$, and let's see $[1, n]$ as $[1, m]^2$.
- 4 servers $S_{00}, S_{01}, S_{10}, S_{11}$. Each server holds F.
- Assume user U wants to retrieve $F_{(i_1, i_2)}$, $1 \leq i_1, i_2 \leq m$.



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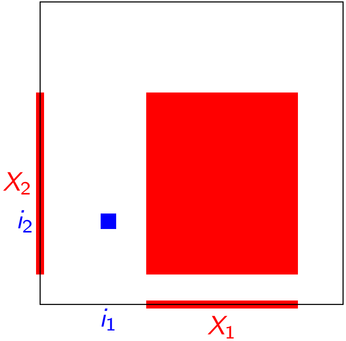

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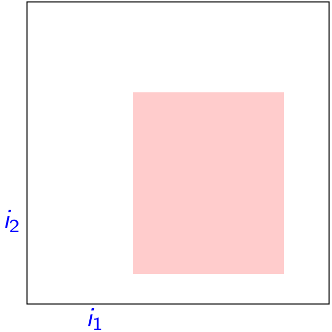

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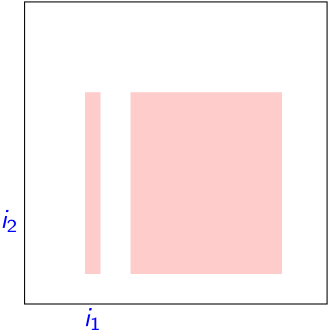

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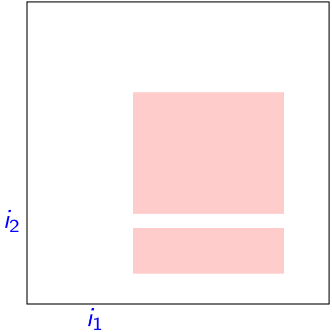

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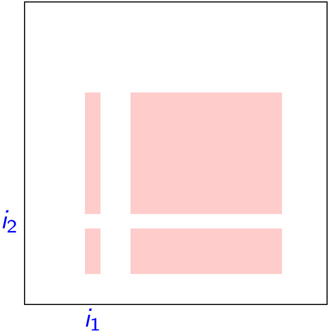

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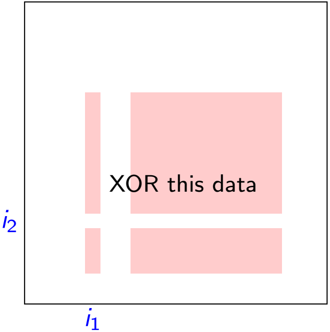

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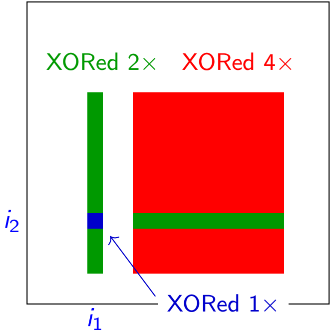

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3. User XORs the 4 received bits and outputs the result.

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With 4 servers:

- Communication: $8\sqrt{n}$ uploaded bits, 4 dowloaded bits,
- Storage: replication of F over 4 servers,
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Generalizable to 2^s servers:

- Communication: $s2^s n^{1/s}$ uploaded bits, 2^s dowloaded bits,
- Storage: replication of F over 2^s servers,
- Complexity: in average, XOR of $n/2^s$ bits for each server's answer; XOR of 2^s bits for the user.

Other notable works

Main ideas:

- *Katz, Trevisan '00.*
  Smooth locally decodable codes give PIR protocols.
- *Fazeli, Vardy, Yaakobi '15.*
  PIR codes. Transforms a replication-based PIR into a coded PIR.
- *Sun, Jafar '16.*
  PIR capacity.
- *El Rouayheb, Freij-Hollanti, Gnilke, Hollanti, Karpuk, Tajeddine '16'17.*
  Optimal constructions according to PIR capacity. Star product construction.

Practicality

Context: file F is frequently queried (*e.g.* a public database.) Notion of *price of privacy*, mainly depending on:

- computational complexity for the servers,
- servers' storage overhead.

Yekhanin (in a survey, '12): "the overwhelming computational complexity of PIR schemes (...) currently presents the main bottleneck to their practical deployment".

Overview of our solution

Basic ideas:

- Encode the file $F \mapsto c \in \mathcal{C}$, **split** c in ℓ parts and share them among the ℓ servers.
- Use **low-weight** parity-check equations of $\mathcal{C}$ to retrieve symbols F_i.

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Practical solution:

- use codes $\mathcal{C}$ based on transversal designs.

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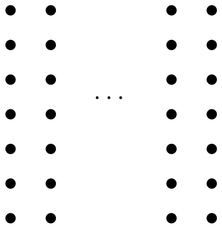
Second instance: with orthogonal arrays

Transversal designs

A **transversal design** $\mathrm{TD}(\ell, s) = (X, \mathcal{B}, \mathcal{G})$ is given by:

- X a set of *points*, $|X| = n = s\ell$,



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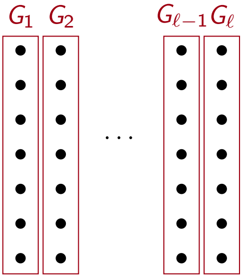

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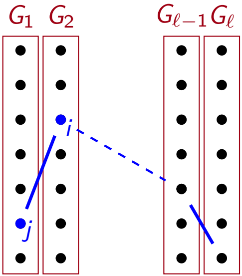
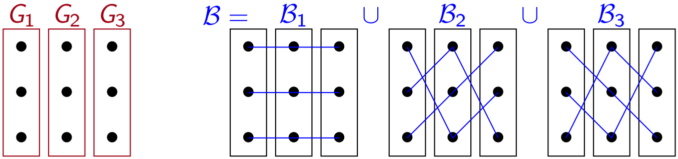
- *blocks* $B \in \mathcal{B}$ satisfying
  - $B \subset X$ and $|B| = \ell$;
  - for all $\{i, j\} \subset X$, $\{i, j\}$ lie:
    either in the same group $G \in \mathcal{G}$,
    or in a unique block $B \in \mathcal{B}$



Examples of TD

- Points X, parallel hyperplanes $\mathcal{G}$ and transversal lines $\mathcal{B}$ in the affine space $\mathbb{A}^m$. For instance, a $\mathrm{TD}(3,3)$:



- Similar construction in $X = \mathbb{P}^m \setminus A$, $\mathrm{codim}(A) = 2$.
- Combinatorial constructions based on orthogonal arrays, on difference sets...

Codes from designs

Let $\mathcal{T}$ be a transversal design $\mathrm{TD}(\ell, s) = (X, \mathcal{B}, \mathcal{G})$.

Its **incidence matrix** M has size $|\mathcal{B}| \times |X|$ and is defined by:

$$M_{i,j} = \begin{cases} 1 & \text{if } x_j \in B_i \\ 0 & \text{otherwise.} \end{cases}$$

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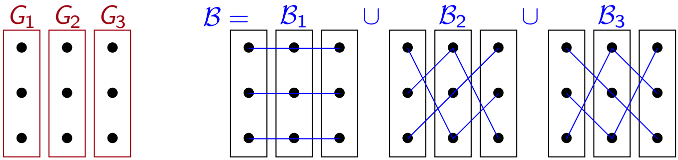
The **code** $\mathcal{C}$ **based on** $\mathcal{T}$ **over** $\mathbb{F}_q$ is the $\mathbb{F}_q$-linear code having M as parity-check matrix ($\mathcal{C}^\perp$ is generated by H).

- $\mathrm{length}(\mathcal{C}) = |X|$,
- $\dim(\mathcal{C}) = \dim(\ker M)$,
- $B \in \mathcal{B} \Rightarrow h \in \mathcal{C}^\perp$, such that $\mathrm{wt}(h_{|G_j}) = 1$, $\forall j = 1, \dots, \ell$.

Example

The transversal design $\mathrm{TD}(3,3)$ represented by:



gives an incidence matrix

$$H = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

whose rank over $\mathbb{F}_3$ is 6. $\implies$ $\mathcal{C}$ is a $[9,3]_3$ code.

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Let $\mathcal{C} \subseteq \mathbb{F}_q^n$ be a code based on a $\mathrm{TD}(\ell, s)$.

- **Initialisation.** User U encodes $F \mapsto c \in \mathcal{C}$, and gives $c_{|G_j}$ to server S_j for $j = 1, \dots, \ell$.
- **To recover $F_i = c_i$:**
  1. User U randomly picks a block $B \in \mathcal{B}$ containing i. Then U defines:

$$q_j = \mathcal{Q}(i)_j := \begin{cases} \text{unique } \in B \cap G_j & \text{if } i \notin G_j \\ \text{a random point in } G_j & \text{otherwise.} \end{cases}$$

  2. each server S_j sends back $a_j = \mathcal{A}_j(q_j, c_{|G_j}) := c_{q_j}$
  3. U recovers

$$-\sum_{j:\, i \notin G_j} c_{q_j} = -\sum_{b \in B \setminus \{i\}} c_{q_j} = c_i$$

Privacy and parameters

Theorem.– If the servers do not collude, then our PIR protocol is information-theoretically private.

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Question: TDs with good k depending on (ℓ, s)?

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A classical TD: points/lines/hyperplanes

Let $\mathcal{T}_A$ be the **classical affine TD**:

- $X = \mathbb{F}_q^m$, $m \geq 2$,
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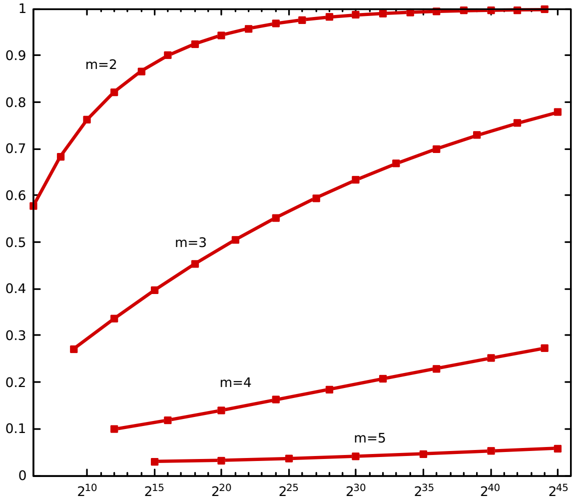
  - its parity-check matrix has q^m columns and q^{2m-2} rows...
  - ... but $\mathcal{C}$ contains $\mathrm{RM}_q(m, q-2)$ which has rate $\simeq 1/m!$,
  - and sometimes it is even larger.

Lower bounds on rates of TD-based codes

rate $R = k/n$



length $n = 2^{em}$

Particular case: $m = 2$

For $m = 2$, $q = p^e$, using Hamada's formula [Ham68] we obtain:

$$n = p^{2e}, \quad k \geq p^{2e} - \binom{p+1}{2}^e, \quad \ell = \sqrt{n}\,.$$

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Moreover, $c_p \nearrow$, with $c_2 = -0.208$ and $c_\infty = 0$.

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Open question:

- is this instance rate-optimal?

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1. The PIR issue

2. Transversal designs for efficient PIR protocols

3. Instances

   First instance: affine transversal designs

   Second instance: with orthogonal arrays

TDs and orthogonal arrays

An *orthogonal array* $\mathrm{OA}(t, \ell, s)$ *of strength* t may be seen as a list of codewords over S, with:

- $|S| = s$,
- length ℓ,
- and dual distance $d^{\perp} = t + 1$

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Construction OA $\rightarrow$ TD :

- $X = S \times [1, \ell]$
- $\mathcal{G} = \{S \times \{i\}, 1 \le i \le \ell\}$

$(a,1)$	$(a,2)$	$(a,3)$
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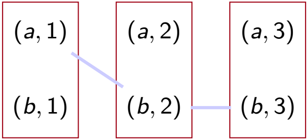

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a	b	b
b	b	a
b	a	b
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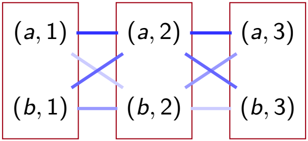

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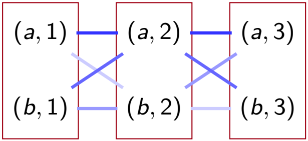

Prop. If $t = 2$, then we obtain a $\mathrm{TD}(\ell, s)$ from an $\mathrm{OA}(t, \ell, s)$.

Resistance to collusions

Experiments: for $t = 2$ and small ℓ and s, the classical affine TD leads to the best code dimension.

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What about $\mathrm{OA}(t, \ell, s)$ **with** $t > 2$**?**

Resulting TD satisfies: for each t-tuple of points lying in t different groups, there is a block which contains them all.

$\Rightarrow$ Our PIR protocol resists $t - 1$ collusive servers.

The *incidence code* construction

Definition.– We call *incidence code* of $\mathcal{C}_0$, denoted $I_q(\mathcal{C}_0)$, the $\mathbb{F}_q$-linear code $\mathcal{C}$ coming from the successive constructions:

$$\mathcal{C}_0 = \mathrm{OA}(t, \ell, s) \quad \mapsto \quad \text{generalized } \mathrm{TD}(\ell, s; t) \quad \mapsto \quad \mathcal{C} = I_q(\mathcal{C}_0)$$

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- $d^\perp(\mathcal{C}_0) - 2$ is the number of collusive servers the protocol resists
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let's use MDS codes for $\mathcal{C}_0$

Incidence codes of Reed-Solomon codes

Example: for $\mathcal{C}_0 = \mathrm{RS}(\mathbb{F}_q, t+1)$,

- $|F| = Rq^2 \log q$ bits, with R the rate of the incidence code
- requires q servers, resists t colluding ones,
- communication complexity $\Theta(q \log q)$,
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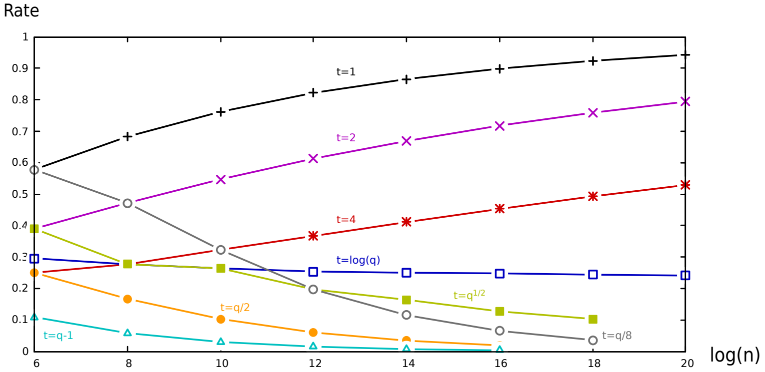

Conclusion

Summary: (server-)efficient PIR protocols can be built with codes based on transversal designs

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Current issues:

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PIR = very dynamic field:

- PIR capacity and optimal constructions,
- PIR on coded databases,
- partial PIR.

Thank you for your attention.
Questions?

Remarks on incidence codes

Proposition.– For any code $\mathcal{C}_0$ of length ℓ over $\mathbb{F}_s$, the incidence code $I_q(\mathcal{C}_0)$ is an $[n, k]_q$ code with:

- $n = s\ell$,
- $\ell - 1 \leq k \leq n - \Omega(\sqrt{n})$.

Proposition.– Let H be the parity-check matrix of $I_q(\mathcal{C}_0)$. Then,

$$HH^T = \ell J - D(\mathcal{C}_0)\,,$$

where J is the all-1 matrix and

$$D(\mathcal{C}_0)_{c,c'} = d(c, c'), \quad \forall c, c' \in \mathcal{C}_0\,.$$

Divisible codes for efficient PIR protocols

A p-divisible code is a code whose codewords' weights are divisible by p.

Corollary.– If $\mathcal{C}_0$ is p-divisible for $p = \mathrm{char}(\mathbb{F}_q)$, then:

$$k = \dim I_q(\mathcal{C}_0) \geq \frac{n-1}{2}\,.$$

Furthermore, if $p \mid \ell$, then:

$$HH^T = 0 \quad \Rightarrow \quad \mathcal{C}^\perp \subseteq \mathcal{C}$$

Theorem.– If there exists a p-divisible code $\mathcal{C}_0$ of length ℓ and dual distance $t+2$, then there exists a PIR protocol resisting to t colluding servers, with rate $\gtrsim 1/2$.

Question.– Do there exist projective ($d^\perp \geq 3$) p-divisible codes of length ℓ over $\mathbb{F}_q$ (with $q \gg \ell$, or $d^\perp$ large)?