Codes with locality: constructions and applications to cryptographic protocols

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Séminaire UVSQ

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Outline

1. Codes with locality
   - Locality in coding theory, examples
   - Lifted projective Reed-Solomon codes
   - A combinatorial point of view

2. Private information retrieval from transversal designs
   - Private information retrieval (PIR)
   - Transversal designs and codes
   - A new PIR construction
   - Instances

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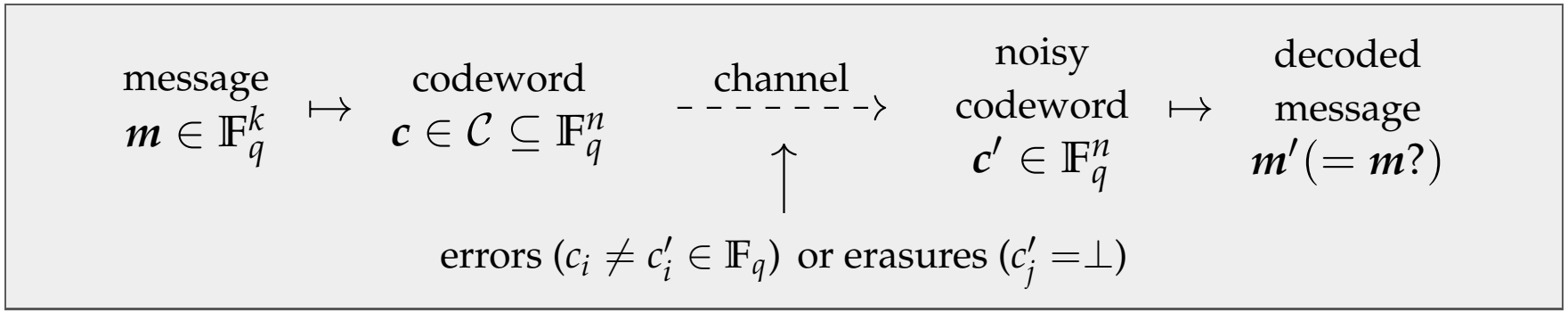

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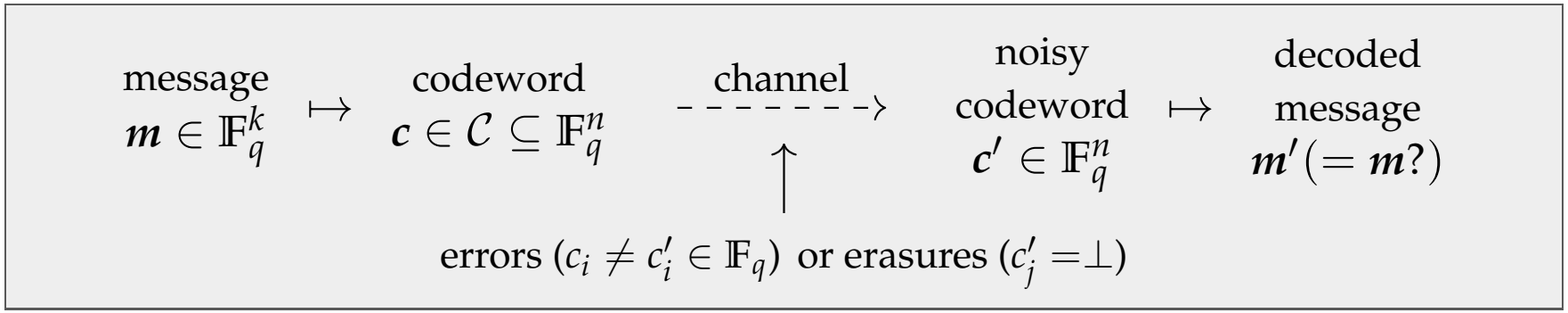

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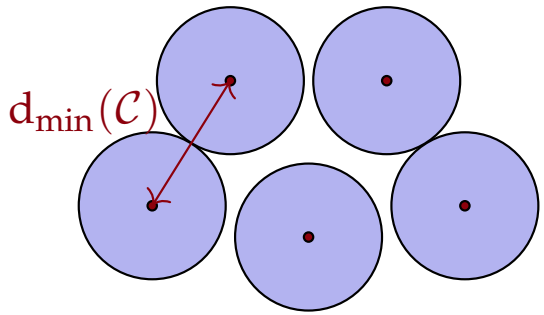

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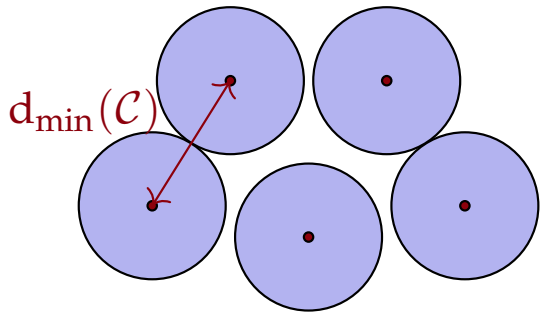


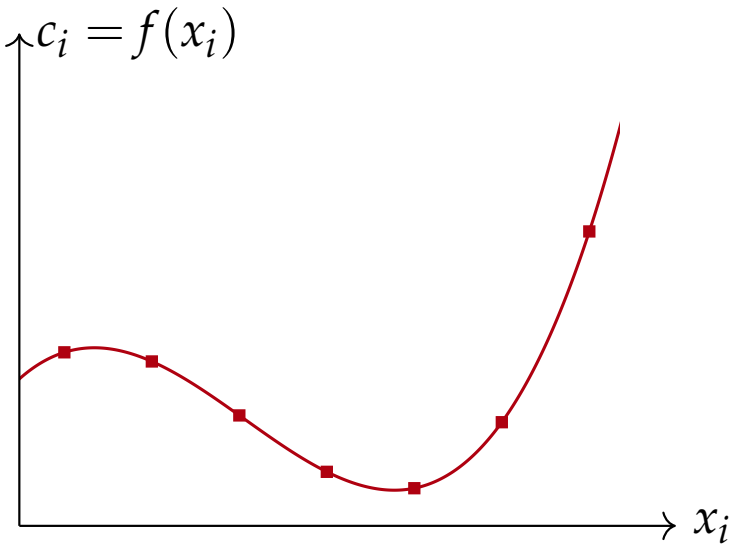
Singleton bound (code is MDS if bound is achieved):

$$k + d \leq n + 1\,.$$

Typical example: Reed-Solomon codes

Definition (Reed-Solomon code). Let $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{F}_q^n$, pairwise distinct.

$$\mathrm{RS}_q(r, \boldsymbol{x}) := \{(f(x_1), \ldots, f(x_n)), f \in \mathbb{F}_q[X], \deg f \leq r\}$$



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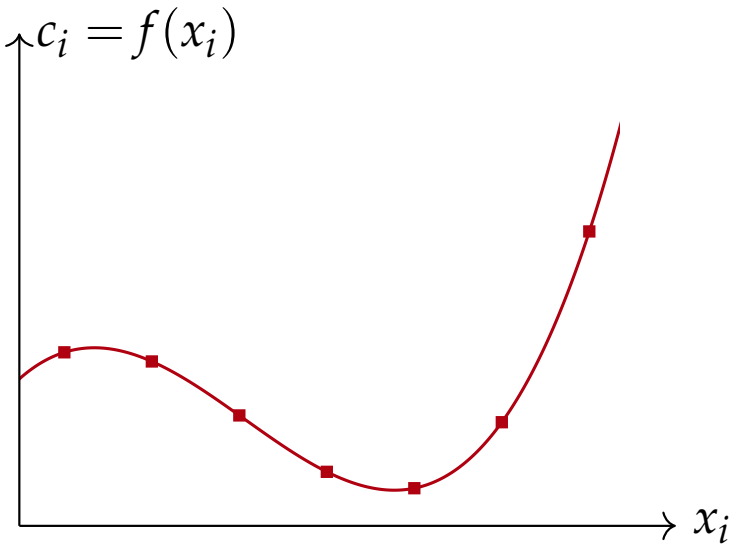

- Dimension $k = r + 1$
- Minimum distance $\mathrm{d}_{\min} = n - r$
  $\Rightarrow$ MDS
- Can decode any b errors and e erasures
  $\rightarrow$ if $e + 2b < \mathrm{d}_{\min}$
  $\rightarrow$ in time $\Theta(n \log^3 n)$.

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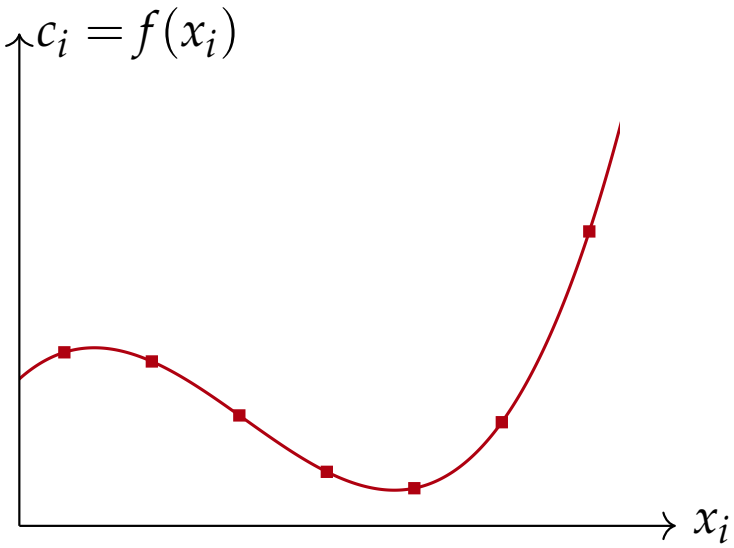

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In this talk,

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Goal: sublinear-time correction of some symbols of $\boldsymbol{c} \in \mathcal{C}$.

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- **locality** $\ell \leq n$,
- failure probability $\varepsilon \in (0,1)$,
- admissible fraction of errors $\delta \in (0,1)$,

if there exists a **probabilistic algorithm** $\mathcal{D}$ such that, for every $y \in \mathbb{F}_q^n$ and $c \in \mathcal{C}$ satisfying $\mathrm{d}(y,c) \leq \delta n$ and for every $1 \leq i \leq n$:

- $\Pr(\mathcal{D}^{(y)}(i) = c_i) \geq 1 - \varepsilon$;
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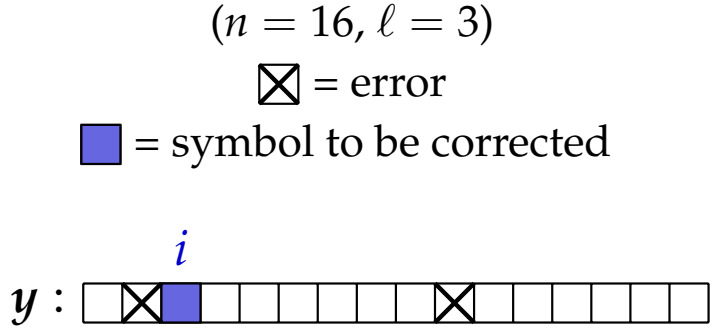

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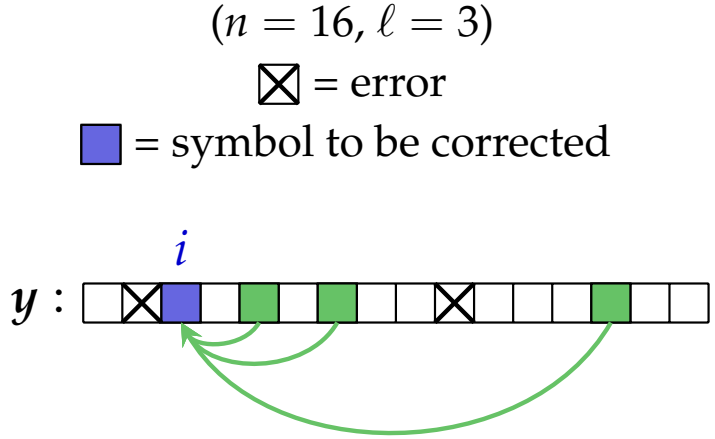

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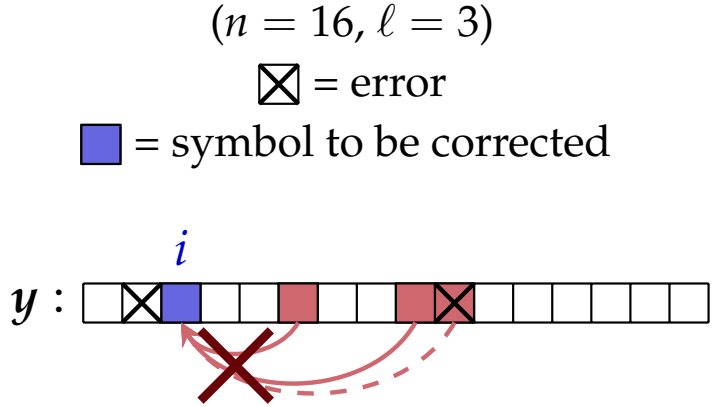

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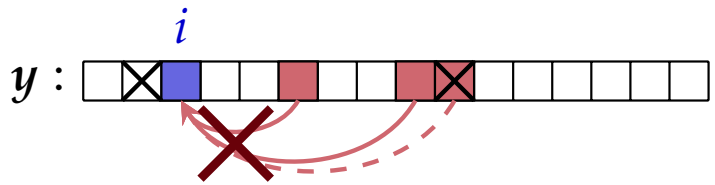
$(n = 16, \ell = 3)$

⊠ = error

■ = symbol to be corrected

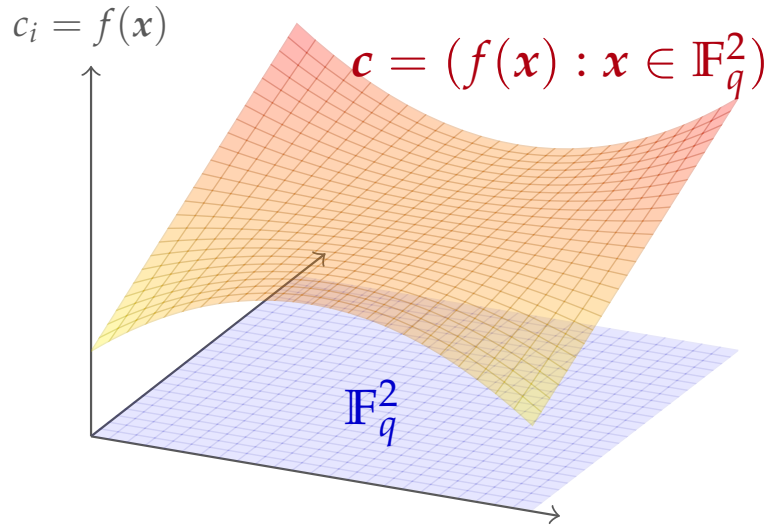


Goals:

- $\ell \ll n$
- $\varepsilon = \mathcal{O}(\delta)$, ideally $\varepsilon = \mathcal{O}(1)$
- $k = \dim \mathcal{C}$ large

Example: Reed-Muller codes

$$\mathrm{RM}_q(m,r) := \{(f(\boldsymbol{x}) : \boldsymbol{x} \in \mathbb{F}_q^m), f \in \mathbb{F}_q[X_1,\ldots,X_m], \deg f \leq r\}$$



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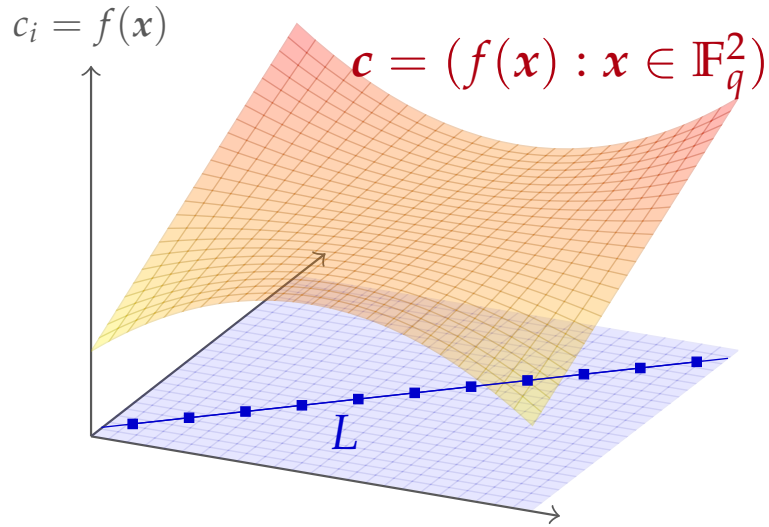

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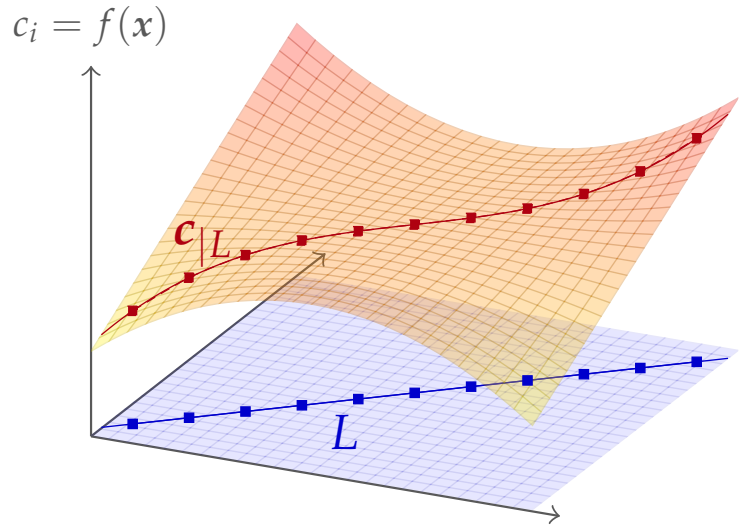
Then, the **restriction** of $\boldsymbol{c}$ to L (*or to* ϕ):

$$\boldsymbol{c}_{|L} := ((f \circ \phi)(t) : t \in \mathbb{F}_q) \in \mathrm{RS}_q(r)$$



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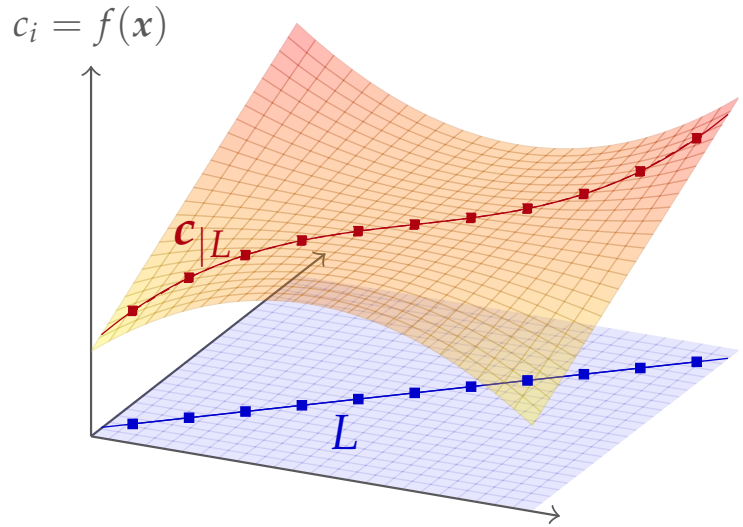

Local correction of $\boldsymbol{y} \in \mathbb{F}_q^{\mathbb{F}_q^m}$ at coordinate $\boldsymbol{i} \in \mathbb{F}_q^m$:

1. Pick at random a line $L \subset \mathbb{F}_q^m$ such that $\boldsymbol{i} \in L$.
2. Correct $\boldsymbol{y}_{|L}$ as a noisy $\mathrm{RS}_q(r)$ codeword, and output $\tilde{y}_{\boldsymbol{i}}$.

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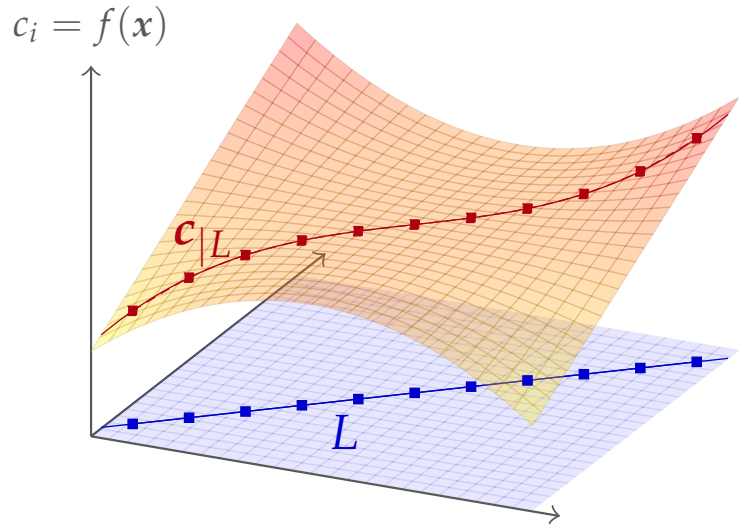

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$$\begin{aligned}(f \circ \phi)(T) &= (aT + b)^2(cT + d)^2 \\ &= (a^2T^2 + b^2)(c^2T^2 + d^2) \\ &= (ac)^2T^4 + (ad + bc)^2T^2 + (bd)^2 \\ &= (ad + bc)^2T^2 + (ac)^2T + (bd)^2 \mod (T^4 - T)\end{aligned}$$

$\Rightarrow$ for every ϕ, the "restriction" $(f \circ \phi)(T)$ can be interpolated as a univariate polynomial of degree 2

High-rate construction: lifted codes (2)

- $\mathbb{A}^m := \mathbb{F}_q^m$ $\qquad\qquad$ $\mathrm{ev}_{\mathbb{A}^m}(f) := (f(\boldsymbol{x}) : \boldsymbol{x} \in \mathbb{F}_q^m) \in \mathbb{F}_q^{\mathbb{A}^m}$
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Definition (lifted Reed-Solomon code [GKS13] reformulated).

$$\mathrm{Lift}(\mathrm{RS}_q(r), m) := \{\mathrm{ev}_{\mathbb{A}^m}(f), f \in \mathbb{F}_q[\boldsymbol{X}] \mid \forall \phi \in \mathrm{Aff}(m), \mathrm{ev}_{\mathbb{A}^1}(f \circ \phi) \in \mathrm{RS}_q(r)\}$$

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What about the dimension/rate?

Theorem (characteristic 2) [GKS13]. For every $m \geq 2$ and $0 < \alpha < 1$, there exists $0 < \gamma < 1$ and a prime power $q > 0$ such that $\mathrm{Lift}(\mathrm{RS}_q((1-\gamma)q), m)$ is locally correctable with $\ell = n^{1/m}$, $\varepsilon = \Theta_{m,\alpha}(\delta)$, and has rate

$$R \geq 1 - \alpha\,.$$

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For $m = 2$, $q = 2^e$ and $r = (1 - 2^{-c})q - 1$,

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Theorem [characteristic p, asymptotic length $n = p^{2e}$, $e \to \infty$].
For $m = 2$, $q = p^e \to \infty$ and $r = (1 - p^{-c})q - 1 \to \infty$,

$$R_{(e\to\infty)} = 1 - \left(1 + \frac{1}{p+2}\right)\left(\frac{1 + 1/p}{2}\right)^c + \frac{1}{p+2}\left(\frac{1}{p^2}\right)^c.$$

Degree sets

Lifted codes are **monomial**, *i.e.* generated by evaluations of monomials

$$\mathrm{ev}_{\mathbb{A}^m}(X_1^{d_1} \dots X_m^{d_m}) = \mathrm{ev}_{\mathbb{A}^m}(\boldsymbol{X}^{\boldsymbol{d}})$$

Degree set of a monomial code:

$$\mathrm{Deg}(\mathcal{C}) := \{\boldsymbol{d} \in [0, q-1]^m, \mathrm{ev}_{\mathbb{A}^m}(\boldsymbol{X}^{\boldsymbol{d}}) \in \mathcal{C}\}$$

Example for $\mathcal{C} = \mathrm{RM}(m, r)$:

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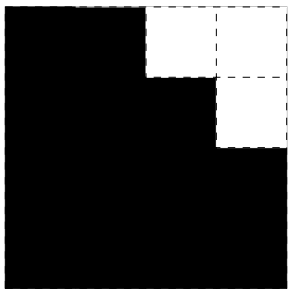
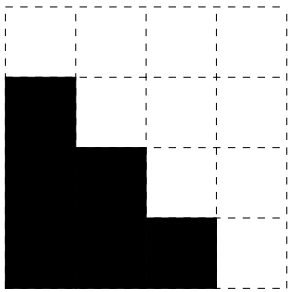
A **representation** for $m = 2$:



$\mathrm{RM}_4(2,4)$



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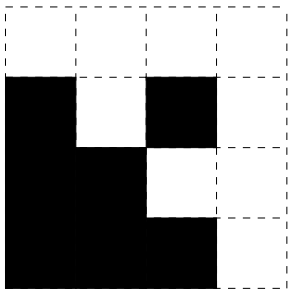
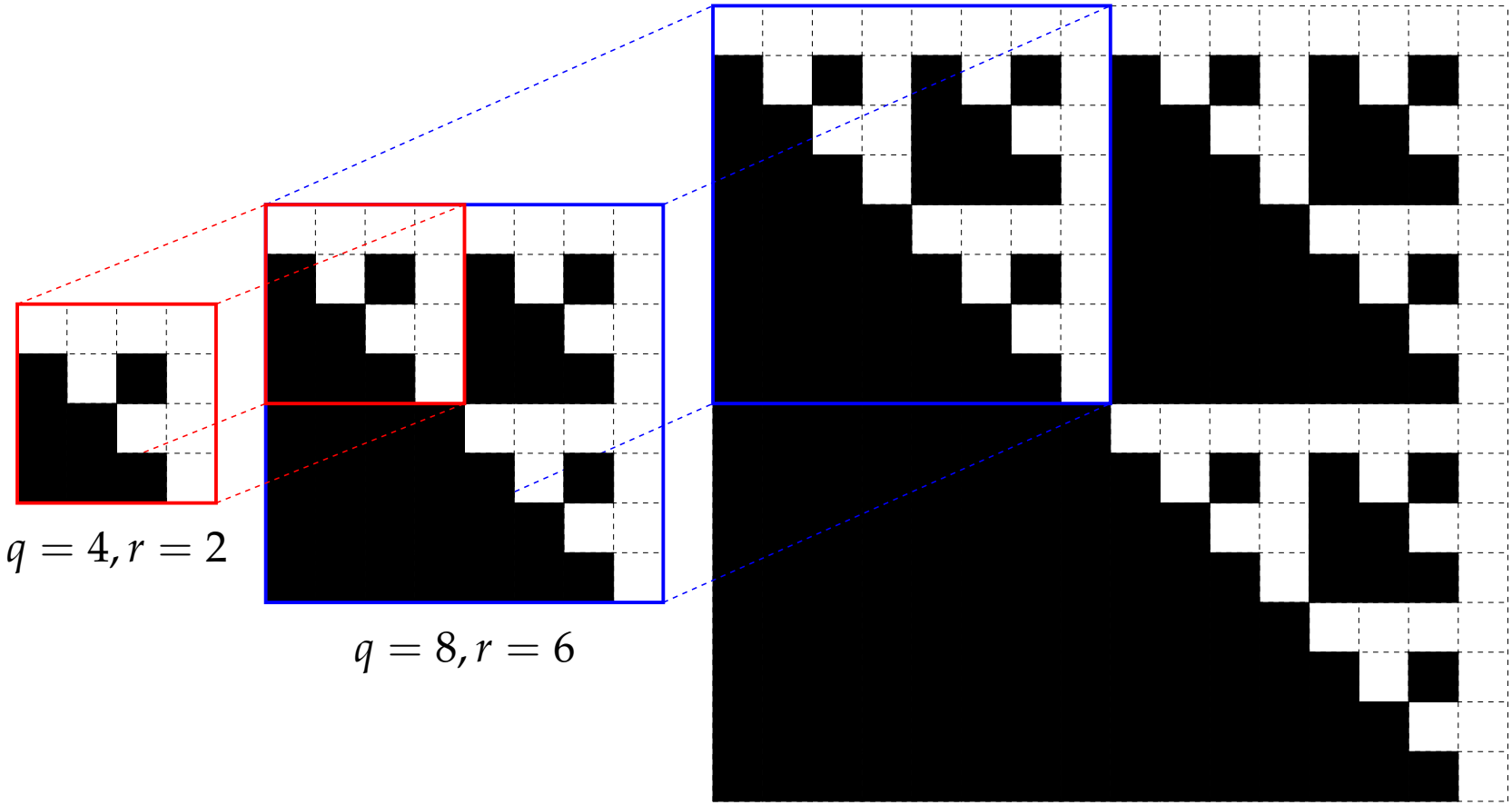


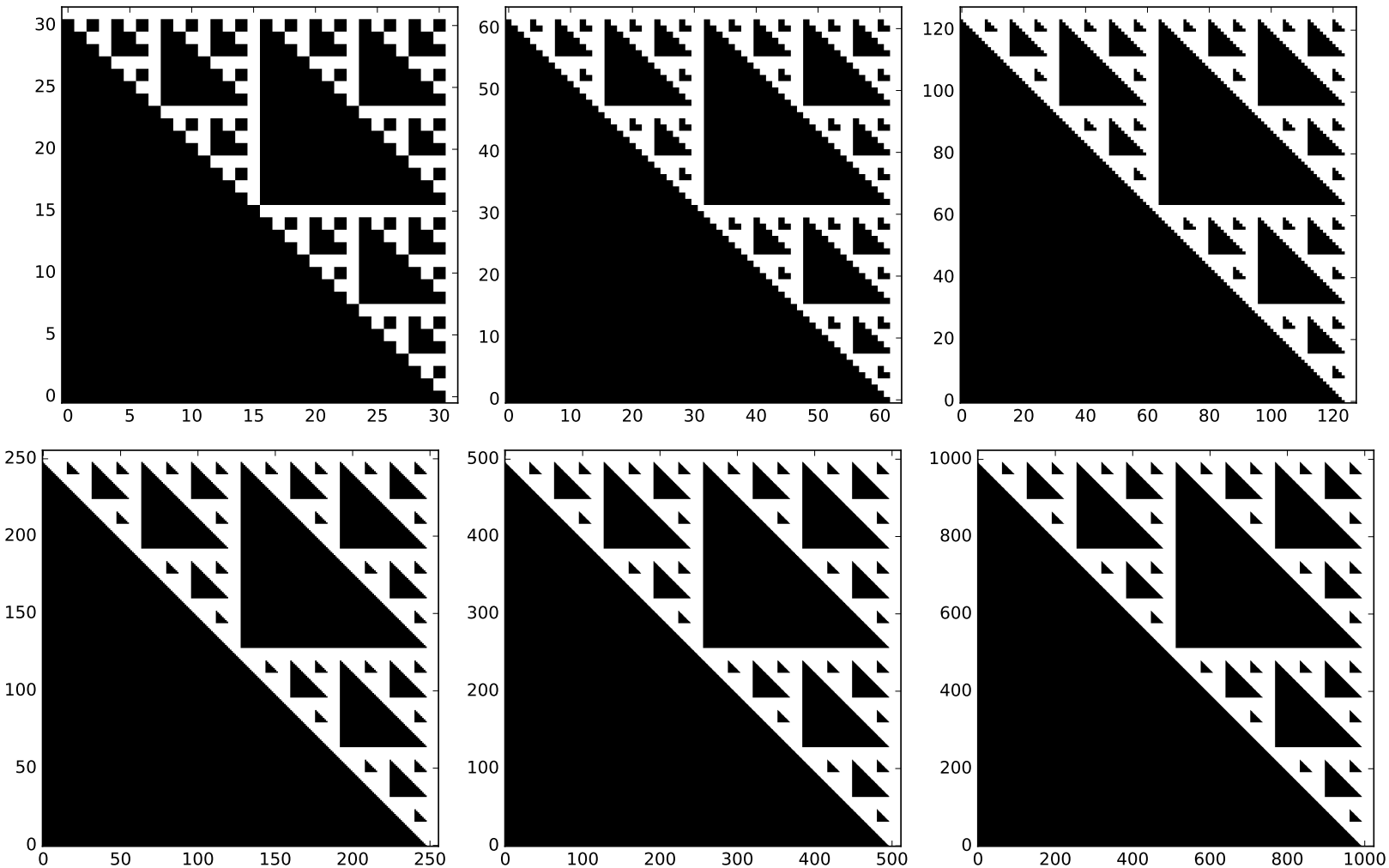
$\mathrm{Lift}(\mathrm{RS}_4(2),2)$

“Fractal” representation of degree sets (1)



“Fractal” representation of degree sets (2)

Degree set of $\mathrm{Lift}(\mathrm{RS}_{2^e}((1-2^{-c})2^e-1),2)$ for fixed $c=5$ and increasing $e \geq 5$.



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Motivation

Why would we consider lifted codes over **projective spaces**?

- projective versions of Reed-Solomon and Reed-Muller codes **already exist**
- lifted projective RS codes would have slightly **larger length**
- relations between affine and projective RM codes via **puncturing** and **shortening**, *e.g.*

$$0 \to \mathrm{RM}_q(m, k-1) \to \mathrm{PRM}_q(m, k) \xrightarrow{\pi} \mathrm{PRM}_q(m-1, k) \to 0\,.$$

where π is the restriction map $\mathbb{P}^m \to \mathbb{P}^{m-1}$

Projective spaces

Projective space:

$$\mathbb{P}^m := \mathbb{A}^{m+1} / \sim$$

where $\boldsymbol{a} \sim \boldsymbol{b}$ iff $\exists \lambda \in \mathbb{F}_q^\times, \boldsymbol{a} = \lambda \boldsymbol{b}$

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We get:

$$f(\boldsymbol{u}) := f(0, \ldots, 0, 1, *, \ldots, *) \in \mathbb{F}_q$$

$$\mathrm{ev}_{\mathbb{P}^m}(f) := (f(\boldsymbol{u}) : \boldsymbol{u} \in \mathbb{P}^m) \in \mathbb{F}_q^{\mathbb{P}^m}$$

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Example. Projective Reed-Solomon code:

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Remarks:

- $\mathrm{ev}_{\mathbb{P}^1}(f \circ \phi) \neq \mathrm{ev}_{\mathbb{P}^m}(f)_{|\phi(\mathbb{P}^1)}$ due to the choice of representative
- fortunately $\mathrm{ev}_{\mathbb{P}^1}(f \circ \phi) = \boldsymbol{w} \star \mathrm{ev}_{\mathbb{P}^m}(f)_{|\phi(\mathbb{P}^1)}$, and $\boldsymbol{w}$ is **independent of** f.

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Details in:
Lifted Projective Reed-Solomon Codes, L., DCC, to appear
10.1007/s10623-018-0552-8

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Lifted codes when $r = q - 2$

Remark. Assume $r = q - 2$. Then,

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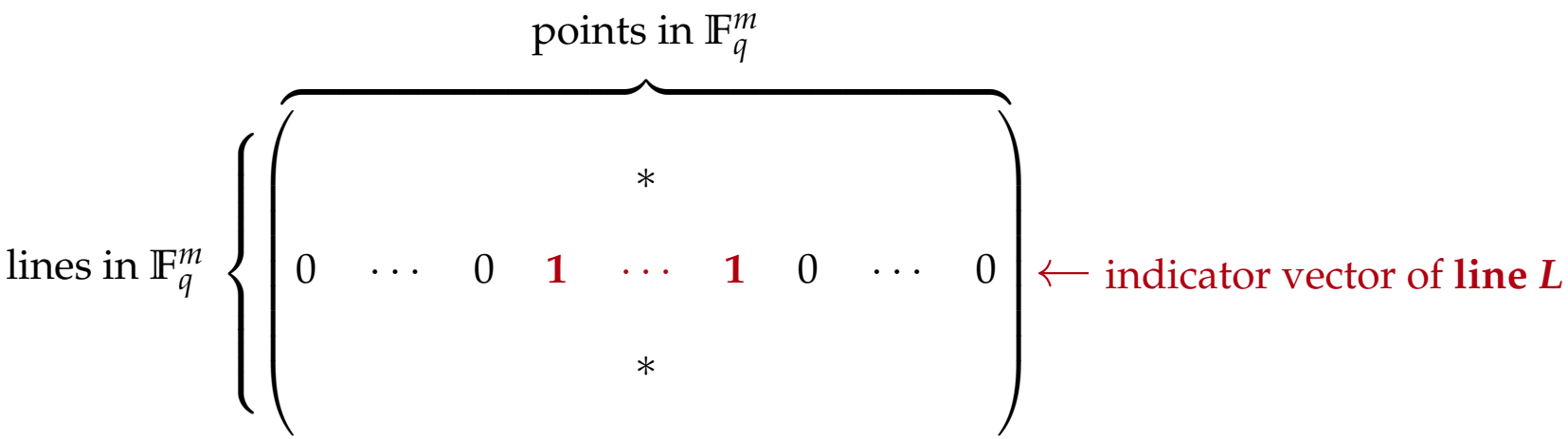
Parity-check matrix for $\mathrm{Lift}(\mathrm{RS}_q(q-2), m)$:



Block designs

Point-line incidences in the affine space form a 2**-design.**

Definition. A *t-design* of parameters (v, k, λ) consists in:

- a set X of points, $|X| = v$,
- a set $\mathcal{B}$ of blocks $B \subset X$, $|B| = k$

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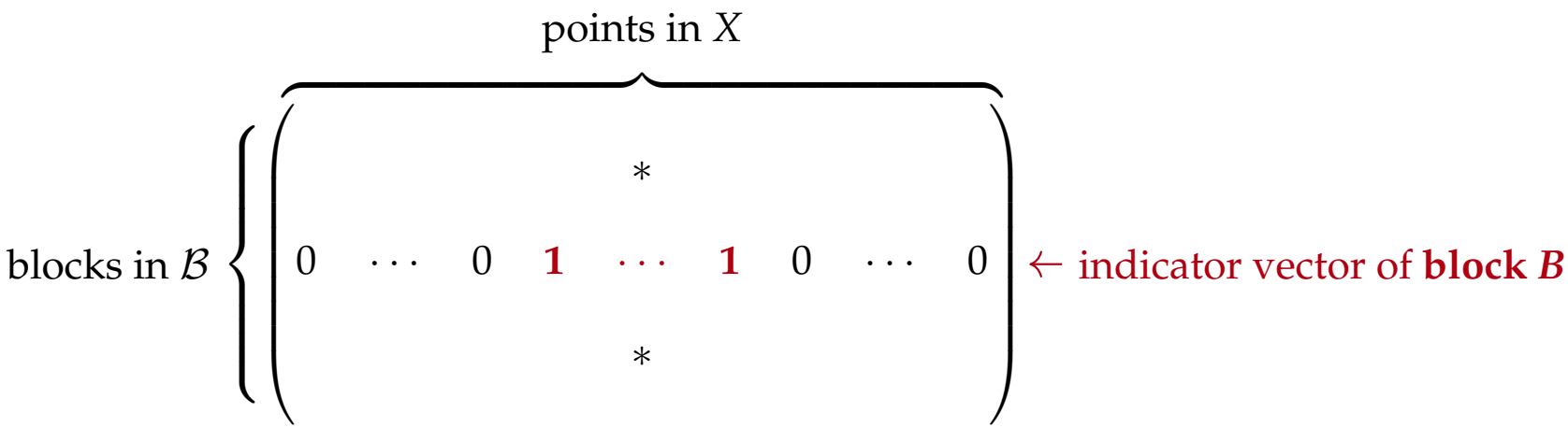
Incidence matrix of a design:



Codes based on designs, and generalisation

The **code based on the design** $\mathcal{D} = (X, \mathcal{B})$ is the code $\mathcal{C} = \mathrm{Code}(\mathcal{D}) \subseteq \mathbb{F}_q^X$ admitting the incidence matrix of $\mathcal{D}$ as a parity-check matrix.

$$\mathrm{Code}(\mathcal{D}) = \{\boldsymbol{c} \in \mathbb{F}_q^X \mid \forall B \in \mathcal{B}, \sum_{x \in B} c_x = 0\}$$

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Let $\mathcal{L} = (\mathcal{L}_B : B \in \mathcal{B})$ be a family of codes indexed by blocks $B \in \mathcal{B}$. The **generalised design-based code**, based on $(\mathcal{D}, \mathcal{L})$ is

$$\mathrm{Code}(\mathcal{D}, \mathcal{L}) := \{\boldsymbol{c} \in \mathbb{F}_q^X \mid \forall B \in \mathcal{B}, \boldsymbol{c}_{|B} \in \mathcal{L}_B\}.$$

Remark. $\mathrm{Code}(\mathcal{D}, \mathcal{L}) = \mathrm{Code}(\mathcal{D})$ if every code in $\mathcal{L}$ is a parity-check code.

Design-based codes and LCCs

- $\mathcal{D}$ be a t-$(n, \ell+1, \lambda)$-design
- $0 < \tau < 1$
- $\mathcal{L} = (\mathcal{L}_B : B \in \mathcal{B})$ s.t. every code in $\mathcal{L}$ corrects $\lfloor \tau\ell \rfloor$ errors and 1 erasure.

Algorithm. Local correction of $\boldsymbol{y} \in \mathbb{F}_q^X$ at $i \in X$

- Pick uniformly at random a block $B \in \mathcal{B}$ such that $i \in X$.
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Proposition [$t = 3$]. For every $\delta < \tau - \sqrt{2/\ell}$, $\mathrm{Code}(\mathcal{D}, \mathcal{L})$ is a $(\ell, \delta, \varepsilon)$-LCC where

$$\varepsilon = \frac{\delta(1-\delta)}{(\tau-\delta)^2} \cdot \frac{1}{\ell}\,.$$

Future works

Design-based codes allow to get rid of probabilistic decoders in the definition of locally correctable codes

$\rightarrow$ "combinatorial" coding-theoretic version of LCCs

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Remaining issues:

- families of 3-designs with high dimension?
- best instances $(\mathcal{D}, \mathcal{L})$ prescribed design parameters (n, ℓ, λ)?

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Remark:

- PIR $\neq$ anonymity (hidden user)
- PIR $\neq$ encryption (hidden data)

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File F encoded and stored on ℓ servers $S_1, \ldots, S_\ell$.

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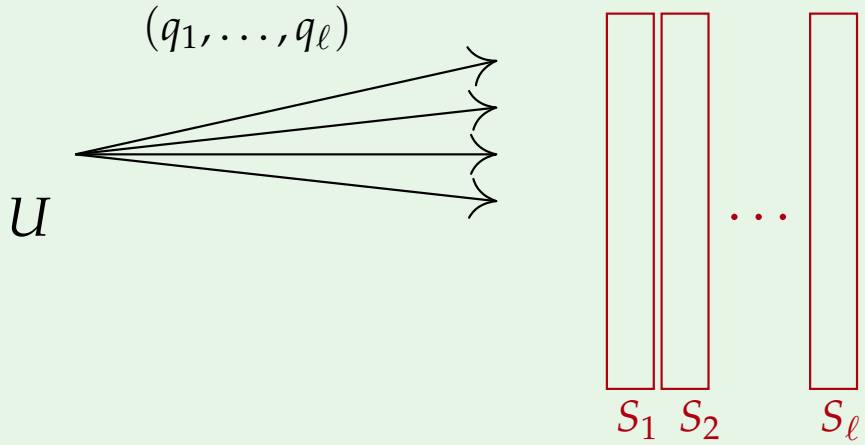

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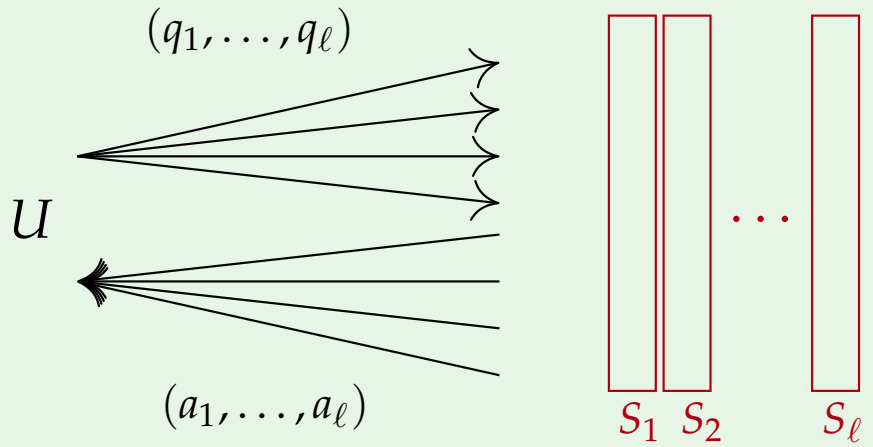

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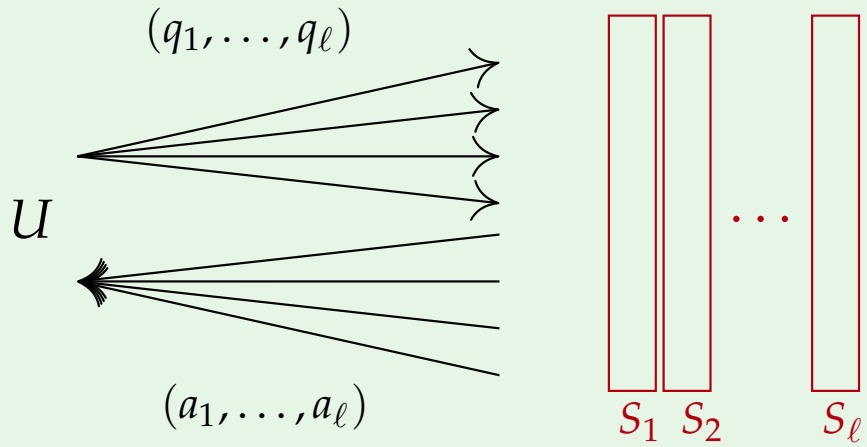

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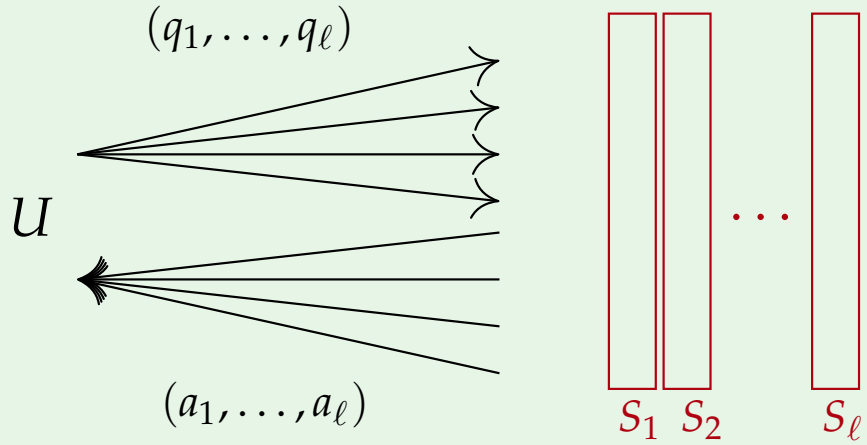

Information-theoretic privacy: $\mathrm{I}(i\,; q_j) = 0,\ \forall j = 1, \ldots, \ell.$

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Usual goals for PIR:

- Low communication complexity
- Low storage overhead for the servers (if coded)
- Low computation complexity for algorithms $\mathcal{A}$ (server) and $\mathcal{R}$ (user)

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Notion of **price of privacy** for the servers, mainly depends on:

- computational complexity for the servers,
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Motivation

Usual goals for PIR:

- Low communication complexity
- Low storage overhead for the servers (if coded)
- Low computation complexity for algorithms $\mathcal{A}$ (server) and $\mathcal{R}$ (user)

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Yekhanin (survey, '12): "the **overwhelming computational complexity** of PIR schemes (...) currently presents the **main bottleneck** to their practical deployment".

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   - Locality in coding theory, examples
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2. Private information retrieval from transversal designs
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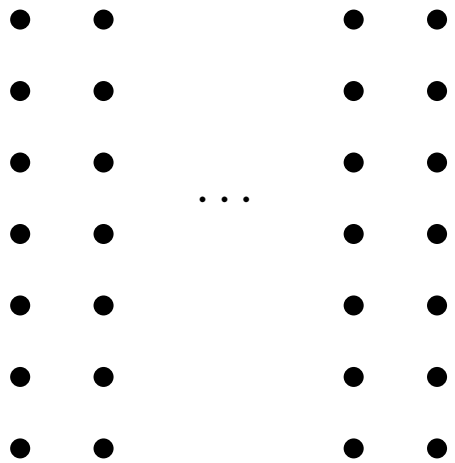
Transversal designs

A **transversal design** $\mathrm{TD}(\ell, s) = (X, \mathcal{B}, \mathcal{G})$ is given by:

- X a set of *points*, $|X| = n = s\ell$,



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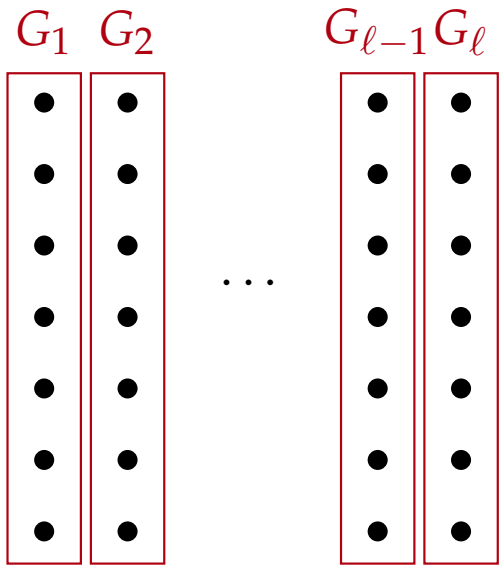

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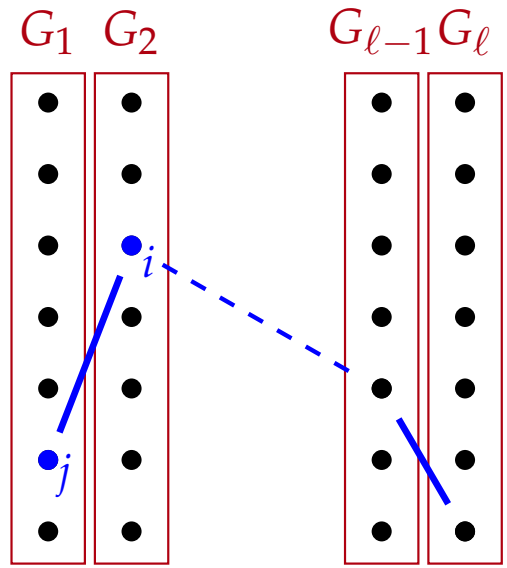

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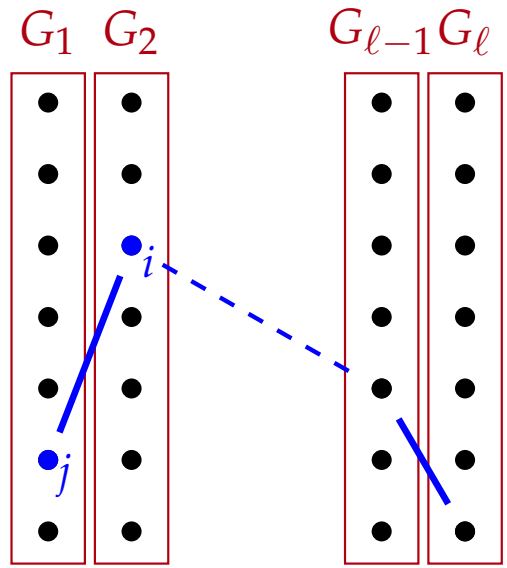
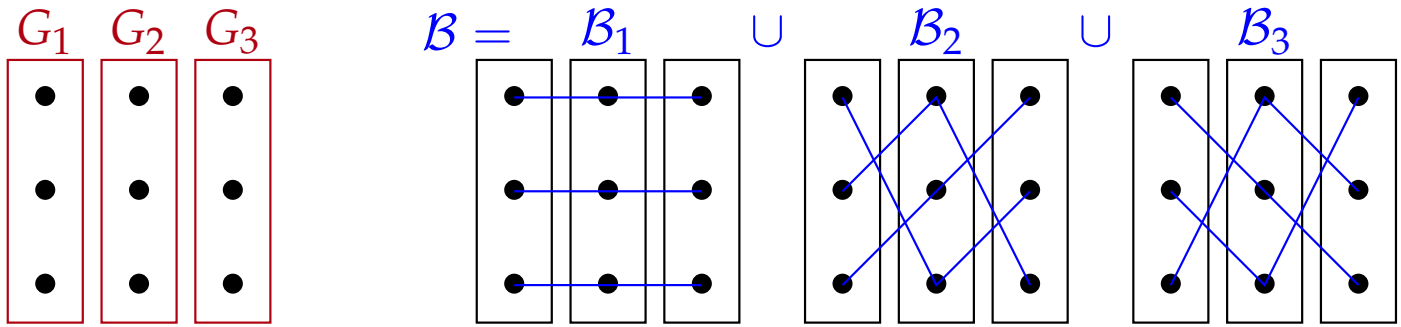


Its incidence matrix (between points and blocks) defines a code.

Example

The transversal design TD$(3,3)$ represented by:



gives an incidence matrix

$$H = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Its rank over $\mathbb{F}_3$ is 6 $\quad\Longrightarrow\quad$ the associated code $\mathcal{C}$ is a $[9,3]_3$ code.

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Let $\mathcal{C} \subseteq \mathbb{F}_q^n$ be a code based on a TD(ℓ, s).

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- **Initialisation.** User U encodes $F \mapsto \boldsymbol{c} \in \mathcal{C}$, and gives $\boldsymbol{c}_{|G_j}$ to server S_j.
- **To recover** $F_i = c_i$**:**
  1. User U randomly picks a block $B \in \mathcal{B}$ containing i. Then U defines:

$$q_j = \mathcal{Q}(i)_j := \begin{cases} \text{unique } \in B \cap G_j & \text{if } i \notin G_j \\ \text{a random point in } G_j & \text{otherwise.} \end{cases}$$

  2. each server S_j sends back $a_j = \mathcal{A}_j(q_j, \boldsymbol{c}_{|G_j}) := c_{q_j}$
  3. U recovers

$$-\sum_{j:\, i \notin G_j} c_{q_j} = -\sum_{b \in B \setminus \{i\}} c_b = c_i$$

Privacy and parameters

Theorem. If the servers do not collude, then our PIR protocol is information-theoretically private.

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- communication complexity: $\ell(\log s + \log q)$ bits
- computational complexity:
  - $\mathcal{O}(1)$ **for the response algorithm** $\mathcal{A}$ (somewhat optimal)
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Question: Transversal designs with good k depending on (ℓ, s)?

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Instances with geometric designs

$\mathcal{T}_A$, the **classical affine transversal design**:

- $X = \mathbb{F}_q^m$, $m \geq 2$,
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Question: are they the best instances?

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An **orthogonal array** $\mathrm{OA}(t, \ell, s)$ of strength t may be seen as a list A of codewords

- over a finite set S, $|S| = s$,
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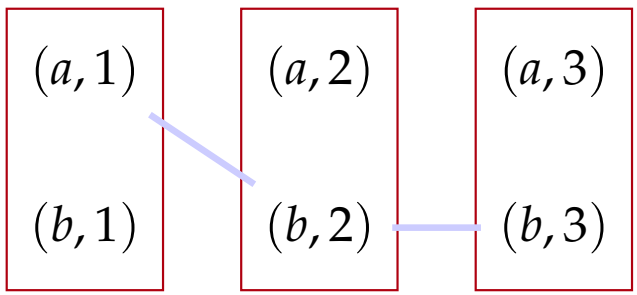

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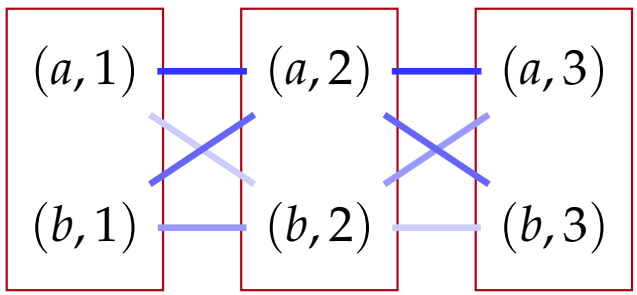

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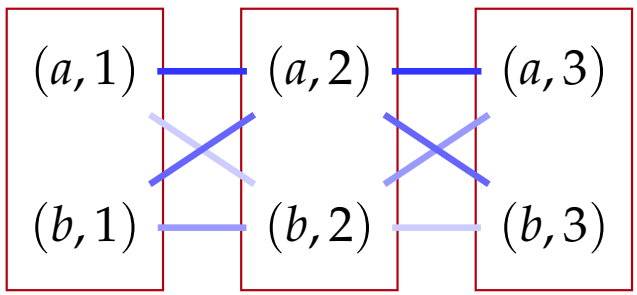

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Details in:

Private Information Retrieval from Transversal Designs, L., IEEE TIT, to appear

Conclusion

- Codes with local properties gained interest
  - theoretically: PCP theorem, etc.
  - in practice: storage of large files on distributed storage systems or p2p networks
  - more recently STARKs, etc.
- A combinatorial point of view (through designs) could help their analysis
- Cryptographic applications: private information retrieval (PIR), proofs of retrievability (PoR)