On the construction of PIR schemes

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Outline

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Problem statement

Private information retrieval (PIR):

Given a **remote** database $F \in \Sigma^M$ and $i \in [1, M]$,
can we **retrieve** the entry/file F_i,
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Trivial solution: full download.

Definition of PIR

Introduced in:

Private Information Retrieval. Chor, Goldreich, Kushilevitz, Sudan. FOCS. **1995**.

Database F stored (in some way) on n servers $S_1, \ldots, S_n$,
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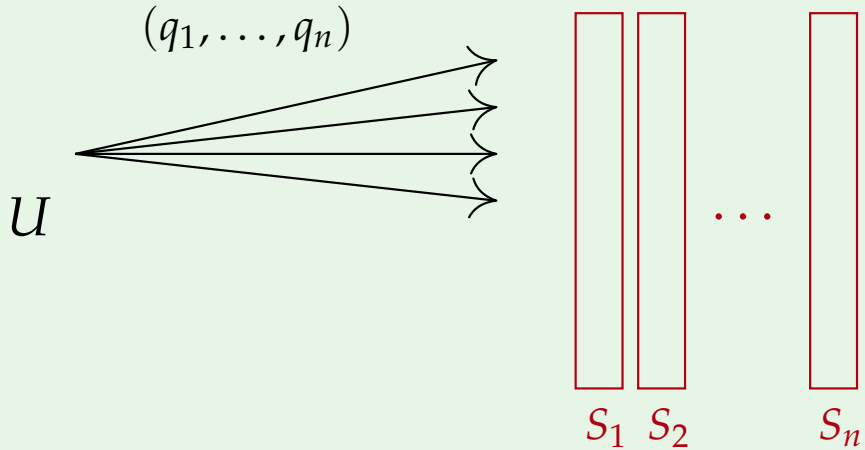

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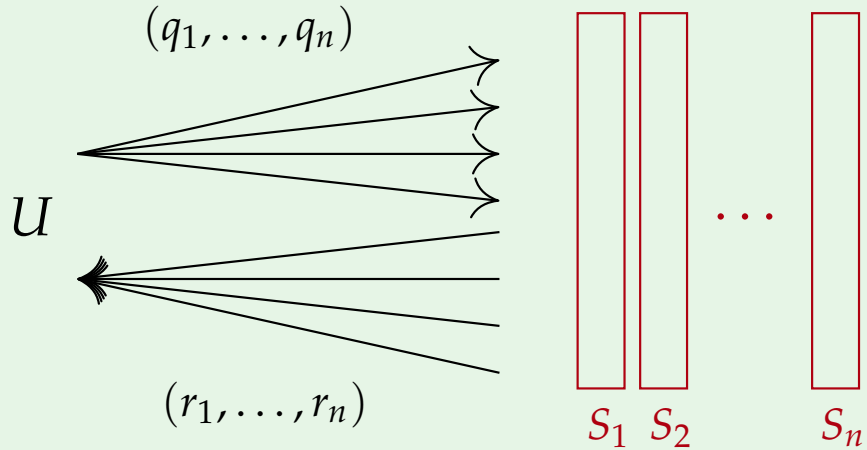

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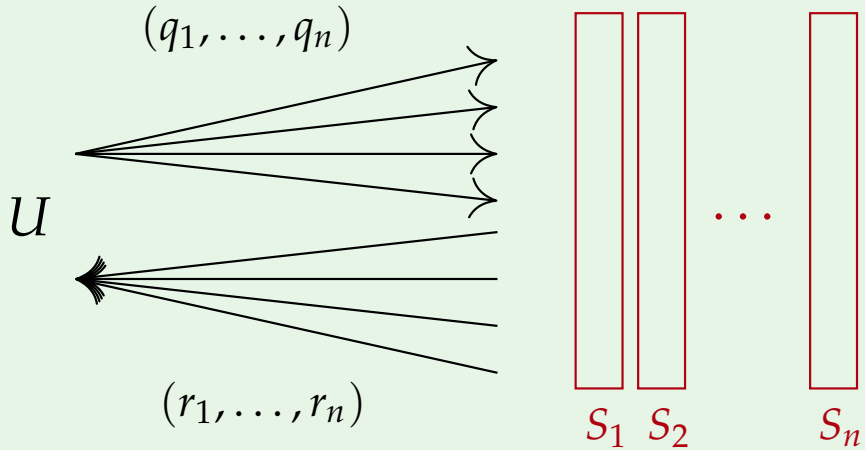
3. U recovers $F_i = \mathcal{R}(\boldsymbol{q}, \boldsymbol{r}, i)$



Privacy

A **collusion of servers**: set of servers $\{S_j : j \in T\}$, where $T \subset [1, n]$, which exchange information about queries, data, etc.

$$t := \max\{|T|, T \subseteq [1, n] \text{ is a collusion}\} \geq 1$$

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- **Information-theoretic privacy:**

$$\mathrm{I}(i; \boldsymbol{q}_{|T}) = 0, \quad \forall T \subseteq [1, n], |T| \leq t.$$

- **Computational privacy:** by varying the index i, distributions of queries $\boldsymbol{q}_{|T} = \mathcal{Q}(i)_{|T}$ are computationally indistinguishable.

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Theorem [CGKS95, CG97]. If $t = n$ (in particular if $n = 1$), then:

- for IT-privacy, **no better solution than full download**,
- computational privacy is possible (but remains **expensive** as of now).

Main parameters of PIR schemes

We focus on **IT-privacy**
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Parameters to be taken into account:

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Several possible **settings**:

- bounded vs. unbounded number of entries in the database
- replicated database vs. coded database
- small entries vs. large entries
- dynamic database vs. static database
- unresponsive or byzantine servers

Seminal work [CGKS'95-98]

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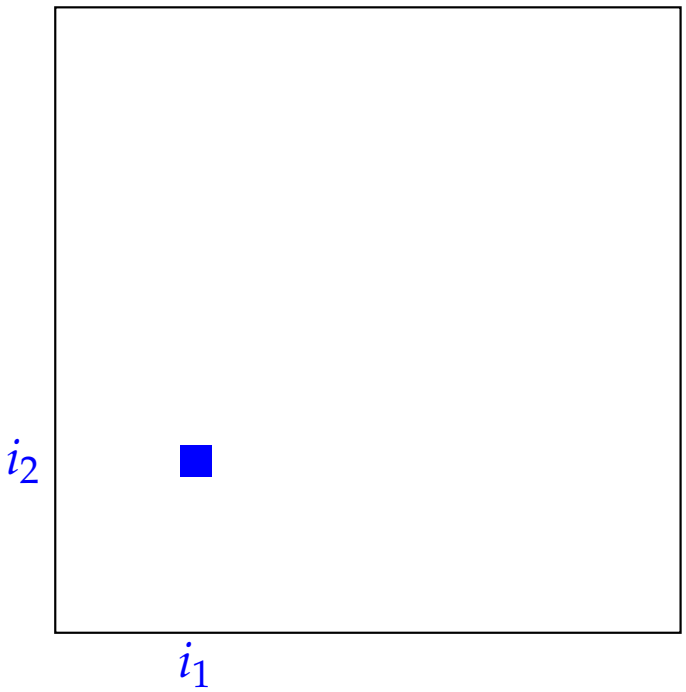
Settings:

- $|F| = M$ bits, with $M = L^2$, and $[1, M] \simeq [1, L]^2$.
- $n = 4$ servers $S_{00}, S_{01}, S_{10}, S_{11}$, each storing a replica of F.
- **Goal:** retrieve $F_i = F_{(i_1,i_2)}$, for $1 \leq i_1, i_2 \leq L$.



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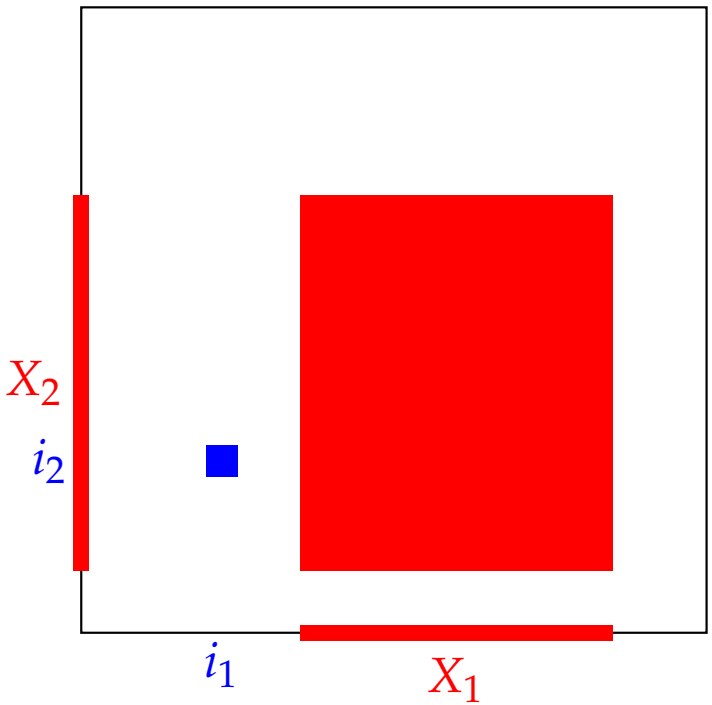

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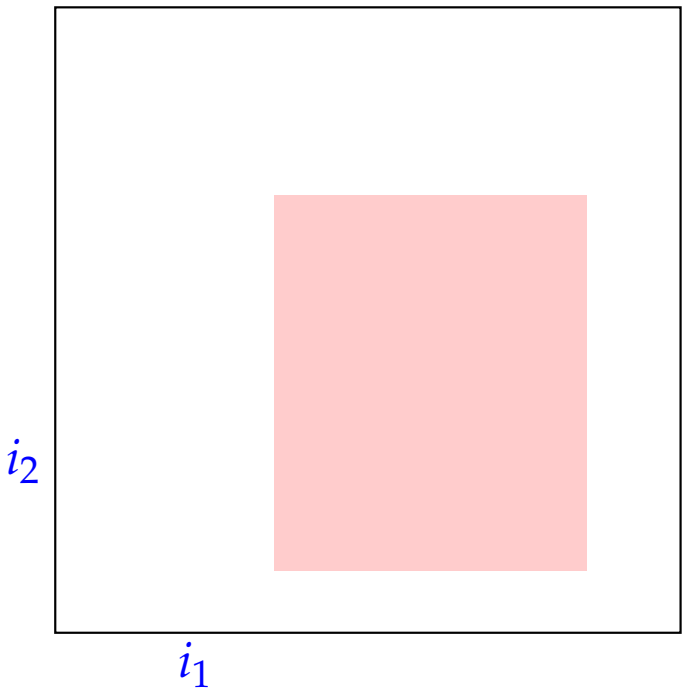

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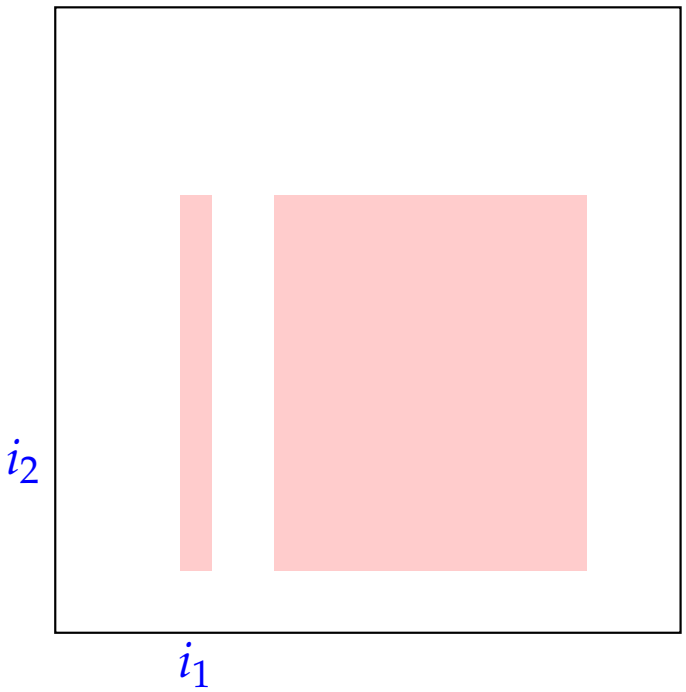

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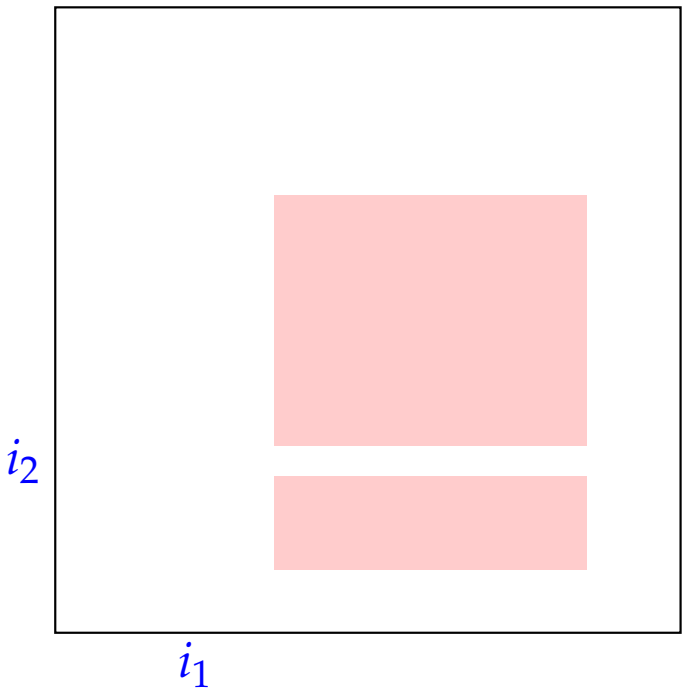

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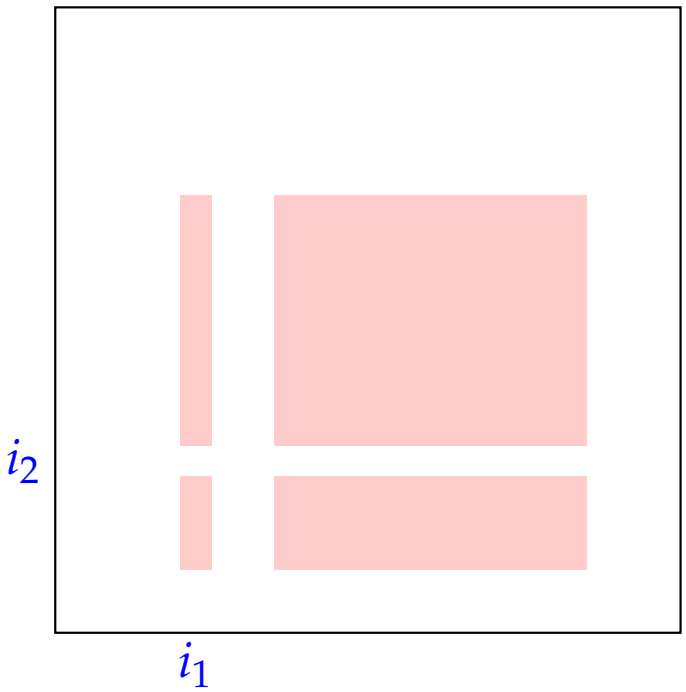

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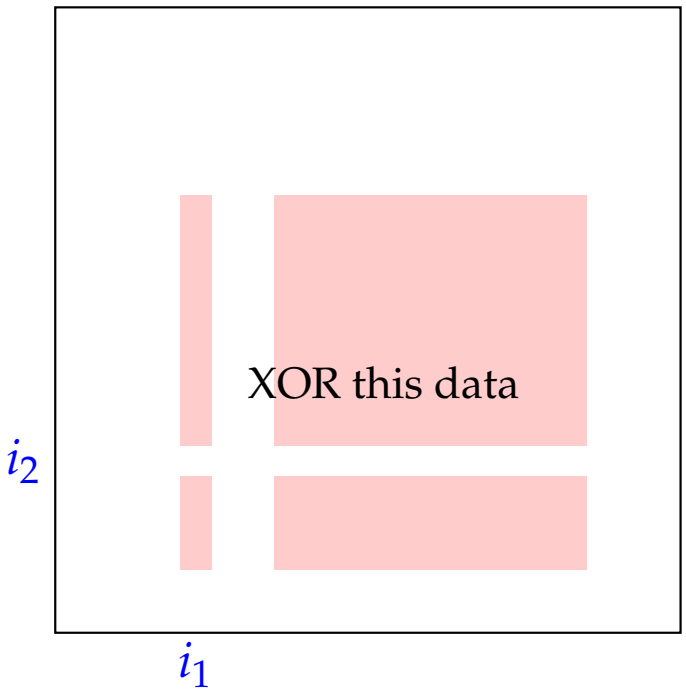

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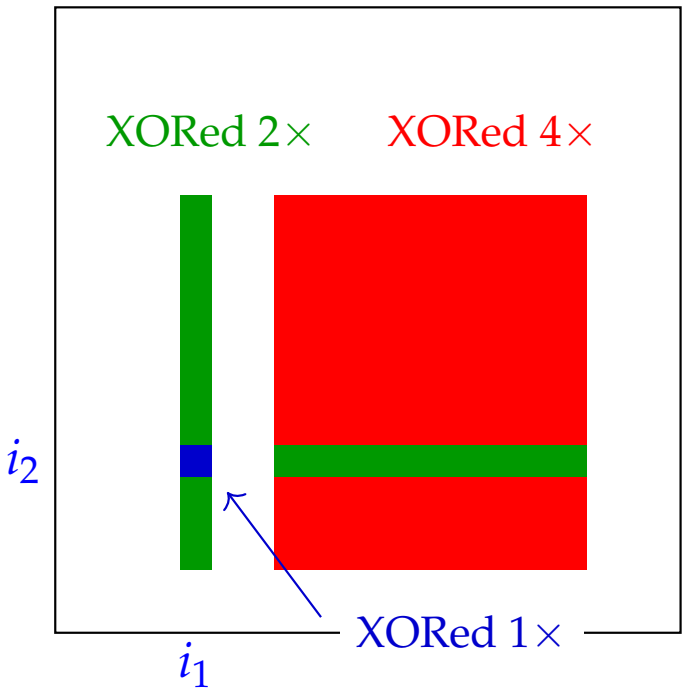

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3. User XORs the 4 bits and retrieves F_i.

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Correct, and **secure** if no collusion.

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Generalisable to $n = 2^b$ servers:

- **Communication:** $b2^b M^{1/b} = n \log(n) M^{1/\log(n)}$ uploaded bits, n downloaded bits,
- **Storage:** replication of F over n servers,
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(Short) state of the art

- 1995: first definition [CGKS95]
- 2000: reduction from smooth locally decodable codes [KT00]
- 2000-10's: many improvements
  - PIR with 3 servers and subpolynomial communication [Yek08, Efr09]
  - PIR with 2 servers and subpolynomial communication [DG16]
  - lower storage overhead with *PIR codes* [FVY15]
- 2016-now: capacity-achieving schemes, schemes dedicated to storage systems
  - capacity of PIR [SJ17, BU18]
  - (nearly) capacity-achieving schemes [SRR14, CHY15, TR16, ...]

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Storage systems use codes to cope with node failures.

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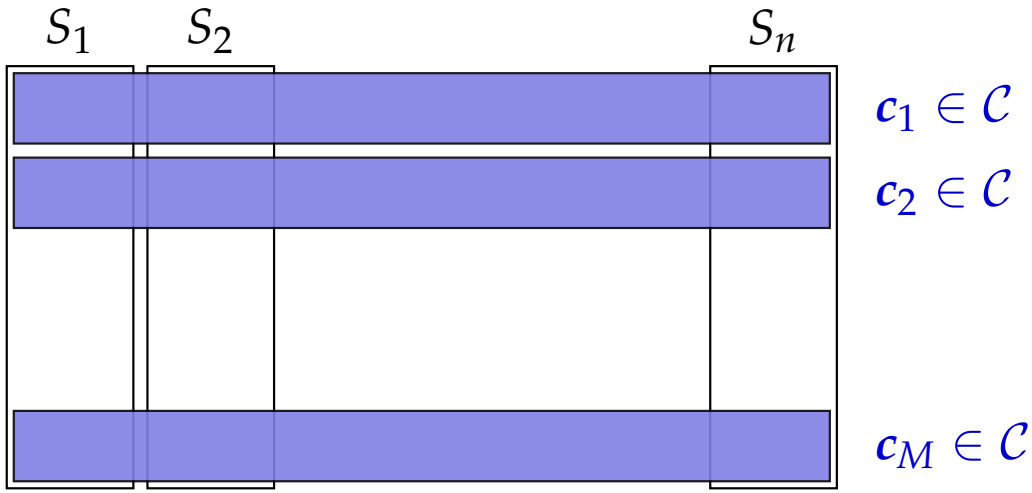
Given a code $\mathcal{C}$ of length n:



Example: Reed-Solomon storage systems

Definition (Reed-Solomon code). Let $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{F}_q^n$, pairwise distinct.

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- every codeword $\boldsymbol{c} \in \mathcal{C}$ can be reconstructed from any k-subset of coordinates of $\boldsymbol{c}$,
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File storage:

$$\text{a file} \quad F_i \in \Sigma \simeq \mathbb{F}_{q^s}^k \quad \text{is encoded into} \quad \boldsymbol{c}_i \in \mathrm{RS}_q(k,n) \otimes \mathbb{F}_{q^s}$$

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Main assumption (can be discussed):

$$s \gg M$$

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Usual goal (assuming $s \gg M$): a large *PIR rate*

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Next, we present a PIR scheme for RS-coded databases.

- Originally [TR16], then extended and reformulated [TGKFH18, TGR18].
- Scalable.
- Optimal PIR rate for $t = 1$ and $M \to \infty$.
- PIR rate conjectured optimal for $M \to \infty$.

[TR16] *PIR from MDS Coded Data in Distributed Storage Systems*. Tajeddine, El Rouayheb. ISIT. **2016**.
[TGKFH18] *Robust PIR from Coded Systems with Byzantine and Colluding Servers*. Tajeddine, Gnilke, Karpuk, Freij-Hollanti, Hollanti. ISIT. **2018**.
[TGR18] *PIR from MDS Coded Data in Distributed Storage Systems*. Tajeddine, Gnilke, El Rouayheb. IEEE-TIT. **2018**.

The protocol: query generation

Notation:

$$\boldsymbol{a} \star \boldsymbol{b} := (a_1 b_1, \ldots, a_n b_n)$$
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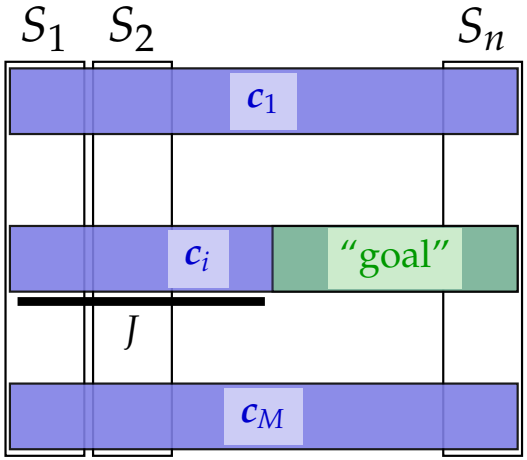

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$\mathcal{C} \subseteq \mathbb{F}_q^n$ the *storage code*, $\boldsymbol{C} \in \mathcal{C}^M$ the coded database

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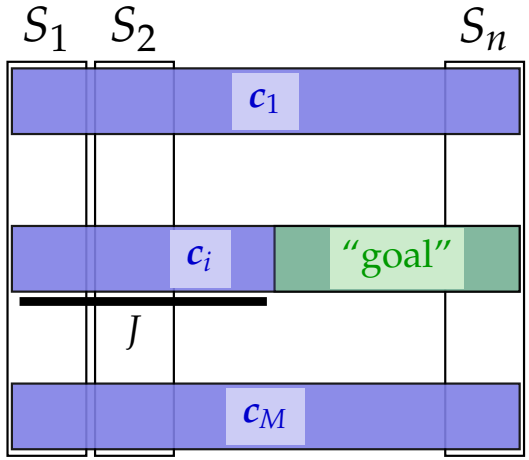

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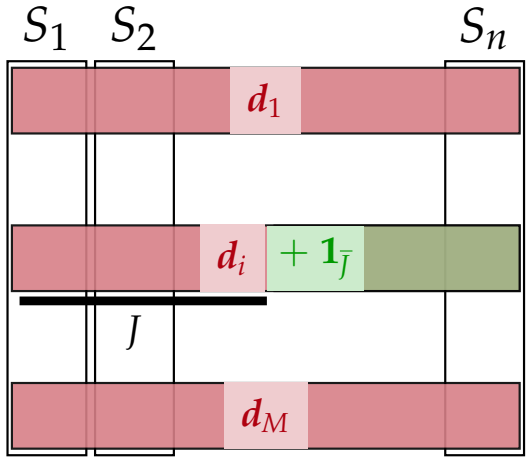
Queries:

1. the user generates at random M words $\boldsymbol{d}_1, \ldots, \boldsymbol{d}_M \in \mathcal{D}$ and defines $\boldsymbol{Q}$ as follows:
2. the j-th column of $\boldsymbol{Q}$ is sent to server S_j



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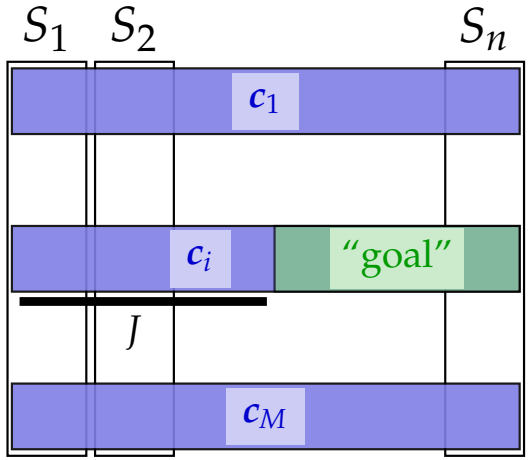
$$\mathcal{C} \star \mathcal{C}' := \langle \{ \boldsymbol{c} \star \boldsymbol{c}' \mid \boldsymbol{c} \in \mathcal{C}, \boldsymbol{c}' \in \mathcal{C}' \} \rangle$$



System parameters:

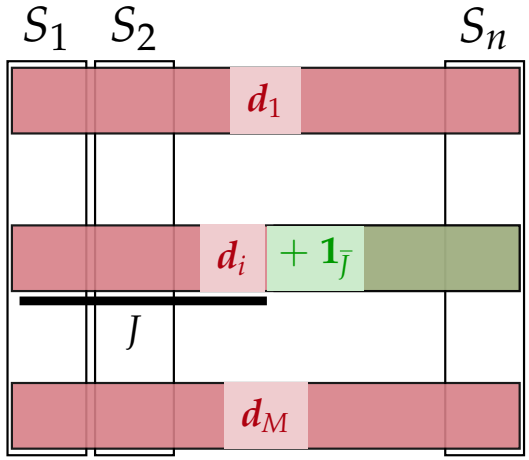
$\mathcal{C} \subseteq \mathbb{F}_q^n$ the *storage code*, $\boldsymbol{C} \in \mathcal{C}^M$ the coded database

$\mathcal{D} \subseteq \mathbb{F}_q^n$ a *query code* of dual distance $d^\perp(\mathcal{D}) = t + 1$

$J \subseteq [1, n]$ an information set for $\mathcal{C} \star \mathcal{D}$, and $\bar{J} := [1, n] \setminus J$

Queries:

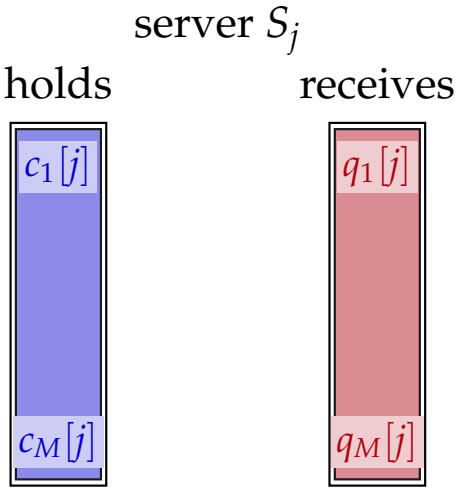
1. the user generates at random M words $\boldsymbol{d}_1, \ldots, \boldsymbol{d}_M \in \mathcal{D}$ and defines $\boldsymbol{Q}$ as follows:
2. the j-th column of $\boldsymbol{Q}$ is sent to server S_j



Remark: queries remain private against collusions of servers of size $\leq t$.

The protocol: server answers and reconstruction

Server answers: server S_j receives as a query a column $\boldsymbol{Q}^{(j)} \in \mathbb{F}_q^M$ of $\boldsymbol{Q}$,



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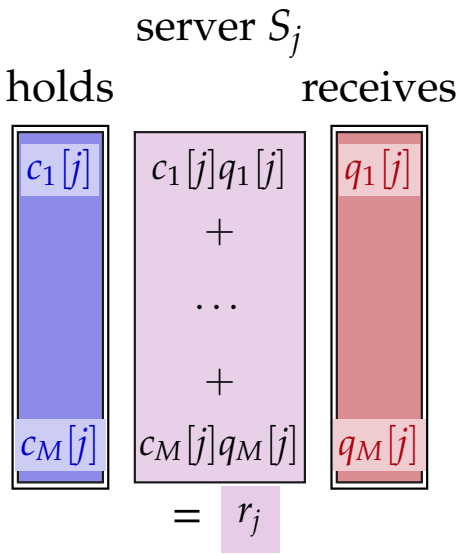
$$r_j = \langle \boldsymbol{Q}^{(j)}, \boldsymbol{C}^{(j)} \rangle \in \mathbb{F}_q.$$



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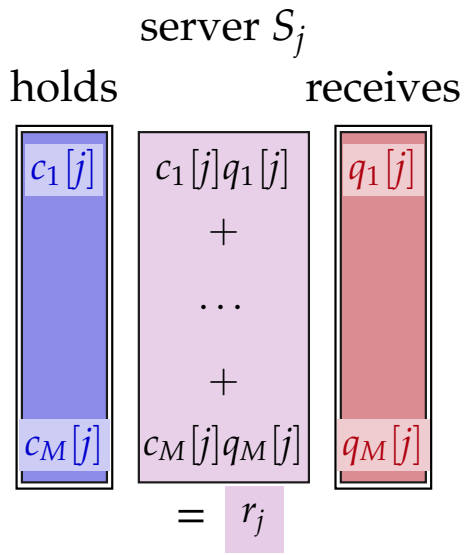
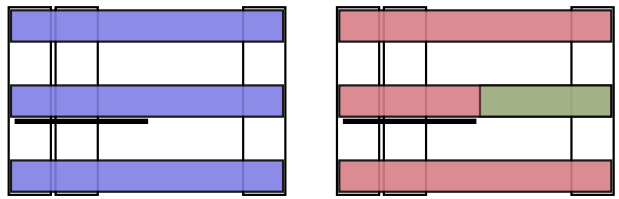


Reconstruction:



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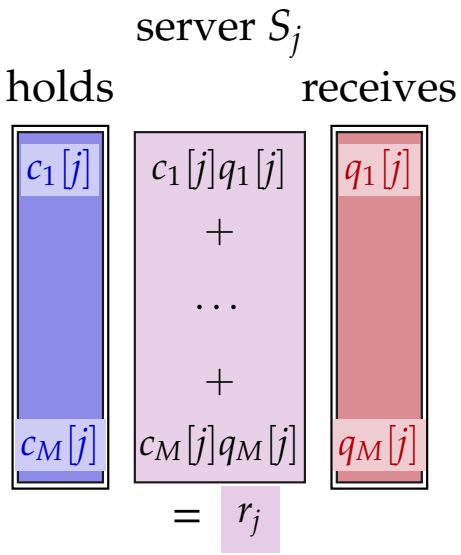
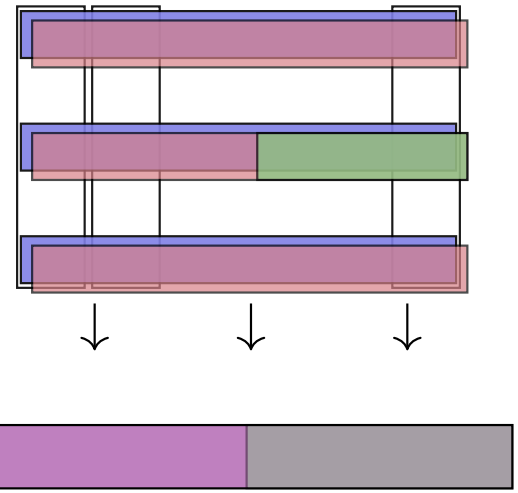


Reconstruction: The user collects

$$\boldsymbol{r} = (r_1, \ldots, r_n) = \underbrace{\sum_{m=1}^{M} \boldsymbol{d}_m \star \boldsymbol{c}_m}_{\in \mathcal{C} \star \mathcal{D}} + \underbrace{\mathbf{1}_{\bar{J}} \star \boldsymbol{c}_i}_{= \boldsymbol{c}_i \text{ on } \bar{J}}$$



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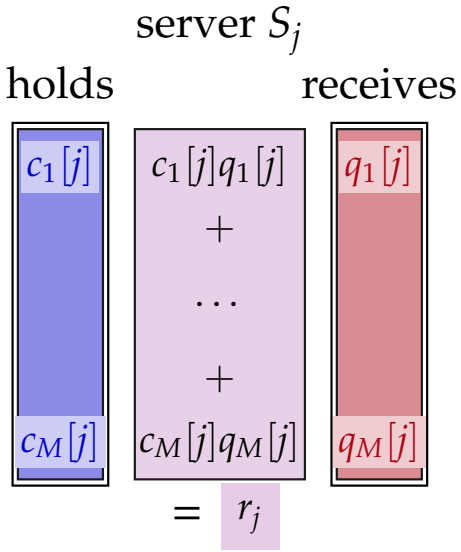

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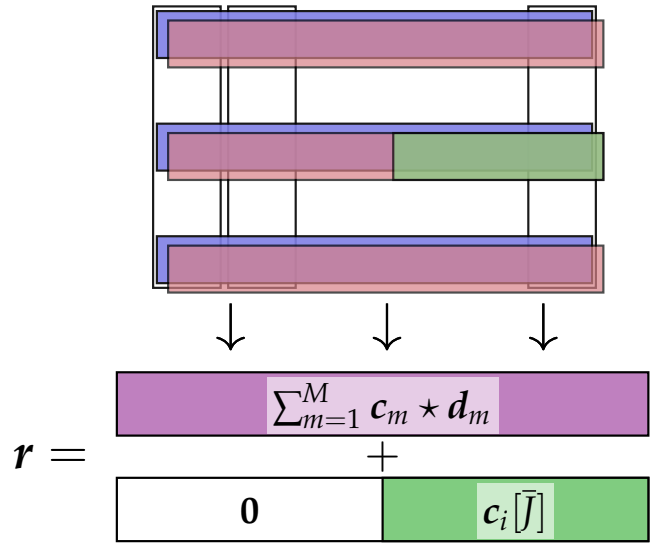
and interpolates on J to recover

- $\sum_{m=1}^{M} \boldsymbol{d}_m \star \boldsymbol{c}_m$,
- then $\boldsymbol{c}_i[|\bar{J}]$.



Analysis

Features for 1 run of the protocol.

- download cost: n symbols over $\mathbb{F}_{q^s}$
- upload cost: an $(M \times n)$-matrix over $\mathbb{F}_q$ (negligible if $s \gg M$)
- retrieval of $|\bar{J}| = n - \dim(\mathcal{C} \star \mathcal{D})$ symbols of the desired file
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Otherwise, **striping** methods allow to achieve the same PIR rate.

Outline

Context

Previous schemes:

- low communication complexity
- computationally inefficient (linear in $|F| = \sum_{m=1}^{M} |F_m|$)

Our goal:

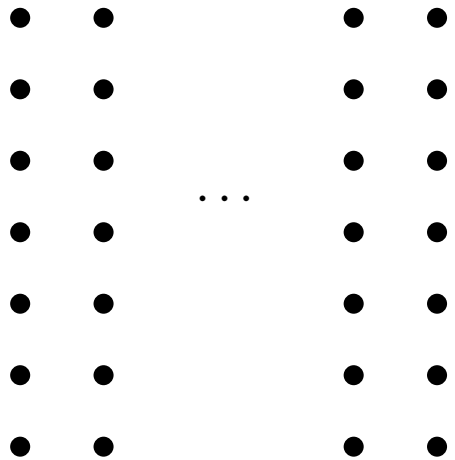
- optimal computation ($|r_j|$ for each server S_j)
- remove the assumption $s \gg M$
- moderate communication complexity

Outline

Transversal designs

A **transversal design** $\mathrm{TD}(n,s) = (X, \mathcal{B}, \mathcal{G})$ is given by:

- X a set of *points*, $|X| = N = ns$,



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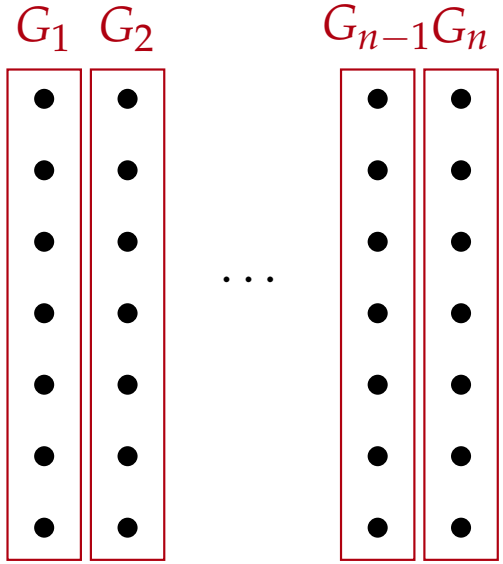

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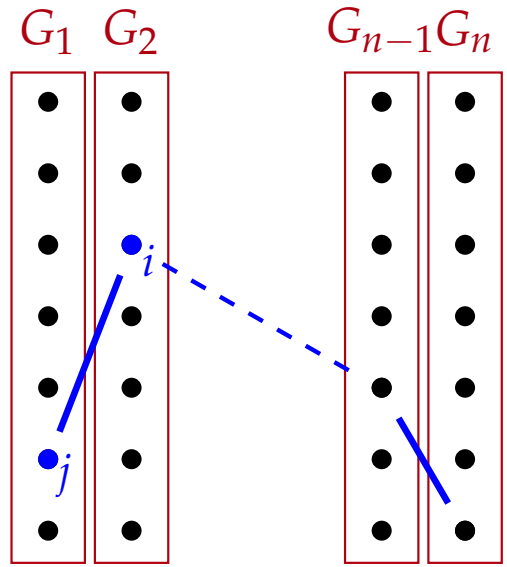
- *blocks* $B \in \mathcal{B}$ satisfying
  - $B \subset X$ and $|B| = n$;
  - for all $\{i,j\} \subset X$, $\{i,j\}$ lie:
    either in a single group $G \in \mathcal{G}$,
    or in a unique block $B \in \mathcal{B}$



Codes from designs

Let $\mathcal{T}$ be a transversal design $\mathrm{TD}(n,s) = (X, \mathcal{B}, \mathcal{G})$.

Its **incidence matrix** $\boldsymbol{M}$ has size $|\mathcal{B}| \times |X|$ and is defined by:

$$M_{i,j} = \begin{cases} 1 & \text{if } x_j \in B_i \\ 0 & \text{otherwise.} \end{cases}$$

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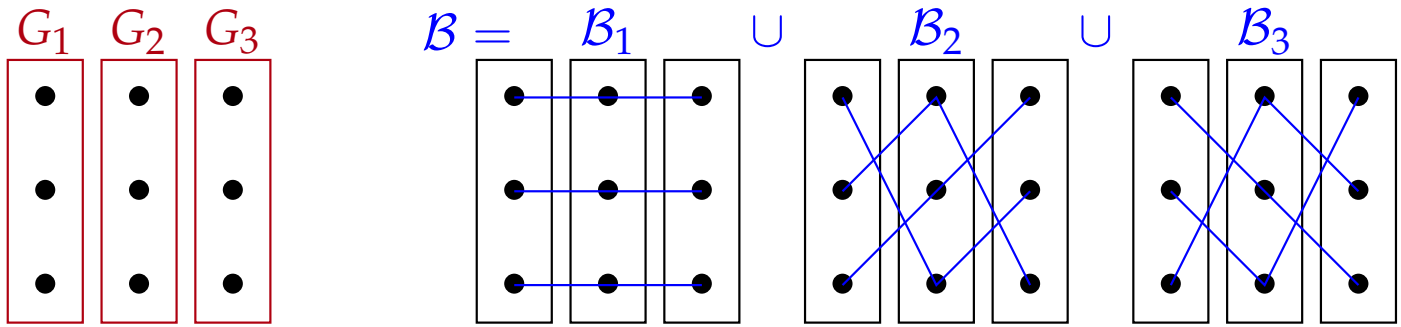
The **code** $\mathcal{C}$ **based on** $\mathcal{T}$ **over** $\mathbb{F}_q$ is the $\mathbb{F}_q$-linear code admitting $\boldsymbol{M}$ as a parity-check matrix ($\mathcal{C}^\perp$ is generated by $\boldsymbol{M}$).

- $\mathrm{length}(\mathcal{C}) = |X|$,
- $\dim(\mathcal{C}) = \dim(\ker \boldsymbol{M})$,
- every $B \in \mathcal{B}$ gives an $\boldsymbol{h} \in \mathcal{C}^\perp$ such that $\mathrm{wt}(\boldsymbol{h}_{|G_j}) = 1$, $\forall j = 1, \ldots, n$.

Example

The transversal design TD$(3,3)$ represented by:



gives an incidence matrix

$$M = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Its rank over $\mathbb{F}_3$ is 6 $\quad\Longrightarrow\quad$ the associated code $\mathcal{C}$ is a $[9,3]_3$ code.

Outline

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Let $\mathcal{C} \subseteq \mathbb{F}_q^N$ be a code based on a $\mathrm{TD}(n, s)$.

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The PIR scheme

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- **Initialisation.** User U encodes $F \mapsto \boldsymbol{c} \in \mathcal{C}$, and gives $\boldsymbol{c}_{|G_j}$ to server S_j.
- **To recover** $F_i = c_i$, with $i \in X$:
  1. User U randomly picks a block $B \in \mathcal{B}$ containing i.
     Then U defines:

$$q_j = \mathcal{Q}(i)_j := \begin{cases} \text{unique } \in B \cap G_j & \text{if } i \notin G_j \\ \text{a random point in } G_j & \text{otherwise.} \end{cases}$$

  2. Each server S_j sends back c_{q_j}
  3. U recovers

$$c_i = -\sum_{j:\, i \notin G_j} c_{q_j} = -\sum_{b \in B \setminus \{i\}} c_b$$

Privacy and parameters

Theorem. This PIR protocol is information-theoretically private.

Proof:

- the only server which holds F_i received a random query;
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- communication complexity: $n \log s$ uploaded bits, $n \log q$ downloaded bits
- computational complexity:
  - **only 1 read for each server** (somewhat optimal)
  - $\leq n$ additions over $\mathbb{F}_q$ for the user
- storage overhead: $(ns - M) \log q$ bits, where $M = \dim(\mathcal{C})$

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Question: transversal designs with good $\dim(\mathcal{C})$ depending on (n, s)?

Outline

Instances with geometric designs

$\mathcal{T}_A$, the **classical affine transversal design**:

- $X = \mathbb{F}_q^m$, $m \geq 2$,
- $\mathcal{G}$ a set of q disjoint hyperplanes partitionning X,
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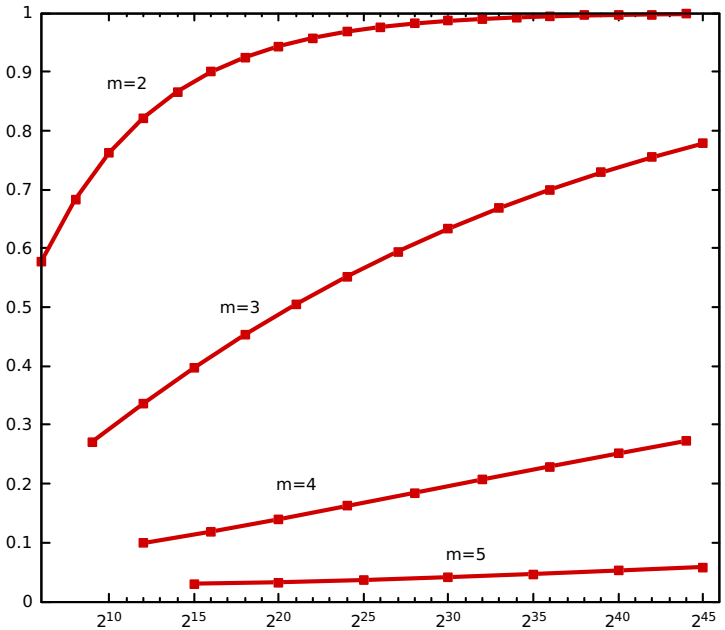

length $N = ns = 2^{em}$

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Private information retrieval:

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