Elliptic operator with Dirichlet data and associated semigroup

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Contents

1 Resolvent and spectral properties of elliptic operators

On a smooth bounded open set Ω of $\mathbb{R}^d$, we consider elliptic second-order operator P_0 given by

$$P_0 = \sum_{1\leq i,j\leq d} D_i(p^{ij}(x)D_j), \qquad \text{with} \quad \sum_{1\leq i,j\leq d} p^{ij}(x)\xi_i\xi_j \geq C|\xi|^2. \tag{1.1}$$

where $p^{ij} \in \mathscr{C}^\infty(\mathbb{R}^d; \mathbb{R})$ with furthermore $p^{ij} = p^{ji}$, $1 \leq i, j \leq d$. In addition we shall impose Dirichlet boundary conditions, that is, the trace of the solution at the boundary $\partial\Omega$.

1.1 Basic properties of second-order elliptic operators

Here we recall some well-known facts on elliptic operators such as P_0. In particular, in the case of homogeneous Dirichlet boundary conditions, we recall that P_0 is maximal monotone and has a spectral decomposition with a Hilbert basis of eigenfunctions.

We consider the following problem

$$P_0 u + \lambda u = f, \quad \text{for } \lambda \in \mathbb{R}.$$

Assume first f is smooth and there exists a smooth solution with $u_{|\partial\Omega} = 0$. Picking a second function v, also satisfying $v_{|\partial\Omega} = 0$, upon multiplying the equation by $\overline{v}$, integrating over Ω, and performing integrations by part we find $a(u, v) = (f, v)_{L^2(\Omega)}$ with the sesquilinear form $a(.,.)$ given by

$$a(u, v) = \sum_{1 \leq i,j \leq d} (p^{ij} D_i u, D_j v)_{L^2(\Omega)} + \lambda (u, v)_{L^2(\Omega)}.$$

Invoking first the ellipticity of p and second the Poincaré inequality on $H_0^1(\Omega)$ we have

$$\sum_{1 \leq i,j \leq d} (p^{ij} D_i v, D_j v)_{L^2(\Omega)} \gtrsim \|Dv\|^2_{(L^2(\Omega))^d} \gtrsim \|v\|^2_{H^1(\Omega)}. \tag{1.2}$$

Thus there exists $\lambda_0 < 0$ such that $(u, v) \mapsto a(u, v)$ is coercive on $H_0^1(\Omega)$ for $\lambda > \lambda_0$. The value λ_0 is given by the best possible constant in the following Poincaré inequality

$$\sum_{1 \leq i,j \leq d} (p^{ij} D_i v, D_j v)_{L^2(\Omega)} \geq C \|v\|^2_{L^2(\Omega)}. \tag{1.3}$$

If now $f \in H^{-1}(\Omega)$, as $v \mapsto \langle f, \overline{v} \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}$ is continuous on $H_0^1(\Omega)$, the Lax-Milgram theorem (see e.g. [4, 2]), yields the existence and the uniqueness of a solution $u \in H_0^1(\Omega)$ such that

$$a(u, v) = (f, v)_{L^2(\Omega)}, \qquad \forall v \in H_0^1(\Omega). \tag{1.4}$$

and we have

$$\|u\|_{H_0^1(\Omega)} \eqsim \|f\|_{H^{-1}(\Omega)},$$

with the H_0^1-norm given by

$$\|u\|^2_{H_0^1(\Omega)} = \sum_{1\le i,j\le d} (p^{ij} D_i u, D_j u)_{L^2(\Omega)}. \tag{1.5}$$

which is equivalent to the usual H^1-norm,

$$\|u\|^2_{H^1(\Omega)} = \|u\|^2_{L^2(\Omega)} + \sum_{1\le i\le d} \|D_i u\|^2_{L^2(\Omega)}, \tag{1.6}$$

by the Poincaré inequality.

One says that u is a weak solution of the elliptic problem

$$P_0 u + \lambda u = f, \quad \text{for } \lambda > \lambda_0,\ f \in H^{-1}(\Omega). \tag{1.7}$$

Observe that (1.4) is the Euler-Lagrange equation associated with the minimisation of the functional

$$J(u) = \frac{1}{2} a(u,u) - (f,u)_{L^2(\Omega)},$$

over $H_0^1(\Omega)$. The weak formulation (1.4) is thus also called the variational formulation of the elliptic problem.

If now $f \in L^2(\Omega)$ in (1.4)–(1.7), and if the boundary is $\mathscr{C}^2$ (which is the case here), the solution $u \in H_0^1(\Omega)$ given above is in fact in $H^2(\Omega)$. Moreover,

$$\|u\|_{H^2(\Omega)} \eqsim \|f\|_{L^2(\Omega)}. \tag{1.8}$$

Hence, in the case $f \in L^2(\Omega)$ the weak solution is in fact classical and satisfies $P_0 u + \lambda u = f$ in $L^2(\Omega)$. Finally, if $m \in \mathbb{N}$, if the boundary is $\mathscr{C}^{m+2}$ and if $f \in H^m(\Omega)$, then $u \in H^{m+2}(\Omega)$ and we have

$$\|u\|_{H^{m+2}(\Omega)} \eqsim \|f\|_{H^m(\Omega)}. \tag{1.9}$$

We refer to [4, Section 8.4] and [2, Section 9.6] for proofs.

We define the unbounded operator $\mathsf{P}_0 : L^2(\Omega) \to L^2(\Omega)$, with domain $D(\mathsf{P}_0) = H^2(\Omega) \cap H_0^1(\Omega)$, given by

$$\mathsf{P}_0 u = P_0 u, \quad u \in D(\mathsf{P}_0).$$

From the elements reviewed above we see that the H^2-norm, viz.,

$$\|u\|_{H^2(\Omega)} = \Big(\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} + \sum_{j,k} \|D^2_{jk} u\|^2_{L^2(\Omega)} \Big)^{1/2},$$

the graph norm, viz., $\|u\|_{D(\mathsf{P}_0)} = \big(\|u\|^2_{L^2(\Omega)} + \|\mathsf{P}_0 u\|^2_{L^2(\Omega)}\big)^{1/2}$, or simply the norm $\|\mathsf{P}_0 u\|_{L^2(\Omega)}$ are all equivalent on the space $D(\mathsf{P}_0)$, that is,

$$\|u\|_{H^2(\Omega)} \eqsim \|u\|_{D(\mathsf{P}_0)} \eqsim \|\mathsf{P}_0 u\|_{L^2(\Omega)}, \quad u \in D(\mathsf{P}_0), \tag{1.10}$$

and they make it a Hilbert space. In particular, $D(\mathsf{P}_0)$ is a closed subspace of $H^2(\Omega)$.

Observe that $\mathscr{C}^\infty_c(\Omega) \subset D(\mathsf{P}_0)$. However, the closure of $\mathscr{C}^\infty_c(\Omega)$ for the H^2-norm is the space $H^2_0(\Omega)$, that is the H^2-functions u on Ω such that the (well-defined) traces $u_{|\partial\Omega}$ and $\partial_\nu u_{|\partial\Omega}$ vanish. Since $H^2_0(\Omega)$ is strictly included in $D(\mathsf{P}_0)$, we see that the space $\mathscr{C}^\infty_c(\Omega)$ is not dense in $D(\mathsf{P}_0)$. We recall that yet the space $\mathscr{C}^\infty_c(\Omega)$ is dense in $H^1_0(\Omega)$ (see e.g. [1]).

We observe that we have $(\mathsf{P}_0 u, u)_{L^2(\Omega)} \geq 0$ for all $u \in D(\mathsf{P}_0)$. From the properties gathered above we have the following result.

Proposition 1. *The operator* $(\mathsf{P}_0, D(\mathsf{P}_0))$ *is maximal monotone.*

Note that the domain of P_0 is dense in $L^2(\Omega)$ from general results on Sobolev space. In fact, a general argument can also be invoked, as a maximal monotone operator on a Hilbert space has a dense domain. Note also that P_0 is a closed operator from the estimation (1.8).

1.2 Spectral properties

From the symmetry of P_0, viz.,

$$(\mathsf{P}_0 u, v)_{L^2(\Omega)} = (u, \mathsf{P}_0 v)_{L^2(\Omega)}, \qquad u, v \in D(\mathsf{P}_0),$$

we have the following result by Proposition 28 in the notes on semigroup theory.

Lemma 2. *The operator* $(\mathsf{P}_0, D(\mathsf{P}_0))$ *is selfadjoint on* $L^2(\Omega)$.

From the elliptic results recalled above, the value 0 is in the resolvent set $\rho(\mathsf{P}_0)$ of P_0: P_0 is a bijection from $D(\mathsf{P}_0) \subset L^2(\Omega)$ onto $L^2(\Omega)$ and the map $\mathsf{P}_0^{-1} : L^2(\Omega) \to D(\mathsf{P}_0)$ is bounded. We then set $\mathsf{R}_0 = \iota \circ \mathsf{P}_0^{-1}$ where ι is the natural injection of $H^2(\Omega)$ into $L^2(\Omega)$. As ι is a compact map by the Rellich-Kondrachov theorem [2, Theorem 9.16], so is R_0. We say that P_0 has a compact resolvent on $L^2(\Omega)$. In what follows we shall often omit to write the map ι explicitly.

From the symmetry of P_0, we conclude that R_0 is selfadjoint on $L^2(\Omega)$. As R_0 is injective, the spectral decomposition of compact selfadjoint operators on separable Hilbert spaces yields the existence of a nonincreasing sequence of real eigenvalues $(m_j)_{j\in\mathbb{N}} \subset (0, +\infty)$ (counted with their multiplicity) that converges to 0 and an associated sequence of eigenfunctions, denoted by $(\phi_j)_{j\in\mathbb{N}}$, that forms a Hilbert basis of $L^2(\Omega)$:

$$\forall u \in L^2(\Omega),\ \exists (u_j)_j \in \ell^2(\mathbb{C}),\ u = \sum_{j\in\mathbb{N}} u_j\phi_j \ \text{ in } L^2(\Omega).$$

Moreover $u_j = (u,\phi_j)_{L^2(\Omega)}$, $\|u\|^2_{L^2(\Omega)} = \sum_{j\in\mathbb{N}} |u_j|^2$, and for $u, v \in L^2(\Omega)$,

$$(u,v)_{L^2(\Omega)} = \sum_{j\in\mathbb{N}} u_j\overline{v}_j, \qquad u_j = (u,\phi_j)_{L^2(\Omega)},\ v_j = (v,\phi_j)_{L^2(\Omega)}.$$

We have $\mathsf{R}_0(\phi_j) = m_j\phi_j$. In particular $\phi_j \in D(\mathsf{P}_0)$ and equivalently we have $\phi_j = m_j\mathsf{P}_0\phi_j$. Note that we concluded above that the eigenvalues of R_0 are positive because of the injectivity of R_0 and the nonnegativity of P_0, viz. $(\mathsf{P}_0 u, u)_{L^2(\Omega)} \geq 0$.

We have

$$\mathsf{R}_0 u = \sum_{j\in\mathbb{N}} u_j m_j \phi_j, \quad \text{for } u = \sum_{j\in\mathbb{N}} u_j\phi_j \in L^2(\Omega).$$

As $D(\mathsf{P}_0)$ is the range of R_0 setting $\mu_j = m_j^{-1}$, $j \in \mathbb{N}$, the domain $D(\mathsf{P}_0)$ is characterized by

$$D(\mathsf{P}_0) = H^2(\Omega) \cap H^1_0(\Omega) = \big\{v = \sum_{j\in\mathbb{N}} v_j\phi_j;\ (v_j\mu_j)_j \in \ell^2(\mathbb{C})\big\},$$

and, with this characterization and the fact that P_0 is closed, we then have

$$\mathsf{P}_0 v = \sum_{j\in\mathbb{N}} \mu_j v_j \phi_j, \quad v = \sum_{j\in\mathbb{N}} v_j\phi_j \in D(\mathsf{P}_0). \tag{1.11}$$

As a summary, we have here obtained the classical result of the existence of a Hilbert basis $(\phi_j)_{j\in\mathbb{N}} \subset L^2(\Omega)$, formed by eigenfunctions of the operator P_0, associated with the eigenvalues $(\mu_j)_{j\in\mathbb{N}}$, sorted here as an nondecreasing sequence:

$$\mathsf{P}_0\phi_j = \mu_j\phi_j, \quad j \in \mathbb{N}, \qquad \text{with } \ 0 < \mu_0 \leq \mu_1 \leq \cdots \leq \mu_k \leq \cdots \tag{1.12}$$

The following asymptotic result is known as the Weyl law for the sequence of eigenvalues $(\mu_j)_{j\in\mathbb{N}}$.

Theorem 3 (Weyl law)**.** *Define $J_\mu = \#\{j \in \mathbb{N};\ \mu_j \leq \mu\}$. We have*

$$J_\mu \sim (2\pi)^{-d}\omega_d\,|\Omega|\,\mu^{d/2}, \quad \text{as } r \to \infty,$$

where ω_d is the volume of the Euclidean unit ball, that is, $\omega_d = \pi^{d/2}/\Gamma(1+d/2)$, with Γ the gamma function.

We refer for example to [3, Chapter 6, Theorems 16 and 18] or to [5, Theorem 8.16].

Remark 4. An equivalent formulation is the following asymptotic formula:

$$\mu_j \sim (2\pi)^2\big(\omega_d\,|\Omega|\big)^{-2/d} j^{2/d} = 4\pi\big(\Gamma(1+d/2)/|\Omega|\big)^{2/d} j^{2/d}, \quad \text{as } j \to \infty. \tag{1.13}$$

1.3 A Sobolev scale and operator extensions

For the analysis of the semigroup generated by the operator $(\mathsf{P}_0, D(\mathsf{P}_0))$ (and some of its extensions) carried out below the proper functional framework needs to be introduced. To that purpose, we define some adapated spaces of Sobolev type.

With the above spectral family, the space $H^1_0(\Omega)$ is characterized by the following proposition.

Proposition 5. *We have the following equivalence*

$$u \in H^1_0(\Omega) \quad \Leftrightarrow \quad u \in L^2(\Omega) \text{ and } (\mu_j^{1/2} u_j)_j \in \ell^2(\mathbb{C}), \quad u_j = (u, \phi_j)_{L^2(\Omega)}.$$

In particular, the inner product $(u, v) \mapsto \sum_{j\in\mathbb{N}} \mu_j u_j \overline{v_j}$ *gives the usual Hilbert space structure on* $H^1_0(\Omega)$*, with* $v_j = (v, \phi_j)_{L^2(\Omega)}$*. We also have*

$$\|Du\|^2_{L^2(\Omega)} \eqsim \sum_{1\leq i,j\leq d} (p^{ij} D_i u, D_j u)_{L^2(\Omega)} = \|u\|^2_{H^1_0(\Omega)} = \sum_{j\in\mathbb{N}} \mu_j |u_j|^2, \tag{1.14}$$

recalling (1.5)*.*

Proof. As recalled in (1.6), the H^1-norm is given by $\|u\|_{H^1(\Omega)} = (\|u\|^2_{L^2(\Omega)} + \|Du\|^2_{L^2(\Omega)})^{1/2}$. On $H^1_0(\Omega)$ an equivalent norm is simply $\|Du\|_{L^2(\Omega)}$ by the Poincaré inequality. In (1.5) we defined the following norm

$$\|u\|^2_{H^1_0(\Omega)} = \sum_{1\leq i,j\leq d} (p^{ij} D_i u, D_j u)_{L^2(\Omega)}$$

on $H^1_0(\Omega)$ that is equivalent to $\|Du\|_{L^2(\Omega)}$.

Above we recalled that $\mathscr{C}^\infty_c(\Omega) \subset D(\mathsf{P}_0) \subset H^1_0(\Omega)$ implying the density of $D(\mathsf{P}_0)$ in $H^1_0(\Omega)$. If $u \in D(\mathsf{P}_0)$ one has $\mathsf{P}_0 u = \sum_{j\in\mathbb{N}} \mu_j u_j \phi_j$ and

$$\sum_{1\leq i,j\leq d} (p^{ij} D_i u, D_j u)_{L^2(\Omega)} = (\mathsf{P}_0 u, u)_{L^2(\Omega)} = \sum_{j\in\mathbb{N}} \mu_j |u_j|^2. \tag{1.15}$$

Hence, on $D(\mathsf{P}_0)$, the norm $\|u\|_{H^1_0(\Omega)}$ is equivalent to that associated with the inner product $(u, v) \mapsto \sum_{j\in\mathbb{N}} u_j \overline{v_j} \mu_j$.

Consider $v \in H^1_0(\Omega)$ and $(v^{(n)}) \subset D(\mathsf{P}_0)$ is such that $v^{(n)} \to v$ in $H^1_0(\Omega)$. One has

$$v = \sum_{j\in\mathbb{N}} v_j \phi_j, \qquad v^{(n)} = \sum_{j\in\mathbb{N}} v_j^{(n)} \phi_j,$$

with $V^{(n)} = (\mu_j^{1/2} v_j^{(n)})_{j\in\mathbb{N}} \in \ell^2(\mathbb{C})$, $n \in \mathbb{N}$. With (1.15) we see that $(V^{(n)})_{n\in\mathbb{N}}$ is a Cauchy sequence in $\ell^2(\mathbb{C})$. Thus, there exists $(w_j)_{j\in\mathbb{N}}$ such that $(\mu_j^{1/2} w_j)_{j\in\mathbb{N}} \in \ell^2(\mathbb{C})$ and

$$\sum_{j\in\mathbb{N}} \mu_j |v_j^{(n)} - w_j|^2 \to 0 \qquad \text{as } n \to \infty.$$

In particular for each $j \in \mathbb{N}$ one has $v_j^{(n)} \to w_j$ as $n \to \infty$. Yet, $v_j^{(n)} = (v^{(n)}, \phi_j)_{L^2(\Omega} \to (v, \phi_j)_{L^2(\Omega} = v_j$ meaning that $v_j = w_j$. Hence, v is such that $(\mu_j^{1/2} v_j)_{j\in\mathbb{N}} \in \ell^2(\mathbb{C})$.

Let now $v \in L^2(\Omega)$ be such that $(\mu_j^{1/2} v_j)_{j\in\mathbb{N}} \in \ell^2(\mathbb{C})$. For $n \in \mathbb{N}$, set $v^{(n)} = \sum_{j\le n} v_j \phi_j$. One has $v^{(n)} \in D(\mathsf{P}_0)$. With the norm equivalence given in (1.15) one finds that

$$\|v^{(n)} - v^{(m)}\|^2_{H_0^1(\Omega)} \eqsim \sum_{j=n+1}^{m} \mu_j |v_j|^2, \qquad n \le m,$$

implying that $(v^{(n)})_{n\in\mathbb{N}}$ is a Cauchy sequence in $H_0^1(\Omega)$. Since it converges to v in $L^2(\Omega)$ it shows that $v \in H_0^1(\Omega)$. ■

Remark 6. Note that the $\sqrt{\mu_0}$ coincides with the optimal constant in the Poincaré inequality since we have

$$\|u\|^2_{H_0^1(\Omega)} = \sum_{j\in\mathbb{N}} |u_j|^2 \mu_j \ge \mu_0 \|u\|^2_{L^2(\Omega)},$$

with equality in the case $u = \phi_0$.

With the description of $H_0^1(\Omega)$ by means of the spectral family $(\phi_j)_j$ we can recover classical characterizations of $H_0^1(\Omega)$ functions.

Proposition 7. *Let $u \in L^2(\Omega)$ be such that*

$$|(u, \mathsf{P}_0 v)_{L^2(\Omega)}| \le L \|v\|_{H_0^1(\Omega)},$$

for some $L > 0$ and all $v \in D(\mathsf{P}_0)$. Then, $u \in H_0^1(\Omega)$ and $\|u\|_{H_0^1(\Omega)} \le L$.

Proof. We have $u = \sum_{j\in\mathbb{N}} u_j \phi_j$ with $(u_j)_j \in \ell^2(\mathbb{C})$. For $v \in D(\mathsf{P}_0)$, $v = \sum_{j\in\mathbb{N}} v_j \phi_j$, with $(\mu_j v_j)_j \in \ell^2(\mathbb{C})$ we have $(u, \mathsf{P}_0 v)_{L^2(\Omega)} = \sum_{j\in\mathbb{N}} \mu_j u_j \overline{v}_j$. Letting $N \in \mathbb{N}$ and choosing $v_j = u_j$ for $j \in \{0, \dots, N\}$, and $v_j = 0$ for $j \ge N+1$, we find

$$\sum_{0\le j\le N} \mu_j |u_j|^2 \le L \|v\|_{H_0^1(\Omega)} = L \Big(\sum_{0\le j\le N} \mu_j |u_j|^2 \Big)^{1/2}.$$

This gives $\big(\sum_{0\le j\le N} \mu_j |u_j|^2 \big)^{1/2} \le L$, which yields the conclusion. ■

The space $H^{-1}(\Omega)$ denotes the dual space of $H_0^1(\Omega)$. Instead of identifying $H_0^1(\Omega)$ and $H^{-1}(\Omega)$ by the Riesz theorem through the scalar product on $H_0^1(\Omega)$, one usually uses the space $L^2(\Omega)$ as a *pivot* space. This is possible because $H_0^1(\Omega)$ is dense in $L^2(\Omega)$. Then, we have

$$H_0^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-1}(\Omega),$$

where both injections have a dense range. We may then write, for $u \in L^2(\Omega)$,

$$\langle u, \overline{v} \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = (u, v)_{L^2(\Omega)} = \sum_{j \in \mathbb{N}} u_j \overline{v}_j, \qquad v \in H_0^1(\Omega).$$

If $u \in H^{-1}(\Omega)$ we set $u_j = \langle u, \overline{\phi_j} \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}$. Considering the norm given on $H_0^1(\Omega)$ by Proposition 5, this leads to the following characterization of the space $H^{-1}(\Omega)$.

Proposition 8. *If $u \in H^{-1}(\Omega)$ then $(u_j \mu_j^{-1/2})_j \in \ell^2(\mathbb{C})$ and*

$$u = \lim_{n \to \infty} \sum_{j=0}^{n} u_j \phi_j \quad \text{in } H^{-1}(\Omega).$$

Conversely, if $(w_j)_j \subset \mathbb{C}$ is such that $(w_j \mu_j^{-1/2})_j \in \ell^2(\mathbb{C})$ then the sequence of L^2-functions $\big(\sum_{j=0}^{n} w_j \phi_j \big)_{n \in \mathbb{N}}$ converges to some u in $H^{-1}(\Omega)$ and $w_j = \langle u, \phi_j \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}$. We thus write

$$H^{-1}(\Omega) = \big\{ u = \sum_{j \in \mathbb{N}} u_j \phi_j ;\ (u_j \mu_j^{-1/2})_j \in \ell^2(\mathbb{C}) \big\}.$$

With the pivot space $L^2(\Omega)$, the duality between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$ reads

$$\langle u, \overline{v} \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \sum_{j \in \mathbb{N}} u_j \overline{v}_j,$$

for $u = \sum_{j \in \mathbb{N}} u_j \phi_j \in H^{-1}(\Omega)$ and $v = \sum_{j \in \mathbb{N}} v_j \phi_j \in H_0^1(\Omega)$, that is, $(u_j \mu_j^{-1/2})_j \in \ell^2(\mathbb{C})$ and $(v_j \mu_j^{1/2})_j \in \ell^2(\mathbb{C})$.

With the characterizations of $H_0^1(\Omega)$ and $H^{-1}(\Omega)$ given above through the spectral family, we can introduce the unbounded operator $\mathsf{P}_{-1} : H^{-1}(\Omega) \to H^{-1}(\Omega)$, with domain $D(\mathsf{P}_{-1}) = H_0^1(\Omega)$, given by

$$\mathsf{P}_{-1} u = \sum_{j \in \mathbb{N}} \mu_j u_j \phi_j, \quad u = \sum_{j \in \mathbb{N}} u_j \phi_j \in H_0^1(\Omega).$$

From (1.11), P_{-1} is an extension of P_0 to $H^{-1}(\Omega)$. We have

$$\| \mathsf{P}_{-1} u \|_{H^{-1}(\Omega)} = \| u \|_{H_0^1(\Omega)}, \qquad u \in H_0^1(\Omega). \tag{1.16}$$

For $u \in H_0^1(\Omega)$ the action of P_0 makes no sense in general. However, $P_0 u$ is well defined in $H^{-1}(\Omega)$ in the sense of distributions.

Proposition 9. *Let $u \in H_0^1(\Omega)$. We have $P_0 u = \mathsf{P}_{-1}\, u$ in $H^{-1}(\Omega)$.*

Proof. The proof uses that $H^{-1}(\Omega)$ is a space of distributions since $\mathscr{C}_c^\infty(\Omega)$ is dense in $H_0^1(\Omega)$. Let $\varphi \in \mathscr{C}_c^\infty(\Omega)$. On the one hand, we naturally have

$$\langle P_0 u, \varphi \rangle_{\mathscr{D}'(\Omega), \mathscr{C}_c^\infty(\Omega)} = \langle u, P_0 \varphi \rangle_{\mathscr{D}'(\Omega), \mathscr{C}_c^\infty(\Omega)}$$

by the symmetry of P_0. On the other hand, we write

$$\begin{aligned} \langle \mathsf{P}_{-1} u, \varphi \rangle_{\mathscr{D}'(\Omega), \mathscr{C}_c^\infty(\Omega)} &= \langle \mathsf{P}_{-1} u, \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \langle u, \mathsf{P}_{-1} \varphi \rangle_{H_0^1(\Omega), H^{-1}(\Omega)} \\ &= \langle u, \mathsf{P}_0 \varphi \rangle_{L^2(\Omega), L^2(\Omega)} = \langle u, P_0 \varphi \rangle_{\mathscr{D}'(\Omega), \mathscr{C}_c^\infty(\Omega)}, \end{aligned}$$

since $\varphi \in D(\mathsf{P}_0)$. Hence, $P_0 u = \mathsf{P}_{-1} u$ in $\mathscr{D}'(\Omega)$ and thus this equality holds in $H^{-1}(\Omega)$. ■

With the above duality we may then write

$$\begin{aligned} (u, v)_{H_0^1(\Omega)} &= \sum_{1 \le i,j \le d} (p^{ij} D_i u, D_j v)_{L^2(\Omega)} = \sum_{j \in \mathbb{N}} \mu_j u_j \overline{v}_j \\ &= \langle \mathsf{P}_{-1} u, \overline{v} \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \langle P_0 u, \overline{v} \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}, \end{aligned} \tag{1.17}$$

using Proposition 9.

Proposition 10. *Let $u \in H^{-1}(\Omega)$ be such that*

$$|\langle u, \overline{v} \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}| \le L \|v\|_{L^2(\Omega)},$$

for some $L > 0$ and all $v \in H_0^1(\Omega)$. Then, $u \in L^2(\Omega)$ and $\|u\|_{L^2(\Omega)} \le L$.

The proof can be adapted from that of Proposition 7.

For $s \ge 0$, we introduce the unbounded operator $\mathsf{P}_0^s : L^2(\Omega) \to L^2(\Omega)$, with domain

$$D(\mathsf{P}_0^s) = \big\{ u = \sum_{j \in \mathbb{N}} u_j \phi_j;\ (\mu_j^s u_j)_j \in \ell^2(\mathbb{C}) \big\} \subset L^2(\Omega),$$

given by

$$\mathsf{P}_0^s u = \sum_{j \in \mathbb{N}} \mu_j^s u_j \phi_j, \quad u = \sum_{j \in \mathbb{N}} u_j \phi_j \in D(\mathsf{P}_0^s).$$

We naturally equip the space $D(\mathsf{P}_0^s)$ with the following scalar product and associated norm that endows it with a Hilbert space structure:

$$(u, v)_{D(\mathsf{P}_0^s)} = \sum_{j \in \mathbb{N}} u_j \overline{v}_j \mu_j^s, \qquad \|u\|_{D(\mathsf{P}_0^s)}^2 = \sum_{j \in \mathbb{N}} |u_j|^2 \mu_j^s.$$

This norm is equivalent to the graph norm on $D(\mathsf{P}_0^s)$. We have $\mathsf{P}_0^0 = \mathrm{Id}_{L^2(\Omega)}$ with $D(\mathsf{P}_0^0) = L^2(\Omega)$ and the case $s = 1$ is consistent with the domain of the operator P_0 on $L^2(\Omega)$. Note that $D(\mathsf{P}_0^{1/2}) = H_0^1(\Omega)$ and

$$(u, v)_{H_0^1(\Omega)} = \langle P_0 u, \overline{v} \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = (\mathsf{P}_0^{1/2} u, \mathsf{P}_0^{1/2} v)_{L^2(\Omega)}, \quad u, v \in H_0^1(\Omega). \tag{1.18}$$

using (1.17), and

$$\|u\|_{H_0^1(\Omega)}^2 = \langle P_0 u, \overline{u} \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \|\mathsf{P}_0^{1/2} u\|_{L^2(\Omega)}^2, \quad u \in H_0^1(\Omega). \tag{1.19}$$

Note that this is precisely the norm defined in (1.5) by Proposition 5. Note also that we have

$$(u, v)_{H_0^1(\Omega)} = (\mathsf{P}_0 u, v)_{L^2(\Omega)} \quad \text{if } u \in D(\mathsf{P}_0),\ v \in H_0^1(\Omega). \tag{1.20}$$

In the case $s = k \in \mathbb{N}$, P_0^k and $D(\mathsf{P}_0^k)$ correspond to the iterated operators and domains for the elliptic operator P_0, that is, $D(\mathsf{P}_0^{k+1}) = \{u \in D(\mathsf{P}_0^k);\ \mathsf{P}_0 u \in D(\mathsf{P}_0^k)\}$.

Note that for the Hilbert basis $(\phi_j)_j$ introduced above we have

$$\phi_j \in \cap_{s \geq 0} D(\mathsf{P}_0^s), \quad j \in \mathbb{N}.$$

For $s < 0$ we can define the following bounded operator on $L^2(\Omega)$:

$$\mathsf{P}_0^s u = \sum_{j \in \mathbb{N}} \mu_j^{-s} u_j \phi_j, \qquad u = \sum_{j \in \mathbb{N}} u_j \phi_j \in L^2(\Omega).$$

In fact, if $s < 0$, the operator P_0^s is compact. With this notation we have[1] $\mathsf{R}_0 = \mathsf{P}_0^{-1}$. We observe that $\mathsf{P}_0^{-1/2}$ maps $L^2(\Omega)$ onto $H_0^1(\Omega) = D(\mathsf{P}_0^{1/2})$ isometrically and we have $\mathsf{P}_0^{-1/2} = (\mathsf{P}_0^{1/2})^{-1}$. Noting that $\|\mathsf{P}_0^{-1/2} u\|_{L^2(\Omega)} = \|u\|_{H^{-1}(\Omega)}$ we see also that $\mathsf{P}_0^{-1/2}$ can be (uniquely) extended to an isometry from $H^{-1}(\Omega)$ onto $L^2(\Omega)$.

Arguing as we did above for the $H_0^1(\Omega)$-$H^{-1}(\Omega)$ duality with $L^2(\Omega)$ as a pivot space, we may then obtain the following result.

Proposition 11. *Let $s \geq 0$. We denote by $D(\mathsf{P}_0^s)'$ the dual of $D(\mathsf{P}_0^s)$. We have*

$$D(\mathsf{P}_0^s) \hookrightarrow L^2(\Omega) \hookrightarrow D(\mathsf{P}_0^s)',$$

where both injections have a dense range. For $u \in D(\mathsf{P}_0^s)'$, if we set $u_j = \langle u, \phi_j \rangle_{D(\mathsf{P}_0^s)', D(\mathsf{P}_0^s)}$, the space $D(\mathsf{P}_0^s)'$ is characterized as follows

$$u \in D(\mathsf{P}_0^s)' \quad \Leftrightarrow \quad (u_j \mu_j^{-s})_j \in \ell^2(\mathbb{C}),$$

[1] Here, as we omit the operator ι defined above, in the case $s = -1$, we can identify P_0^s and the operator P_0^{-1} defined at the beginning of Section 1.2.

and $u = \sum_{j\in\mathbb{N}} u_j \phi_j$ *in* $D(\mathsf{P}_0^s)'$. *Finally, we have*

$$\langle u, \overline{v}\rangle_{D(\mathsf{P}_0^s)', D(\mathsf{P}_0^s)} = \sum_{j\in\mathbb{N}} u_j \overline{v}_j,$$

for $u = \sum_{j\in\mathbb{N}} u_j\phi_j \in D(\mathsf{P}_0^s)'(\Omega)$ *and* $v = \sum_{j\in\mathbb{N}} v_j\phi_j \in D(\mathsf{P}_0^s)$, *that is,* $(u_j\mu_j^{-s})_j \in \ell^2(\mathbb{C})$ *and* $(v_j\mu_j^s)_j \in \ell^2(\mathbb{C})$.

Definition 12. For $s \geq 0$, we set $K^s(\Omega) = D(\mathsf{P}_0^{s/2})$ and, for $s < 0$, we set $K^s(\Omega) = D(\mathsf{P}_0^{-s/2})'$.

We have

$$K^s(\Omega) \hookrightarrow K^{s'}(\Omega) \quad \text{for } s \geq s',$$

where the injection has a dense range. From what is presented above, for all $s \in \mathbb{R}$, we have

$$K^s(\Omega) = \{u = \sum_{j\in\mathbb{N}} u_j\phi_j;\ (\mu_j^{s/2} u_j)_j \in \ell^2(\mathbb{C})\}.$$

On $K^s(\Omega)$ the following inner product and associated norm

$$(u, v)_{K^s(\Omega)} = \sum_{j\in\mathbb{N}} \mu_j^s u_j \overline{v_j}, \quad \|u\|^2_{K^s(\Omega)} = \sum_{j\in\mathbb{N}} \mu_j^s |u_j|^2 < \infty,$$

with $u_j = \langle u, \phi_j\rangle_{K^s(\Omega), K^{-s}(\Omega)}$ and $v_j = \langle v, \phi_j\rangle_{K^s(\Omega), K^{-s}(\Omega)}$, yield a Hilbert space structure. For $s \geq 0$, if $u \in L^2(\Omega)$ and $v \in K^s(\Omega)$, we recover the *pivot rôle* played by $L^2(\Omega)$:

$$\langle u, \overline{v}\rangle_{K^{-s}(\Omega), K^s(\Omega)} = (u, v)_{L^2(\Omega)} = \sum_{j\in\mathbb{N}} u_j \overline{v_j}.$$

For $r \in \mathbb{R}$ and $s \geq 0$, with the spectral family $(\phi_j)_j$, we defined the *unbounded* operator $\mathsf{P}_r^s : K^r(\Omega) \to K^r(\Omega)$, with domain $D(\mathsf{P}_r^s) = K^{r+2s}(\Omega)$ given by

$$\mathsf{P}_r^s u = \sum_{j\in\mathbb{N}} \mu_j^s u_j \phi_j, \quad u = \sum_{j\in\mathbb{N}} u_j\phi_j \in K^{r+2s}(\Omega). \tag{1.21}$$

We have

$$\|\mathsf{P}_r^s u\|_{K^r(\Omega)} = \|u\|_{K^{r+2s}(\Omega)}, \qquad u \in K^{r+2s}(\Omega). \tag{1.22}$$

If $r \geq 0$, the operator P_r^s is a restriction of P_0^s to $K^r(\Omega) \subset K^0(\Omega) = L^2(\Omega)$. If $r < 0$, the operator P_r^s is an extension of P_0^s to $K^r(\Omega) \supset L^2(\Omega)$.

For $r \in \mathbb{R}$ and $s < 0$, we define the *bounded* operator $\mathsf{P}_r^s : K^r(\Omega) \to K^r(\Omega)$ also given by (1.21). We have

$$\|\mathsf{P}_r^s u\|_{K^{r+2|s|}(\Omega)} = \|u\|_{K^r(\Omega)}.$$

We may then state the following results whose proof is elementary from what precedes.

Proposition 13. *Let $r, s, \sigma \in \mathbb{R}$.*

1. *If $s \geq 0$ and $\sigma \geq r+2s$ we have $K^{\sigma}(\Omega) \subset D(\mathsf{P}_r^s) = K^{r+2s}(\Omega)$ and $\mathsf{P}_r^s\big(K^{\sigma}(\Omega)\big) = K^{\sigma-2s}(\Omega) \subset K^r(\Omega)$. Moreover, we have*

$$\|\mathsf{P}_r^s(u)\|_{K^{\sigma-2s}(\Omega)} = \|u\|_{K^{\sigma}(\Omega)}, \qquad u \in K^{\sigma}(\Omega).$$

2. *If $s < 0$ and $\sigma \geq r$ we have $K^{\sigma}(\Omega) \subset D(\mathsf{P}_r^s) = K^r(\Omega)$ and $\mathsf{P}_r^s\big(K^{\sigma}(\Omega)\big) = K^{\sigma+2|s|}(\Omega) \subset K^{\sigma}(\Omega)$. Moreover, we have*

$$\|\mathsf{P}_r^s(u)\|_{K^{\sigma+2|s|}(\Omega)} = \|u\|_{K^{\sigma}(\Omega)}, \qquad u \in K^{\sigma}(\Omega).$$

The selfadjointness property further extends to P_0^s.

Lemma 14. *Let $s \in \mathbb{R}$ and let $u, v \in D(\mathsf{P}_0^s)$. We have*

$$(\mathsf{P}_0^s u, v)_{L^2(\Omega)} = (u, \mathsf{P}_0^s v)_{L^2(\Omega)}.$$

We also have the following results.

Lemma 15. *Let $r, s \in \mathbb{R}$ and let $u \in K^{r+2s}(\Omega)$ and $v \in K^{-r}(\Omega)$. We have*

$$\langle \mathsf{P}_r^s u, \overline{v} \rangle_{K^r(\Omega), K^{-r}(\Omega)} = \langle u, \mathsf{P}_{-r-2s}^s \overline{v} \rangle_{K^{r+2s}(\Omega), K^{-r-2s}(\Omega)}.$$

Lemma 16. *Let $u \in K^{r+2}(\Omega)$. We have*

$$\|u\|_{K^{r+1}(\Omega)}^2 = (\mathsf{P}_r u, u)_{K^r(\Omega)}.$$

We finish this section by further analyzing the properties of the functions in $K^k(\Omega)$.

Proposition 17. *Let $k \in \mathbb{N}$. We have $K^k(\Omega) = D(\mathsf{P}_0^{k/2}) \subset H^k(\Omega)$ and*

$$K^k(\Omega) = \{u \in H^k(\Omega);\ \mathsf{P}_0^j u \in H_0^1(\Omega),\ j = 0, \ldots, E[(k-1)/2]\},$$

Moreover, there exists $C > 0$ such that $C^{-1}\|u\|_{H^k(\Omega)} \leq \|u\|_{K^k(\Omega)} \leq C\|u\|_{H^k(\Omega)}$. In fact, $K^k(\Omega)$ is closed linear subspace of $H^k(\Omega)$.

Here, we denote by $E[.]$ the integer part of a real number.

Proof. The property holds for $k = 0, 1, 2$. We proceed by induction and assume that the property holds for $k-1$ and k for some $k \in \mathbb{N}$, with $k \geq 2$. Let then $u \in K^{k+1}(\Omega)$. We thus have $\mathsf{P}_0 u \in K^{k-1}(\Omega)$ from the results given above. We thus have $\mathsf{P}_0^{j+1} u \in H_0^1(\Omega)$ for $j = 0, \ldots, E[(k-2)/2] = E[k/2] - 1$, that is $\mathsf{P}_0^j u \in H_0^1(\Omega)$

for $j = 1, \dots, E[k/2]$. We also have $u \in H_0^1(\Omega)$, since $u \in K^k(\Omega) \subset K^1(\Omega)$ as $k \geq 2$. Moreover, we have $\mathsf{P}_0 u \in H^{k-1}(\Omega)$. By (1.9), we have $u \in H^{k+1}(\Omega)$ and

$$\|u\|_{H^{k+1}(\Omega)} \eqsim \|\mathsf{P}_0 u\|_{H^{k-1}(\Omega)} \eqsim \|\mathsf{P}_0 u\|_{K^{k-1}(\Omega)} = \|u\|_{K^{k+1}(\Omega)}.$$

We have thus found

$$K^{k+1}(\Omega) \subset \{u \in H^{k+1}(\Omega);\ \mathsf{P}_0^j u \in H_0^1(\Omega),\ j = 0, \dots, E[k/2]\}.$$

Let now $u \in H^{k+1}(\Omega)$ be such that $\mathsf{P}_0^j u \in H_0^1(\Omega)$, for $j = 0, \dots, E[k/2]$. Thus $u \in D(\mathsf{P}_0)$ and $v = \mathsf{P}_0 u \in H^{k-1}(\Omega)$. As $\mathsf{P}_0^j v \in H_0^1(\Omega)$, for $j = 0, \dots, E[k/2] - 1 = E[(k-2)/2]$, we find that $\mathsf{P}_0 u \in K^{k-1}(\Omega)$. This implies that $u \in K^{k+1}(\Omega)$ from the results given above. This concludes the proof. ■

Proposition 18. *Let $\alpha \in \mathscr{C}^\infty(\overline{\Omega})$. If $u \in K^k(\Omega)$, then $\alpha u \in K^k(\Omega)$ for $k = 0, 1, 2$.*

This is a consequence of the following lemma that follows from the smoothness of the coefficients of the operator P_0.

Lemma 19. *Let $\alpha \in \mathscr{C}^\infty(\overline{\Omega})$ and let $k \in \mathbb{N}$. If $u \in H^{k+2}(\Omega)$, we then have $v = \sum_{1 \leq i,j \leq d} D_i(p^{ij}(x) D_j)(\alpha u) \in H^k(\Omega)$ and $\|v\|_{H^k(\Omega)} \leq C \|u\|_{H^{k+2}(\Omega)}$, where the constant C only depends on α and the coefficients p^{ij}.*

Next, we observe that if $u \in K^3(\Omega)$ and α as above, we have $\alpha u \in D(\mathsf{P}_0) = K^2(\Omega)$ by Proposition 18, and

$$\mathsf{P}_0(\alpha u) = P_0(\alpha u) = \alpha P_0 u + \big(\sum_{1 \leq i,j \leq d} D_i(p^{ij} D_j)\alpha \big) u + 2 \sum_{1 \leq i,j \leq d} p^{ij}(x)(D_i \alpha)(D_j u).$$

While the first two term are in $H_0^1(\Omega)$, the last sum is in $H^1(\Omega)$ but not in $H_0^1(\Omega)$ in general, meaning then that $\alpha u \notin K^3(\Omega)$ by Proposition 17. Yet, we note that, at the boundary, ∇u is colinear to the normal vector to $\partial\Omega$, $n = (n_1, \dots, n_d)$, as u vanishes at $\partial\Omega$. Consequently, if we have

$$\sum_{1 \leq i,j \leq d} p^{ij}(x) D_i \alpha_{|\Omega} n_j = 0,$$

we find that $\mathsf{P}_0(\alpha u)$ vanishes at the boundary. Hence, in this case we have $\alpha u \in K^3(\Omega)$ by Proposition 17.

For higher orders in the Sobolev scale we may simply write the following result.

Lemma 20. *If $\alpha \in \mathscr{C}^\infty(\overline{\Omega})$ and if α is flat at all orders at $\partial\Omega$ then, for any $k \in \mathbb{N}$ and $u \in K^k(\Omega)$ we have $\alpha u \in K^k(\Omega)$ and $\|\alpha u\|_{K^k(\Omega)} \leq C \|u\|_{K^k(\Omega)}$, where the constant $C > 0$ is only dependent upon the function α.*

2 The parabolic semigroup

The unbounded operator P_0 on $L^2(\Omega)$ with dense domain $D(\mathsf{P}_0) = H^2(\Omega) \cap H_0^1(\Omega)$ is maximal monotone by Proposition 1. Then, with the Lumer-Philips theorem we have the following result that states the well-posedness of the parabolic equation associated with the operator P_0.

Theorem 21. *Let $T \in \mathbb{R}_+ \cup \{+\infty\}$. The operator P_0 generates C_0-semigroup of contraction $S(t) = e^{-t\mathsf{P}_0}$ on $L^2(\Omega)$. If $y^0 \in D(\mathsf{P}_0)$, then $y(t) = S(t)y^0$ is the unique solution in*

$$\mathscr{C}^0([0,T]; D(\mathsf{P}_0)) \cap \mathscr{C}^1([0,T]; L^2(\Omega)),$$

such that $y(0) = y^0$ and

$$\frac{d}{dt} y(t) + \mathsf{P}_0 y(t) = 0$$

holds in $L^2(\Omega)$ for all $0 \leq t \leq T$.

Here, $[0,T]$ means $[0,+\infty)$ if $T = +\infty$.

We recall that Sobolev spaces $K^s(\Omega)$ as introduced in Section 1.3 are given by $K^s(\Omega) = D(\mathsf{P}_0^{s/2})$ for $s \geq 0$ and $K^s(\Omega) = D(\mathsf{P}_0^{-s/2})'$ for $s < 0$.

As P_0 is moreover selfadjoint by Lemma 2, then, the stronger version of the Lumer-Philips theorem adapted to Hilbert spaces yields the following result.

Theorem 22. *Let $T \in \mathbb{R}_+ \cup \{+\infty\}$. The semigroup $S(t)$ is analytic and for $y^0 \in L^2(\Omega)$, the function $y(t) = S(t)y^0$ is in*

$$\mathscr{C}^0([0,T]; L^2(\Omega)) \cap \mathscr{C}^\infty((0,T]; K^s(\Omega)), \quad s \in \mathbb{R},$$

and is such that

$$y(0) = y^0 \quad \text{and} \quad \frac{d}{dt} y(t) + \mathsf{P}_0 y(t) = 0 \text{ holds in } L^2(\Omega) \text{ for } 0 < t \leq T. \tag{2.1}$$

Moreover, $y(t) = S(t)y^0$ is the unique solution of (2.1) in

$$\mathscr{C}^0([0,T]; L^2(\Omega)) \cap \mathscr{C}^1((0,T]; L^2(\Omega)) \cap \mathscr{C}^0((0,T]; D(\mathsf{P}_0)).$$

Here, $[0,T]$ (resp. $(0,T]$) means $[0,+\infty)$ (resp. $(0,+\infty)$) if $T = +\infty$. Observe that the C_0-semigroup $S(t)$ is selfadjoint since the generator is selfadjoint.

The above theorems are consequences of general results on semigroups. Here, in the particular case of the operator P_0 and of the semigroup $S(t)$ it generates, using the spectral representation $S(t)$ given in Section 2.1 by mean of the Hilbert basis introduced in Section 1.2, we can recover all the results of Theorem 22 in a quite elementary way, only invoking few aspects of semigroup theory. A reader experienced with semigroup theory can readily skip this section. In Section 2.1 we give however a spectral representation of the semigroup $S(t)$.

2.1 Spectral representation of the semigroup

We recall that the Hilbert basis of $L^2(\Omega)$ introduced in Section 1.2 is composed of eigenfunctions $(\phi_j)_{j\in\mathbb{N}}$ of P_0, with $(\mu_j)_{j\in\mathbb{N}} \subset \mathbb{R}$ for associated eigenvalues. In particular, if $u \in L^2(\Omega)$ we have $u = \sum_{j\in\mathbb{N}} u_j\phi_j$, with $u_j = (u, \phi_j)_{L^2(\Omega)}$. It is quite simple to obtain the form of the semigroup $S(t)$ within this spectral family according to the following lemma.

Lemma 23. *Let $T \in \mathbb{R}_+ \cup \{+\infty\}$. Let $y^0 \in L^2(\Omega)$ and let*

$$t \mapsto y(t) \in \mathscr{C}^0([0,T]; L^2(\Omega)) \cap \mathscr{C}^1((0,T]; L^2(\Omega)) \cap \mathscr{C}^0((0,T]; D(\mathsf{P}_0))$$

be such that $y(0) = y^0$ and such that

$$\frac{d}{dt}y(t) + \mathsf{P}_0 y(t) = 0 \text{ holds in } L^2(\Omega) \text{ for } 0 < t \leq T. \tag{2.2}$$

If we set $y_j(t) = e^{-t\mu_j}(y^0, \phi_j)_{L^2(\Omega)}$, for $t \geq 0$ and $j \in \mathbb{N}$, then $(y_j(t))_j \in \mathscr{C}^0([0,T], \ell^2(\mathbb{C}))$ and

$$y(t) = \sum_{j\in\mathbb{N}} y_j(t)\phi_j, \quad t \geq 0,$$

with convergence in $L^2(\Omega)$.

The action of the semigroup $S(t)$ generated by P_0 on $L^2(\Omega)$ and given by Theorems 21 and 22 is thus given by,

$$S(t)u = \sum_{j\in\mathbb{N}} e^{-t\mu_j}(u, \phi_j)_{L^2(\Omega)}\phi_j, \qquad u \in L^2(\Omega),\ t \geq 0, \tag{2.3}$$

where the series convergence is to be understood in $L^2(\Omega)$.

Note that the result of Lemma 23 is also to be understood as an uniqueness result for the semigroup equation $\frac{d}{dt}y(t) + \mathsf{P}_0 y(t) = 0$.

Proof. Let $t > 0$. As $\mathsf{P}_0 y(t) \in L^2(\Omega)$ we have $y(t) \in D(\mathsf{P}_0)$. We set $z_j(t) = (y(t), \phi_j)_{L^2(\Omega)}$, for $t \geq 0$. Then $(z_j(t))_j \in \mathscr{C}^0([0,T], \ell^2(\mathbb{C}))$. We have, for $t, t' > 0$

$$(t'-t)^{-1}\big(z_j(t') - z_j(t)\big) = \big((t'-t)^{-1}\big(y(t') - y(t)\big), \phi_j\big)_{L^2(\Omega)}.$$

As $y \in \mathscr{C}^1((0,T]; L^2(\Omega))$, letting $t' \to t$, we find that $z_j(t)$ is differentiable for $t > 0$ and

$$\frac{d}{dt}z_j(t) = (\frac{d}{dt}y(t), \phi_j)_{L^2(\Omega)} = -(\mathsf{P}_0\, y(t), \phi_j)_{L^2(\Omega)} = -(y(t), \mathsf{P}_0\phi_j)_{L^2(\Omega)},$$

by Lemma 2, as $\phi_j \in D(\mathsf{P}_0)$. Since $\mathsf{P}_0\phi_j = \mu_j\phi_j$, we obtain

$$\frac{d}{dt}z_j(t) = -\mu_j(y(t), \phi_j)_{L^2(\Omega)} = -\mu_j z_j(t).$$

Consequently $z_j(t) = y_j(t)$ for any $t \geq 0$, which concludes the proof. ∎

2.2 Well-posedness: an elementary proof

As mentioned above, we provide here a simple proof of Theorems 21 and 22, based on the decomposition (2.3) of the semigroup $S(t)$ in the spectral family $(\phi_j)_{j\in\mathbb{N}}$. Lemma 23 is to be treated as the uniqueness part of both theorems.

With Lemma 23, for $t \geq 0$, for $u \in L^2(\Omega)$, we define the map

$$\Sigma(t)u = \sum_{j\in\mathbb{N}} e^{-\mu_j t} u_j \phi_j, \quad u_j = (u, \phi_j)_{L^2(\Omega)}.$$

As $(u_j)_j \in \ell^2(\mathbb{C})$, so is $(e^{-\mu_j t} u_j)_j$, implying that $\Sigma(t)u \in L^2(\Omega)$.

Lemma 24. *The map $\Sigma(t)$ is the strongly continuous contraction semigroup $S(t)$ generated by the unbounded operator $(\mathsf{P}_0, D(\mathsf{P}_0))$ on $L^2(\Omega)$.*

Proof. Let $u \in L^2(\Omega)$. We write $u = \sum_{j\in\mathbb{N}} u_j\phi_j$ with $(u_j) \in \ell^2(\mathbb{C})$. Observe that we have $\|\Sigma(t)u\|^2_{L^2(\Omega)} = \sum_{j\in\mathbb{N}} e^{-2\mu_j t}|u_j|^2 \leq \sum_{j\in\mathbb{N}} |u_j|^2 = \|u\|^2_{L^2(\Omega)}$, implying that $\Sigma(t)$ is in $\mathscr{L}(L^2(\Omega))$ and moreover of contraction type. Observe also that $\Sigma(t)$ satisfies the following semigroup properties

$$\Sigma(0) = \mathrm{Id}_{L^2(\Omega)}, \quad \Sigma(t) \circ \Sigma(t') = \Sigma(t + t').$$

With $u \in L^2(\Omega)$ as above, we write $\Sigma(t)u - u = \sum_{j\in\mathbb{N}}(e^{-\mu_j t} - 1)u_j\phi_j$, yielding $\|\Sigma(t)u - u\|^2_{L^2(\Omega)} = \sum_{j\in\mathbb{N}}(1 - e^{-\mu_j t})^2|u_j|^2$. As for each $j \in \mathbb{N}$, we have $e^{-\mu_j t} - 1 \to 0$ as $t \to 0^+$, and as $0 \leq 1 - e^{-\mu_j t} \leq 1$, the Lebesgue dominated-convergence theorem (for the counting measure) implies that $\Sigma(t)u \to u$ in $L^2(\Omega)$ as $t \to 0^+$ for all $u \in L^2(\Omega)$. Considering the definition of a C_0-semigroup, we have obtained that $\Sigma(t)$ is such a semigroup.

We now prove that P_0 with domain $D(\mathsf{P}_0) = K^2(\Omega)$ is the generator of $\Sigma(t)$. As the map that associates a semigroup to its generator is injective this allows one to conclude the $\Sigma(t)$ is *the* C_0-semigroup generated by $(\mathsf{P}_0, D(\mathsf{P}_0))$.

For the time being, we denote by $A : L^2(\Omega) \to L^2(\Omega)$, with $D(A) \subset L^2(\Omega)$, the generator of $\Sigma(t)$. Let $u \in L^2(\Omega)$ such that, moreover, the limit $\lim_{t\to 0^+}(u - \Sigma(t)u)/t$ exists in $L^2(\Omega)$. We denote by v this limit and we have $v = \sum_{j\in\mathbb{N}} v_j\phi_j$ with $(v_j = (v, \phi_j))_j \in \ell^2(\mathbb{C})$. Then, $u \in D(A)$ and $Au = v$. We have

$$\|(u - \Sigma(t)u)/t - v\|^2_{L^2(\Omega)} = \sum_{j\in\mathbb{N}} \left|(1 - e^{-\mu_j t})u_j/t - v_j\right|^2.$$

For all $j \in \mathbb{N}$, we thus have $(1 - e^{-\mu_j t})u_j/t - v_j \to 0$ as $t \to 0^+$, meaning that $v_j = \mu_j u_j$. Hence, if $u \in D(A)$ then $u \in D(\mathsf{P}_0)$ and $Au = \mathsf{P}_0 u$.

Conversely, let us consider $u \in D(\mathsf{P}_0)$; we have $(\mu_j u_j)_j \in \ell^2(\mathbb{C})$. Then, $\mathsf{P}_0 u = \sum_{j\in\mathbb{N}} \mu_j u_j \phi_j$. We then have

$$\|(u - \Sigma(t)u)/t - \mathsf{P}_0 u\|^2_{L^2(\Omega)} = \sum_{j\in\mathbb{N}} \left|(1 - e^{-\mu_j t})u_j/t - \mu_j u_j\right|^2.$$

As for each $j \in \mathbb{N}$, we have $(1-e^{-\mu_j t})/t-\mu_j \to 0$ as $t \to 0^+$ and as $(1-e^{-\mu_j t})/t \le \mu_j$ for $t>0$, the Lebesgue dominated-convergence theorem implies $(u-\Sigma(t)u)/t \to \mathsf{P}_0 u$ in $L^2(\Omega)$ as $t \to 0^+$ for all $u \in D(\mathsf{P}_0)$. We thus conclude that the domain of the generator A of $\Sigma(t)$ is precisely $D(\mathsf{P}_0)$ and that A coincides with P_0. ■

From semigroup properties, if $y^0 \in D(\mathsf{P}_0)$ and $y(t) = \Sigma(t)y^0 = S(t)y^0$, then $\frac{d}{dt}y(t) + \mathsf{P}_0 y(t) = 0$ is satisfied in $L^2(\Omega)$ for $t \ge 0$, and $y(t) \in \mathscr{C}^0([0,T]; D(\mathsf{P}_0))$ and $\frac{d}{dt}y(t) \in \mathscr{C}^0([0,T]; L^2(\Omega))$. This concludes the second proof of Theorem 21.

The next lemma concludes the proof Theorem 22.

Lemma 25. *Let $y^0 \in L^2(\Omega)$ and $y(t) = S(t)y^0 \in \mathscr{C}([0,+\infty); L^2(\Omega))$. For any $s \ge 0$, we have $y(t) \in \mathscr{C}^\infty((0,+\infty); K^s(\Omega))$ and $\frac{d}{dt}y(t) = -\mathsf{P}_0 y(t)$ in $K^s(\Omega)$ for $t > 0$. Moreover, $\left(\frac{d}{dt}\right)^k y(t) = (-\mathsf{P}_0)^k y(t)$ in $K^s(\Omega)$ for $t>0$.*

Proof. We write $y^0 = \sum_{j\in\mathbb{N}} y_j^0 \phi_j$ with $(y_j^0)_j \in \ell^2(\mathbb{C})$. We pick $s \ge 0$. First, let us consider $t > 0$. We have $\|y(t)\|^2_{K^s(\Omega)} = \sum_{j\in\mathbb{N}} \mu_j^s e^{-2\mu_j t}|y_j^0|^2 \le C_t \sum_{j\in\mathbb{N}} |y_j^0|^2 = C_t\|y^0\|^2_{L^2(\Omega)}$, implying that $y(t) \in \cap_{r\in\mathbb{R}} K^r(\Omega)$. Second, let $t>0$ and $h \in \mathbb{R}$ such that $t+h>0$. We write $y(t+h)-y(t) = \sum_{j\in\mathbb{N}}(e^{-\mu_j h}-1)e^{\mu_j t} y_j^0 \phi_j$. As $y(t+h)-y(h) \in K^s(\Omega)$ we find

$$\|y(t+h)-y(t)\|^2_{K^s(\Omega)} = \sum_{j\in\mathbb{N}} (1-e^{-\mu_j h})^2 \mu_j^s e^{2\mu_j t}|y_j^0|^2.$$

As $(1-e^{-\mu_j h})^2\mu_j^s e^{2\mu_j t}$ converges to zero as $h \to 0$ and is bounded by some constant $C_{s,t}$ independent of j, the Lebesgue dominated-convergence theorem (for the counting measure) implies that $\|y(t+h)-y(t)\|_{K^s(\Omega)} \to 0$ as $h \to 0$. We thus have $y \in \mathscr{C}^0((0,+\infty); K^s(\Omega))$ for any $s \ge 0$.

We now proceed by induction and assume that $y \in \mathscr{C}^k((0,+\infty); K^s(\Omega))$, for any $s \ge 0$, for some $k \in \mathbb{N}$. For $t>0$, and $h \in \mathbb{R}$ such that $t+h>0$, we write, in $K^s(\Omega)$, for some $s > 0$,

$$\|h^{-1}(y(t+h)-y(t)) + \mathsf{P}_0 y(t)\|^2_{K^s(\Omega)} = \sum_{j\in\mathbb{N}} \left|h^{-1}(e^{-\mu_j h}-1)+\mu_j\right|^2 \mu_j^s e^{-2\mu_j t}|y_j^0|^2.$$

Note that $\mathsf{P}_0 y(t) \in K^s(\Omega)$ as $y(t) \in K^{s+2}(\Omega) \subset D(\mathsf{P}_0)$ by the induction hypothesis and Proposition 13. As we have $|h^{-1}(e^{-\mu_j h}-1)-\mu_j| \lesssim \mu_j$, the Lebesgue dominated-convergence theorem yields that $\frac{d}{dt}y(t) + \mathsf{P}_0 y(t) = 0$ in $K^s(\Omega)$ if $t>0$. With the induction hypothesis we have $\mathsf{P}_0 y \in \mathscr{C}^k((0,+\infty); K^s(\Omega))$ for any $s \ge 0$, implying that $y \in \mathscr{C}^{k+1}((0,+\infty); K^s(\Omega))$. We thus have $y \in \mathscr{C}^\infty((0,+\infty); K^s(\Omega))$.

Similarly, we prove that $\|h^{-1}(\mathsf{P}_0^k y(t+h) - \mathsf{P}_0^k y(t)) + \mathsf{P}_0^{k+1} y(t)\|_{K^s(\Omega)} \to 0$ as $h \to 0$, for any $s \ge 0$, implying that $\frac{d}{dt}\mathsf{P}_0^k y(t) + \mathsf{P}_0^{k+1} y(t) = 0$ in $K^s(\Omega)$, which allows one to concludes that $\left(\frac{d}{dt}\right)^k y(t) = (-\mathsf{P}_0)^k y(t)$, in $K^s(\Omega)$, for $t>0$. ■

2.3 Additional properties of the parabolic semigroup

We have the following bounds for the semigroup, expressing in particular the natural decay of the L^2-norm of the solution.

Proposition 26. *The semigroup $S(t)$ maps $L^2(\Omega)$ into $L^2(\Omega)$ with*

$$\|S(t)\|_{\mathscr{L}(L^2(\Omega))} \leq e^{-\mu_0 t}, \tag{2.4}$$

and moreover, for some $C > 0$, if $t > 0$,

$$\|S(t)\|_{\mathscr{L}(L^2(\Omega),H_0^1(\Omega))} \leq C/\sqrt{t}, \qquad \|S(t)\|_{\mathscr{L}(L^2(\Omega),D(\mathsf{P}_0))} \leq C/t.$$

In addition, $S(t)$ can be uniquely extended to $H^{-1}(\Omega)$ and there exists $C > 0$ such that $\|S(t)\|_{\mathscr{L}(H^{-1}(\Omega),L^2(\Omega))} \leq C/\sqrt{t}$ if $t > 0$.

Proof. Let $u \in L^2(\Omega)$, with $u = \sum_{j\in\mathbb{N}} u_j \phi_j$. We have

$$\|S(t)u\|^2_{L^2(\Omega)} = \sum_{j\in\mathbb{N}} |u_j|^2 e^{-2\mu_j t} \leq e^{-2\mu_0 t} \sum_{j\in\mathbb{N}} |u_j|^2 = e^{-2\mu_0 t}\|u\|^2_{L^2(\Omega)}.$$

We also have

$$\|S(t)u\|^2_{H_0^1(\Omega)} = \sum_{j\in\mathbb{N}} \mu_j |u_j|^2 e^{-2\mu_j t} \leq \frac{1}{t} \sup_{[0,+\infty)} (xe^{-2x})\|u\|^2_{L^2(\Omega)}.$$

The other operator norm estimates can be proven similarly. ∎

More generally, we have the following result.

Proposition 27. *For $r < 0$ the semigroup $S(t)$ can be uniquely extended as a map from $K^r(\Omega)$ into itself. For $r \geq 0$, the restriction of $S(t)$ to $K^r(\Omega) \subset L^2(\Omega)$ maps $K^r(\Omega)$ into itself.*

For any $r \in \mathbb{R}$, if $u = \sum_{j\in\mathbb{N}} u_j\phi_j \in K^r(\Omega)$, that is, with $(\mu_j^{r/2} u_j)_j \in \ell^2(\mathbb{C})$, then $S(t)u = \sum_{j\in\mathbb{N}} e^{-\mu_j t} u_j \phi_j \in K^r(\Omega)$ for $t \geq 0$ and $S(t)u \in \cap_{s\in\mathbb{R}} K^s(\Omega)$ for $t > 0$. Moreover, if $s \geq 0$, there exists $C_{s,r} > 0$ such that

$$\|S(t)\|_{\mathscr{L}(K^r(\Omega),K^{r+s}(\Omega))} \leq C_{s,r}\ t^{-s/2}, \quad t > 0.$$

If $s = 0$ then one has $\|S(t)\|_{\mathscr{L}(K^r(\Omega))} \leq e^{-\mu_0 t}$, for all $r \in \mathbb{R}$.

To avoid cumbersome notation the extension or restriction of the semigroup $S(t)$ to $K^r(\Omega)$ is also denoted by $S(t)$, for all values of $r \in \mathbb{R}$.

Proof. We only prove that $S(t)$ can be extended to $K^r(\Omega)$ if $r < 0$ and that, in this case, $S(t)u = \sum_{j\in\mathbb{N}} e^{-\mu_j t} u_j \phi_j$ if $u = \sum_{j\in\mathbb{N}} u_j\phi_j$ with $(\mu_j^{r/2} u_j)_j \in \ell^2(\mathbb{C})$. The rest of the proof is similar to that of Proposition 26.

Let thus $r < 0$. If $u \in K^0(\Omega) = L^2(\Omega)$ we have $u = \sum_{j\in\mathbb{N}} u_j\phi_j$, with $u_j = (u, \phi_j)_{L^2(\Omega)}$, and $S(t)u = \sum_{j\in\mathbb{N}} e^{-\mu_j t} u_j\phi_j \in K^0(\Omega)$ for $t \geq 0$. Observe that we have and $u_j = \langle u, \phi_j\rangle_{K^r(\Omega),K^{-r}(\Omega)}$ and

$$\alpha_j(t) = (S(t)u, \phi_j)_{L^2(\Omega)} = \langle S(t)u, \phi_j\rangle_{K^r(\Omega),K^{-r}(\Omega)} = e^{-\mu_j t} u_j.$$

We thus have $(\mu_j^{r/2}\alpha_j(t))_j \in \ell^2(\mathbb{C})$ for $t \geq 0$ and

$$\|S(t)u\|^2_{K^r(\Omega)} = \sum_{j\in\mathbb{N}} \mu_j^r |\alpha_j(t)|^2 \leq \sum_{j\in\mathbb{N}} \mu_j^r |u_j|^2 = \|u\|^2_{K^r(\Omega)}.$$

As $K^0(\Omega)$ is dense in $K^r(\Omega)$ (since $r \leq 0$ here), we see that $S(t)$ can be uniquely extended to $K^r(\Omega)$ and, if $u = \sum_{j\in\mathbb{N}} u_j\phi_j$ in $K^r(\Omega)$, we have $S(t)u = \sum_{j\in\mathbb{N}} e^{-\mu_j t} u_j\phi_j$, with convergence occurs in $K^r(\Omega)$. ∎

With the above results we see that the C_0-semigroup $S(t)$ is differentiable for $t > 0$.

Arguing as in the proof of Lemma 24 we obtain the following result.

Lemma 28. *Let $r \in \mathbb{R}$. The bounded operator $S(t) : K^r(\Omega) \to K^r(\Omega)$ is a C_0-semigroup. It is generated by the unbounded operator $(\mathsf{P}_r, D(\mathsf{P}_r))$ on $K^r(\Omega)$.*

We can state an equivalent version of Lemma 23 and Theorem 22.

Theorem 29. *Let $r \in \mathbb{R}$ and $y^0 \in K^r(\Omega)$. Let also $T \in \mathbb{R}_+ \cup \{+\infty\}$. The function $y(t) = S(t)y^0$ is in*

$$\mathscr{C}^0([0,T]; K^r(\Omega)) \cap \mathscr{C}^\infty((0,T]; K^s(\Omega)), \quad s \in \mathbb{R},$$

and is such that

$$y(0) = y^0 \quad \text{and} \quad \frac{d}{dt} y(t) + \mathsf{P}_r\, y(t) = 0 \text{ holds in } K^r(\Omega) \text{ for } 0 < t \leq T. \tag{2.5}$$

Moreover, $y(t) = S(t)y^0$ is the unique solution of (2.5) in

$$\mathscr{C}^0([0,T]; K^r(\Omega)) \cap \mathscr{C}^1((0,T]; K^r(\Omega)) \cap \mathscr{C}^0((0,T]; K^{r+2}(\Omega)).$$

If we set $y_j(t) = e^{-t\mu_j}\langle y^0, \phi_j\rangle_{K^r(\Omega),K^{-r}(\Omega)}$, for $t \geq 0$ and $j \in \mathbb{N}$, then $(\mu_j^{r/2} y_j(t))_j \in \mathscr{C}^0([0,T], \ell^2(\mathbb{C}))$ and

$$y(t) = \sum_{j\in\mathbb{N}} y_j(t)\phi_j, \quad t \geq 0,$$

with convergence in $K^r(\Omega)$.

We recall that $[0,T]$ (resp. $(0,T]$) means $[0,+\infty)$ (resp. $(0,+\infty)$) if $T=+\infty$.

Observe that if $r,s\in\mathbb{R}$, then we have

$$\mathsf{P}_r^s S(t)u = S(t)\mathsf{P}_r^s u, \qquad u\in D(\mathsf{P}_r^s),\ t\geq 0. \tag{2.6}$$

Above it was mentioned that $S(t)$ is selfadjoint on $L^2(\Omega)$. Similarly, using $L^2(\Omega)$ as a pivot space, with Proposition 27 we obtain the following result.

Proposition 30. *Let $s\in\mathbb{R}$, $u\in K^s(\Omega)$, and $v\in K^{-s}(\Omega)$. We have, for $t\geq 0$,*

$$\langle S(t)u,\overline{v}\rangle_{K^s(\Omega),K^{-s}(\Omega)} = \langle u, S(t)\overline{v}\rangle_{K^s(\Omega),K^{-s}(\Omega)}.$$

If moreover $t>0$, then for $s,r\in\mathbb{R}$, $u\in K^s(\Omega)$, and $v\in K^r(\Omega)$, we have

$$\langle S(t)u,\overline{v}\rangle_{K^{-r}(\Omega),K^r(\Omega)} = \langle u, S(t)\overline{v}\rangle_{K^s(\Omega),K^{-s}(\Omega)}.$$

The second statement makes perfect sense by Proposition 27.

If $\mathbb{S}(t)$ is some semigroup on a Banach space X, for every $x\in X$, we have $\mathbb{S}(t)x\to x$ in X as $t\to 0^+$. Note that we do *not* have $\|\mathbb{S}(t)-\mathrm{Id}_X\|_{\mathscr{L}(X)}\to 0$ in general, as this is equivalent to having a bounded generator. In the present case of the parabolic semigroup, we however have $\|S(t)-\mathrm{Id}\|_{\mathscr{L}(D(\mathsf{P}_0),L^2(\Omega))}=\mathcal{O}(t)$ for $t>0$. This is stated in the following proposition in a more general form.

Proposition 31. *Let $r,s\in\mathbb{R}$ with $0\leq s\leq 2$. There exists $C>0$ such that $\|S(t)-\mathrm{Id}\|_{\mathscr{L}(K^{r+s}(\Omega),K^r(\Omega))}\leq Ct^{s/2}$, for $t\geq 0$.*

Proof. Let $u\in K^{r+s}(\Omega)$. Then $u=\sum_{j\in\mathbb{N}} u_j\phi_j$ with $(\mu_j^{(r+s)/2}u_j)_j\in\ell^2(\mathbb{C})$. For $t>0$ we write $S(t)u-u=\sum_{j\in\mathbb{N}}(e^{-\mu_j t}-1)u_j\phi_j$. Thus, we have

$$\|S(t)u-u\|^2_{K^r(\Omega)} = \sum_{j\in\mathbb{N}}(1-e^{-\mu_j t})^2\mu_j^r|u_j|^2 \leq t^s\sum_{j\in\mathbb{N}}\mu_j^{r+s}|u_j|^2 = t^s\|u\|^2_{K^{r+s}}.$$

as $0\leq 1-e^{-\alpha}\leq \alpha^{s/2}$ for $\alpha\geq 0$, as $0\leq s/2\leq 1$. ∎

Further regularity results and bounds are given by the following proposition.

Proposition 32. *Let $r\in\mathbb{R}$. If $y^0\in K^r(\Omega)$ and $y(t)=S(t)y^0$ then*

$$y\in\mathscr{C}([0,+\infty);K^r(\Omega))\cap\mathscr{C}^\infty((0,+\infty);K^s(\Omega)),\ s\in\mathbb{R},$$

and $y\in L^2(0,+\infty;K^{r+1}(\Omega))\cap H^1(0,\infty;K^{r-1}(\Omega))$. Moreover, there exists $C>0$ such that

$$\|y\|_{L^2(0,+\infty;K^{r+1}(\Omega))}+\|y\|_{H^1(0,+\infty;K^{r-1}(\Omega))}\leq C\|y^0\|_{K^r(\Omega)}.$$

In particular, the equation $\frac{d}{dt}y+\mathsf{P}_{r-1}\,y=0$ holds in $L^2(0,+\infty;K^{r-1}(\Omega))$.

Proof. One way to prove this result is to use the spectral representation (2.3) of the semigroup. If $y^0 \in K^r(\Omega)$ with $y^0 = \sum_{j\in\mathbb{N}} y_j^0 \phi_j$ where $(\mu_j^{r/2} y_j^0)_{j\in\mathbb{N}} \in \ell^2(\mathbb{C})$ we have

$$y(t) = S(t)y^0 = \sum_{j\in\mathbb{N}} e^{-\mu_j t} y_j^0 \phi_j,$$

and by (1.14) we have

$$\|y\|^2_{L^2(0,+\infty;K^{r+1}(\Omega))} = \sum_{j\in\mathbb{N}} \int_0^{+\infty} \mu_j^{r+1} e^{-2\mu_j t} |y_j^0|^2 dt = \frac{1}{2} \sum_{j\in\mathbb{N}} \mu_j^r |y_j^0|^2 = \frac{1}{2} \|y^0\|^2_{K^r(\Omega)},$$

yielding $y \in L^2(0,T;K^{r+1}(\Omega))$.

Alternatively, for $t > 0$, as $y \in \mathscr{C}^\infty((0,+\infty), K^s(\Omega))$ for any $s \in \mathbb{R}$, we can compute

$$0 = \big(\frac{d}{dt} y(t) + \mathsf{P}_r y(t), y(t)\big)_{K^r(\Omega)} = \frac{1}{2}\frac{d}{dt} \|y(t)\|^2_{K^r(\Omega)} + (\mathsf{P}_0 y(t), y(t))_{K^r(\Omega)}.$$

For $T > 0$, integrating for $t \in (0,T)$, we find

$$\frac{1}{2}\|y(T)\|^2_{K^r(\Omega)} + \int_0^T (\mathsf{P}_r y(t), y(t))_{K^r(\Omega)}\, dt = \frac{1}{2}\|y^0\|^2_{K^r(\Omega)}.$$

By Lemma 16 we write

$$\frac{1}{2}\|y(T)\|^2_{K^r(\Omega)} + \|y\|^2_{L^2(0,T;K^{r+1}(\Omega))} = \frac{1}{2}\|y^0\|^2_{K^r(\Omega)}.$$

We then let $T \to +\infty$ and, by the K^r-norm decay given by Proposition 27, we obtain the same equality as above.

Now we write, as $\frac{d}{dt} y(t) + \mathsf{P}_r y(t) \in K^r(\Omega)$ for $t > 0$,

$$\begin{aligned}
\|y\|^2_{H^1(0,+\infty;K^{r-1}(\Omega))} &= \|y\|^2_{L^2(0,+\infty;K^{r-1}(\Omega))} + \Big\|\frac{d}{dt} y\Big\|^2_{L^2(0,+\infty;K^{r-1}(\Omega))} \\
&\lesssim \|y\|^2_{L^2(0,+\infty;K^{r+1}(\Omega))} + \|\mathsf{P}_r y\|^2_{L^2(0,+\infty;K^{r-1}(\Omega))} \\
&\lesssim \|y\|^2_{L^2(0,+\infty;K^{r+1}(\Omega))} + \|\mathsf{P}_{r-1} y\|^2_{L^2(0,+\infty;K^{r-1}(\Omega))} \\
&\lesssim \|y\|^2_{L^2(0,+\infty;K^{r+1}(\Omega))}
\end{aligned}$$

by (1.22), which gives the second estimation from the previous one. We conclude that $y \in H^1(0,+\infty;K^{r-1}(\Omega))$. ■

Remark 33. By abuse of notation one often writes that the equation $\frac{d}{dt} y + \mathsf{P}_0 y = 0$ holds in $L^2(0,+\infty;K^{r-1}(\Omega))$.

Particular and important cases that are often used in practice are the following ones ($r = 0$ and $r = 1$).

Corollary 34. *If $y^0 \in L^2(\Omega)$ and $y(t) = S(t)y^0$ then*

$$y \in \mathscr{C}([0,+\infty); L^2(\Omega)) \cap \mathscr{C}^\infty((0,+\infty); K^s(\Omega)), \ \ s \in \mathbb{R},$$

and

$$y \in L^2(0,+\infty; H_0^1(\Omega)) \cap H^1(0,+\infty; H^{-1}(\Omega)).$$

Moreover, there exists $C > 0$ such that

$$\|y(t)\|_{L^2(0,+\infty;H_0^1(\Omega))} + \|y(t)\|_{H^1(0,+\infty;H^{-1}(\Omega))} \le C\|y^0\|_{L^2(\Omega)}.$$

This implies in particular that the equation $\frac{d}{dt}y + \mathsf{P}_{-1}\, y = 0$ holds in $L^2(0,+\infty; H^{-1}(\Omega))$.

Corollary 35. *If $y^0 \in H_0^1(\Omega)$ and $y(t) = S(t)y^0$ then*

$$y \in \mathscr{C}([0,+\infty); H_0^1(\Omega)) \cap \mathscr{C}^\infty((0,+\infty); K^s(\Omega)), \ \ s \in \mathbb{R},$$

and

$$y \in L^2(0,+\infty; D(\mathsf{P}_0)) \cap H^1(0,\infty; L^2(\Omega)).$$

Moreover, there exists $C > 0$ such that

$$\|y\|_{L^2(0,+\infty;D(\mathsf{P}_0))} + \|y\|_{H^1(0,+\infty;L^2(\Omega))} \le C\|y^0\|_{L^2(\Omega)}.$$

In particular the equation $\frac{d}{dt}y + \mathsf{P}_0 y = 0$ holds in $L^2((0,+\infty)\times\Omega)$.

We conclude this section with the following uniqueness result.

Proposition 36. *Let $T \in \mathbb{R}_+ \cup \{+\infty\}$ and let $r \in \mathbb{R}$. If y is in*

$$\mathscr{C}^0([0,T]; K^{r+1}(\Omega)) \cap L^2(0,T; K^{r+2}(\Omega)) \cap H^1(0,T; K^r(\Omega))$$

and satisfies $y(0) = 0$ and

$$\frac{d}{dt}y + \mathsf{P}_r y = 0 \ \ \text{in } L^2(0,T; K^r(\Omega)),$$

then $y = 0$.

We recall that $[0,T]$ (resp. $(0,T]$) means $[0,+\infty)$ (resp. $(0,+\infty)$) if $T = +\infty$.

Proof. We have $\mathsf{P}_r y \in L^2(0,T; K^r(\Omega)) \subset L^2(0,T; K^{r-1}(\Omega))$. Note that we have $\mathsf{P}_r y = \mathsf{P}_{r-1} y \in \mathscr{C}^0([0,T]; K^{r-1}(\Omega))$. We thus find that $\frac{d}{dt}y \in \mathscr{C}^0([0,T]; K^{r-1}(\Omega))$. This implies that $y \in \mathscr{C}^1([0,T]; K^{r-1}(\Omega))$.

As $y \in \mathscr{C}^1([0,T]; K^{r-1}(\Omega)) \cap \mathscr{C}^0([0,T]; K^{r+1}(\Omega))$, the equation

$$\frac{d}{dt}y(t) + \mathsf{P}_{r-1}\, y(t) = 0$$

holds in $K^{r-1}(\Omega)$ for $0 < t \le T$. Since $y(0) = 0$, by the second part of Theorem 29 with $r-1$ in place of r, we obtain the result. ∎

3 The nonhomogeneous parabolic Cauchy problem

We now consider the nonhomogeneous parabolic equation

$$\frac{d}{dt}y + \mathsf{P}_0 y = f, \quad y_{|t=0} = y^0, \tag{3.1}$$

The mild solution is given by the Duhamel formula

$$y(t) = S(t)y^0 + \int_0^t S(t-\sigma)f(\sigma)d\sigma.$$

If $f \in L^1_{\text{loc}}(0,\infty; D(\mathsf{P}_0))$ The second term is called the Duhamel term.

3.1 Properties of the Duhamel term

Let $r \in \mathbb{R}$. For $f \in L^2(0,T; K^r(\Omega))$, with the properties of the semigroup $S(t)$ on $K^r(\Omega)$ we can define

$$\Psi_r(f)(t) = \int_0^t S(t-\sigma)f(\sigma)\,d\sigma, \quad t \geq 0. \tag{3.2}$$

Theorem 37. *Let $T \in \mathbb{R}^+ \cup \{+\infty\}$. The map Ψ_r maps linearly and continuously $L^2(0,T; K^r(\Omega))$ into*

$$\mathscr{C}^0([0,T]; K^{r+1}(\Omega)) \cap L^2(0,T; K^{r+2}(\Omega)) \cap H^1(0,T; K^r(\Omega)).$$

If $f \in L^2(0,T; K^r(\Omega))$ then

$$\frac{d}{dt}\Psi_r(f) + \mathsf{P}_r\,\Psi_r(f) = f.$$

Proof. We set $\mathscr{G} = L^\infty(0,T; K^{r+1}(\Omega)) \cap L^2(0,T; K^{r+2}(\Omega)) \cap H^1(0,T; K^r(\Omega))$ equiped with the norm

$$z \mapsto \|z\|_{L^\infty(0,T;K^{r+1}(\Omega))} + \|z\|_{L^2(0,T;K^{r+2}(\Omega))} + \|z\|_{H^1(0,T;K^r(\Omega))}.$$

We first consider $f \in \mathscr{C}([0,T]; K^r(\Omega))$ as this space is dense in $L^2(0,T; K^r(\Omega))$. Then, for all $t \in [0,T]$ we have $f(t) = \sum_{j\in\mathbb{N}} f_j(t)\phi_j$ with $f_j(t) = \langle f(t), \phi_j \rangle_{K^r(\Omega),K^{-r}(\Omega)}$ and $(\mu_j^{r/2} f_j(t))_j \in \mathscr{C}([0,T]; \ell^2(\mathbb{C}))$. For $N \in \mathbb{N}$, we set $f^N(t) = \sum_{j=0}^N f_j(t)\phi_j$, we have $\lim_{N\to\infty} f^N = f$ in $L^2(0,T; K^r(\Omega))$. Hence, the space

$$\mathscr{E} := \Big\{ g = \sum_{j=0}^N g_j(t)\phi_j;\ N \in \mathbb{N},\ (\mu_j^{r/2} g_j(t))_j \subset \mathscr{C}([0,T]; \ell^2(\mathbb{C})) \Big\}$$

is dense in $L^2(0,T;K^r(\Omega))$. Note that we have

$$\mathscr{E} \subset \bigcap_{s\in\mathbb{R}} \mathscr{C}([0,T];K^s(\Omega)),$$

Recalling that $\phi_j \in \cap_{s\in\mathbb{R}} K^s(\Omega)$ (see Section 1.3). We consider $g = \sum_{j=0}^N g_j(t)\phi_j \in \mathscr{E}$ and we set, for any $s \in \mathbb{R}$,

$$z(t) = \Psi_s(g)(t) = \int_0^t S(t-\sigma)g(\sigma)d\sigma = \sum_{j=0}^N \int_0^t e^{(\sigma-t)\mu_j} g_j(\sigma)\, d\sigma\, \phi_j.$$

With the uniform continuity property of Proposition 31 we have $S(t+h) - S(t) = \mathcal{O}(h)$ in $\mathscr{L}(K^{s+2}(\Omega), K^s(\Omega))$ for any $s \in \mathbb{R}$. We then see that $z \in \mathscr{C}^\infty([0,T];K^s(\Omega))$ for any $s \in \mathbb{R}$. Moreover, we find

$$\frac{d}{dt} z(t) + \mathsf{P}_s\, z(t) = g(t), \quad t \geq 0,$$

for any $s \in \mathbb{R}$. We may thus write

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|z(t)\|^2_{K^{r+1}(\Omega)} + \|z(t)\|^2_{K^{r+2}(\Omega)} &= \big(\frac{d}{dt} z(t), z(t)\big)_{K^{r+1}(\Omega)} + (\mathsf{P}_{r+1}\, z(t), z(t))_{K^{r+1}(\Omega)}\\
&= (g(t), z(t))_{K^{r+1}(\Omega)}\\
&\leq \|g(t)\|_{K^r(\Omega)} \|z(t)\|_{K^{r+2}(\Omega)},
\end{aligned}$$

yielding, after integration with respect to time t, using that $z(0) = 0$,

$$\begin{aligned}
\frac{1}{2}\|z(t)\|^2_{K^{r+1}(\Omega)} + \|z\|^2_{L^2(0,t;K^{r+2}(\Omega))} &\leq \int_0^t \|g(\sigma)\|_{K^r(\Omega)} \|z(\sigma)\|_{K^{r+2}(\Omega)} d\sigma\\
&\leq \|g\|_{L^2(0,t;K^r(\Omega))} \|z\|_{L^2(0,t;K^{r+2}(\Omega))}, \quad t \in [0,T].
\end{aligned}$$

With the Young inequality we obtain

$$\|z\|_{L^\infty(0,T;K^{r+1}(\Omega))} + \|z\|_{L^2(0,T;K^{r+2}(\Omega))} \lesssim \|g\|_{L^2(0,T;K^r(\Omega))}. \tag{3.3}$$

From the equation satisfied by z,

$$\frac{d}{dt} z(t) + \mathsf{P}_r\, z(t) = g(t), \quad t \geq 0, \tag{3.4}$$

we also have $\|z\|_{H^1(0,T;K^r(\Omega))} \lesssim \|g\|_{L^2(0,T;K^r(\Omega))}$. From the density of $\mathscr{E}$ in $L^2(0,T;K^r(\Omega))$, these estimates show that Ψ_r maps $L^2(0,T;K^r(\Omega))$ continuously into $\mathscr{G}$.

If $(f^N)_N \subset \mathscr{E}$ converges to $f \in L^2(0,T;K^r(\Omega))$, then $\Psi_r(f^N)$ converges to $\Psi_r(f)$ in $\mathscr{G}$. From (3.4) one finds that

$$\frac{d}{dt}\Psi_r(f) + \mathsf{P}_r\, \Psi_r(f) = f \ \text{ in } L^2(0,T;K^r(\Omega)).$$

As $\Psi_r(f^N) \in \mathscr{C}^0([0,T];K^{r+1}(\Omega))$, the estimate in (3.3) shows that $\Psi_r(f) \in \mathscr{C}^0([0,T];K^{r+1}(\Omega))$, by uniform convergence. ■

We state the regularity result in the case $r = 0$.

Corollary 38. *Let $T \in \mathbb{R}^+ \cup \{+\infty\}$. The map Ψ_0 given in (3.2) in the case $r = 0$ maps linearly and continuously $L^2((0,T) \times \Omega)$ into*

$$\mathscr{C}^0([0,T]; H_0^1(\Omega)) \cap L^2(0,T; H^2(\Omega) \cap H_0^1(\Omega)) \cap H^1(0,T; L^2(\Omega)).$$

3.2 Abstract solutions of the nonhomogeneous semigroup equations

Observe that the regularity of the Duhamel term $\Psi_r(f)$ for $f \in L^2(0,T; K^r(\Omega))$ given by Theorem 37 coincides with that of the free evolution term $S(t)y^0$ if $y^0 \in K^{r+1}(\Omega)$ according to Theorem 29 and Proposition 32:

$$S(t)y^0 \in \mathscr{C}([0,+\infty); K^{r+1}(\Omega)) \cap L^2(0,+\infty; K^{r+2}(\Omega)) \cap H^1(0,\infty; K^r(\Omega)).$$

However, the Duhamel term does not exhibit the same degree of regularization as the term $S(t)$. In general, $\Psi_r(f) \notin \mathscr{C}^\infty((0,+\infty); K^s(\Omega))$.

This observation gives a natural regularity level for both the initial condition y^0 and the source term to state an existence and uniqueness result for a solution of the equation $\frac{d}{dt}y + \mathsf{P}_r y = f$, for some $r \in \mathbb{R}$.

Theorem 39. *Let $T \in \mathbb{R}^+ \cup \{+\infty\}$ and $r \in \mathbb{R}$. Let $f \in L^2(0,T; K^r(\Omega))$ and $y^0 \in K^{r+1}(\Omega)$. There exists a unique function $y \in \mathscr{C}^0([0,T]; K^{r+1}(\Omega)) \cap L^2(0,T; K^{r+2}(\Omega)) \cap H^1(0,T; K^r(\Omega))$ that is solution of the parabolic equation*

$$\frac{d}{dt}y + \mathsf{P}_r y = f$$

in $L^2(0,T; K^r(\Omega))$ and satisfies moreover $y(0) = y^0$. The solution is given by

$$y(t) = S(t)y^0 + \int_0^t S(t-\sigma)f(\sigma)d\sigma.$$

Moreover, there exists $C > 0$ such that

$$\|y\|_{L^\infty(0,T;K^{r+1}(\Omega))} + \|y\|_{L^2(0,T;K^{r+2}(\Omega))} + \Big\|\frac{d}{dt}y\Big\|_{L^2(0,T;K^r(\Omega))} \leq C\Big(\|y^0\|_{K^{r+1}(\Omega)} + \|f\|_{L^2(0,T;K^r(\Omega))}\Big).$$

Remark 40. Let $s \in \mathbb{R}$ and $s' > 0$. If $y^0 \in K^{s+s'+1}(\Omega)$ and $f \in L^2([0,T]; K^{s+s'}(\Omega))$ then Theorem 39 applies both in the cases $r = s$ and $r = s + s'$. Uniqueness shows that the two obtained solutions coincide. If fact, both are given by the same Duhamel formula.

Proof of Theorem 39. First, we address uniqueness. Assume that there are two solutions in $\mathscr{C}^0([0,T]; K^{r+1}(\Omega)) \cap L^2(0,T; K^{r+2}(\Omega)) \cap H^1(0,T; K^r(\Omega))$. Then, their difference $z(t)$ lies in that space and is solution to $\frac{d}{dt}z + \mathsf{P}_r z = 0$ in $L^2(0,T; K^r(\Omega))$ and $z(0) = 0$. By Proposition 36 we find that $z = 0$.

Second, we address existence. If we set

$$y(t) = S(t)y^0 + \Psi_r(f)(t),$$

we see by Theorem 29, Proposition 32, and Theorem 37, using the linearity of the equation, that

$$y \in \mathscr{C}^0([0,T]; K^{r+1}(\Omega)) \cap L^2(0,T; K^{r+2}(\Omega)) \cap H^1(0,T; K^r(\Omega))$$

and that

$$\frac{d}{dt}y + \mathsf{P}_r y = f$$

holds in $L^2(0,T; K^r(\Omega))$. ∎

3.3 Strong solutions

In general, one calls a strong solution a function $y \in \mathscr{C}^0([0,T]; H_0^1(\Omega)) \cap L^2(0,T; D(\mathsf{P}_0)) \cap H^1(0,T; L^2(\Omega))$ that solves the equation

$$\frac{d}{dt}y + \mathsf{P}_0 y = f$$

in $L^2((0,T) \times \Omega)$. Its definition does not require the use of the Sobolev scale $\big(K^r(\Omega)\big)_{r\in\mathbb{R}}$ of Section 1.3. All terms in the equation are functions in Ω. The uniqueness and the existence of such strong solution under regularity assumptions for the initial condition y^0 and the source term f are given by Theorem 39 in the case $r = 0$, which we write explicitly in the following corollary.

Corollary 41 (strong solutions - first version)**.** *Let $T \in \mathbb{R}^+ \cup \{+\infty\}$. Let $f \in L^2((0,T)\times\Omega)$ and $y^0 \in H_0^1(\Omega)$. There exists a unique function $y \in \mathscr{C}^0([0,T]; H_0^1(\Omega)) \cap L^2(0,T; D(\mathsf{P}_0)) \cap H^1(0,T; L^2(\Omega))$ that is solution of the parabolic equation*

$$\frac{d}{dt}y + \mathsf{P}_0 y = f$$

in $L^2((0,T) \times \Omega)$ and satisfies moreover $y(0) = y^0$. The solution is given by

$$y(t) = S(t)y^0 + \int_0^t S(t-\sigma)f(\sigma)d\sigma.$$

Moreover, there exists $C > 0$ such that

$$\|y\|_{L^\infty(0,T;H_0^1(\Omega))} + \|y\|_{L^2(0,T;D(\mathsf{P}_0))} + \Big\|\frac{d}{dt}y\Big\|_{L^2((0,T)\times\Omega)} \\ \leq C\Big(\|y^0\|_{H_0^1(\Omega)} + \|f\|_{L^2((0,T)\times\Omega)}\Big).$$

The term "strong solution" is sometimes used for more regular solutions, namely solution that lie in $\mathscr{C}^0([0,T];D(\mathsf{P}_0))$. They are given by Theorem 39 in the case $r = 1$.

Corollary 42 (strong solutions - second version)**.** *Let $T \in \mathbb{R}^+ \cup \{+\infty\}$. Let $f \in L^2(0,T;H_0^1(\Omega))$ and $y^0 \in D(\mathsf{P}_0)$. There exists a unique function $y \in \mathscr{C}^0([0,T];D(\mathsf{P}_0)) \cap L^2(0,T;K^3(\Omega)) \cap H^1(0,T;H_0^1(\Omega))$ that is solution of the parabolic equation*

$$\frac{d}{dt}y + \mathsf{P}_0 y = \frac{d}{dt}y + \mathsf{P}_1 y = f$$

in $L^2(0,T;H_0^1(\Omega))$ and satisfies moreover $y(0) = y^0$. The solution is given by

$$y(t) = S(t)y^0 + \int_0^t S(t-\sigma)f(\sigma)d\sigma.$$

Moreover, there exists $C > 0$ such that

$$\|y\|_{L^\infty(0,T;D(\mathsf{P}_0))} + \|y\|_{L^2(0,T;K^3(\Omega))} + \Big\|\frac{d}{dt}y\Big\|_{L^2(0,T;H_0^1(\Omega))} \\ \leq C\Big(\|y^0\|_{D(\mathsf{P}_0)} + \|f\|_{L^2(0,T;H_0^1(\Omega))}\Big).$$

If one further assumes that $f \in \mathscr{C}^0((0,T];L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$, since $y \in \mathscr{C}^0([0,T];D(\mathsf{P}_0))$ the semigroup equation further gives that $y \in \mathscr{C}^1((0,T];L^2(\Omega))$. We then obtain a classical solution for an abstract nonhomogeneous semigroup equation.

3.4 Weak solutions

For a regularity lower than that of strong solutions as introduced in Section 3.3, with $y^0 \in L^2(\Omega)$ and $f \in L^2(0,T;H^{-1}(\Omega))$, Theorem 39 for $r = -1$ yields the existence and uniqueness of solution in $\mathscr{C}^0([0,T];L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega))$. One is often inclined to use a weak formulation to characterize these solutions.

Definition 43. Let $T \in \mathbb{R}_+ \cup \{+\infty\}$. Let $y^0 \in L^2(\Omega)$ and $f \in L^2(0,T;H^{-1}(\Omega))$. One says that $y \in \mathscr{C}^0([0,T];L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$ is a weak solution to the parabolic equation

$$\frac{d}{dt}y + \mathsf{P}_0 y = f, \qquad y(0) = y^0,$$

if we have

$$(y(t),\psi)_{L^2(\Omega)} + (y,\psi)_{L^2(0,t;H_0^1(\Omega))} = (y^0,\psi)_{L^2(\Omega)} + \int_0^t \langle f(\sigma),\overline{\psi}\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\,d\sigma,$$

for all $\psi \in H_0^1(\Omega)$ and for all $t \in [0,T]$.

We recall that the H_0^1-norm is given by (1.19), yielding

$$\begin{aligned}(y,\psi)_{L^2(0,t;H_0^1(\Omega))} &= \int_0^t (\mathsf{P}_0^{1/2} y(\sigma), \mathsf{P}_0^{1/2}\psi)_{L^2(\Omega)}\,d\sigma \\ &= \int_0^t \langle P_0\, y(\sigma),\overline{\psi}\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\,d\sigma = \int_0^t \langle y(\sigma), P_0\,\overline{\psi}\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\,d\sigma\end{aligned}$$

In fact, it is equivalent if one chooses $\psi \in D(P_0)$ yielding the form

$$(y(t),\psi)_{L^2(\Omega)} + (y,P_0\psi)_{L^2((0,t)\times\Omega)} = (y^0,\psi)_{L^2(\Omega)} + \int_0^t \langle f(\sigma),\overline{\psi}\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\,d\sigma,$$

for all $t \in [0,T]$.

Theorem 44. *Let $T \in \mathbb{R}_+ \cup \{+\infty\}$. Let $y^0 \in L^2(\Omega)$ and $f \in L^2(0,T;H^{-1}(\Omega))$. There exists a unique weak solution y to the parabolic equation*

$$\frac{d}{dt}y + \mathsf{P}_0 y = f, \qquad y(0) = y^0,$$

in the sense of Definition 43. It coincides with the solution of the semigroup equation

$$\frac{d}{dt}y + \mathsf{P}_{-1}\, y = f, \qquad y(0) = y^0,$$

given by Theorem 39 in the case $r = -1$. In particular we have

$$y \in \mathscr{C}^0([0,T];L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega)).$$

By Remark 40 we see that a weak solution associated with data with the following regularity, $y^0 \in H_0^1(\Omega)$ and $f \in L^2((0,T)\times\Omega)$, is in fact a strong solution as given by Corollary 41.

Proof. First, we address uniqueness. Assume that there are two solutions in $\mathscr{C}^0([0,T];L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$. Then, their difference $z(t)$ lies in that space and is solution to

$$(z(t),\psi)_{L^2(\Omega)} + (z,\psi)_{L^2(0,t;H_0^1(\Omega))} = 0, \qquad \psi \in H_0^1(\Omega),\ t \in [0,T],$$

which we write

$$\begin{aligned}0 &= (z(t),\psi)_{L^2(\Omega)} + \int_0^t \langle \mathsf{P}_{-1} z(\sigma),\overline{\psi}\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\,d\sigma \\ &= \Big\langle z(t) + \int_0^t \mathsf{P}_{-1}\, z(\sigma)\,d\sigma, \overline{\psi}\Big\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}.\end{aligned}$$

As a result, we have, in $H^{-1}(\Omega)$,

$$z(t) + \int_0^t \mathsf{P}_{-1}\, z(\sigma) = 0, \quad 0 \le t \le T.$$

As $\mathsf{P}_{-1}\, z \in L^2(0,T;H^{-1}(\Omega))$ we find that $z \in H^1(0,T;H^{-1}(\Omega))$ and $\frac{d}{dt}z + \mathsf{P}_{-1}\, z = 0$ holds in $L^2(0,T;H^{-1}(\Omega))$. With Proposition 36 we conclude that $z = 0$.

Second, we address existence. Let

$$y \in \mathscr{C}^0([0,T];L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega))$$

be the solution to $\frac{d}{dt}y + \mathsf{P}_{-1}\, y = f$ in $L^2(0,T;H^{-1}(\Omega))$ and $y(0) = y^0$, as given by Theorem 39. We then see that

$$\Big\langle \frac{d}{dt}y(t), \overline{\psi}\Big\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + \langle \mathsf{P}_{-1}\, y(t), \overline{\psi}\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = \langle f(t), \overline{\psi}\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}$$

holds in $L^2(0,T)$. As $y \in L^2(0,T;H_0^1(\Omega))$, we observe that for almost every $t \in (0,T)$

$$\langle \mathsf{P}_{-1}\, y(t), \overline{\psi}\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = (y(t), \psi)_{H_0^1(\Omega)}.$$

Since $\langle y, \overline{\psi}\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \in H^1(0,T)$ and $\frac{d}{dt}\langle y, \overline{\psi}\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = \big\langle \frac{d}{dt}y(t), \overline{\psi}\big\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}$ we find that, for all $t \in (0,T)$,

$$\langle y(t), \overline{\psi}\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} - \langle y(0), \overline{\psi}\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + \int_0^t (y(\sigma), \psi)_{H_0^1(\Omega)}\, d\sigma$$
$$= \int_0^t \langle f(\sigma), \overline{\psi}\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\, d\sigma.$$

Since $y(t) \in \mathscr{C}([0,T],L^2(\Omega))$ we find that $\langle y(t), \overline{\psi}\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = (y(t), \psi)_{L^2(\Omega)}$ implying that y is a weak solution in the sense of Definition 43. ■

We observe that one can simply use solutions in the sense of distributions to define weak solutions.

Proposition 45. *Let $T \in \mathbb{R}_+ \cup \{+\infty\}$. Let $y^0 \in L^2(\Omega)$ and $f \in L^2(0,T;H^{-1}(\Omega))$. There exists a unique $y \in \mathscr{C}^0([0,T];L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$ such that*

$$\partial_t y + P_0 y = f \text{ in } \mathscr{D}'((0,T)\times\Omega), \qquad y(0) = y^0. \tag{3.5}$$

It coincides with the unique solution of the semigroup equation

$$\frac{d}{dt}y + \mathsf{P}_{-1}\, y = f, \qquad y(0) = y^0,$$

given by Theorem 39 (in the case $r = -1$) and thus with the unique weak solution given in Theorem 44.

Note that in (3.5) the occurence of the operator P_0 acting on y in the sense of distribution, not to be confused with the unbounded operator P_0.

Proof. We first treat uniqueness. Let $y \in \mathscr{C}^0([0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega))$ be such that

$$\frac{d}{dt}y + P_0 y = 0 \text{ in } \mathscr{D}'((0,T)\times\Omega), \qquad y(0) = 0.$$

As we have $y \in L^2(0,T; H_0^1(\Omega))$ we have $P_0 y = \mathsf{P}_{-1} y \in L^2(0,T; H^{-1}(\Omega))$, yielding $\partial_t y \in L^2(0,T; H^{-1}(\Omega))$ and thus $y \in H^1(0,T; H^{-1}(\Omega))$. Thus $\frac{d}{dt}y + \mathsf{P}_{-1} y = 0$. As $y_{|t=0} = 0$, the uniqueness part of Theorem 39 in the case $r = -1$ gives $y \equiv 0$.

Conversely, if y is the solution given by Theorem 39 we have $\frac{d}{dt}y + \mathsf{P}_{-1} y = f \in L^2(0,T; H^{-1}(\Omega))$. As $\mathsf{P}_{-1} y(t) = P_0 y(t)$ for allmost all $t \in (0,T)$ we have $\partial_t y + P_0 y = f$ in $\mathscr{D}'((0,T)\times\Omega)$. ∎

Weak solutions are also called *solutions by transposition* because of the alternative formulation given in the following proposition.

Proposition 46. *Let $T \in \mathbb{R}_+ \cup \{+\infty\}$. Let $y^0 \in L^2(\Omega)$ and $f \in L^2(0,T; H^{-1}(\Omega))$. There exists a unique $y \in \mathscr{C}^0([0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega))$ such that*

$$(y(t), \varphi(t))_{L^2(\Omega)} + \int_0^t \langle y(\sigma), (-\partial_\sigma + \mathsf{P}_{-1})\overline{\varphi}(\sigma)\rangle_{H_0^1(\Omega), H^{-1}(\Omega)} d\sigma$$
$$= (y^0, \varphi(0))_{L^2(\Omega)} + \int_0^t \langle f(\sigma), \overline{\varphi}(\sigma)\rangle_{H^{-1}(\Omega), H_0^1(\Omega)}\, d\sigma, \quad (3.6)$$

for all $\varphi \in \mathscr{C}^0([0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega)) \cap H^1(0,T; H^{-1}(\Omega))$ and for all $t \in [0,T]$. It coincides with the unique solution of of the semigroup equation

$$\frac{d}{dt}y + \mathsf{P}_{-1}\, y = f, \qquad y(0) = y^0,$$

given by Theorem 39 (in the case $r = -1$) and thus with the unique weak solution given in Theorem 44.

Proof. First, we treat uniqueness. Let $y \in \mathscr{C}^0([0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega))$ be such that (3.6) holds. But choosing φ constant with respect to t we find that the property of Definition 43 is fulfilled. Hence, y is the weak solution given in Theorem 44.

Conversely, let y be the solution given by Theorem 39 in the case $r = -1$. We then see that

$$\langle \frac{d}{dt} y(t), \overline{\varphi}(t)\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \langle \mathsf{P}_{-1}\, y(t), \overline{\varphi}(t)\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \langle f(t), \overline{\varphi}(t)\rangle_{H^{-1}(\Omega), H_0^1(\Omega)}$$

holds in $L^2(0,T)$. As the identity

$$\left\langle \frac{d}{dt} y(t), \overline{\varphi}(t) \right\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \frac{d}{dt} \left\langle y(t), \overline{\varphi}(t) \right\rangle_{L^2(\Omega), L^2(\Omega)} - \left\langle y(t), \frac{d}{dt} \overline{\varphi}(t) \right\rangle_{H_0^1(\Omega), H^{-1}(\Omega)}$$

holds in $L^2(0,T)$. We see that an adaptation of the proof of Theorem 44 shows that y is indeed a solution of (3.6). ■

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