Some elements of functional analysis

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Contents

Here, X and Y will denote Banach spaces with their norms denoted by $\|.\|_X$, $\|.\|_Y$, or simply $\|.\|$ when there is no ambiguity.

1 Linear operators in Banach spaces

An operator A from X to Y is a linear map on its domain, a linear subspace of X, to Y. One denotes by $D(A)$ the domain of this operator. An operator from X to Y is thus characterized by its domain and how it acts on this domain. Operators defined this way are usually referred to as *unbounded operators*. One writes $(A, D(A))$ to denote the operator along with its domain. The set of linear operators from X to Y is denoted by $\mathscr{L}(X, Y)$.

If $D(A)$ is dense in X the operator is said to be densely defined. If $D(A) = X$ one says that the operator A is *on* X to Y.

The range of the operator is denoted by $\operatorname{Ran}(A)$, that is,

$$\operatorname{Ran}(A) = \{Ax;\ x \in D(A)\} \subset Y,$$

and its kernel, $\ker(A)$, is the set of all $x \in D(A)$ such that $Ax = 0$.

The graph of A, $G(A)$, is given by

$$G(A) = \{(x, Ax);\; x \in D(A)\} \subset X \times Y.$$

We naturally endow $X \times Y$ with the norm $\|(x,y)\|^2_{X\times Y} = \|x\|^2_X + \|y\|^2_Y$ which makes $X \times Y$ a Banach space. One says that A is a closed operator if its graph $G(A)$ is a closed subset of $X \times Y$ for this norm. The so-called graph norm on $D(A)$ is given by

$$\|x\|^2_{D(A)} = \|x\|^2_X + \|Ax\|^2_Y = \|(x, Ax)\|^2_{X\times Y}.$$

The operator A is closed if and only if the space $D(A)$ is complete for the graph norm $\|.\|_{D(A)}$.

If a linear operator A from X to Y is injective, one can define the operator A^{-1} from Y to X such that

$$D(A^{-1}) = \mathrm{Ran}(A), \quad \mathrm{Ran}(A^{-1}) = D(A), \quad A^{-1}A = \mathrm{Id}_{D(A)}, \quad AA^{-1} = \mathrm{Id}_{\mathrm{Ran}(A)}\,.$$

One says that A is invertible and A^{-1} is called the inverse operator.

If $(A_1, D(A_1)), (A_2, D(A_2))$ are two linear operators from X to Y one defines that the operator $B = A_1 + A_2$ with domain $D(A_1) \cap D(A_2)$.

2 Continuous and bounded operators

An linear operator A from X to Y is said to be continuous if it is continuous at every $x \in D(A)$ or equivalently if it continuous at $x = 0$. This is equivalent to having $M > 0$ such that $\|Ax\|_Y \leq M\|x\|_X$ for all $x \in D(A)$. One says that A is a *bounded* operator. The positive number

$$M = \sup_{\substack{x\in D(A)\\ x\neq 0}} \frac{\|Ax\|_Y}{\|x\|_X},$$

is called the bound of A, and denoted by $\|A\|_{\mathscr{L}(X,Y)}$ or simply $\|A\|$.

Note that linear operator from X to Y that fails to be continuous are such that

$$\sup_{\substack{x\in D(A)\\ x\neq 0}} \frac{\|Ax\|_Y}{\|x\|_X} = +\infty.$$

This justifies the name *unbounded* for general linear operators from X to Y.

Theorem 1 (closed-graph theorem)**.** *Let A be such that $D(A)$ is a closed linear subspace in X. Then, A is bounded if and only if A is a closed operator.*

For a proof see for instance [4].

Remark 2. While one aspect of the proof of the closed-graph theorem is involved and based on the Baire lemma, one can also easily prove the following statements: if A is closed and A is bounded then $D(A)$ is a closed linear subspace in X. Hence, if a A is densely defined, closed and bounded then $D(A) = X$: the operator is bounded *on* X to Y.

Note also that any bounded operator A with domain $D(A)$ can be uniquely extended to $\overline{D(A)}$, as a bounded operator with the same bound, thus leading to a closed operator.

We shall denote by $\mathscr{B}(X, Y)$ the set of bounded operators A on X to Y, that is, such that $D(A) = X$. In the main text, if we speak of a bounded operator $A : X \to Y$ without any mention of its domain, this means that $D(A) = X$, that is, A is on X to Y.

Remark 3. Following the above remark assume that A is a closed linear operator from X to Y that is invertible and such that A^{-1} is bounded. As A^{-1} is also closed for obvious reasons we find that $\operatorname{Ran}(A) = D(A^{-1})$ is a closed subset of Y.

3 Spectrum of a linear operator in a Banach space

We consider here a linear operator from X to itself. One says that $\lambda \in \mathbb{C}$ is in the resolvent set $\rho(A)$ of an linear operator A from X to X if the operator $\lambda \operatorname{Id} - A$ is injective, and the inverse operator $(\lambda \operatorname{Id} - A)^{-1}$ has a dense domain $D\big((\lambda \operatorname{Id} - A)^{-1}\big) = \operatorname{Ran}(\lambda \operatorname{Id} - A)$ in X and is bounded. If $\lambda \in \rho(A)$ then we set the resolvent operator as $R_\lambda(A) = (\lambda \operatorname{Id} - A)^{-1}$. The spectrum is then simply the complement set of $\rho(A)$ in $\mathbb{C}$. We denote it by $\operatorname{sp}(A)$.

The spectrum of a linear operator is often separated in three disjoint sets:

1. The *point spectrum* that gathers all $\lambda \in \mathbb{C}$ such that the operator $\lambda \operatorname{Id} - A$ is not injective. Such a complex number λ is called an eigenvalue of A and the dimension of the the kernel $\ker(\lambda \operatorname{Id} - A)$ is the geometric multiplicity associated with this eigenvalue. An element of $\ker(\lambda \operatorname{Id} - A)$ is called an eigenvector or, often, an eigenfunction in the case the Banach space X is a function space.

2. The *continuous spectrum* that gathers all $\lambda \in \mathbb{C}$ such that the operator $\lambda \operatorname{Id} - A$ is injective, has a dense image, but its inverse $(\lambda \operatorname{Id} - A)^{-1}$ is not bounded.

3. The *residual spectrum* that gathers all $\lambda \in \mathbb{C}$ such that the operator $\lambda \operatorname{Id} - A$ is injective but does not have a dense image.

In the case A is a closed operator, if $\lambda \in \rho(A)$ then $D(R_\lambda(A)) = \operatorname{Ran}(\lambda \operatorname{Id} - A) = X$ (see Remark 3). Hence, in this case, $\lambda \in \rho(A)$ if and only if $\lambda \operatorname{Id} - A$ is injective and $\operatorname{Ran}(\lambda \operatorname{Id} - A) = X$ because of the closed graph theorem (Theorem 1). For $\lambda_0 \in \rho(A)$, if we set $L_0 = (\lambda_0 \operatorname{Id} - A)^{-1}$, then L_0 is a bounded operator on X and we may write

$$\lambda \operatorname{Id} - A = (\lambda_0 \operatorname{Id} - A)\big(\operatorname{Id} + (\lambda - \lambda_0)L_0\big).$$

For $|\lambda - \lambda_0| < \|L_0\|^{-1}$ one then finds that $\operatorname{Id} + (\lambda - \lambda_0)L_0$ is itself invertible with a bounded inverse. Consequently, the resolvent set is an open set in $\mathbb{C}$ and the spectrum is closed. Moreover, one finds that on $\rho(A)$, the map $\lambda \mapsto R_\lambda(A)$ is holomorphic. We refer the reader for instance to Chapter 3.6 in [4].

4 Adjoint operator

If X' be the dual space of a Banach space X, that is, the linear space of bounded linear forms on X, we equip X' with the strong topology associated with the norm

$$\|x^*\|_{X'} = \sup_{\substack{x \in X \\ \|x\|_X \leq 1}} |\langle x^*, x \rangle|.$$

With this topology X' is a Banach space.

If A is a linear operator from X to Y densely defined, one sets

$$D(A^*) = \big\{ y^* \in Y';\ \exists C > 0,\ \forall x \in D(A),\ |\langle y^*, Ax \rangle_{Y',Y}| \leq C \|x\|_X \big\}.$$

If $y^* \in D(A^*)$, there exists a unique $x^* \in X'$ such that

$$\langle y^*, Ax \rangle_{Y',Y} = \langle x^*, x \rangle_{X',X}, \qquad x \in D(A).$$

Uniqueness follows from the density of $D(A)$ in X. One then sets $A^* y^* = x^*$, which defines a linear operator A^* from Y^* to X^* with domain $D(A^*)$.

Proposition 4. *The operator $(A^*, D(A^*))$ is a closed operator.*

Proposition 5. *If the operator A is a bounded on X to Y then $D(A^*) = Y'$ and A^* is a bounded operator on Y' to X'. Moreover $\|A\|_{\mathscr{L}(X,Y)} = \|A^*\|_{\mathscr{L}(Y',X')}$.*

5 Fredholm operators

Let A be a linear closed operator from X to Y. The nullity of A, $\operatorname{nul} A$, is defined as the dimension of $\ker(A)$. The deficiency of A, $\operatorname{def} A$, is defined as the dimension of $Y/\overline{\operatorname{Ran}(A)}$. Both $\operatorname{nul} A$ and $\operatorname{def} A$ take value in $\mathbb{N} \cup \{\infty\}$.

Definition 6. A linear operator A from X to Y is said to be Fredholm if

1. it is closed;
2. $\operatorname{Ran}(A)$ is closed;
3. both $\operatorname{nul} A$ and $\operatorname{def} A$ are finite.

One then sets the index of A as $\operatorname{ind}(A) = \operatorname{nul} A - \operatorname{def} A$.

5.1 Characterization of bounded Fredholm operators

We denote by $F\mathscr{B}(X,Y)$ the space of Fredholm operators that are bounded on X into Y. The following result states that those operators are the operators in $B(X,Y)$ that have an inverse up to remainder operators that are compact.

Theorem 7. *Let $A \in B(X,Y)$. It is Fredholm if and only if there exists $S \in B(Y,X)$ such that*

$$SA = \operatorname{Id}_X + K^\ell, \qquad AS = \operatorname{Id}_Y + K^r, \tag{5.1}$$

where $K^\ell \in B(X,X)$ and $K^r \in B(Y,Y)$ are compact operators. In particular, S is Fredholm and $\operatorname{ind}(A) = -\operatorname{ind}(S)$.

For the proof we shall need the following lemma.

Lemma 8. *Let $A \in B(X,Y)$ and $K \in B(X,X_1)$ be compact, with X, Y and X_1 Banach spaces, and $C > 0$ such that*

$$\|x\|_X \leq C\big(\|Ax\|_Y + \|Kx\|_{X_1}\big), \tag{5.2}$$

for $x \in X$. Then, $\operatorname{Ran}(A)$ is closed.

Proof. Let $(y_n)_n \subset \operatorname{Ran}(A)$ be a converging sequence in Y. Set $y = \lim y_n$ and consider a sequence $(x_n)_n \subset X$ such that $Ax_n = y_n$. Set also $X_0 = \ker A$.

First, assume that $d_n = \operatorname{dist}(x_n, X_0)$ is bounded, say $d_n \leq R$. Thus, for any $n \in \mathbb{N}$ there exists $\tilde{x}_n \in X_0$ such that $\|x_n - \tilde{x}_n\|_X \leq R+1$. Replacing x_n by $x_n - \tilde{x}_n$ we have found $(x_n)_n \subset X$ such that $Ax_n = y_n$ with $(x_n)_n$ bounded.

Then, $(Kx_{\varphi(n)})_n$ converges in X_1, for some increasing function $\varphi : \mathbb{N} \to \mathbb{N}$. With (5.2) we have

$$\|x_{\varphi(n)} - x_{\varphi(m)}\|_X \lesssim \|A(x_{\varphi(n)} - x_{\varphi(m)})\|_Y + \|K(x_{\varphi(n)} - x_{\varphi(m)})\|_{X_1},$$

implying that $(x_{\varphi(n)})_n$ is a Cauchy sequence in X complete. We denote by x its limit and, as A is bounded, we find $Ax = \lim Ax_{\varphi(n)} = y$.

Second, we assume that $d_n = \mathrm{dist}(x_n, X_0)$ is unbounded. By contradiction, we prove that this second case does not occur, which yields the conclusion.

If fact, up to a subsequence we have $d_n \geq 1$ and $\lim d_n = +\infty$. For any $n \in \mathbb{N}$ there exists $\tilde{x}_n \in X_0$ such that $d_n \leq \|x_n - \tilde{x}_n\|_X \leq d_n + 1$ and we set $z_n = x_n - \tilde{x}_n$. Naturally, we have $\mathrm{dist}(z_n, X_0) = d_n$. If we set $u_n = z_n/\|z_n\|_X$ we have $\mathrm{dist}(u_n, X_0) = d_n/\|z_n\|_X$ yielding $\mathrm{dist}(u_n, X_0) \geq d_n/(d_n+1)$. Using that $t \mapsto t/(t+1)$ is increasing on $[0, +\infty)$ we find that $\mathrm{dist}(u_n, X_0) \geq 1/2$.

We now see that $Au_n = y_n/\|z_n\|_X$ converges to 0 as $\lim \|z_n\|_X = +\infty$ and that $(Ku_{\psi(n)})_n$ converges in X_1, for some increasing function $\psi : \mathbb{N} \to \mathbb{N}$. With (5.2) we have

$$\|u_{\psi(n)} - u_{\psi(m)}\|_X \lesssim \|A(u_{\psi(n)} - u_{\psi(m)})\|_Y + \|K(u_{\psi(n)} - u_{\psi(m)})\|_{X_1},$$

implying that $(u_{\psi(n)})_n$ is a Cauchy sequence in X. Set $u = \lim u_{\psi(n)}$. By continuity, we have $Au = 0$, meaning that $u \in X_0$ in contradition with $\mathrm{dist}(u_n, X_0) \geq 1/2$ obtained above. ∎

Lemma 9. *Let X be a Banach space and $K \in B(X, X)$ be compact. Then $\ker(\mathrm{Id} + K)$ is finite dimensional.*

Proof. If $x \in \ker(\mathrm{Id} + K)$, we have $x = -K(x)$. In particular, the unit ball in $\ker(\mathrm{Id} + K)$ is the image of bounded set by the compact operator K. It follows that the unit ball of $\ker(\mathrm{Id} + K)$ is compact and thus, by the Riesz theorem, $\ker(\mathrm{Id} + K)$ is finite dimensional. ∎

Proof of Theorem 7. First, assume that (5.1) holds. The first identity gives $\ker(A) \subset \ker(\mathrm{Id} + K^\ell)$, with the latter space finite dimensional by Lemma 9.

From the first equality in (5.1) we deduce

$$\|x\|_X \lesssim \|Ax\|_Y + \|K^\ell x\|_X.$$

By Lemma 8 this implies that $\mathrm{Ran}(A)$ is closed. As $Y/\mathrm{Ran}(A) \cong \mathrm{Ran}(A)^\perp$, proving $\mathrm{codim}\,\mathrm{Ran}(A) < \infty$ amounts to proving that $\mathrm{Ran}(A)^\perp$ is finite dimensional. From the second equality in (5.1) we have $\mathrm{Ran}(\mathrm{Id}_Y + K^r) \subset \mathrm{Ran}(A)$ and thus $\mathrm{Ran}(A)^\perp \subset \mathrm{Ran}(\mathrm{Id}_Y + K^r)^\perp$. By Corollary 2.18 in [1] we have $\mathrm{Ran}(\mathrm{Id}_Y + K^r)^\perp = \ker(\mathrm{Id}_{Y'} + (K^r)^*)$ and the latter space is finite dimensional by Lemma 9.

Second, assume that A is Fredholm. As $\dim \ker A < \infty$, there exists $\tilde{X}$ a closed linear subspace of X that is a complementary subspace of $\ker A$, that is, $\tilde{X} \oplus \ker A = X$ in the algebraic sense and moreover the projections associated with this direct sum are continuous. Similarly, as $\mathrm{codim}\,\mathrm{Ran}(A) < \infty$ and as $\mathrm{Ran}(A)$ is closed, there exists also Z complementary subspace of $\mathrm{Ran}(A)$ in Y. We refer for instance [1, Section 2.4]. Observe that the projections $\Pi_{\ker A}$ onto $\ker A$ and Π_Z onto Z associated with the above direct sums are compact since $\dim \ker A < \infty$ and $\dim Z < \infty$.

We consider the bijective map $\tilde{A} : \tilde{X} \to \mathrm{Ran}(A)$ given by $\tilde{A}x = Ax$. As $\tilde{X}$ and $\mathrm{Ran}(A)$ are Banach spaces if equipped with the norms inherited from X and Y, the open map theorem shows that $\tilde{A}$ is an isomorphism. We denote by $\tilde{S}$ its inverse map and we set $S = \tilde{S}(\mathrm{Id}_Y - \Pi_Z)$ We then find that

$$AS = A\tilde{S}(\mathrm{Id}_Y - \Pi_Z) = \tilde{A}\tilde{S}(\mathrm{Id}_Y - \Pi_Z) = \mathrm{Id}_Y - \Pi_Z.$$

We also write

$$SA = \tilde{S}A = \tilde{S}A(\mathrm{Id}_X - \Pi_{\ker A}) = \tilde{S}\tilde{A}(\mathrm{Id}_X - \Pi_{\ker A}) = \mathrm{Id}_X - \Pi_{\ker A},$$

which concludes the proof. ■

Proposition 10. *Let $A \in B(X, Y)$. It is Fredholm if and only if there exist $K_1 \in B(X, Z_1)$ and $K_2 \in B(Y', Z_2)$ both compact, with Z_1 and Z_2 Banach spaces, and $C > 0$ such that*

$$\|x\|_X \leq C\big(\|Ax\|_Y + \|K_1 x\|_{Z_1}\big), \quad \|y^*\|_{Y'} \leq C\big(\|A^* y^*\|_{X'} + \|K_2 y^*\|_{Z_2}\big),$$

for $x \in X$ and $y^* \in Y'$.

Remark 11. The first part of the proof shows that one can use the compact operators $K_1 = K^\ell \in \mathscr{B}(X, X)$ and $K_2 = (K^r)^* \in \mathscr{B}(Y', Y')$, that are given by Theorem 7. Then one has $Z_1 = X$ and $Z_2 = Y'$.

Proof. By Theorem 7, if A is Fredholm, there exists S bounded from Y to X such that

$$SA = \mathrm{Id}_X + K^\ell, \qquad AS = \mathrm{Id}_Y + K^r,$$

with $K^\ell : X \to X$ and $K^r : Y \to Y$ both compact operators. With the first identity we obtain

$$\|x\|_X \lesssim \|Ax\|_Y + \|K^\ell x\|_X.$$

With the second identity we compute $S^* A^* = \mathrm{Id}_{Y'} + (K^r)^*$, yielding

$$\|y^*\|_{Y'} \lesssim \|A^* y^*\|_{X'} + \|(K^r)^* y^*\|_{Y'}.$$

Conversely, if $\|x\|_X \lesssim \|Ax\|_Y + \|K_1 x\|_{Z_1}$, for some $K_1 : X \to Z_1$ compact, we consider a sequence $(x_n)_n \subset \ker(A)$ such that $\|x_n\|_X = 1$. Then, up to a subsequence, $(K_1 x_n)_n$ converges in Z_1. Writing

$$\|x_n - x_m\|_X \lesssim \|K_1 x_n - K_1 x_m\|_{Z_1},$$

we find that $(x_n)_n$ is a Cauchy sequence and thus converges as X is a complete. The unit ball of $\ker(A)$ is thus compact; $\ker(A)$ is thus finite dimensional by the Riesz theorem.

Similarly we find that $\ker(A^*)$ is finite dimensional. As $\mathrm{Ran}(A)$ is closed by Lemma 8 we have $\mathrm{Ran}(A) = \overline{\mathrm{Ran}(A)} = \ker(A^*)^\perp$ by Corollary 2.18 in [1] implying that $\operatorname{codim}\mathrm{Ran}(A) < \infty$ as $\operatorname{codim}\mathrm{Ran}(A) = \dim(X/\mathrm{Ran}(A)) = \dim\ker(A^*)$. ■

Corollary 12. *Let $A \in F\mathscr{B}(X,Y)$ and F be a closed subspace of X. Then $A(F)$ is closed.*

Proof. As $A \in F\mathscr{B}(X,Y)$, we have the estimations of Proposition 10 and the first one applies to $A_{|F}$. By Lemma 8 we conlude that $\mathrm{Ran}(A_{|F}) = A(F)$ is closed in Y. ■

The set $F\mathscr{B}(X,Y)$ of bounded Fredholm operators has some important topological properties.

Theorem 13. *The set $F\mathscr{B}(X,Y)$ is open in $\mathscr{B}(X,Y)$.*

Proof. Let $A \in F\mathscr{B}(X,Y)$. By Proposition 10 and Remark 11 we have

$$\|x\|_X \lesssim \|Ax\|_Y + \|K^\ell x\|_X, \quad \|y^*\|_{Y'} \lesssim \|A^*y^*\|_{X'} + \|(K^r)^*y^*\|_{Y'}.$$

with the compact operators $K^\ell \in B(X,X)$ and $K^r \in B(Y,Y)$ given by Theorem 7. With these two inequalities we see that there exists $\varepsilon > 0$ such that

$$\|x\|_X \lesssim \|(A+B)x\|_Y + \|K^\ell x\|_X, \quad \|y^*\|_{Y'} \lesssim \|(A+B)^*y^*\|_{X'} + \|(K^r)^*y^*\|_{Y'}.$$

for $B \in (X,Y)$ such that $\|B\|_{\mathscr{L}(X,Y)} \leq \varepsilon$. By Proposition 10 we then find that $A + B \in F\mathscr{B}(X,Y)$. ■

Theorem 14. *The maps $F\mathscr{B}(X,Y) \to \mathbb{N}$*

$$\mathrm{nul} : A \mapsto \dim \mathit{ker}(A) \quad \text{and} \quad \mathrm{def} : A \mapsto \operatorname{codim}\mathrm{Ran}(A)$$

are both upper semi-continuous. Moreover, the index map, $\mathrm{ind} = \mathrm{nul} - \mathrm{def}$, is constant in each connected component of $F\mathscr{B}(X,Y)$.

Proof. Let $A \in F\mathscr{B}(X,Y)$. As we have $\mathrm{nul}\,A = \dim\ker(A) < \infty$, $\mathrm{def}\,A = \operatorname{codim}\mathrm{Ran}(A) < \infty$, and $\mathrm{Ran}(A)$ is closed, there exists $\tilde{X}$ and $\tilde{Y}$ that are complementary $\ker(A)$ and $\mathrm{Ran}(A)$ in X and Y respectively, that is,

$$\tilde{X} \oplus \ker(A) = X \quad \text{and} \quad \mathrm{Ran}(A) \oplus \tilde{Y} = Y, \tag{5.3}$$

with moreover $\tilde{X}$ and $\tilde{Y}$ closed (see Section 2.4 in [1]). Set $Z = \tilde{X} \oplus \tilde{Y}$. For $T \in \mathscr{B}(X,Y)$ we define $\kappa_T \in \mathscr{B}(Z,Y)$ given by

$$\kappa_T(x+y) = Tx + y, \quad x \in \tilde{X},\ y \in \tilde{Y}.$$

Observe that κ_A is bijective. Note that $\|\kappa_{T_1} - \kappa_{T_2}\|_{\mathscr{L}(Z,Y)} \leq \|T_1 - T_2\|_{\mathscr{L}(X,Y)}$. Thus, for $T \in \mathscr{B}(X,Y)$ chosen such that $\|T - A\|_{\mathscr{L}(X,Y)} \leq \varepsilon$, the operator κ_T is also bijective, since the set of bounded invertible operators from Z into Y is open in $B(Z,Y)$, and $T \in F\mathscr{B}(X,Y)$ by Theorem 13, for $\varepsilon > 0$ chosen sufficiently small.

Below, T is chosen such that $\|T - A\|_{\mathscr{L}(X,Y)} \leq \varepsilon$. We have $\kappa_T(\tilde{X} \times \{0\}) = T(\tilde{X})$ and, as κ_T is an isomorphism, we have

$$\operatorname{codim} T(\tilde{X}) = \operatorname{codim} \kappa_T(\tilde{X} \times \{0\}) = \dim \tilde{Y}. \tag{5.4}$$

As $T(\tilde{X}) \subset \operatorname{Ran}(T)$, we thus find that

$$\operatorname{def} T = \operatorname{codim} \operatorname{Ran}(T) \leq \dim \tilde{Y} = \operatorname{codim} \operatorname{Ran}(A) = \operatorname{def} A, \tag{5.5}$$

meaning that the map $T \mapsto \operatorname{def} T$ is upper semicontinuous at A.

As κ_T is an isomorphism, we find that $\ker(T) \cap \tilde{X} = \{0\}$. Since $\ker(T)$ is finite dimensional, $\operatorname{codim} \tilde{X} < \infty$, and $\tilde{X}$ closed, there exists a finite dimensional complementary subspace $\hat{X}$ of $\tilde{X} \oplus \ker(T)$, that is,

$$X = \hat{X} \oplus \tilde{X} \oplus \ker(T). \tag{5.6}$$

Note that $T(\hat{X} \oplus \tilde{X}) = T(\hat{X}) \oplus T(\tilde{X}) = T(X)$ yielding

$$\operatorname{codim} T(X) + \dim T(\hat{X}) = \operatorname{codim} T(\tilde{X}),$$

that is, $\operatorname{def} T + \dim \hat{X} = \operatorname{def} A$, since $\hat{X} \cap \ker T = \{0\}$ and using (5.4)–(5.5). In turn, from (5.3) and (5.6) we conclude that $\operatorname{nul} T + \dim \hat{X} = \operatorname{nul} A$. Together, these last two equalities show that $\operatorname{ind}(T) = \operatorname{ind}(A)$.

Finally, as we now have $\operatorname{nul} A - \operatorname{nul} T = \operatorname{def} A - \operatorname{def} T$, we also conclude that the map $T \mapsto \operatorname{nul} T$ is upper semicontinuous at A. ■

Given a bounded operator $A \in F\mathscr{B}(X,Y)$, we saw above that small perturbations $B \in \mathscr{B}(X,Y)$ neither affect its Fredholm property nor its index as one remains in the connected component of A in $F\mathscr{B}(X,Y)$. The following Proposition shows if $K \in \mathscr{B}(X,Y)$ is compact, then $A + K$ also remains in the connected component of A, without any size assumption on K.

Theorem 15. *Let $A \in F\mathscr{B}(X,Y)$ and $K \in \mathscr{B}(X,Y)$ be compact. Then $A + K \in F\mathscr{B}(X,Y)$ and* $\operatorname{ind}(A+K) = \operatorname{ind}(A)$.

Proof. As A is Fredholm, by Proposition 10 there exist $K_1 \in B(X, Z_1)$ and $K_2 \in B(Y', Z_2)$ both compact, with Z_1 and Z_2 Banach spaces, and $C > 0$ such that

$$\|x\|_X \lesssim \|Ax\|_Y + \|K_1 x\|_{Z_1}, \quad \|y^*\|_{Y'} \lesssim \|A^* y^*\|_{X'} + \|K_2 y^*\|_{Z_2}.$$

We thus have

$$\begin{aligned}\|x\|_X &\lesssim \|(A+K)x\|_Y + \|Kx\|_Y + \|K_1 x\|_{Z_1},\\ \|y^*\|_{Y'} &\lesssim \|(A+K)^* y^*\|_{X'} + \|K^* y^*\|_{X'} + \|K_2 y^*\|_{Z_2}.\end{aligned}$$

We then define

$$\tilde{K}_1 : X \to Y \oplus Z_1 \qquad \tilde{K}_2 : Y' \to X' \oplus Z_2$$

given by $\tilde{K}_1(x) = K(x) + K_1(x)$ and $\tilde{K}_2(y^*) = K^*(y^*) + K_2(y^*)$. Both are compact and we have

$$\|x\|_X \lesssim \|(A+K)x\|_Y \|\tilde{K}_1 x\|_{Y \oplus Z_1}, \quad \|y^*\|_{Y'} \lesssim \|(A+K)^* y^*\|_{X'} + \|\tilde{K}_2 y^*\|_{X' \oplus Z_2}.$$

We thus find that $A + K$ is Fredholm by the converse part of Proposition 10.

Next, for the same reason $A + tK \in F\mathscr{B}(X, Y)$ for any $t \in [0, 1]$. This implies that $A + K$ and A lie in the same connected component of $F\mathscr{B}(X, Y)$. Their index is thus the same by Theorem 14. ■

6 Linear operators in Hilbert spaces

Let H be a Hilbert space. By the Riesz theorem there exists an isomorphism $J : H' \to H$ such that

$$\langle u^*, u \rangle_{H',H} = (Ju^*, u)_H, \qquad u^* \in H', \; u \in H,$$

which allows one to identify H' with H.

Let H_1 and H_2 be two Hilbert spaces. For an unbounded operator A from H_1 to H_2 with dense domain, its adjoint operator, as defined in Section 4, can then be uniquely identified with an operator from H_2 to H_1, that we also denote by A^*, with domain

$$D(A^*) = \left\{ v \in H_2; \; \exists C > 0, \; \forall u \in D(A), \; |(v, Au)_{H_2}| \leq C \|u\|_{H_1} \right\} \subset H_2,$$

and such that

$$(Au, v)_{H_2} = (u, A^* v)_{H_1}, \quad u \in D(A) \subset H_1, \; v \in D(A^*) \subset H_2.$$

As a Hilbert space is reflexive we have the following result (see e.g. Theorem III.5.29 in [4]).

Proposition 16. *Let $(A, D(A))$ be a closed and densely defined operator from H_1 to H_2. Then, the operator $(A^*, D(A^*))$ from H_2 to H_1 is also closed and densely defined. Moreover $A^{**} = A$.*

We also have the following result (see Theorem III.5.30 in [4]).

Proposition 17. *Let $(A, D(A))$ be a closed and densely defined operator from H_1 to H_2. If A^{-1} exists and is bounded on H_2 to H_1, then $(A^*)^{-1}$ exists and is bounded on H_1 to H_2 and $(A^*)^{-1} = (A^{-1})^*$. Moreover, $\|(A^*)^{-1}\|_{\mathscr{L}(H_1,H_2)} = \|A^{-1}\|_{\mathscr{L}(H_2,H_1)}$.*

If A is an unbounded operator from a Hilbert space H into itself, the operator is said to be symmetric if one has

$$(Au, v)_H = (u, Av)_H, \quad u, v \in D(A).$$

If its domain is dense, A^* is well defined, $D(A) \subset D(A^*)$ and A^* coincides with A on $D(A)$. One usually writes $(A, D(A)) \subset (A^*, D(A^*))$. The operator A is furthermore said to be selfadjoint if $D(A) = D(A^*)$: we then have $(A, D(A)) = (A^*, D(A^*))$.

Observe that a symmetric operator may not be selfadjoint. Consider for instance the operator A given by $Au = \Delta u$ with domain $D(A) = \mathscr{C}_c^\infty(\Omega) \subset H = L^2(\Omega)$, for Ω a bounded open set in $\mathbb{R}^d$. The operator A is symmetric as one has $(Au, v)_{L^2} = (u, Av)_{L^2}$, for all $u, v \in D(A)$. One sees readily that $H^2(\Omega) \subset D(A^*)$. The operator is not selfadjoint. A useful criterium is the following result; we refer to [5, Theorem 8.3] for a proof.

Theorem 18. *Let $(A, D(A))$ with dense domain be a symmetric linear operator on a Hilbert space H. The following three statements are equivalent:*

1. *$(A, D(A))$ is selfadjoint.*
2. *A is closed and $\ker(A^* + i) = \ker(A^* - i) = \{0\}$.*
3. *$\mathrm{Ran}(A + i) = \mathrm{Ran}(A - i) = H$.*

For a bounded operator A on a Hilbert space H, its adjoint operator yields a bounded operator on H by what precedes and Proposition 5.

The following lemma due to [3, 2] is based on the closed-graph theorem and allows one to quantify the inclusion of the ranges of two operators.

Lemma 19. *Let $K_1 : H_1 \to H$ and $K_2 : H_2 \to H$ with H, H_1, and H_2 Hilbert spaces and K_1 and K_2 linear and bounded. The following statements are equivalent:*

1. *We have $\mathrm{Ran}(K_1) \subset \mathrm{Ran}(K_2)$;*

2. There exists a bounded linear map $\Phi : H_1 \to H_2$ such that $K_1 = K_2 \circ \Phi$;

3. There exists $C_0 \geq 0$ such that

$$\|K_1^* z\|_{H_1} \leq C_0 \|K_2^* z\|_{H_2}, \qquad z \in H. \tag{6.1}$$

Moreover, if the second statement holds, then one can choose $C_0 = \|\Phi\|_{\mathscr{L}(H_1,H_2)}$ in (6.1). Conversely, if $C_0 \geq 0$ is the best possible constant for which (6.1) holds, then there exists Φ as in the second statement such that $\|\Phi\|_{\mathscr{L}(H_1,H_2)} = C_0$.

Proof. We start by proving that the first statement implies the second statement. Note that the converse is obvious.

We thus assume that $\mathrm{Ran}(K_1) \subset \mathrm{Ran}(K_2)$. Let $u_1 \in H_1$. Then

$$L(u_1) = \{u \in H_2;\ K_1(u_1) = K_2(u)\}$$

is a nonempty closed affine subspace of H_2. We denote by $\Phi(u_1)$ the orthogonal projection of 0 onto $L(u_1)$, characterized as the unique element w of $L(u_1)$ such that

$$\forall v \in L(u_1), \quad (w, v - w)_{H_2} = 0. \tag{6.2}$$

Observe that $L(u_1) = \Phi(u_1) + \ker(K_2)$ and (6.2) means that $\Phi(u_1)$ is orthogonal to $\ker(K_2)$.

The operator $\Phi : H_1 \to H_2$ is linear and we prove that the graph of Φ is closed. In fact, consider two sequences $(u_1^{(n)})_n \subset H_1$, $(u_2^{(n)})_n \subset H_2$ such that

$$u_2^{(n)} = \Phi(u_1^{(n)}), \quad u_1^{(n)} \underset{n\to\infty}{\to} u_1 \text{ in } H_1, \quad u_2^{(n)} \underset{n\to\infty}{\to} u_2 \text{ in } H_2.$$

Then $K_1(u_1^{(n)}) = K_2(u_2^{(n)})$ giving in the limit $K_1(u_1) = K_2(u_2)$. Moreover $u_2^{(n)}$ is orthogonal to $\ker(K_2)$ giving in the limit u_2 orthogonal to $\ker(K_2)$. Hence $u_2 = \Phi(u_1)$, that is the graph of Φ is closed. The closed graph theorem then implies that Φ is a bounded operator. There exists $C_0 > 0$ such that $\|\Phi(u_1)\|_{H_2} \leq C_0 \|u_1\|_{H_1}$.

Having proven the equivalence of the first two statements, we now show that they imply the inequality of the third statement. Note that having $K_2(u_2) = K_1(u_1)$ implies

$$(u_1, K_1^*(z))_{H_1} = (u_2, K_2^*(z))_{H_2}, \qquad z \in H.$$

For $z \in H$ we set $u_1 = K_1^*(z)$ and set $u_2 = \Phi(u_1)$, with Φ as defined above. This gives

$$\|K_1^*(z)\|_{H_1}^2 \leq \|\Phi(u_1)\|_{H_2} \|K_2^*(z)\|_{H_2} \leq C_0 \|K_1^*(z)\|_{H_1} \|K_2^*(z)\|_{H_2},$$

which gives $\|K_1^*(z)\|_{H_1} \leq C_0\|K_2^*(z)\|_{H_2}$ for $z \in H$.

Finally, we prove that the third statement implies the second one. We thus assume that there exists $C_0 > 0$ such that

$$\forall z \in H,\ \|K_1^*(z)\|_{H_1} \leq C_0\|K_2^*(z)\|_{H_2}. \tag{6.3}$$

Let $u_1 \in H_1$. We define the linear map

$$\begin{aligned} \Psi : \operatorname{Ran}(K_2^*) \subset H_2 &\to \mathbb{C} \\ K_2^*(z) &\mapsto (K_1^*(z), u_1)_{H_1}. \end{aligned}$$

This map is well defined. In fact if $w = K_2^*(z) = K_2^*(z')$ then $K_1^*(z) = K_1^*(z')$ by (6.3). We have, for $w = K_2^*(z)$,

$$|\Psi(w)| \leq \|K_1^*(z)\|_{H_1}\|u_1\|_{H_1} \leq C_0\|w\|_{H_2}\|u_1\|_{H_1},$$

that is, the map Ψ is bounded and $\|\Psi\| \leq C_0\|u_1\|_{H_1}$. We then denote by $\tilde{\Psi}$ the map $\overline{\operatorname{Ran}(K_2^*)} \to \mathbb{C}$ that uniquely extends Ψ to the Hilbert space $\overline{\operatorname{Ran}(K_2^*)}$ endowed with the inner product on H_2. We have $\|\tilde{\Psi}\| = \|\Psi\| \leq C_0\|u_1\|_{H_1}$. By the Riesz theorem, there exists $u_2 \in \overline{\operatorname{Ran}(K_2^*)}$ such that $\|u_2\|_{H_2} = \|\Psi\| \leq C_0\|u_1\|_{H_1}$ and

$$\tilde{\Psi}(w) = (w, u_2)_{H_2}, \qquad w \in \overline{\operatorname{Ran}(K_2^*)}.$$

We define the map $\Phi : H_1 \to H_2$ by $u_2 = \Phi(u_1)$. It is linear and bounded. For $z \in H$, we set $w = K_2^*(z)$ and we obtain

$$(K_1^*(z), u_1)_{H_1} = (K_2^*(z), \Phi(u_1))_{H_2},$$

and thus for all $z \in H$ we have $(z, K_1(u_1))_H = (z, K_2(\Phi(u_1)))_H$. That is, $K_1(u_1) = K_2(\Phi(u_1))$. The proof is complete. ∎

A corollary is the following result that characterizes the surjectivity of a bounded operator.

Corollary 20. *Let $K : H_1 \to H$ with H_1, H Hilbert spaces and K linear and bounded. The following statements are equivalent.*

1. *We have* $\operatorname{Ran}(K) = H$*;*

2. *There exists a bounded linear map* $\Phi : H \to H_1$ *such that* $\operatorname{Id}_H = K \circ \Phi$*.*

3. *There exists* $C_0 \geq 0$ *such that*

$$\|x\|_H \leq C_0\|K^*x\|_{H_1}. \tag{6.4}$$

Moreover, if the second statement holds, then one can choose $C_0 = \|\Phi\|_{\mathscr{L}(H,H_2)}$ *in* (6.4)*. Conversely, if* $C_0 \geq 0$ *is the best possible constant for which* (6.4) *holds, then there exists* Φ *as in the second statement such that* $\|\Phi\|_{\mathscr{L}(H,H_2)} = C_0$*.*

Remark 21. Note that if we decide to not identify the Hilbert spaces H and H_1 with their respective dual spaces we then obtain the characterization

$$\|x\|_{H'} \leq C_0 \|K^* x\|_{H_1'}. \tag{6.5}$$

We may also decide to identify H_1 with its dual and not H and *vice versa*.

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