Normal geodesic coordinates

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We denote by $P(x, D)$ a general second-order elliptic operator with a principal part of the form

$$P_0 = \sum_{1 \le i,j \le d} D_i(p^{ij}(x) D_j), \qquad \text{with} \quad \sum_{1 \le i,j \le d} p^{ij}(x)\xi_i\xi_j \ge C|\xi|^2, \tag{0.1}$$

where $p^{ij} \in \mathscr{C}^\infty(\mathbb{R}^d; \mathbb{R})$ is such that $p^{ij} = p^{ji}$, $1 \le i, j \le d$. The elliptic operator under consideration is then

$$P = P_0 + \sum_{1 \le i \le d} b^i(x) D_i + c(x),$$

where $b^i, c \in L^\infty(\mathbb{R}^d)$, $1 \le i \le d$. We denote by p the principal symbol of P given by

$$p(x, \xi) = \sum_{1 \le i,j \le d} p^{ij}(x)\xi_i\xi_j.$$

We now consider a smooth function ψ defined in a neighborhood V of x^0 and we assume that $d\psi(x^0) \ne 0$ in V. We set

$$S = \{x \in V;\ \psi(x) = \psi(x^0)\}, \quad V_+ = \{x \in V;\ \psi(x) \ge \psi(x^0)\}.$$

The hypersurface S is smooth in V.

Theorem 1. *There exist X an open neighborhood of x^0, Y an open neighborhood of 0 in $\mathbb{R}^d$, and a $\mathscr{C}^\infty$-diffeomorphism $\kappa : X \to Y$, such that:*

1. *We have $\kappa(x^0) = 0$ and*

$$\kappa(V_+ \cap X) = \{y \in Y;\ y_d \ge 0\} \ \ \textit{and} \ \ \kappa(S \cap X) = \{y \in Y;\ y_d = 0\}.$$

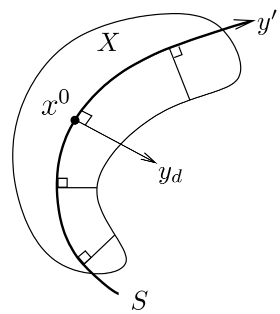

Figure 1: Local normal geodesic coordinates for the Laplace operator $\Delta = \partial_1^2 + \cdots + \partial_d^2$ in $\mathbb{R}^d$.

2. *In the local coordinates $y = \kappa(x)$, the operator P takes the form*
$$P = D_d^2 + \sum_{i,j=1}^{d-1} \tilde{p}^{ij}(y) D_i D_j + \sum_{i=1}^{d} \tilde{b}^i(y) D_i + \tilde{c}(y),$$
where the coefficients $\tilde{p}^{ij}$ are $\mathscr{C}^\infty$ in Y and the coefficients $\tilde{b}^i$ and $\tilde{c}$ are in $L^\infty(Y)$.

3. *Moreover there exists $C > 0$ such that*
$$\sum_{i,j=1}^{d-1} \tilde{p}^{ij}(y)\eta_i\eta_j \geq C|\eta'|^2, \quad \eta' = (\eta_1, \ldots, \eta_{d-1}),\ y \in Y.$$

Proof of Theorem 1. The proof is made of several steps.

Preliminary remarks Each step of the proof is associated with the construction of a change of variables, that is a smooth diffeomorphism. For simplicity, at each step, the original and final variables with be denoted by x and y respectively. At each step we shall start from an open neighborhood V of x^0. The diffeomorphism κ that will be built for that step will then map X, a possibly smaller open neighborhood of x^0, onto an open set Y of $\mathbb{R}^d$.

If $x \mapsto y = \kappa(x) = (\kappa_1(x), \ldots, \kappa_d(x))$ is the built change of variables we then have the following relation $D_{x_i} = \sum_{j=1}^d (\partial_{x_i}\kappa_j(x)) D_{y_j}$. We denote by $\kappa'(x) = d\kappa(x)$ the differential of κ which can be identified with its Jacobian matrix of κ at x. If Q denotes the differential operator P after the action of the change of variables[1], that is,
$$P(f \circ \kappa) = (Qf) \circ \kappa, \quad f \in \mathscr{C}^\infty(Y),$$

[1]In the course of the proof we shall not use the same letter for P and Q, as is commonly done, for avoid any confusion.

then if $q(y, \eta)$ denotes the principal symbol of Q we have

$$q(\kappa(x), \eta) = p(x, {}^t\kappa'(x)\eta), \quad x \in X, \eta \in \mathbb{R}^d.$$

In particular $q(y, \eta)$ is a positive quadratic form, uniformly w.r.t. $y \in Y$. In the course of the proof, at every step we shall ignore first- and zero-order terms as they are only required to have bounded coefficients, which is, and as they do not appear in the smooth principal symbols. At each step P_0 will denote the principal part of the operator obtained at the previous step, and Q_0 will denote the principal part of P after change of variables.

Step 1 By reordering the variables we can assume $\partial_d \psi(x^0) \neq 0$ in V. We then define the following change of variables $y = \kappa(x)$ by

$$\begin{cases} y_j = x_j - x_j^0 & \text{for } j = 1, \ldots, d-1, \\ y_d = \psi(x) - \psi(x^0). \end{cases}$$

With the local diffeomorphism theorem, there exits an open neighborhood X of x^0 such that κ is a smooth diffeomorphism of X onto $Y = \kappa(X)$. Moreover we have $\kappa(x_0) = 0$ and

$$\kappa(V_+ \cap X) = \{y \in Y;\ y_d \geq 0\} \ \text{ and } \ \kappa(S \cap X) = \{y \in Y;\ y_d = 0\}.$$

With this step we have preserved the assumptions of the theorem and we have achieved the first point in the statement of the theorem. The next two steps will not affect this property.

Step 2 We now have with $x^0 = 0$, $S = \{x \in V;\ x_d = 0\}$ and $\psi(x) = x_d$.

For this second step, we aim to write P_0 in the new variables under the form

$$Q_0 = D_{y_d}^2 + 2 \sum_{i=1}^{d-1} q^{id}(y) D_{y_i} D_{y_d} + \sum_{i,j=1}^{d-1} q^{ij}(y) D_{y_i} D_{y_j}.$$

If compared to the previous step, we thus want to also enforce $q(y, e_d) = 1$, for all $y \in Y$, where $e_d = (0, \ldots, 0, 1)$, i.e.,

$$p(x, {}^t\kappa'(x) e_d) = \sum_{i,j=1}^{d} p^{ij}(x) (\partial_i \kappa_d(x))(\partial_j \kappa_d(x)) = p(x, d\kappa_d(x)) = 1, \quad x \in X. \quad (0.2)$$

We thus obtain an equation that solely involves the coordinate function κ_d, in the form of an Eikonal equation. Then, an admissible change of variables is for example

$$y_j = \kappa_j(x) = x_j \ \text{ for } j = 1, \ldots, d-1, \qquad y_d = \kappa_d(x),$$

with κ_d solution to (0.2) and such that $\kappa_d(x) = 0$ if $x_d = 0$ and $\partial_{x_d}\kappa_d(0) = p^{dd}(0)^{-1/2}$. The existence of a local smooth solution is given in Proposition 2 below, using that $p(0, p^{dd}(0)^{-1/2}e_d) = 1$ and $\partial_{\xi_d}p(0, e_d) \neq 0$ as $p^{dd}(0) > 0$.

As $\partial_d\kappa_d(0) \neq 0$, this holds in a neighborhood of 0. This implies, as in the first step, with the local diffeomorphism theorem, that there exits an open neighborhood X of x^0 such that κ is a smooth diffeomorphism of X onto $Y = \kappa(X)$. Note that the sets $\{x_d = 0\}$ and $\{x_d \geq 0\}$ are changed into $\{y_d = 0\}$ and $\{y_d \geq 0\}$ respectively.

Note that this second step has preserved the assumptions of the theorem.

Step 3 Now we have $x_0 = 0$, $S = \{x_d = 0\}$, $\psi(x) = x_d$ and moreover P_0 takes the form

$$P_0 = D_{x_d}^2 + 2\sum_{i=1}^{d-1} p^{id}(x)D_{x_i}D_{x_d} + \sum_{i,j=1}^{d-1} p^{ij}(x)D_{x_i}D_{x_j}.$$

We then write P_0 in the following form

$$P_0 = (D_{x_d} + \sum_{i=1}^{d-1} p^{id}(x)D_{x_j})^2 + \sum_{i,j=1}^{d-1} q^{ij}(x)D_{x_i}D_{x_j}, \tag{0.3}$$

where the coefficients q^{ij} are related to the coefficients p^{ij} in a smooth way that needs not made explicit here. From the positivity of $p(x, \xi)$ uniformly w.r.t. x in V we find that there exists $C > 0$ such that

$$\sum_{i,j=1}^{d-1} q^{ij}(x)\xi_i\xi_j \geq C|\xi'|^2 \tag{0.4}$$

Now build a diffeomorphism $x \mapsto \kappa(x) = y$ so has to have

$$D_{y_d} = D_{x_d} + \sum_{i=1}^{d-1} p^{id}(x)D_{x_j},$$

and D_{y_i} as a linear combination of $D_{x_1}, \dots, D_{x_{d-1}}$ for $i = 1, \dots, d-1$. In fact, we shall built κ^{-1} by considering the following differential system

$$\begin{cases} \dot{x}_i = p^{id}(x), \quad x_j(0) = y_j \quad \text{for } i = 1, \dots, d-1, \\ \dot{x}_d = 1, \quad x_d(0) = 0. \end{cases} \tag{0.5}$$

We denote the solution of System (0.5) by $x(y', t)$ with $y' = (y_1, \dots, y_{d-1})$. We define κ^{-1} by $\kappa^{-1}(y) = x(y', y_d)$ which is a local diffeomorphism that maps an open neighborhood Y of 0 into an open neighborhood $X \subset V$ of 0 by the Cauchy-Lipschitz theorem. Through this change of variables we have

$$D_{y_i} = \sum_{j=1}^{d} (\partial_{y_i}\kappa_j^{-1}(y))D_{x_j},$$

which yields

$$D_{y_d} = \sum_{j=1}^{d} (\partial_{y_d} \kappa_j^{-1}(y)) D_{x_j} = \sum_{j=1}^{d} \partial_{y_d} x_j(y', y_d) D_{x_j} = D_{x_d} + \sum_{i=1}^{d-1} p^{id}(x) D_{x_j},$$

and,

$$D_{y_i} = \sum_{j=1}^{d} (\partial_{y_i} x_j(y'; y_d)) D_{x_j} = \sum_{j=1}^{d-1} (\partial_{y_i} x_j(y'; y_d)) D_{x_j}, \qquad i = 1, \ldots, d-1,$$

as $\partial_{y_i} x_d(y', t) = 0$ for $i = 1, \ldots, d-1$ since $x_d(y', t) = t$.

Note that the matrix $(\partial_{y_i} x_j(y', y_d))_{1 \le i,j \le d-1}$ is the identity for $y_d = 0$ according to the initial conditions in System (0.5). Thus, for Y chosen sufficiently small it remains invertible. The vector fields D_{x_i}, $i = 1, \ldots, d-1$, are then transformed into $\sum_{j=1}^{d-1} c_{ij}(y) D_{y_j}$.

Following (0.3), in the new variable y, P_0 takes the form

$$Q_0 = D_{y_d}^2 + \sum_{i,j=1}^{d-1} \tilde{p}^{ij}(y) D_{y_i} D_{y_j}.$$

Point 2 of the statement of the theorem is achieved. From the positivity of $(q^{ij}(x))_{1 \le i,j \le d-1}$ in (0.4) we deduce that a similar positivity holds for $(\tilde{p}^{ij}(y))_{1 \le i,j \le d-1}$ which gives point 3 of the statement of the theorem .

Finally, observe that since $\dot{x}_d = 1 > 0$ and $x_d(y', 0) = 0$, we see that the sets $\{x_d = 0\}$ and $\{x_d > 0\}$ are transformed in $\{y_d = 0\}$ and $\{y_d > 0\}$ respectively in a neighborhood of 0. ■

As above, we write $x = (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$ and similarly $\xi = (\xi', \xi_d)$ the associated cotangent vectors.

Proposition 2. *Let $q(x, \xi)$ be a smooth real function defined in a neighborhood of $(0, \eta)$ in $\mathbb{R}^d \times \mathbb{R}^d$ such that $q(0, \eta) = 0$ and $\partial_{\xi_d} q(0, \eta) \neq 0$. Let $f \in \mathscr{C}^\infty(\mathbb{R}^{d-1})$ be real valued and such that $d_{x'} f(0) = \eta'$. Then, there exists a neighborhood U of $(0, \eta)$ and $g \in \mathscr{C}^\infty(U)$, real valued, such that $q(x, dg(x)) = 0$, for $x \in U$, and the boundary condition*

$$g(x', 0) = f(x'), \quad \text{for } (x', 0) \in U, \qquad \text{and} \quad dg(0) = \eta.$$

For a proof we refer to [1, Theorem 6.4.5].

References

[1] Hörmander, L. *The Analysis of Linear Partial Differential Operators*, second ed., vol. I. Springer-Verlag, 1990.