Some elements of semigroup theory

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Semigroup theory is at the heart of the understanding of many evolution equations that can be put in the form

$$\frac{d}{dt}x(t) + Ax(t) = f(t), \quad t > 0, \qquad x(t) = x_0, \tag{0.1}$$

with $x(t)$ and x_0 in a proper function space, usually a Banach space, denoted by X below, if not a Hilbert space, with A an unbounded operator on X, with dense domain, and f a function of the time variable t taking values in X. First, in Sections 1 and 2, we review the case of a homogeneous equation, that is $f \equiv 0$. Second, in Section 3 we consider the more general case of an inhomogeneous equation. In particular, we provide the necessary form of the solution, given by the so-called mild solution based on the Duhamel formula. Third, in Section 4 we show how some results improve in the case the function space X is a Hilbert space.

1 Strongly continuous semigroups

Consider the homogeneous equation associated with the evolution problem (0.1), that is,

$$\frac{d}{dt}x(t) + Ax(t) = 0, \quad t > 0, \qquad x(t) = x_0, \tag{1.1}$$

Under proper assumptions on A we can write the solution in the form $x(t) = \mathbb{S}(t)x_0$, where $\mathbb{S}(t) : X \to X$ is a bounded operator. As some sort of differentiation with respect to time is expected in (1.1), a minimal assumption is then that

$$\mathbb{S}(0)x = x \quad \text{and} \quad t \mapsto \mathbb{S}(t)x \text{ be continuous for all } x \in X. \tag{1.2}$$

With $t \mapsto x(t)$ solution to (1.1), if the evolution problem is *well posed*, we expect from uniqueness that solving the following problem, for some $t_0 \geq 0$,

$$\frac{d}{dt}y(t) + Ay(t) = 0, \quad t > t_0, \qquad y(t_0) = x(t_0), \tag{1.3}$$

yield a solution that satisfies $y(t) = x(t)$ for $t \geq t_0$. In particular, this implies the following property

$$\mathbb{S}(t + t') = \mathbb{S}(t) \circ \mathbb{S}(t'), \text{ for } t, t' \in [0, +\infty). \tag{1.4}$$

Properties (1.2) and (1.4) are precisely the starting point of semigroup theory in Banach spaces.

1.1 Definition and basic properties

Let X be a Banach space.

Definition 1. A family $\mathbb{S}(t)$ of bounded operators on X, indexed by $t \in [0, +\infty)$ is called a semigroup if:

$$\mathbb{S}(0) = \mathrm{Id}_X \quad \text{and} \quad \mathbb{S}(t + t') = \mathbb{S}(t) \circ \mathbb{S}(t') \text{ for } t, t' \in [0, +\infty). \tag{1.5}$$

The semigroup is called strongly continuous if, moreover, for all $x \in X$ we have $\lim_{t \to 0^+} \mathbb{S}(t)x = x$. One says that $\mathbb{S}(t)$ is a C_0-semigroup for short.

For each C_0-semigroup there exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that

$$\|\mathbb{S}(t)\|_{\mathscr{L}(X)} \leq Me^{\omega t}, \tag{1.6}$$

by the uniform boundedness principle [3, Theorem 1.2.2]. It follows that the map $(t, x) \mapsto \mathbb{S}(t)x$ is continuous from $[0, +\infty) \times X$ into X. A C_0-semigroup $\mathbb{S}(t)$ is said to be bounded if there exists $M \geq 1$ such that $\|\mathbb{S}(t)\|_{\mathscr{L}(X)} \leq M$, for $t \geq 0$. In the case $M = 1$ one says that the C_0-semigroup is of contraction.

We define the unbounded linear operator A from X to X, with domain

$$D(A) = \{x \in X;\ \lim_{t \to 0^+} t^{-1}(x - \mathbb{S}(t)x) \text{ exists}\}, \tag{1.7}$$

and given by

$$Ax = \lim_{t \to 0^+} t^{-1}(x - \mathbb{S}(t)x), \qquad x \in D(A). \tag{1.8}$$

For basic aspects of unbounded operators we refer to [1].

This operator $(A, D(A))$ is called the generator of the C_0-semigroup. One can prove that the generator of a C_0-semigroup has a dense domain in X and is a closed operator [3, Corollary 1.2.5] (see [1] for the notion of closed operators).

The domain $D(A)$ can be equipped with the graph norm

$$\|x\|_{D(A)} = \|x\|_X + \|A(x)\|_X.$$

Since A is closed one finds that $(D(A), \|.\|_{D(A)})$ is complete.

Note that the map

$$\mathbb{S}(t) \mapsto A \tag{1.9}$$

is injective [3, Theorem 1.2.6]. The following proposition shows that computing $\mathbb{S}(t)x$ yields a solution of an evolution equation.

Proposition 2. *Let $T \in \mathbb{R}_+ \cup \{+\infty\}$ and let $x \in D(A)$. We have $u(t) = \mathbb{S}(t)x \in \mathscr{C}^0([0,T], D(A)) \cap \mathscr{C}^1([0,T], X)$ and*

$$\frac{d}{dt}u(t) + Au(t) = 0, \quad 0 \leq t \leq T, \qquad u(0) = x. \tag{1.10}$$

Moreover, $\mathbb{S}(t)x$ is the unique solution to (1.10) in $\mathscr{C}^0([0,T], D(A)) \cap \mathscr{C}^1([0,T], X)$. In addition, we have $A\mathbb{S}(t)x = \mathbb{S}(t)Ax$.

Here, $[0,T]$ means $[0, +\infty)$ if $T = +\infty$.

Proof. Let $x \in D(A)$. We write, for $h > 0$,

$$h^{-1}\big(\operatorname{Id}_X - \mathbb{S}(h)\big)\mathbb{S}(t)x = \mathbb{S}(t)h^{-1}\big(\operatorname{Id}_X - \mathbb{S}(h)\big)x \to_{h \to 0^+} \mathbb{S}(t)Ax,$$

as $\mathbb{S}(t)$ is bounded on X. Hence, we have $\mathbb{S}(t)x \in D(A)$ and $A\mathbb{S}(t)x = \mathbb{S}(t)Ax$ by (1.7)–(1.8). Moreover, we have the following right derivative $\frac{d^+}{dt}\mathbb{S}(t)x = -A\mathbb{S}(t)x$. To compute the left derivative we write, for $h > 0$,

$$h^{-1}(\mathbb{S}(t)x - \mathbb{S}(t-h)x) = \mathbb{S}(t-h)h^{-1}(\mathbb{S}(h)x - x).$$

Since $h^{-1}(\mathbb{S}(h)x - x) \to -Ax$ as $h \to 0^+$ and since $(t, x) \mapsto \mathbb{S}(t)x$ is continuous on $[0, T] \times X$ we obtain

$$h^{-1}(\mathbb{S}(t)x - \mathbb{S}(t-h)x) \underset{h\to 0^+}{\to} -\mathbb{S}(t)Ax.$$

We thus obtain $\frac{d}{dt}\mathbb{S}(t)x + A\mathbb{S}(t)x = 0$. From $A\mathbb{S}(t)x = \mathbb{S}(t)Ax$ and the continuity of $t \mapsto \mathbb{S}(t)y$ for all $y \in X$ we conclude that $\mathbb{S}(t)x \in \mathscr{C}^0([0, T], D(A))$ and finally using the equation we have $\mathbb{S}(t)x \in \mathscr{C}^1([0, T], X)$.

Uniqueness. Let $u \in \mathscr{C}^0([0, T], D(A)) \cap \mathscr{C}^1([0, T], X)$ be a solution to (1.10) satisfying $u(0) = x$. Let $0 < s \le T$. For $t \in [0, s]$ we set $v(t) = \mathbb{S}(s-t)u(t)$. With the first part we have

$$\frac{d}{dt}v(t) = \mathbb{S}(s-t)\frac{d}{dt}u(t) + \mathbb{S}(s-t)Au(t) = 0.$$

We thus find $\mathbb{S}(s)x = \mathbb{S}(s)u(0) = v(0) = v(s) = u(s)$. ■

We provide also the following result that shows that integration with respect to time yields a gain of regularity.

Lemma 3. *For $x \in X$ and $T > 0$, we have $\int_0^T \mathbb{S}(t)x\,dt \in D(A)$ and*

$$\mathbb{S}(T)x - x + A\int_0^T \mathbb{S}(t)x\,dt = 0.$$

Proof. For $h > 0$ we compute

$$\begin{aligned} F_h &= h^{-1}(\mathrm{Id}_X - \mathbb{S}(h))\int_0^T \mathbb{S}(t)x\,dt = h^{-1}\int_0^T \big(\mathbb{S}(t) - \mathbb{S}(t+h)\big)x\,dt \\ &= h^{-1}\int_0^h \big(\mathbb{S}(t)x - \mathbb{S}(t)\mathbb{S}(T)x\big)\,dt. \end{aligned}$$

With the continuity of $t \mapsto \mathbb{S}(t)x$ and $t \mapsto \mathbb{S}(t)\mathbb{S}(T)x$, the fundamental theorem of calculus yields the result, by (1.7)–(1.8). ■

We shall use the following version in which a smooth window function is introduced.

Lemma 4. *For $x \in X$ and $\chi \in \mathscr{C}_c^\infty(0, \infty)$ we have*

$$\int_0^\infty \chi(t)\mathbb{S}(t)x\,dt \in D(A) \quad \textit{and} \quad A\int_0^\infty \chi(t)\mathbb{S}(t)x\,dt = \int_0^\infty \chi'(t)\mathbb{S}(t)x\,dt.$$

We see that the result of Lemma 3 formally coincides with that of Lemma 4 in the case $\chi = \mathbf{1}_{(0,T)}$.

Proof. For $h > 0$ we compute

$$F_h = h^{-1}(\mathrm{Id}_X - \mathbb{S}(h)) \int_0^\infty \chi(t)\mathbb{S}(t)x\,dt = h^{-1} \int_0^\infty \chi(t)\big(\mathbb{S}(t) - \mathbb{S}(t+h)\big)x\,dt.$$

Observe that we have

$$\int_0^\infty \chi(t)\mathbb{S}(t+h)x\,dt = \int_h^\infty \chi(t-h)\mathbb{S}(t)x\,dt = \int_0^\infty \chi(t-h)\mathbb{S}(t)x\,dt,$$

because of the support of χ. We thus obtain

$$F_h = h^{-1} \int_0^\infty \big(\chi(t) - \chi(t-h)\big)\mathbb{S}(t)x\,dt.$$

With the continuity of $t \mapsto \mathbb{S}(t)x$, the Lebesgue dominated-convergence theorem yields

$$\lim_{h\to 0^+} F_h = \int_0^\infty \chi'(t)\mathbb{S}(t)x\,dt.$$

Consequently, by (1.7)–(1.8), we obtain the result. ■

Observe that if $\mathbb{S}(t)$ is a C_0-semigroup and $z \in \mathbb{C}$ then $e^{zt}\mathbb{S}(t)$ satisfies (1.5). The following proposition is then clear from what precedes.

Proposition 5. *Let $\mathbb{S}(t)$ be a C_0-semigroup and $z \in \mathbb{C}$. Then $e^{zt}\mathbb{S}(t)$ is also a C_0-semigroup and its generator is $A - z\,\mathrm{Id}_X$.*

Note that, because of the uniqueness of the generator of a C_0-semigroup [3, Theorem 1.2.6], conversely, if A generates a C_0-semigroup, then $A - z\,\mathrm{Id}_X$ is the generator of a C_0-semigroup, namely $e^{zt}\mathbb{S}(t)$.

1.2 The Hille-Yosida theorem

The next natural question is to wonder if an unbounded operator on X is the generator of a C_0-semigroup. The Hille-Yosida theorem is central in the semigroup theory, providing a clear answer to this question. We refer to [3, Theorem 1.3.1] for a proof.

Theorem 6. *Let $(A, D(A))$ be a linear unbounded operator on a Banach space X. It generates a C_0-semigroup of contraction if and only if:*

1. *A is closed and $D(A)$ is dense in X.*

2. *The resolvent set $\rho(A)$ of A contains $(-\infty, 0)$ and we have the following estimate:*

$$\|R_\lambda(A)\|_{\mathscr{L}(X)} \leq 1/|\lambda|, \quad \lambda < 0, \qquad R_\lambda(A) = (\lambda\,\mathrm{Id}_X - A)^{-1}.$$

For the notions of closed operators, resolvent set, spectrum, and resolvent operator $R_\lambda(A)$, we refer to [1, 2, 5].

The previous result is limited to contraction C_0-semigroups. The following corollary provides a characterization of all generators of C_0-semigroups; we refer to [3, Theorem 1.5.3] for a proof.

Corollary 7. *Let $(A, D(A))$ be a linear unbounded operator on a Banach space X. It generates a C_0-semigroup $\mathbb{S}(t)$ such that $\|\mathbb{S}(t)\|_{\mathscr{L}(X)} \le M e^{\omega t}$, for some $M \ge 1$ and $\omega \in \mathbb{R}$, if and only if:*

1. *A is closed and $D(A)$ is dense in X.*

2. *The resolvent set $\rho(A)$ of A contains $(-\infty, -\omega)$ and we have the following estimate:*

$$\|R_\lambda(A)^n\|_{\mathscr{L}(X)} \le M/|\omega+\lambda|^n, \quad \lambda < -\omega,\ n \in \mathbb{N}^*, \qquad R_\lambda(A) = (\lambda \operatorname{Id}_X - A)^{-1}.$$

The Hille-Yosida theorem has the following simple consequence.

Corollary 8. *Let $(A, D(A))$ be the generator of a bounded C_0-semigroup $\mathbb{S}(t)$, that is, $\|\mathbb{S}(t)\|_{\mathscr{L}(X)} \le M$, for $t \ge 0$, for some $M > 0$. Then, its spectrum satisfies* $\operatorname{sp}(A) \subset \{z \in \mathbb{C}; \operatorname{Re} z \ge 0\}$.

Proof. Let $b \in \mathbb{R}$, the C_0-semigroup $e^{ibt}\mathbb{S}(t)$ is generated by $A - ib \operatorname{Id}_X$. As $e^{ibt}\mathbb{S}(t)$ satisfies $\|e^{ibt}\mathbb{S}(t)\|_{\mathscr{L}(X)} \le M$, for $t \ge 0$, the conclusion follows from Corollary 7 in the case $\omega = 0$. ■

1.3 The Lumer-Phillips theorem

The Lumer-Phillips theorem provides another characterisation of generators of contraction semigroups.

Let X' be the dual space of X equipped with the strong topology (see [1, 4]). For $x \in X$ we set

$$F(x) = \{\phi \in X';\ \phi(x) = \langle \phi, x\rangle_{X',X} = \|x\|_X^2 = \|\phi\|_{X'}^2\}, \tag{1.11}$$

which is not empty by the Hahn-Banach theorem.

Definition 9. A linear unbounded operator $(A, D(A))$ on X is said to be monotone (or accretive) if for all $x \in D(A)$, $x \ne 0$, there exists $\phi \in F(x)$ such that $\operatorname{Re}\langle \phi, Ax\rangle_{X',X} \ge 0$.

Definition 10. A linear unbounded operator $(A, D(A))$ on X is said to be maximal monotone if it is monotone and if moreover there exists $\lambda_0 > 0$ such that the range of $\lambda_0 \operatorname{Id}_X + A$, $\operatorname{Ran}(\lambda_0 \operatorname{Id}_X + A)$, is equal to X,

The Lumer-Philips theorem reads as follows.

Theorem 11. *Let $(A, D(A))$ be a linear unbounded operator. It generates a C_0-semigroup of contraction if and only if*

1. *A has a dense domain.*
2. *A is maximal monotone.*

A proof based on the Hille-Yosida theorem directly follows from Lemmata 13 and 14 given below.

Remark 12. Observe that there is no need to assume that the operator A is closed in the converse part of the Lumer-Philips theorem as in the Hille-Yosida theorem. In fact, as proven below, a maximal monotone operator is closed (see Lemma 14). In the case of a reflexive Banach space, the dense domain assumption may be dropped in the converse part of the Lumer-Philips theorem: a maximal monotone operator has a dense domain; see [3, Theorem 1.4.6] (see also [1, Proposition 7.1] for the Hilbert space case).

The next lemma gives a characterization of monotone operators.

Lemma 13. *An unbounded operator $(A, D(A))$ on X is monotone if and only if*

$$\|(\lambda \operatorname{Id}_X + A)x\|_X \geq \lambda \|x\|_X, \quad x \in D(A) \text{ and } \lambda > 0. \tag{1.12}$$

Proof. First, we assume that A is monotone. Let $\lambda > 0$ and $x \in D(A)$. We may then write, for some $\phi \in F(x)$,

$$\begin{aligned}\lambda \|x\|_X^2 &\leq \lambda \operatorname{Re}\langle \phi, x\rangle_{X',X} + \operatorname{Re}\langle \phi, Ax\rangle_{X',X} = \operatorname{Re}\langle \phi, \lambda x + Ax\rangle_{X',X} \\ &\leq \|\phi\|_{X'} \|\lambda x + Ax\|_X = \|x\|_X \, \|\lambda x + Ax\|_X,\end{aligned}$$

yielding (1.12).

Second, we assume that (1.12) holds. Let $x \in D(A)$ with $x \neq 0$. For $\lambda > 0$ we let $\phi_\lambda \in F(\lambda x + Ax)$. By (1.12), we have $\lambda x + Ax \neq 0$ and thus $\phi_\lambda \neq 0$. We normalize it setting $\psi_\lambda = \phi_\lambda / \|\phi_\lambda\|_X$. We then have $\|\lambda x + Ax\|_X = \langle \psi_\lambda, \lambda x + Ax\rangle_{X',X}$. We may thus write, with (1.12),

$$\begin{aligned}\lambda \|x\|_X \leq \|(\lambda \operatorname{Id}_X + A)x\|_X &= \langle \psi_\lambda, \lambda x + Ax\rangle_{X',X} \\ &= \lambda \langle \psi_\lambda, x\rangle_{X',X} + \langle \psi_\lambda, Ax\rangle_{X',X}.\end{aligned}$$

As $\|\psi_\lambda\|_{X'} = 1$ the conclusion is twofold:

$$\langle \psi_\lambda, Ax\rangle_{X',X} \geq 0, \qquad \langle \psi_\lambda, x\rangle_{X',X} \geq \|x\|_X - \|Ax\|_X/\lambda, \quad \lambda > 0.$$

As the unit ball of X' is compact for the weak star topology[1] by the Banach-Alaoglu theorem, there exists $\psi \in X'$ with $\|\psi\|_{X'} \leq 1$ and an increasing sequence $(\lambda_n)_n$ that diverges to $+\infty$ such that $\psi_{\lambda_n} \xrightharpoonup{*} \psi$ implying

$$\langle \psi, Ax\rangle_{X',X} \geq 0, \qquad \langle \psi, x\rangle_{X',X} \geq \|x\|_X.$$

This yields $\|\psi\|_{X'} = 1$. We set $\phi = \|x\|_X \psi$ and we have $\phi \in F(x)$ and $\langle \phi, Ax\rangle_{X',X} \geq 0$. As $x \neq 0$ is arbitrary in $D(A)$, this yields that A is monotone. ∎

The value of $\lambda_0 > 0$ in Definition 10 is not of great significance. In fact, we have the following result.

Lemma 14. *Let A be a maximal monotone operator on X. Then, A is closed and for all $\lambda > 0$ the operator $\lambda \operatorname{Id}_X + A$ is bijective from $D(A)$ onto X. Moreover, if $\lambda > 0$, its inverse, $(\lambda \operatorname{Id}_X + A)^{-1}$, is a bounded operator and we have the following estimation $\|(\lambda \operatorname{Id}_X + A)^{-1}\|_{\mathscr{L}(X)} \leq \lambda^{-1}$.*

Proof. Let $\lambda > 0$. The injectivity of $\lambda \operatorname{Id}_X + A$ follows from Lemma 13.

As A is maximal monotone, there exists $\lambda_0 > 0$ such that $\lambda_0 \operatorname{Id}_X + A$ is also surjective. Its inverse $(\lambda_0 \operatorname{Id}_X + A)^{-1}$ is thus well defined on X. By Lemma 13 we have $\|(\lambda_0 \operatorname{Id}_X + A)^{-1}\|_{\mathscr{L}(X)} \leq \lambda_0^{-1}$. By the closed-graph theorem (see [1]), the graph of $(\lambda_0 \operatorname{Id}_X + A)^{-1}$ is closed in $X \times X$ and thus so is the graph of A.

We now prove that if $\lambda \operatorname{Id}_X + A$ is surjective then so is $\lambda' \operatorname{Id}_X + A$ for any λ' such that $0 < \lambda' < 2\lambda$. The proof is based on the Banach contraction fixed point theorem. By induction, starting with $\lambda = \lambda_0$ we then reach the conclusion that $\lambda \operatorname{Id}_X + A$ is onto for any $\lambda > 0$ and then the boundedness of its inverse follows from Lemma 13.

Let $\lambda, \lambda' > 0$ be such that $\lambda \operatorname{Id}_X + A$ is onto and $0 < \lambda' < 2\lambda$. Let $y \in X$. We wish to find $x \in X$ such that $\lambda' x + Ax = y$. This reads $\lambda x + Ax = y + (\lambda - \lambda')x$ and thus thus we have $x = (\lambda \operatorname{Id}_X + A)^{-1}(y + (\lambda - \lambda')x)$, meaning that we seek a fixed point for the bounded affine map $H : x \mapsto (\lambda \operatorname{Id}_X + A)^{-1}(y + (\lambda - \lambda')x)$. By the computation above we have $\|(\lambda \operatorname{Id}_X + A)^{-1}\|_{\mathscr{L}(X)} \leq 1/\lambda$, we thus find

$$\|H(x) - H(x')\|_X \leq |1 - \lambda'/\lambda| \, \|x - x'\|_X.$$

As $0 < |1 - \lambda'/\lambda| < 1$, the Banach contraction fixed point theorem applies. ∎

[1] The weak star topology is often referred to as the $\sigma(E', E)$ topology en E' (see e.g. [4, 1]).

2 Differentiable and analytic semigroups

Above, we have considered C_0-semigroups. If the continuity assumption of a semigroup $\mathbb{S}(t)$ is reinforced, say by assuming uniform continuity instead of strong continuity: $\|\mathbb{S}(t) - \mathrm{Id}_X\|_{\mathscr{L}(X)} \to 0$ as $t \to 0^+$. Then, one can prove that the generator A of $\mathbb{S}(t)$ is a bounded operator on X and that one simply has $\mathbb{S}(t) = \exp(-tA) = \sum_{n\geq 0}(-tA)^n/n!$ (see e.g. [3, Section 1.1]).

Other regularity assumptions with respect to t can be made, yet not making the semigoup becoming a trivial exponential of a bounded operator.

Definition 15. A C_0-semigroup $\mathbb{S}(t)$ is called differentiable for $t > t_0$ if for all $x \in X$ the map $t \mapsto \mathbb{S}(t)x$ is differentiable for $t > t_0$.

If $x \in D(A)$, then $t \mapsto \mathbb{S}(t)x$ is differentiable for $t \geq 0$ and $\frac{d}{dt}\mathbb{S}(t)x = -A\mathbb{S}(t)x$ by Proposition 2. If the semigroup is differentiable for $t > t_0$, this means that this property extends to all $x \in X$ if $t > t_0$. The following proposition states properties of differentiable semigroups; see [3, Lemma 2.4.2] for a proof.

Proposition 16. *Let $\mathbb{S}(t)$ be a differentiable semigroup for $t > t_0$. Then for $t > nt_0$, $n \in \mathbb{N}$, $\mathbb{S}(t)$ maps X in $D(A^n)$, and for all $x \in X$, the map $t \to \mathbb{S}(t)x$ is n times differentiable, and $\left(\frac{d}{dt}\right)^n\mathbb{S}(t) = (-A)^n\mathbb{S}(t) \in \mathscr{L}(X)$. Moreover, $t \to \mathbb{S}(t) \in \mathscr{C}^n(((n+1)t_0, +\infty), \mathscr{L}(X))$.*

Note that if $t_0 = 0$ then $\mathbb{S}(t) \in \mathscr{C}^k((0, +\infty), \mathscr{L}(X, D(A^\ell)))$, for any $k, \ell \in \mathbb{N}$.

Like many other semigroup properties, the differentiability of a semigroup can be characterized through a resolvent estimate; see [3, Theorem 2.4.8] for a proof.

Theorem 17. *Let $\mathbb{S}(t)$ be a C_0-semigroup that satisfies $\|\mathbb{S}(t)\|_{\mathscr{L}(X)} \leq Me^{\omega t}$ with A as its generator. The semigroup $\mathbb{S}(t)$ is differentiable for $t > 0$ if and only if for every $b > 0$ there exist $a > 0$ and $C_0 > 0$ such that*

$$\Sigma_b = \{z \in \mathbb{C};\ \operatorname{Re} z \leq -a + b\log|\operatorname{Im} z|\} \subset \rho(A),$$

and

$$\|z\,\mathrm{Id}_X - A\|_{\mathscr{L}(X)} \leq C_0|\operatorname{Im} z| \qquad z \in \Sigma_b,\ \operatorname{Re} z \geq -\omega.$$

We now consider analytic semigroups. For $0 < \theta < \pi/2$, we set

$$\Sigma_\theta = \{z;\ |\arg(z)| \leq \theta\} \cup \{0\}.$$

Definition 18. A map $\mathbb{S} : \Sigma_\theta \to \mathscr{L}(X)$ is said to be an analytic semigroup on Σ_θ if

1. $z \mapsto \mathbb{S}(z)$ is analytic on $\Sigma_\theta \setminus \{0\}$ in the topology of $\mathscr{L}(X)$.

2. $\mathbb{S}(0) = \mathrm{Id}_X$ and $\lim_{\substack{z\to 0\\ z\in\Sigma_\theta\setminus\{0\}}} \mathbb{S}(z)x = x$ for all $x \in X$;

3. $\mathbb{S}(z_1 + z_2) = \mathbb{S}(z_1)\mathbb{S}(z_2)$, for all $z_1, z_2 \in \Sigma_\theta$.

The following result yields a characterization of an analytic semigroup; we refer to [3, Theorem 2.5.2] for a proof.

Theorem 19. *Let $\mathbb{S}(t)$ be a C_0-semigroup on X and let A be its generator such that $0 \in \rho(A)$ and $\|\mathbb{S}(t)\|_{\mathscr{L}(X)} \le M$, for some $M > 0$. The following statements are equivalent.*

1. *There exists $0 < \theta < \pi/2$ such that $\mathbb{S}(t)$ can be extended as an analytic semigroup on Σ_θ and is such that $\|\mathbb{S}(z)\|_{\mathscr{L}(X)} \le M'$ for $z \in \Sigma_\theta$ for some $M' > 0$.*

2. *There exists $C > 0$ such that*
$$\sigma + i\tau \in \rho(A) \quad \text{and} \quad \|(\sigma + i\tau - A)^{-1}\|_{\mathscr{L}(X)} \le C/|\tau|, \qquad \sigma < 0,\ \tau \neq 0.$$

3. *There exist $0 < a < \pi/2$ and $C > 0$ such that*
$$z \in \rho(A) \quad \text{and} \quad \|(z - A)^{-1}\|_{\mathscr{L}(X)} \le C/|z|,$$
if $z \neq 0$ and $\pi/2 - a < \arg(z) < 3\pi/2 + a$.

4. *The semigroup $\mathbb{S}(t)$ is differentiable for $t > 0$ and there exists $C > 0$ such that*
$$\|A\mathbb{S}(t)\|_{\mathscr{L}(X)} \le C/t, \quad t > 0.$$

3 Mild solution of the inhomogeneous Cauchy problem

In what precedes, we have seen that semigroups can be used to solve the abstract homogeneous Cauchy problem in a Banach space X:
$$\frac{d}{dt}u(t) + Au(t) = 0 \text{ for } t > 0 \quad u(0) = u_0.$$
In fact, if A generates a semigroup on X and if $u_0 \in D(A)$ then the unique solution in $\mathscr{C}^0([0,+\infty), D(A))$ is given by $u(t) = \mathbb{S}(t)u_0$ by proposition 2. Note that $u(t) \in \mathscr{C}^1([0,+\infty), X)$ and moreover the equation is even satisfied for $t \ge 0$. In the case of a selfadjoint generator on a Hilbert space this can be extended to $x \in X$ by

Corollary 27. Then, the unique solution is in $\mathscr{C}^0([0,+\infty),X)\cap\mathscr{C}^k((0,\infty),D(A^\ell))$, $k,\ell\in\mathbb{N}$ and note that the equation is only satisfied for $t>0$.

We are now interested into solving a nonhomogeneous abstract Cauchy problem of the form

$$\frac{d}{dt}u(t)+Au(t)=f(t)\in X \text{ for } t>0 \quad \text{and} \quad u(0)=u_0. \tag{3.1}$$

Here, A is assumed to generate a C_0-semigroup on X.

A classical solution is a function

$$u\in\mathscr{C}([0,\infty),X)\cap\mathscr{C}^1((0,\infty);X),$$

such that $u(t)\in D(A)$ for $t>0$ and that satisfies (3.1), that is, both the equation for $t>0$ and the initial condition. If $u(t)$ is such a solution, we choose $T>0$ and we set $w(t)=\mathbb{S}(T-t)u(t)$. We have, for $t>0$,

$$\frac{d}{dt}w(t)=\mathbb{S}(T-t)\big(\frac{d}{dt}u(t)+Au(t)\big)=\mathbb{S}(T-t)f(t).$$

If $f_{|(0,T)}\in L^1(0,T;X)$ we find

$$u(T)=w(T)=w(0)+\int_0^T\mathbb{S}(T-t)f(t)\,dt=\mathbb{S}(T)u_0+\int_0^T\mathbb{S}(T-t)f(t)\,dt.$$

This is precisely the Duhamel formula in an abstract setting. We thus have the following uniqueness result.

Proposition 20. *If $f\in L^1_{\mathrm{loc}}(0,\infty;X)$ and if u is a classical solution it is given by*

$$u(t)=\mathbb{S}(t)u_0+\int_0^t\mathbb{S}(t-s)f(s)\,ds,\quad t\geq 0$$

We are thus naturally led to the following definition.

Definition 21. Let $f\in L^1_{\mathrm{loc}}(0,\infty;X)$ and $u_0\in X$. The function

$$u(t)=\mathbb{S}(t)u_0+\int_0^t\mathbb{S}(t-s)f(s)\,ds,\quad t\geq 0$$

is such that $u\in\mathscr{C}([0,\infty);X)$ and is called the mild solution of the inhomogeneous abstract Cauchy problem (3.1).

If $f\in L^1_{\mathrm{loc}}(0,\infty;D(A))\cap\mathscr{C}((0,\infty);X)$ and $u_0\in D(A)$ then the mild solution is a classical solution as can be readily checked.

4 The case of a Hilbert space.

In the case the space X is a Hilbert space. It can be identified with its dual space X' by the Riesz theorem. Out of habit, we shall denote by H the Hilbert rather than by X.

Then if $x \in H$ we have, for the set defined in (1.11), $F(x) = \{x\}$. We then have the following definition in agreement with Definition 9.

Definition 22. A linear unbounded operator $(A, D(A))$ on H is said to be monotone (or accretive) if for all $x \in D(A)$ one has $\operatorname{Re}(Ax, x)_H \geq 0$.

In the Hilbert case the Lumer-Phillips theorem reads, in agreement with Remark 12.

Theorem 23. *Let A be a linear unbounded operator. It generates a C_0-semigroup of contraction if and only if it is maximal monotone.*

Proposition 24. *Let $(A, D(A))$, be an unbounded operator on H, that generates a C_0-semigroup $\mathbb{S}(t)$. Then, the operator $(A^*, D(A^*))$ generates a semigroup $\Sigma(t)$ and $\Sigma(t) = \mathbb{S}(t)^*$.*

Recall that if $(A, D(A))$ generates a semigroup then it is closed and densely defined. This allows one to properly define its adjoint operator (see [1]).

Proof. The operator $(A^*, D(A^*))$ is closed and densely defined [1]. By Corollary 7, consequence of the Hille-Yosida theorem (Theorem 6), there exists $M \geq 1$ and $\omega \in \mathbb{R}$ such that, for $\lambda < -\omega$, the operator $R_\lambda(A) = (\lambda \operatorname{Id}_X - A)^{-1}$ exists and is bounded on H and moreover

$$\|R_\lambda(A)^n\|_{\mathscr{L}(H)} \leq M/|\omega + \lambda|^n.$$

By Theorem III.5.30 in [2], the operator $R_\lambda(A^*)^n = \big((\lambda \operatorname{Id}_X - A^*)^{-1}\big)^n$ exists and we have

$$\|R_\lambda(A^*)^n\|_{\mathscr{L}(H)} = \|R_\lambda(A)^n\|_{\mathscr{L}(H)} \leq M/|\omega + \lambda|^n.$$

Hence, by Corollary 7, the operator $(A^*, D(A^*))$ generates a C_0-semigroup.

We denote by $\Sigma(t)$ the semigroup generated by A^*. Let $t > 0$ and let $x \in D(A)$ and $x' \in D(A^*)$. For $s \in [0, t]$ we introduce the function

$$h(s) = (\mathbb{S}(t-s)x, \Sigma(s)x')_H.$$

We compute $h'(s) = -(A\mathbb{S}(t-s)x, \Sigma(s)x')_H + (\mathbb{S}(t-s)x, A^*\Sigma(s)x')_H = 0$, as $\mathbb{S}(t-s)x \in D(A)$ and $\Sigma(s)x' \in D(A^*)$. We thus have $h(t) = h(0)$, that is,

$$(x, \Sigma(t)x')_H = (\mathbb{S}(t)x, x')_H, \qquad x \in D(A),\ x' \in D(A^*).$$

Since the domains of the two operators are dense in H we conclude that we have

$$(x, \Sigma(t)x')_H = (\mathbb{S}(t)x, x')_H, \qquad x, x' \in H,$$

which yields the conclusion. ■

In the case of a self adjoint operator $(A, D(A))$, that is, $D(A^*) = D(A)$ and $A = A^*$, we have the following consequence.

Corollary 25. *Let $(A, D(A))$ be a selfadjoint operator on H. If it generates a C_0-semigroup $\mathbb{S}(t)$ on H then $\mathbb{S}(t)^* = \mathbb{S}(t)$.*

If the operator $(A, D(A))$ is selfadjoint, the conclusions of the Lumer-Philips theorem are even stronger; we refer to [1, Theorem 7.7] for a proof.

Theorem 26. *Let A be a linear unbounded selfadjoint operator on a Hilbert space H. It generates an* analytic *semigroup $\mathbb{S}(t)$ of contraction if and only if it is maximal monotone.*

From Proposition 16 and Theorem 19 we have the following properties.

Corollary 27. *Let $\mathbb{S}(t)$ be the analytic semigroup of contraction generated by A unbounded and selfadjoint on H a Hilbert space. Then:*

1. *We have $t \mapsto \mathbb{S}(t) \in \mathscr{C}^k((0, \infty); \mathscr{L}(H, D(A^\ell)))$, for any $k, \ell \in \mathbb{N}$.*
2. *There exists $C > 0$ such that $\|A\mathbb{S}(t)\|_{\mathscr{L}(H)} \leq C/t$.*
3. *For all $x \in H$, there exists a unique solution in*

$$\mathscr{C}^0([0, \infty); H) \cap \mathscr{C}^1((0, \infty); H) \cap \mathscr{C}^0((0, \infty); D(A))$$

   to $\frac{d}{dt}u(t) + Au(t) = 0$, for $t > 0$. It is given by $u(t) = \mathbb{S}(t)x$.

4. *Moreover $t \mapsto \mathbb{S}(t)x \in \mathscr{C}^0([0, \infty); H) \cap \mathscr{C}^k((0, \infty), D(A^\ell))$, $k, \ell \in \mathbb{N}$.*

The important property, as compared to the result of Theorems 11 and 23, lays in the fact that here the function $u(t) = \mathbb{S}(t)x$ solves the equation $\frac{d}{dt}u(t) + Au(t) = 0$ for $t > 0$ not only for $x \in D(A)$ but also for $x \in H$, as the semigroup is differentiable for $t > 0$.

The following result is often handy to assess that an operator is selfadjoint in the framework of semigroup theory. We refer to [1, Proposition 7.6].

Proposition 28. *Let A be an unbounded operator on H a Hilbert space that is symmetric, in the sense that $(Ax, y)_H = (x, Ay)_H$ for $x, y \in D(A)$. If A is maximal monotone then A is selfadjoint, that is, $D(A^*) = D(A)$ and $A = A^*$.*

If A is a symmetric unbounded operator with dense domain then one has $D(A) \subset D(A^*)$. Hence, A^* also has a dense domain, but it may happen that $D(A) \neq D(A^*)$. A simple example is the operator $A : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ with domain $D(A) = \mathscr{C}_c^\infty(\mathbb{R})$, defined by $Au = u''$. We see that A is symmetric and yet we have $D(A^*) = H^2(\mathbb{R})$.

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