Transfer theorems for multivariate generating functions (MGF)

Journée MathSTIC — 13 June 2023 — Université Paris Nord Linxiao Chen

Univariate generating functions

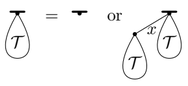
$(F_n)_{n\geq 0}$ $\qquad$ $F(x)=\sum_{n\geq 0} F_n\cdot x^n$

sequence of numbers $\qquad$ formal power series $\longrightarrow$ analytic function (in a neighborhood of $\mathbf{0}$)

Analytic combinatorics (... schematically)

Example: $\mathcal{T}=\{\text{rooted plan tree}\}$

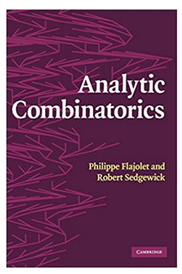
$n=$ number of edges



$F(x)=1+x\,F(x)^2$ $\qquad$ $F(x)=\dfrac{1-\sqrt{1-4x}}{2x}$ $\qquad$ $F_n=\dfrac{1}{n+1}\binom{2n}{n}$ $\qquad$ $F_n\sim\dfrac{4^n}{\sqrt{\pi n^3}}$

Combinatorial specification	Functional equation	Generating function	Expression of coefficients	Coefficient asymptotics
		$F(x)=\dfrac{1}{1-x-x^2-x^3}$	$F_n=$ 	$F_n\sim???$

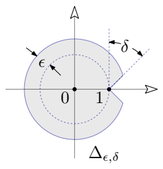
Transfer theorem for univariate generating functions



p. 393 $\longrightarrow$

$$\begin{cases} f(z)=(1-z)^{-\alpha} & \longrightarrow\quad f_n=\dfrac{n^{\alpha-1}}{\Gamma(\alpha)}+\cdots & \text{(Theorem VI.1)}\\ f(z)=O((1-z)^{-\alpha}) & \longrightarrow\quad f_n=O(n^{\alpha-1}) & \text{(Theorem VI.3 }(i)\text{)}\\ f(z)=o((1-z)^{-\alpha}) & \longrightarrow\quad f_n=o(n^{\alpha-1}) & \text{(Theorem VI.3 }(ii)\text{)}\\ f(z)\sim(1-z)^{-\alpha} & \longrightarrow\quad f_n\sim\dfrac{n^{\alpha-1}}{\Gamma(\alpha)} & \text{(Corollary VI.1).}\end{cases}$$

The important requirement is that the function should have an isolated singularity (the condition of Δ–analyticity) and that the asymptotic property of the function near its singularity should be valid in an area of the complex plane extending beyond the disc of convergence of the original series, (in a Δ–domain). Extensions to logarithmic



Transfer theorem (Flajolet–Odlyzko 1990)

Assume:

- $f(x)=A(x)+c\cdot(1-x)^{-\alpha}+R(x)\qquad(c\neq 0$ and $-\alpha\notin\mathbb{N})$
- $f(x)$ is analytic on some $\Delta_{\epsilon,\delta}$
- $A(x)$ is a polynomial in x.
- $R(x)=o((1-x)^{-\alpha})\qquad$ when $x\to 1$ in $\Delta_{\epsilon,\delta}$

Then:

$$[x^n]f(x)=\ 0\ +c\cdot\frac{n^{\alpha-1}}{\Gamma(\alpha)}+o(n^{\alpha-1})$$

Multivariate generating functions

$(F_{m,n})_{m,n\geq 0}$ $\qquad$ $F(x,y)=\sum_{m,n\geq 0} F_{m,n}\cdot x^m y^n$

Two-step asymptotics

$$\begin{array}{ccccccc} F_{0,0} & F_{0,1} & F_{0,2} & F_{0,3} & \dots & \longrightarrow & \sim C_0\cdot n^a\cdot\rho^n\\ F_{1,0} & F_{1,1} & F_{1,2} & F_{1,3} & \dots & \longrightarrow & \sim C_1\cdot n^a\cdot\rho^n\\ F_{2,0} & F_{2,1} & F_{2,2} & F_{2,3} & \dots & \longrightarrow & \sim C_2\cdot n^a\cdot\rho^n\\ F_{3,0} & F_{3,1} & F_{3,2} & F_{3,3} & \dots & \longrightarrow & \sim C_3\cdot n^a\cdot\rho^n\\ \vdots & \vdots & \vdots & \vdots & \ddots & & \vdots \end{array}$$

$$C_m\sim K\cdot m^b\cdot\tau^m$$

$$F_{m,n}\underset{n\to\infty}{\sim} C_m\cdot n^a\cdot\rho^n\underset{m\to\infty}{\sim}(K\cdot m^b\cdot\tau^m)\cdot n^a\cdot\rho^n$$

By the diagonal argument, if $N_m\to\infty$ fast enough, then

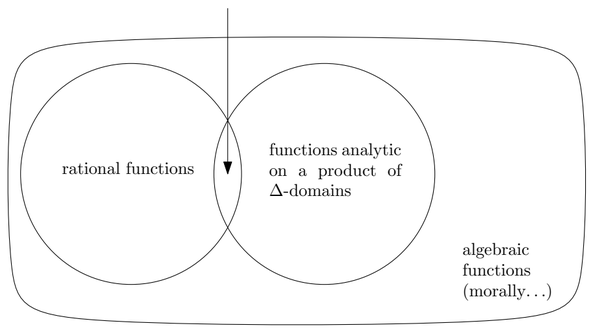
$$F_{m,N_m}\underset{m\to\infty}{\sim}(K\cdot m^b\cdot\tau^m)\cdot {N_m}^a\cdot\rho^{N_m}$$

Theorem (C' 2022+) Assume:

- $f(x,y)=\frac{G(x,y)}{H(x,y)}$ where G,H are coprime polynomials.
- $f(x,y)$ is analytic on some $\Delta_{\epsilon,\delta}\times\Delta_{\epsilon,\delta}$
- $H(1,1)=0$

Then:

$f(x,y)=R_1(x)R_2(y)$ for some univariate rational functions R_1, R_2.



Transfer theorem for MGF no. 1 (Gao–Wormald 2000)

Assume:

$$f(x,y)=A_1(x,y)+A_2(x,y)+c(y)\cdot(1-x)^{-\alpha}(1-y)^{-\beta}+R(x,y)\qquad(\alpha\notin\mathbb{N})$$

- $A_1(x,y)$ is a polynomial in x,
  $A_2(x,y)$ is a polynomial in y.
- $c(y)$ is analytic in some $\Delta_{\epsilon,\delta}$,
  and $c(y)=c(1)+o(1-y)$ in $\Delta_{\epsilon,\delta}$ with $c(1)\neq 0$
- $R(x,y)$ is analytic on $\Delta_{\epsilon,\delta}\times\mathbb{D}$,
  $|R(x,y)|\leq M\cdot|1-x|^{-A'}\cdot(1-|y|)^{-B'}$ uniformly on $\Delta_{\epsilon,\delta}\times\mathbb{D}$,
  and for some $A<\alpha$ and $\gamma>0$,
  $R(x,y)=O\left(|1-x|^{-A}(1-|y|)^{-B}\right)$ when $(x,y)\to(1,1)$ in $\Delta_{\epsilon,\delta}\times\mathbb{D}$ while $\frac{1-x}{(1-y)^\gamma}\to 0$.

Then:

$$[x^m y^n]f(x,y)=\ 0\ +c(1)\cdot\frac{m^{\alpha-1}}{\Gamma(\alpha)}\frac{n^{\beta-1}}{\Gamma(\beta)}+O(m^{\alpha-1}n^{\beta-2})$$

when $m,n\to\infty$ and $\dfrac{\log m}{\log n}\to 0$.

Transfer theorem for MGF no. 3 (C' 2022+)

Assume:

$$f(x,y)=A_1(x,y)+A_2(x,y)+H(1-x,1-y)+o(H(1-x,1-y))$$

- $f(x,y)$ is analytic on some $\Delta_{\epsilon,\delta}\times\Delta_{\epsilon,\delta}$,
  and $|f(x,y)|\leq M(|1-x|^{-A}+|1-y|^{-B})$ uniformly on $\Delta_{\epsilon,\delta}\times\Delta_{\epsilon,\delta}$.
- $A_1(x,y)$, $A_2(x,y)$ and $H(1-x,1-y)$ satisfy the same bound as $f(x,y)$, but only locally near $(x,y)=(1,1)$.
- $A_1(x,y)$ is analytic with respect to x in a neighborhood of 1,
  $A_2(x,y)$ $\qquad\qquad$ y
- $H(a^{\theta_1}u,a^{\theta_2}v)=a^{\theta_0}H(u,v)$ for all $a>0$. $\qquad(\theta_1,\theta_2>0$ and $\theta_0\in\mathbb{R})$

Then:

$$[x^m y^n]f(x,y)=\ 0\ +I(\lambda_1,\lambda_2)\cdot N^{-(\theta_0+\theta_1+\theta_2)}+O\left(N^{-(\theta_0+\theta_1+\theta_2)}\right)$$

when $m=\lambda_1 N^{\theta_1}$, $n=\lambda_2 N^{\theta_2}$ and $N\to\infty$.

where $I(\lambda_1,\lambda_2)=\left(\dfrac{1}{2\pi i}\right)^2\iint H(u,v)e^{\lambda_1 u+\lambda_2 v}\mathrm{d}u\mathrm{d}v$
is the inverse Laplace transform of $H(u,v)$.

Transfer theorem for MGF no. 2 (Pemantle-Wilson 2003)

Assume:

$f(x,y)=\dfrac{G(x,y)}{H(x,y)}$ where G,H are coprime polynomials. $\qquad\lambda_1>0,\quad\lambda_2>0$

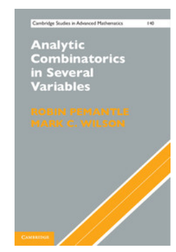
The system $\left\{H(x,y)=0,\ \frac{x\partial_x H(x,y)}{\lambda_1}=\frac{y\partial_y H(x,y)}{\lambda_2}\right\}$ has a solution (u,v) such that

- There is no other solution (x,y) such that $|x|\leq|u|$ and $|y|\leq|v|$.
- $\gamma:=\dfrac{u\partial_x H(u,v)}{\lambda_1}=\dfrac{v\partial_y H(u,v)}{\lambda_2}$ is nonzero.
- $\mathcal{M}:=\dfrac{1}{\lambda_1}+\dfrac{1}{\lambda_2}+\dfrac{1}{\gamma}\left(\dfrac{u^2\partial_x^2 H(u,v)}{\lambda_1^2}+2\dfrac{uv\partial_x\partial_y H(u,v)}{\lambda_1\lambda_2}+\dfrac{v^2\partial_y^2 H(u,v)}{\lambda_2^2}\right)$ is nonzero.

Then:

$$[x^m y^n]f(x,y)=u^{-m}v^{-n}\cdot\left(\frac{-G(u,v)}{\lambda_1\lambda_2\gamma\cdot\sqrt{2\pi\mathcal{M}}}\frac{1}{\sqrt{N}}+O\Big(\frac{1}{N^{3/2}}\Big)\right)$$

when $m=\lambda_1 N$, $n=\lambda_2 N$ and $N\to\infty$.





Example: number of dissections of m-gon with root vertex degree n

$$f(x,y)=\frac{1}{1-y\tau(x)}\quad\text{with}\quad\tau(x)=1-\sqrt{1-x}$$

We have

- $f(x,y)=\dfrac{1}{1-y}-y\dfrac{\sqrt{1-x}}{(1-y)^2}+R(x,y)\qquad(\alpha=-\frac{1}{2},\ \beta=2,\ c(y)=-y)$
- $R(x,y)=\dfrac{(1-x)y^2}{(1-y)^2(1-y\tau(x))}$ is analytic on some $\Delta_{\epsilon,\delta}\times\mathbb{D}$.
- $|R(x,y)|\leq M\cdot|1-x|\cdot(1-|y|)^{-3}$ uniformly on $\Delta_{\epsilon,\delta}\times\mathbb{D}$.

Therefore

$$[x^m y^n]f(x)=-\frac{m^{-3/2}k}{\Gamma(-1/2)\Gamma(2)}+O(m^{-3/2})$$

when $m,n\to\infty$ and $\frac{\log m}{\log n}\to 0$.

Example: parking process on a Galton-Watson tree

$$f(x,y)=1-\frac{\Phi(x)}{y}+H(1-x,1-y)\cdot(1+o(1))\quad\text{with}\quad H(u,v)=(v-\sqrt{u})\sqrt{v+2\sqrt{u}}$$

We have

- $f(x,y)$ is analytic and bounded by $M(|1-x|^{-A}+|1-y|^{-B})$ on some $\Delta_{\epsilon,\delta}\times\Delta_{\epsilon,\delta}$. (non trivial)
- $1-\frac{\Phi(x)}{y}$ is analytic with respect to y in a neighborhood of 1.
- $H(a^2u,av)=a^{3/2}H(u,v).\qquad(\theta_1=2,\ \theta_2=1,\ \theta_0=\frac{3}{2}.)$

Therefore

$$[x^m y^n]f(x,y)=I(\lambda_1,\lambda_2)N^{-9/2}+o(N^{-9/2})$$

when $m=\lambda_1 N^2$, $n=\lambda_2 N$ and $N\to\infty$ with $I(\lambda_1,\lambda_2)=\dfrac{3}{4\pi}(\lambda_1\lambda_2)^{-3/2}e^{-\lambda_2^2/\lambda_1}$

Example: Ising model on random triangulation of disk

$$f(x,y)=\text{Polynomials}+\begin{cases}(1-x)^{3/2}(1-y)^{3/2} & T<T_{\text{critical}}\\ \dfrac{(1-x)^{1/2}(1-y)^{1/2}}{(1-x)^{1/2}+(1-y)^{1/2}} & T>T_{\text{critical}}\\ \dfrac{(1-x)(1-y)}{(1-x)^{1/3}+(1-y)^{1/3}} & T=T_{\text{critical}}\end{cases}+\text{error terms}$$

We have

$$\theta_0=\begin{cases}3\\ \frac{1}{2}\\ \frac{5}{3}\end{cases},\quad I(\lambda_1,\lambda_2)=\begin{cases}\dfrac{225}{64\pi}(\lambda_1\lambda_2)^{-5/2} & T<T_{\text{critical}}\\ \dfrac{3}{4\sqrt{\pi}}(\lambda_1+\lambda_2)^{-5/2} & T>T_{\text{critical}}\\ \dfrac{4}{3\Gamma(-1/3)^2}\displaystyle\int_0^\infty(\lambda_1+r)^{-7/3}(\lambda_2+r)^{-7/3}\mathrm{d}r & T=T_{\text{critical}}\end{cases}$$

Example: binomial coefficients

$$f(x,y)=\frac{1}{1-x-y}=\sum_{m,n\geq 0}\binom{m+n}{n}x^m y^n$$

We have

- The system $\left\{x+y-1=0,\ \frac{x}{\lambda_1}=\frac{y}{\lambda_2}\right\}$ has a unique solution $(u,v)=\left(\dfrac{\lambda_1}{\lambda_1+\lambda_2},\dfrac{\lambda_2}{\lambda_1+\lambda_2}\right)$
- $\gamma=\dfrac{1}{\lambda_1+\lambda_2}>0$ and $\mathcal{M}=\dfrac{\lambda_1+\lambda_2}{\lambda_1\lambda_2}>0$.

Therefore

$$[x^m y^n]f(x,y)=\binom{m+n}{n}=\left(\frac{\lambda_1+\lambda_2}{\lambda_1}\right)^m\left(\frac{\lambda_1+\lambda_2}{\lambda_2}\right)^n\cdot\sqrt{\frac{\lambda_1+\lambda_2}{2\pi\lambda_1\lambda_2}}\frac{1}{\sqrt{N}}\left(1+o\Big(\frac{1}{N}\Big)\right)$$

when $m=\lambda_1 N$, $n=\lambda_2 N$ and $N\to\infty$.