Topics in Probability — Exercise Sheet 1
Solution

Exercise 1. Let X be a d-dimensional Gaussian with mean μ and covariance matrix Σ. Prove that

$$\mathbb{E}\left[e^{i(v;X)}\right] = e^{i(v,\mu)-\frac{(\Sigma v;v)}{2}}$$

where $(\cdot;\cdot)$ denotes the scalar product.

Solution of Exercise 1. By the definition of a d-dimensional Gaussian, (v, X) is a real Gaussian random variable. Moreover, it has mean $\mathbb{E}[(v;X)] = (v;\mu)$ and variance $\mathbb{E}\left[(v;X-\mu)^2\right] = (\Sigma v, v)$. It is well known that the characteristic function of a such Gaussian random variable is

$$\mathbb{E}\left[e^{i(v;X)}\right] = e^{i(v,\mu)-\frac{(\Sigma v;v)}{2}}.$$

Calculation: one can write $(v, X) = (v;\mu) + \sigma_v N$ where $\sigma_v = \sqrt{(\Sigma v, v)}$ and N follows the standard normal distribution. Therefore

$$\begin{aligned}\mathbb{E}\left[e^{i(v;X)}\right] &= \int_{-\infty}^{\infty} e^{i((v;\mu)+\sigma_v x)} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,\mathrm{d}x \\ &= e^{i(v;\mu)-\frac{1}{2}\sigma_v^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-i\sigma_v)^2}\,\mathrm{d}x \\ &= e^{i(v;\mu)-\frac{(\Sigma v;v)}{2}} \int_{-i\sigma_v-\infty}^{-i\sigma_v+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,\mathrm{d}x\ .\end{aligned}$$

Since $|e^{-\frac{x^2}{2}}|$ converges uniformly to zero for $\Im(x) \in [-\sigma_v, 0]$ as $\Re(x) \to \pm\infty$, by Morera's theorem, we can replace the contour of integration $-i\sigma_v + \mathbb{R}$ in the last line by $\mathbb{R}$ without changing the integral. Thus the integral is equal to 1.

Another way to compute $f(\sigma) = \int_{-\infty}^{\infty} e^{i\sigma x} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,\mathrm{d}x$**.** Since $\left|\frac{\mathrm{d}}{\mathrm{d}\sigma} e^{i\sigma x} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\right|$ is bounded uniformly for all $\sigma \in \mathbb{R}$ by the integrable function $\frac{|x|}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, we can differentiate under $\int$ to obtain

$$\begin{aligned}f'(\sigma) &= \int_{-\infty}^{\infty} ixe^{i\sigma x} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,\mathrm{d}x \\ &= \left[\frac{1}{\sqrt{2\pi}} ie^{i\sigma x} \cdot (-e^{-\frac{x^2}{2}})\right]_{-\infty}^{\infty} - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma e^{i\sigma x} \cdot e^{-\frac{x^2}{2}}\,\mathrm{d}x \ = \ -\sigma f(\sigma)\,.\end{aligned}$$

This differential equation, together with $f(0) = 1$, implies that $f(\sigma) = e^{-\frac{\sigma^2}{2}}$.

Exercise 2. Let $X \sim \mathcal{N}(0, 1)$. Prove that for all $x \geq 0$,

$$\sqrt{2\pi}\mathbb{P}(X > x) \leq \left(\frac{1}{x} - \frac{1}{x^3} + \frac{3}{x^5}\right) e^{-\frac{x^2}{2}}.$$

Bonus question: Refine this estimate further infinitely many times. Does it yield a representation of $\mathbb{P}(X > x)$ as convergent series for a fixed x?

Solution of Exercise 2. Similarly to the bounds seen in the lecture, we prove this bound using $\sqrt{2\pi}\mathbb{P}(X > x) = \int_x^{\infty} e^{-\frac{u^2}{2}}\,\mathrm{d}u$ and repeated integration by parts.

$$\begin{aligned}\sqrt{2\pi}\mathbb{P}(X > x) = \int_x^{\infty} \frac{1}{u} \cdot u\, e^{-\frac{u^2}{2}}\,\mathrm{d}u \ &= \ \frac{1}{x} e^{-\frac{x^2}{2}} - \int_x^{\infty} \frac{1}{u^3} \cdot u\, e^{-\frac{u^2}{2}}\,\mathrm{d}u \\ &= \left(\frac{1}{x} - \frac{1}{x^3}\right) e^{-\frac{x^2}{2}} + \int_x^{\infty} \frac{3}{u^4} e^{-\frac{u^2}{2}}\,\mathrm{d}u \\ &= \left(\frac{1}{x} - \frac{1}{x^3} + \frac{3}{x^5}\right) e^{-\frac{x^2}{2}} - \int_x^{\infty} \frac{15}{u^6} e^{-\frac{u^2}{2}}\,\mathrm{d}u\ .\end{aligned}$$

Since the last term is always positive, the desired bound holds.

It is not hard to continue this procedure and prove by induction that for all $x > 0$ and $n \geq 1$,

$$\sqrt{2\pi}\mathbb{P}\left(X > x\right) = \sum_{k=0}^{n-1} (-1)^k \frac{(2k-1)!!}{x^{2k+1}} e^{-\frac{x^2}{2}} + (-1)^n I_n(x)$$

where $(2k-1)!! = (2k-1)\cdot(2k-3)\cdots 1$ for $k \geq 1$ and $(-1)!! = 1$, and

$$I_n(x) = \int_x^\infty \frac{(2n-1)!!}{u^{2n}} e^{-\frac{u^2}{2}} \mathrm{d}u\,.$$

However the series $\sum_{k=0}^\infty (-1)^k \frac{(2k-1)!!}{x^{2k+1}} e^{-\frac{x^2}{2}}$ is not convergent for any fixed x because the absolute value of its k-th term tends to ∞ as $k \to \infty$. (Recall that factorials like $(2k-1)!!$ increase much faster than x^{2k+1} for any fixed x. For example, the first half of the product gives $(2k-1)!! \geq (k-1)^{k/2}$.)

Exercise 3. (Brownian bridge) Denote $\tilde{B}_t := B_t - tB_1$, where B_t is the standard Brownian motion. The process $\tilde{B}_t$ is called the Brownian bridge (from 0 to 0).

(1) Prove that $\tilde{B}_t$ has the same distribution as $\tilde{B}_{1-t}$, and that $\tilde{B}_t$ is independent of B_1 (that is, for any $t_1, \ldots, t_n \in [0,1)$, the vector $(\tilde{B}_{t_1}, \ldots, \tilde{B}_{t_n})$ is independent of B_1);

(2) For a fixed $s \in (0;1)$, let $Y_s(t)$ be the linear interpolation between the values of the Brownian motion B_t at times 0, s, and 1 (that is, $Y_s(t) = \frac{t}{s} B_s$ for $t \leq s$ and $Y_s(t) = \frac{1-t}{1-s} B_s + \frac{t-s}{1-s} B_1$ for $t \geq s$). Prove that the restrictions of the process $B_t - Y_s(t)$ to the intervals $(0;s)$ and $(s;1)$ are independent of each other and of B_s, and have the same distribution as the Brownian bridge, up to a linear change of time and scaling.

Solution of Exercise 3.

(1). The processes in this exercise are implicitly defined for $t \in [0;1]$. It is not hard to see that $\tilde{B}$ is a centered Gaussian process. So its law is determined by its covariance function $(t,s) \mapsto \mathrm{cov}(\tilde{B}_t, \tilde{B}_s) = \mathbb{E}[\tilde{B}_t \tilde{B}_s]$. Recall that $\mathbb{E}[B_t B_s] = \min(t,s)$, therefore

$$\begin{aligned}\mathbb{E}[\tilde{B}_t \tilde{B}_s] = \mathbb{E}[(B_t - tB_1)(B_s - sB_1)] \;&=\; \mathbb{E}[B_t B_s] - t\mathbb{E}[B_1 B_s] - s\mathbb{E}[B_t B_1] + ts\mathbb{E}[B_1^2] \\ &= \min(t,s) - ts\,.\end{aligned}$$

Let us write $\tilde{\mathfrak{A}}_t := \tilde{B}_{1-t}$. The process $\tilde{\mathfrak{A}}$ is Gaussian, centered, and has the same covariance function as $\tilde{B}$:

$$\begin{aligned}\mathbb{E}[\tilde{\mathfrak{A}}_t \tilde{\mathfrak{A}}_s] = \mathbb{E}[\tilde{B}_{1-t}\tilde{B}_{1-s}] &= \min(1-t, 1-s) - (1-t)(1-s) \\ &= (1 - \max(t,s)) - (1 - t - s + ts) \;=\; (t + s - \max(t,s)) - ts \\ &= \min(t,s) - ts\,.\end{aligned}$$

Therefore $\tilde{B}$ and $\tilde{\mathfrak{A}}$, as processes defined on $[0;1]$, have the same law.[1]

$\tilde{B}$ is independent of B_1 because both are Gaussian, and their correlation function vanishes: $\mathbb{E}[\tilde{B}_t B_1] = \mathbb{E}[B_t B_1] - t\mathbb{E}[B_1^2] = 0$ for all $t \in [0;1]$.

[1] Notice that this statement is stronger than the statement that for all $t \geq 0$, the marginals $\tilde{B}_t$ and $\tilde{\mathfrak{A}}_t$ have the same law. For example, by the reflection principle (c.f. next lecture (22 Jan)), $M_t := \sup_{s\in[0,t]} B_s$ and $|B_t|$ have the same law for all $t \geq 0$. However, the processes M and $|B|$ do not have the same law: M is almost surely increasing but $|B|$ is not.

(2). Let $B^{(1)} = (B_t - Y_s(t))_{t\in[0,s]}$ and $B^{(2)} = (B_t - Y_s(t))_{t\in[s,1]}$. Both $B^{(1)}$ and $B^{(2)}$ are centered Gaussian processes. As in (1), we could compute explicitly the correlation functions to show that they are mutually independent and independent of B_s: for $0 \le t \le s \le t' \le 1$, we have

$$\begin{aligned}
\mathbb{E}[B_t^{(1)} B_s] &= \mathbb{E}\left[\Big(B_t - \frac{t}{s}B_s\Big)B_s\right] \;=\; \min(t,s) - \frac{t}{s}s \;=\; t - t \;=\; 0 \\
\mathbb{E}[B_s B_{t'}^{(2)}] &= \mathbb{E}\left[B_s\Big(B_{t'} - \frac{1-t'}{1-s}B_s - \frac{t'-s}{1-s}B_1\Big)\right] \;=\; \min(s,t') - \frac{1-t'}{1-s}s - \frac{t'-s}{1-s}\min(s,1) \;=\; 0 \\
\mathbb{E}[B_t^{(1)} B_{t'}^{(2)}] &= \mathbb{E}\left[\Big(B_t - \frac{t}{s}B_s\Big)B_{t'}^{(2)}\right] \;=\; \mathbb{E}[B_t B_{t'}^{(2)}] \\
&= \mathbb{E}\left[B_t\Big(B_{t'} - \frac{1-t'}{1-s}B_s - \frac{t'-s}{1-s}B_1\Big)\right] \;=\; \min(t,t') - \frac{1-t'}{1-s}t - \frac{t'-s}{1-s}\min(t,1) \;=\; 0
\end{aligned}$$

Alternative argument using the Markov property. $B_t^{(1)} = B_t - \frac{t}{s}B_s$ is a rescaled Brownian bridge (see below). We have seen in (1) that $B^{(1)}$ is independent of B_s. Both $B^{(1)}$ and B_s are deterministic functions of $(B_t)_{t\in[0,s]}$. On the other hand

$$B_t^{(2)} = B_t - \frac{1-t}{1-s}B_s - \frac{t-s}{1-s}B_1 = (B_t - B_s) - \frac{t-s}{1-s}(B_1 - B_s)$$

is a deterministic function of $(B_t - B_s)_{t\in[s,1]}$, which is independent of $(B_t)_{t\in[0,s]}$ by the Markov property of the Brownian motion. Therefore the couple $(B^{(1)}, B_s)$ is independent of $B^{(2)}$.

Recall the following identities in distribution:

$$(B_{t+s} - B_s)_{t\ge0} \overset{(d)}{=} (B_t)_{t\ge0} \qquad \text{and} \qquad (B_{\alpha t})_{t\ge0} \overset{(d)}{=} \big(\sqrt{\alpha}B_t\big)_{t\ge0}$$

for all $\alpha \ge 0$. Then we can express the laws of $B^{(1)}$ and $B^{(2)}$ in terms of that of a Brownian bridge:

$$\begin{aligned}
B^{(1)} &= \left(B_{s\cdot\frac{t}{s}} - \frac{t}{s}B_{s\cdot 1}\right)_{t\in[0,s]} \overset{(d)}{=} \left(\sqrt{s}B_{\frac{t}{s}} - \frac{t}{s}\sqrt{s}B_1\right)_{t\in[0,s]} = \sqrt{s}\tilde{B}_{\frac{\cdot}{s}} \\
\text{and} \qquad B^{(2)} &= \left((B_t - B_s) - \frac{t-s}{1-s}(B_1 - B_s)\right)_{t\in[s,1]} \overset{(d)}{=} \left(B_{t-s} - \frac{t-s}{1-s}B_{1-s}\right)_{t\in[s,1]} \\
&\overset{(d)}{=} \left(\sqrt{1-s}B_{\frac{t-s}{1-s}} - \frac{t-s}{1-s}\sqrt{1-s}B_{1-s}\right)_{t\in[s,1]} = \sqrt{1-s}\tilde{B}_{\frac{\cdot-s}{1-s}}
\end{aligned}$$

Exercise 4. Let $X^{(1)}, X^{(2)}, \dots$ be a sequence of random variables with values in $\mathbb{R}^d$ such that $X^{(i)} \xrightarrow[i\to\infty]{(d)} X$. (Recall that it means that $\mathbb{E}h(X^{(i)}) \to \mathbb{E}h(X)$ for any bounded continuous function $h : \mathbb{R}^d \to \mathbb{R}$.)

(1) Prove that the sequence $X^{(i)}$ is tight, that is, for any $\epsilon > 0$, there exists $R > 0$ such that for all i, $\mathbb{P}\left(|X^{(i)}| \le R\right) \ge 1 - \epsilon$. Here $|\cdot|$ denotes the length of a vector.

(2) Prove that, if a sequence $X^{(i)}$ of Gaussians is tight, then the sequence $|\mathbb{E}X^{(i)}|$ is bounded, and also the elements of the covariance matrices of $X^{(i)}$ are bounded.

(3) Prove that, if $X^{(i)}$ are Gaussians, then $\mathbb{E}X$ exists, and $\mathbb{E}X^{(i)} \to \mathbb{E}X$.

(4) Does the same hold if $\mathbb{E}X^{(i)}$ exists for any i, but without the assumption that $X^{(i)}$ are Gaussian?

Solution of Exercise 4.

(1). For each individual random variable $X^{(i)}$, since $\cup_{R=1}^{\infty}\{|X^{(i)}| \le R\} = \Omega$, there exists $R_i > 0$ such that $\mathbb{P}\left(|X^{(i)}| \le R_i\right) \ge 1-\epsilon$. Similarly, there exists $R_\infty > 0$ such that $\mathbb{P}\left(|X| \le R_\infty\right) \ge 1 - \frac{\epsilon}{2}$. Consider a continuous function $\varphi : [0,\infty) \to [0,1]$ such that $\varphi(x) = 1$ for all $x \le R_\infty$ and $\varphi(x) = 0$ for all $x \ge R_\infty + 1$. Then we have

$$\mathbb{P}\left(|X| \le R_\infty\right) \;\le\; \mathbb{E}\varphi(|X|) \;\le\; \mathbb{P}\left(|X| \le R_\infty + 1\right)$$

Since $h(X) := \varphi(|X|)$ is continuous on $\mathbb{R}^d$, we have $\mathbb{E}\varphi(|X^{(i)}|) \to \mathbb{E}\varphi(|X|)$ as $i \to \infty$. Therefore

$$\liminf_{i\to\infty} \mathbb{P}\left(|X^{(i)}| \le R_\infty + 1\right) \;\ge\; \lim_{i\to\infty} \mathbb{E}\varphi(|X^{(i)}|) \;\ge\; \mathbb{P}\left(|X| \le R_\infty\right) \;\ge\; 1 - \frac{\epsilon}{2}$$

It follows that there exists $i_0 > 0$ such that $\mathbb{P}\left(|X^{(i)}| \le R_\infty + 1\right) \ge 1-\epsilon$ for all $i > i_0$. Therefore with $R = \max(R_0, \dots, R_{i_0}, R_\infty + 1)$, we have $\mathbb{P}\left(|X^{(i)}| \le R\right) \ge 1-\epsilon$ for all $i \ge 0$.

(2). By the symmetry of a Gaussian distribution, we have $\mathbb{P}\left(|X^{(i)}| < |\mathbb{E}X^{(i)}|\right) \le \frac{1}{2}$ for all i. (Notice that we need a strict inequality between $|X|$ and $|\mathbb{E}X|$ to cover the distributions without a density.) If the family $(X^{(i)})_{i\ge 0}$ is tight, then there exists $R > 0$ such that $\mathbb{P}\left(|X^{(i)}| < R\right) \ge \frac{3}{4}$ for all i. It follows that the means $|\mathbb{E}X^{(i)}|$ are bounded by R.

By triangle inequality, the family $(X^{(i)} - \mathbb{E}X^{(i)})_{i\ge 0}$ is also tight. Moreover, it has the same covariance matrices as $(X^{(i)})_{i\ge 0}$. Thus without loss of generality, we can assume that $\mathbb{E}X^{(i)} = 0$ for all i. Let $\Sigma^{(i)}$ be the covariance matrix of $X^{(i)}$. By the Cauchy-Schwarz inequality, $|\Sigma^{(i)}_{p,q}|^2 \le \Sigma^{(i)}_{p,p}\Sigma^{(i)}_{q,q}$, thus it suffices to prove the boundedness of the diagonal terms $(\Sigma^{(i)}_{p,p})_{i\ge 0}$, which are nothing but the variances of the 1-dimensional Gaussians $(X^{(i)}_p)_{i\ge 0}$.

By the scaling property of the Gaussian distribution, we have

$$\mathbb{P}\left(|X^{(i)}_p| \le \sqrt{\Sigma^{(i)}_{p,p}}\right) = \int_{-1}^{1} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \mathrm{d}x \;=:\; c \in (0,1)$$

On the other hand, the tightness of $(X^{(i)})_{i\ge 0}$ implies that there exists $R > 0$ such that

$$\mathbb{P}\left(|X^{(i)}_p| \le R'\right) \;\ge\; \mathbb{P}\left(|X^{(i)}| \le R'\right) > c$$

for all i. It follows that $\sqrt{\Sigma^{(i)}_{p,p}} \le R'$ for all $i \ge 0$.

(3). If $X^{(i)}$ are Gaussians, according to the previous questions, the sequence $(\mathbb{E}X^{(i)}, \Sigma^{(i)})_{i\ge 0}$ is bounded in $\mathbb{R}^d \times \mathbb{R}^{d\times d}$. Let (μ, Σ) be an accumulation point of the sequence. According to the convergence in distribution, we have

$$\mathbb{E}[e^{i(v,X^{(i)})}] = e^{i(v,X^{(i)}) - \frac{1}{2}(\Sigma^{(i)}v,v)} \xrightarrow[i\to\infty]{} \mathbb{E}[e^{i(v,X)}]$$

and therefore $e^{i(v,\mu)-\frac{1}{2}(\Sigma v,v)} = \mathbb{E}[e^{i(v,X)}]$ for all $v \in \mathbb{R}^d$. This implies that the accumulation point (μ,Σ) is unique, and that X is a Gaussian random variable of mean μ and covariance matrix Σ.

(4). No. Without the Gaussian assumption, even if $\mathbb{E}X^{(i)}$ exists for all i and $X^{(i)}$ converges in distribution, the sequence $(\mathbb{E}X^{(i)})_{i\ge 0}$ may diverge. Even if the sequence converge, its limit is not necessarily $\mathbb{E}X$. Here are two counter-examples:

(a) Define $X^{(i)}$ by $\mathbb{P}(X^{(i)} = i) = i^{-\alpha}$ and $\mathbb{P}(X^{(i)} = 0) = 1 - i^{-\alpha}$ for all $i \geq 1$ and some $\alpha > 0$. We have $X^{(i)} \to 0$ in distribution and $\mathbb{E}X^{(i)} = i^{1-\alpha}$ for all $i \geq 1$.

If $\alpha < 1$, then $\mathbb{E}X^{(i)} \to \infty$.
If $\alpha = 1$, then $\mathbb{E}X^{(i)} = 1 \neq \mathbb{E}[\lim X^{(i)}]$.
If $\alpha > 1$, then $\mathbb{E}X^{(i)} \to 0 = \mathbb{E}[\lim X^{(i)}]$.

(b) Let X be any real random variable such that $\mathbb{E}|X| = \infty$ and $X \overset{(d)}{=} -X$. (For example, X could follow the Cauchy distribution: $\mathbb{P}(X \in \mathrm{d}x) = \frac{\mathrm{d}x}{1+x^2}$.) Let $X^{(i)} = X\mathbb{1}_{|X|\leq i}$. It is not hard to see that $X^{(i)} \to X$ in distribution. By symmetry, we have $\mathbb{E}X^{(i)} = 0$ for all $i \geq 0$. But $\mathbb{E}X$ does not exist.

Exercise 5. Recall the notation: $D_n := \{j2^{-n} : j = 1, \ldots, 2^n\}$, $D := \cup_{n=0}^{\infty} D_n$, and $\hat{D}_n := D_n \setminus D_{n-1}$. Recall that for $s \in \hat{D}_n$, we denote

$$\varphi_s(t) = 2^n \int_0^1 \left(\mathbb{1}_{x\in(s-2^{-n},s)} - \mathbb{1}_{x\in(s,s+2^{-n})}\right) \mathrm{d}x = \begin{cases} 1 & \text{if } t = s \\ 0 & \text{if } t \notin (s-2^{-n}, s+2^{-n}) \\ \text{linear} & \text{if } t \in (s-2^{-n}, s) \text{ or } (s, s+2^{-n}) \end{cases}$$

Prove that $\{\varphi_s : s \in D\}$ form a basis of the Dirichlet space $D([0,1])$.

Solution of Exercise 5. Let us write $s_\pm = s \pm 2^{-n}$ for all $s \in D \setminus \{1\}$, where n is fixed by $s \in \hat{D}_n$.

By definition, the Dirichlet space is isomorphic to $L^2([0,1])$ via the mapping $\iota : F \mapsto F'$. So it suffices to prove that $(f_s := 2^{-n}\varphi'_s, s \in D)$ form a basis of $L^2([0,1])$.

Orthogonality. First it is easy to see that $f_s = 1$ is orthogonal to $f_s = \mathbb{1}_{(s_-,s)} - \mathbb{1}_{(s,s_+)}$ for all $s \in D \setminus \{1\}$. Consider $s, r \in D \setminus \{1\}$ with $s \neq r$. Let $n, m \in 0$ be such that $s \in \hat{D}_n$ and $r \in \hat{D}_m$.

- If $n = m$, then $s \neq r$ implies $(s_-, s_+) \cap (r_-, r_+) = \varnothing$. Since f_s is zero outside (s_-, s_+), it follows that $f_s f_r = 0$. In particular, $\int_0^1 f_s f_r = 0$.
- If $n < m$, then by the properties of the dyadic partition, we have either $(s_-, s_+) \cap (r_-, r_+) = \varnothing$, or $(s_-, s_+) \subseteq (r_-, r)$, or $(s_-, s_+) \subseteq (r, r_+)$. In the first case we have $f_s f_r = 0$ as before. In the other two cases, f_r is constant on (s_-, s_+). Therefore $\int_0^1 f_s f_r = f_r(s) \int_0^1 f_s = 0$.

We conclude that $(f_s, s \in D)$ is an orthogonal family in $L^2([0,1])$.

Completeness. It is not hard to see that $\mathrm{span}(f_s, s \in D_n)$ is the space of functions which are constant on the dyadic intervals $((j-1)2^{-n}, j2^{-n})$ for $j = 1, \ldots 2^n$. Any continuous function h on $[0,1]$ can be approximated uniformly, and thus in the sense of the L^2-norm, by functions $h_n \in \mathrm{span}(f_s, s \in D_n)$ when $n \to \infty$. It is well known that the space of continuous functions is dense in $L^2([0,1])$. Therefore $\mathrm{span}(f_s, s \in D)$ is also dense in $L^2([0,1])$.

Exercise 6. Let $f : [0;1] \to \mathbb{R}$ be any continuous funciton such that $f(0) = 0$. Prove that for any $\epsilon > 0$, one has

$$\mathbb{P}\left(\sup_{t\in[0,1]} |f(t) - B_t| < \epsilon\right) > 0$$

Solution of Exercise 6. Let us write $\|f\|_\infty = \sup_{t\in[0,1]} |f(t)|$, and define D, $\hat{D}_n$ and $(\varphi_s, s \in D)$ as in the previous exercise.

Since f is continuous on the compact interval $[0,1]$, we can find $N_0 \in \mathbb{N}$ such that $|t-s| \leq 2^{-N_0}$ implies $|f(t) - f(s)| \leq \frac{\epsilon}{3}$ for all $t, s \in [0,1]$ (uniform continuity). Let f_0 be the function that is equal to f on $\{0\} \cup D_{N_0}$, and piecewise linear between these points, then we have $\|f - f_0\|_\infty \leq \frac{\epsilon}{3}$.

As in the previous exercise, one can check that f_0 is orthogonal to φ_s for all $s \in D \setminus D_{N_0}$. Thus f_0 has a finite expansion in the orthogonal basis $(\varphi_s, s \in D)$:

$$f_0 = \sum_{s \in D_{N_0}} a_s \varphi_s \,.$$

Recall Lévy's construction of the Brownian motion:

$$B_t = \zeta_1 \varphi_1(t) + \sum_{n=1}^{\infty} F_n(t) \qquad \text{where} \qquad F_n(t) = 2^{-\frac{n+1}{2}} \sum_{s \in \hat{D}_n} \zeta_s \varphi_s(t)$$

and $(\zeta_s, s \in D)$ are i.i.d. standard Gaussians. We have seen in the lectures that the sum $\sum_{n=1}^{\infty} \|F_n\|_\infty$ is almost surely finite. Thus there exists $N \in [N_0, \infty)$ such that $\sum_{n=N+1}^{\infty} \|F_n\|_\infty \le \frac{\epsilon}{3}$.

Notice that this N is random. More precisely, we define the random variable N as the minimal integer satisfying the above condition. It is not hard to see that the event $\{N = k\}$ is measurable with respect to the σ-algebra generated by $(\zeta_s, s \in D \setminus D_k)$. Thus it is independent of the variables $(\zeta_s, s \in D_k)$. It follows that:

$$\begin{aligned} \mathbb{P}\left(\|B - f\|_\infty < \epsilon\right) &\ge \mathbb{P}\left(\|\zeta_1\varphi_1 + \sum_{n=1}^{N} F_n - f_0\|_\infty < \frac{\epsilon}{3}\right) \\ &= \sum_{k=N_0}^{\infty} \mathbb{P}\left(N = k\right) \cdot \mathbb{P}\left(\|\zeta_1\varphi_1 + \sum_{n=1}^{k} F_n - f_0\|_\infty < \frac{\epsilon}{3}\right) \end{aligned}$$

By independence of the ζ_s, the second factor in the above sum is strictly positive for each fixed k:

$$\mathbb{P}\left(\|\zeta_1\varphi_1 + \sum_{n=1}^{k} F_n - f_0\|_\infty < \frac{\epsilon}{3}\right) \;\ge\; \mathbb{P}\left(|\zeta_1 - a_1| < \frac{\epsilon}{6}\right) \prod_{n=1}^{k} \prod_{s \in \hat{D}_n} \mathbb{P}\left(\left|2^{-\frac{n+1}{2}}\zeta_s - a_s\right| < \frac{1}{k}\frac{\epsilon}{6}\right) > 0$$

(Let $a_s = 0$ if $s \in D_k \setminus D_{N_0}$.) On the other hand, $\mathbb{P}(N = k) > 0$ for some $k \ge N_0$, because $\sum_{k=N_0}^{\infty} \mathbb{P}(N = k) = 1$. This implies that $\mathbb{P}\left(\|B - f\|_\infty < \epsilon\right) > 0$.[2]

Alternative solution. We have seen above that f can be approximated uniformly by a piecewise linear function f_0. Since the random vector $(B_{j2^{-N}},\, j = 1, \ldots, 2^N)$ has a density, with positive probability we have $|B_{j2^{-N}} - f_0(j2^{-N})| < \frac{\epsilon}{3}$ for all $j = 1, \ldots, 2^N$.

Let Z be the (random) piecewise linear function that interpolates between $(B_{j2^{-N}},\, j = 0, \ldots, 2^N)$. By an extension of Exercise 3, the difference $B - Z$ is the concatenation of 2^N rescaled Brownian bridges which are mutually independent, and independent of the vector $(B_{j2^{-N}},\, j = 1, \ldots, 2^N)$. Therefore it suffices to prove that for the Brownian bridge $\tilde{B}$, we have

$$\mathbb{P}\left(\|\tilde{B}\|_\infty < a\right) > 0$$

for all $a > 0$.

Now consider the special case $s = 1/2$ of Exercise 3. If $\tilde{Y}$ is the piecewise linear function interpolating between $\tilde{B}_0 = 0$, $\tilde{B}_{1/2}$ and $\tilde{B}_1 = 0$, then $\tilde{B} - \tilde{Y}$ is the concatenation of two Brownian bridges $B^{(1)}$ and $B^{(2)}$. More precisely, up to time translation, we have $B^{(1)} \overset{(d)}{=} B^{(2)} \overset{(d)}{=} \frac{1}{\sqrt{2}}(\tilde{B}_{2t})_{t \in [0, \frac{1}{2}]}$, so

$$\mathbb{P}\Big(\sup_{t \in [0, \frac{1}{2}]} |B_t^{(1)}| < a'\Big) \;=\; \mathbb{P}\Big(\sup_{t \in [\frac{1}{2}, 1]} |B_t^{(2)}| < a'\Big) \;=\; \mathbb{P}\left(\|\tilde{B}\|_\infty < \sqrt{2}a'\right)$$

[2] Thanks Changsun CHOI for the idea of this solution.

for any $a' > 0$. Then the independence between $B^{(1)}$, $B^{(2)}$ and $\tilde{B}_{1/2}$ implies that for any $\lambda \in (0,1)$,

$$\begin{aligned}\mathbb{P}\left(\|\tilde{B}\|_\infty < a\right) &\geq \mathbb{P}\left(|\tilde{B}_{1/2}| < (1-\lambda)a,\ \sup_{t\in[0,\frac{1}{2}]} |B_t^{(1)}| < \lambda a \text{ and } \sup_{t\in[\frac{1}{2},1]} |B_t^{(2)}| < \lambda a\right) \\ &= \mathbb{P}\left(|\tilde{B}_{1/2}| < (1-\lambda)a\right) \cdot \mathbb{P}\left(\|\tilde{B}\|_\infty < \sqrt{2}\lambda a\right)^2.\end{aligned}$$

Assume that $\mathbb{P}(\|\tilde{B}\|_\infty < a) = 0$ for some $a > 0$, then iterating the above inequality gives $\mathbb{P}(\|\tilde{B}\|_\infty < (\sqrt{2}\lambda)^n a) = 0$ for any $n \geq 0$. When $\lambda > \frac{1}{\sqrt{2}}$, this implies that $\|\tilde{B}\|_\infty \geq x$ almost surely for arbitrarily large x. This is impossible, so our assumption is false.

Topics in Probability — Exercise Sheet 2
Solution

In all exercises, B_t denotes the standard Brownian motion.

Exercise 1. Prove that almost surely, for every $\epsilon > 0$, B_t has infinitely many zeros in the interval $(0; \epsilon)$.

Solution of Exercise 1. Consider $M_t^+ = \max_{s\in[0,t]} B_t$ and $M_t^- = \min_{s\in[0,t]} B_t$. By the reflexion principle, $\mathbb{P}(M_t^+ > 0) = \mathbb{P}(M_t^- < 0) = \mathbb{P}(|B_t| > 0) = 1$ for any fixed $t > 0$. Since the process M^+ (resp. M^-) is increasing (resp. decreasing), we have almost surely $M_t^- < 0 < M_t^+$ for all $t > 0$.

If $M_t^- < 0 < M_t^+$, then there exist $s^-, s^+ \in (0, t)$ such that $B_{s^-} < 0 < B_{s^+}$, hence by continuity, B has at least one zero in $(0, t)$. It follows that almost surely, B has a zero in $(0, t)$ for all $t > 0$, which implies that $(0, \epsilon)$ contains infinitely many zeros of B.

Alternative solution. Consider $T^+ = \inf\{s > 0 : B_s > 0\}$ and $T^- = \inf\{s > 0 : B_s < 0\}$. By symmetry, T^+ and T^- have the same law. Moreover, $\{T^+ = 0\} \cup \{T^- = 0\} = \Omega$, therefore $\mathbb{P}(T^+ = 0) = \mathbb{P}(T^- = 0) \geq 1/2$.

On the other hand, The event $\{T^+ = 0\}$ is $\mathcal{F}_t^\circ$-measureable for every $t > 0$, therefore $\{T^+ = 0\} \in \mathcal{F}_0^+$. By Blumenthal's 0-1 law and the previous observation, we have $\mathbb{P}(T^+ = 0) = \mathbb{P}(T^- = 0) = 1$.

In other words, almost surely there exists two (random) strictly decreasing sequences $t_n \to 0$ and $s_n \to 0$ such that $B_{t_n} > 0$ and $B_{s_n} < 0$ for all n. It follows that almost surely, B has a sequence of zeros that decrease towards 0, which implies the desired result.

Exercise 2. The Ornstein-Uhlenbeck process is defined by the formula $X_t = e^{-t}B_{e^{2t}}$. Prove that:

(1) The process X_{-t} has the same distribution as X_t;

(2) For every t, X_t is a standard Gaussian;

(3) The process X_t is stationary, that is, for any $t_1 < \cdots < t_n$ and any $s > 0$, the distributions of the random vectors $(X_{t_1}, \ldots, X_{t_n})$ and $(X_{t_1+s}, \ldots, X_{t_n+s})$ are the same;

(4) The process X_t is Markov, that is, for any $t_1 < \cdots < t_n$, the conditional distribution of X_{t_n} given $X_{t_1}, \ldots, X_{t_{n-1}}$ is the same as the conditional distribution of X_{t_n} given $X_{t_{n-1}}$;

(5) Define, for $t \geq 0$, the process $\tilde{B}_t$ by

$$\sqrt{2}\tilde{B}_t = X_t + \int_0^t X_s \mathrm{d}s - X_0\,.$$

Prove that $\tilde{B}_t$ is a standard Brownian motion. Thus, X_t may be viewed as a Brownian motion with a (linear) "drift" towards zero; in the language of stochastic analysis, one has $\mathrm{d}X_t = -X_t\mathrm{d}t + \sqrt{2}\mathrm{d}\tilde{B}_t$.

Solution of Exercise 2. $X = (X_t)_{t\in\mathbb{R}}$ is a centered Gaussian process with almost surely continuous paths. Its law is characterized by its covariance function $K(t, t') := \mathbb{E}[X_t X_{t'}]$, which is

$$K(t,t') \;=\; e^{-t-t'}\mathbb{E}[B_{e^{2t}}B_{e^{2t'}}] \;=\; e^{-t-t'}\min(e^{2t}, e^{2t'}) \;=\; e^{-t-t'+2\min(t,t')} \;=\; e^{-|t-t'|}.$$

(1). We have $K(-t, -t') = K(t, t')$, thus $(X_{-t})_{t\in\mathbb{R}}$ has the same law as X.

(2). For every t, X_t is a centered Gaussian of covariance $K(t, t) = 1$, thus a standard Gaussian.

(3). We have $K(t + s, t' + s) = K(t, t')$, thus $(X_{t+s})_{t\in\mathbb{R}}$ has the same law as X, for all $s \in \mathbb{R}$.

(4). Since the change of time $t \mapsto e^{2t}$ is deterministic and strictly increasing, X inherits the Markov property of the Brownian motion.

More precisely, the Markov property of B says that $B_{\exp(2t_n)} - B_{\exp(2t_{n-1})}$ is independent of $B_{\exp(2t_1)}, \dots, B_{\exp(2t_{n-1})}$, for any $t_1 < \cdots < t_n$. Thus the conditional distribution of $B_{\exp(2t_n)}$ given $B_{\exp(2t_1)}, \dots, B_{\exp(2t_{n-1})}$ is the same as the conditional distribution of $B_{\exp(2t_n)}$ given $B_{\exp(2t_{n-1})}$. After multiplying the deterministic and non-zero factors e^{-t_i}, we get the Markov property of X.[1]

(5). For a general Gaussian process X with continuous paths on $[0,t]$, the integral $I = \int_0^t X_s \mathrm{d}s$ is the almost sure limit of its Riemann sums $I_n = \frac{t}{n}\sum_{k=1}^n X_{kt/n}$. Each I_n is Gaussian, thus according to Exercise 4 of the previous sheet,[2] I is also Gaussian and we have $\mathbb{E}I_n \to \mathbb{E}I$ and $\mathrm{var}(I_n) \to \mathrm{var}(I)$. In other words, the integral of a continous Gaussian process is still Gaussian, and we can compute its mean and variance by exchanging the order of expectation and integral.

When applied to the Ornstein-Uhlenbeck process, we see that $\mathbb{E}\left[\int_0^t X_s \mathrm{d}s\right] = \int_0^t \mathbb{E}\left[X_s\right] \mathrm{d}s = 0$. Thus $\tilde{B}$ is a centered Gaussian process with almost surely continuous paths. To prove that it is a standard Brownian motion, it suffices to show that $\mathbb{E}[\tilde{B}_t \tilde{B}_{t'}] = \min(t,t')$.

Since $\tilde{B}$ depends linearly on X, it straightforward to expand $\mathbb{E}[\tilde{B}_t \tilde{B}_{t'}]$ in terms of K, the covariance function of X. However, to avoid heavy computation, let's first consider the case $t = t'$:

$$2\mathbb{E}[\tilde{B}_t^2] = \mathbb{E}[(X_t - X_0)^2] + 2\mathbb{E}[(X_t - X_0)I] + \mathbb{E}[I^2]\,.$$

In the second the third term, we can exchange expectation with integrals using the same argument as above (or using the Fubini theorem, after justifying integrability). Therefore

$$\begin{aligned}
\mathbb{E}[(X_t - X_0)^2] &= K(t,t) + K(0,0) - 2K(t,0) \;=\; 2(1 - e^{-t})\;, \\
\mathbb{E}[(X_t - X_0)I] &= \int_0^t (K(t,s) - K(0,s))\,\mathrm{d}s \;=\; \int_0^t e^{-(t-s)}\mathrm{d}s - \int_0^t e^{-s}\mathrm{d}s \;=\; 0\;, \\
\text{and} \qquad \mathbb{E}[I^2] &= \int_0^t \int_0^t K(s,s')\,\mathrm{d}s\,\mathrm{d}s' \\
&= 2\int_0^t \mathrm{d}s \int_0^s \mathrm{d}s' e^{-(s-s')} \;=\; 2\int_0^t \mathrm{d}s(1 - e^{-s}) \;=\; 2(t + e^{-t} - 1)\,.
\end{aligned}$$

It follows that $\mathbb{E}[\tilde{B}_t^2] = (1 - e^{-t}) + 0 + (t + e^{-t} - 1) = t$.

Now let $u \geq 0$. Using stationarity of X (Question (3)), we see that $\tilde{B}$ has stationary increments:

$$\sqrt{2}(B_{t+u} - B_t) \;=\; X_{t+u} - X_t + \int_t^{t+u} B_s\,\mathrm{d}s \;\overset{(d)}{=}\; X_t - X_0 + \int_0^u B_s\,\mathrm{d}s \;=\; \sqrt{2}B_u$$

Therefore $\mathbb{E}[\tilde{B}_u^2] = \mathbb{E}[\tilde{B}_{t+u}^2] + \mathbb{E}[\tilde{B}_t^2] - 2\mathbb{E}[\tilde{B}_{t+u}\tilde{B}_t]$. Plug in the variance calculated above and we get $\mathbb{E}[\tilde{B}_{t+u}\tilde{B}_t] = t = \min(t+u, t)$.

Exercise 3. Recall that the Brownian motion reflected at a (possibly random) time τ is defined by

$$\tilde{B}_t = \begin{cases} B_t, & t \leq \tau \\ 2B_\tau - B_t, & t \geq \tau. \end{cases}$$

Give an example of a random time $\tau \in [0;1]$ such that $\tilde{B}_t$ is not a Brownian motion.

Solution of Exercise 3. As mentioned in the lecture, the reflected process $\tilde{B}$ will be a Brownian motion if τ is a stopping time. An archetype of non-stopping times is the last visit times.

[1] Notice that multiplying $B_{\exp(2t_1)}, \dots, B_{\exp(2t_{n-1})}$ by the deterministic factors e^{-t_i} does not change the σ-algebra generated by these random variables, thus does not change the conditional law of $B_{\exp(2t_n)}$ with respect to it.

[2] Recall that almost sure convergence implies convergence in distribution.

Consider $\tau = \sup\{t \in [0,1] : B_t \geq 0\}$, the last time that B_t visits the positive real axis before $t = 1$. (In particular, $B_1 \geq 0$ implies $\tau = 1$.) With a bit of thought we see that for this τ, we have $\tilde{B}_1 \geq 0$ almost surely, which is clearly not the behavior of a Brownian motion.

Exercise 4. Let $M > 0$, and $\alpha \in (\frac{1}{2}; 1]$. An interval $[a, b]$ is called α, M-good if

$$|B_b - B_a| \leq M|b - a|^\alpha.$$

(1) Give an upper bound for the probability that an interval $[a, b]$ is α, M-good.

(2) Given $k, n \in \mathbb{N}$, estimate the probability that among the intervals $I_{n,j} := [2^{-n}j; 2^{-n}(j + 1)]$, where $j = 0, \ldots, 2^n - 1$, there are k consecutive α, M-good intervals.

(3) Prove that for every $\alpha > \frac{1}{2}$ and every M, one can find k such that the above probability is $\leq Cq^n$, where $q < 1$ and C may depend on α and M, but not on n.

(4) Conclude that almost surely, for every $t \geq 0$ and every $\alpha > \frac{1}{2}$, the Brownian motion B_t is not locally α-Hölder continuous at t, that is,

$$\limsup_{s \searrow t} \frac{|B_s - B_t|}{|s - t|^\alpha} = +\infty.$$

Solution of Exercise 4.

(1). Since the probability density function of a standard Gaussian is bounded by $1/\sqrt{2\pi}$, we have

$$\mathbb{P}\left(|B_b - B_a| \leq M|b - a|^\alpha\right) \;=\; \mathbb{P}\left(|\mathcal{N}(0,1)| \leq M|b - a|^{\alpha - \frac{1}{2}}\right) \;\leq\; \frac{2}{\sqrt{2\pi}} \cdot M|b-a|^{\alpha-\frac{1}{2}} \;\leq\; M|b-a|^{\alpha-\frac{1}{2}}.$$

(2). As the increments of B are independent and stationary, for each $j \in \{0, \ldots, 2^n - k\}$, we have

$$\mathbb{P}\left(I_{n,j}, \ldots, I_{n,j+k-1} \text{ are all } \alpha, M\text{-good intervals}\right) \;=\; \mathbb{P}\left(I_{n,j} \text{ is } \alpha, M\text{-good}\right)^k \;\leq\; M^k 2^{-n(\alpha-\frac{1}{2})k}.$$

Thus by union bound, the probability that there exists k consecutive α, M-good intervals among $I_{n,0}, \ldots, I_{n,2^n-1}$ is bounded by $p_n := 2^n \cdot M^k 2^{-n(\alpha-\frac{1}{2})k}$.

(3). Choose $k \geq \frac{2}{\alpha-\frac{1}{2}}$, then we have $p_n = M^k\, 2^{n-k(\alpha-\frac{1}{2})n} \leq M^k(\frac{1}{2})^n$.

(4). With the above choice of k, the bounds $(p_n, n \geq 1)$ are summable. By the Borel–Cantelli lemma, almost surely there are only finitely many n such that there exists k consecutive α, M-good intervals among $I_{n,0}, \ldots, I_{n,2^n-1}$.

On the other hand, assume that for some $t \in [0, 1)$, we have $\limsup_{s \searrow t} \frac{|B_s - B_t|}{|s-t|^\alpha} < \infty$. Then there exists $L, \epsilon > 0$ such that $|B_s - B_t| \leq L|s - t|^\alpha$ for all $s \in (t, t + \epsilon)$. For each n, let $j_0 = \lceil t2^n \rceil$ so that I_{n,j_0} is the first interval of the form $I_{n,j}$ to the right of t. Write $s_i = (j_0 + i)2^{-n}$ $(i = 0, \ldots, k)$ so that we have $I_{n,j_0+i} = [s_i, s_{i+1}]$. Assume that n is large enough so that $s_k \in (t, t + \epsilon)$. Then we have for all $i = 0, \ldots, k - 1$,

$$\begin{aligned} |B_{s_{i+1}} - B_{s_i}| \;&\leq\; |B_{s_{i+1}} - B_t| + |B_{s_i} - B_t| \;\leq\; L|s_{i+1} - t|^\alpha + L|s_i - t|^\alpha \\ &\leq\; 2L \cdot ((k + 1)2^{-n})^\alpha \;=\; 2(k + 1)^\alpha L \cdot |s_{i+1} - s_i|^\alpha. \end{aligned}$$

It follows that for all n large enough, the k consecutive intervals I_{n,j_0+i} $(i = 0, \ldots, k)$ are all $\alpha, 2(k + 1)^\alpha L$-good. But according to the first part of the proof, this can only happen with probability zero.

We conclude that for each $\alpha > 1/2$, almost surely, $\limsup_{s \searrow t} \frac{|B_s - B_t|}{|s-t|^\alpha} = \infty$ for all $t \in [0, 1)$. Since the Brownian motion has stationary increments, this is also true on any interval $[m, m + 1)$. Taking the countable union of these intervals and considering a sequence $\alpha_n \searrow 1/2$, we deduce that almost surely, for all $t \geq 0$ and all $\alpha > \frac{1}{2}$, B is not locally α-Hölder continuous at t.

Topics in Probability — Exercise Sheet 3
Solution

Exercise 1. Recall that, for a C^2-smooth function $f : \Omega \to \mathbb{R}$, where $\Omega \subseteq \mathbb{R}^d$, we defined the Laplacian

$$\Delta f = \sum_{i=1}^{d} \frac{\partial^2}{\partial {x_i}^2} f.$$

Prove that the Laplacian commutes with the rotations: if A is a rotation matrix (i.e. $AA^\top = \mathrm{Id}$), and $f_A(x) := f(Ax)$, then $(\Delta f_A)(x) = (\Delta f)(Ax)$.
Bonus question: prove the converse: if $\mathcal{L}$ is a second-order linear differential operator, i.e. $\mathcal{L}f = a_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} f$ where $a_{i,j} \in \mathbb{R}$, with the same property, then $\mathcal{L} = c\Delta$ for some $c \in \mathbb{R}$.

Solution of Exercise 1. For simplicity, let us denote by ∂_i the partial derivative with respect to the i-th variable. We need to compute the Laplacian of f_A and compare it to Δf.

Direct calculation. We apply the chain rule of derivation to compute

$$\begin{aligned} \partial_i f_A(x) &= \sum_{k=1}^{d} \partial_k f(Ax) \frac{\partial (Ax)_k}{\partial x_i} = \sum_{k=1}^{d} \partial_k f(Ax) A_{k,i}\,, \qquad (1) \\ \partial_i^2 f_A(x) &= \sum_{k=1}^{d} A_{k,i} \partial_i (\partial_k f(Ax)) = \sum_{k=1}^{d} A_{k,i} \sum_{j=1}^{d} A_{j,i} \partial_j \partial_k f(Ax)\,. \end{aligned}$$

We sum the the second equation over i, and change the order of summation:

$$\Delta f_A(x) = \sum_{i=1}^{d}\sum_{j=1}^{d}\sum_{k=1}^{d} A_{k,i} A_{j,i} \partial_j \partial_k f(Ax) = \sum_{j=1}^{d}\sum_{k=1}^{d} \partial_j \partial_k f(Ax) \sum_{i=1}^{d} A_{k,i} A_{j,i}\,.$$

By definition of the matrix product, the last sum is equal to $(AA^\top)_{k,j} = \mathrm{Id}_{k,j}$, which is 1 if $k = j$ and 0 if $k \neq j$. Therefore the right hand side simplifies to

$$\Delta f_A(x) = \sum_{k=1}^{d} \partial_k^2 f(Ax) = (\Delta f)(Ax)\,.$$

Calculation using more linear algebra. Recall that the gradient ∇f (resp. the Hessian matrix $\mathcal{H}f$) of f consists of all the first order (resp. second order) partial derivatives of f arranged into a vector (resp. matrix), namely

$$\nabla f := (\partial_1 f, \ldots, \partial_d f) \qquad \text{and} \qquad \mathcal{H}f := \nabla((\nabla f)^\top) = (\partial_j \partial_i f)_{i,j=1,\ldots,d}$$

The Laplacian is simply the trace of the Hessian matrix: $\Delta f = \mathrm{Tr}(\mathcal{H}f)$.

Let us compute $\mathcal{H}f_A$: first we use the chain rule to derive Equation (1). In terms of the gradient, it means that $\nabla f_A(x) = \nabla f(Ax)\, A$, where the right hand side is the product of the row vector $(\nabla f)(Ax)$ and the matrix A. If we denote by h_A the function $x \mapsto h(Ax)$ (as in the statement of the exercise), then the gradient relation can also be written as $\nabla f_A = (\nabla f A)_A$

We apply this relation to the $\mathbb{R}^d$-valued function $g := (\nabla f A)^\top$: since $(\nabla f_A)^\top = (\nabla f A)_A^\top = g_A$, we have

$$\nabla(\nabla f_A)^\top = \nabla g_A = (\nabla g\, A)_A = (\nabla \left(A^\top (\nabla f)^\top\right) A)_A = (A^\top \nabla (\nabla f)^\top A)_A\,.$$

In other words, $\mathcal{H}f_A = (A^\top \mathcal{H}f\, A)_A$, or equivalently $\mathcal{H}f_A(x) = A^\top \mathcal{H}f(Ax)\, A$. Recall the trace has the property that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ for all $A, B \in \mathbb{R}^{d\times d}$. Thus for any rotation matrix A, we have

$$\Delta f_A(x) \;=\; \mathrm{Tr}(\mathcal{H}f_A(x)) \;=\; \mathrm{Tr}(A^\top \mathcal{H}f(Ax)A) \;=\; \mathrm{Tr}(AA^\top \mathcal{H}f(Ax)) \;=\; \Delta f(Ax)\,.$$

Bonus question. By Schwarz's theorem $\partial_i\partial_j f = \partial_j\partial_i f$, thus we can assume without loss of generality that $a_{i,j} = a_{j,i}$. Let L denote the symmetric matrix $(a_{i,j})_{i,j=1,\dots,d}$. One can check that $\mathcal{L}f = \mathrm{Tr}(L\cdot \mathcal{H}f)$.

If $\mathcal{L}$ has the same property as Δ, namely $\mathcal{L}f_A = (\mathcal{L}f)_A$ for all C^2-smooth function f and rotation matrix A, then using the expression of $\mathcal{H}f_A$ derived above, we have $\mathrm{Tr}(L\cdot A^\top \mathcal{H}f\, A)_A = \mathrm{Tr}(L\cdot \mathcal{H}f)_A$, or equivalently, $\mathrm{Tr}(ALA^\top \cdot \mathcal{H}f) = \mathrm{Tr}(L\cdot \mathcal{H}f)$ for all C^2-smooth function f. Take $f(x) = x_ix_j$ so that $\mathcal{H}f$ is the matrix with 1 at positions (i,j) and (j,i), and 0 everywhere else. Then the trace relation reads: $L_{i,j} + L_{j,i} = (ALA^\top)_{i,j} + (ALA^\top)_{j,i}$. Since both matrices are symmetric, we have $ALA^\top = L$, i.e. $AL = LA$ for all rotation matrix A.

Since L is a real and symmetric matrix, it is diagonalizable. In particular, it has an eigenvalue $c \in \mathbb{R}$ with some eigenvector $x \in \mathbb{R}^d$. For any rotation matrix A, since $c\,Ax = A(Lx) = L(Ax)$, the vector Ax is also an eigenvector of eigenvalue c. It follows that the eigenspace of c is the whole space, that is, $L = c\cdot \mathrm{Id}$.

Exercise 2. Let $\varphi \in C^2(\mathbb{R}_{\ge 0})$. Prove that the funciton $f : x \to \varphi(|x|)$ is harmonic in $\mathbb{R}^d \setminus \{0\}$ if and only if

$$\varphi(s) = \begin{cases} c_1 + c_2 \log s, & d = 2\,; \\ c_1 + c_2 \cdot s^{2-d}, & d \ge 3\,. \end{cases}$$

Solution of Exercise 2. Let $\psi(r) = \varphi(\sqrt{r})$, then ψ is C^2-smooth over $(0,\infty)$ and $f(x) = \psi(|x|^2)$ for all $x \in \mathbb{R}^d$. Write $|x|^2 = x_1^2 + \cdots + x_d^2$ and use the chain rule to get

$$\partial_i f(x) = 2x_i\psi'(|x|^2) \qquad \text{and} \qquad \partial_i^2 f(x) = 4x_i^2\psi''(|x|^2) + 2\psi'(|x|^2)\,.$$

It follows that $\Delta f(x) = 4|x|^2\psi''(|x|^2) + 2d\psi'(|x|^2)$. Thus f is harmonic in $\mathbb{R}^d \setminus \{0\}$ if and only if ψ satisfies that for all $r > 0$,

$$4r\psi''(r) + 2d\psi'(r) = 0$$

This is a first order linear differential equation for ψ', whose solution is of the form $\psi'(r) = a\, r^{-d/2}$ with some $a \in \mathbb{R}$. Integrating again, we get

$$\psi(r) = \begin{cases} c_1 + \frac{1}{2}c_2 \log r, & d = 2 \\ c_1 + c_2 \cdot r^{1-d/2} & d \ge 3 \end{cases} \qquad \text{for some } c_1, c_2 \in \mathbb{R}.$$

Exercise 3. Let $u, v \in C^2(\Omega)$, where Ω is a domain in $\mathbb{R}^d$ with a piecewise smooth boundary. Prove the Green's identities:

$$\int_\Omega v\cdot\Delta u + \int_\Omega \nabla v\cdot\nabla u = \int_{\partial\Omega} v\cdot\partial_n u\,,$$
$$\int_\Omega (v\cdot\Delta u - u\cdot\Delta v) = \int_{\partial\Omega} (v\cdot\partial_n u - u\cdot\partial_n v)\,,$$

where ∂_n denotes the outer normal derivative, i.e. $\partial_n f(x) = \lim_{\delta\to 0}\delta^{-1}(f(x+\delta n_x) - f(x))$, where n_x denotes the unit outer normal vector to $\partial\Omega$ at x.

Hint: if you have taken the vector calculus course, you can reduce it to the Gauss theorem $\int_\Omega \operatorname{div}(\vec{w}) = \int_{\partial\Omega} \vec{w}\cdot\vec{n}$. If not, try the case $d=2$ and Ω a rectangle with sides parallel to the axes, then Ω a trapezoid, then argue how to extend the result to the general case.

Solution of Exercise 3.

Using the Gauss theorem. The definition of the normal derivative can be read as: $\partial_n f(x) = \frac{\mathrm{d}}{\mathrm{d}\delta} f(x+\delta n_x)|_{\delta=0}$. Using the chain rule, we check that $\partial_n f = n\cdot\nabla f$, where $\cdot$ denotes the scalar product. In particular, $v\,\partial_n u = (v\nabla u)\cdot n$. Recall that the divergence of a $\mathbb{R}^d$-valued function $\vec{w} = (w_1,\dots,w_d)$ is defined by $\operatorname{div}(\vec{w}) = \sum_{i=1}^d \partial_i w_i$. So for scalar functions u, v, we have

$$\operatorname{div}(v\nabla u) \;=\; \sum_{i=1}^d \partial_i(v\partial_i u) \;=\; \sum_{i=1}^d (\partial_i v)(\partial_i u) + v\partial_i^2 u \;=\; \nabla v\cdot\nabla u + v\Delta u$$

Applying the Gauss theorem to $\vec{w} = v\nabla u$ gives the first identity.

The second identity can be obtained by taking the difference between the first identity with the version of itself where the roles of u and v are exchanged.

Idea of proof of the Gauss theorem. First we consider the case $d=2$ and that Ω is the retangle $(a,b)\times(c,d)$. Assume that $\vec{w}(x) = (w_1(x_1,x_2), w_2(x_1,x_2))$ is a C^1-smooth function on $\overline{\Omega}$. Then we can check that the Gauss theorem is simply a consequence of the fundamental theorem of calculus:

$$\begin{aligned}
\int_\Omega \operatorname{div}(\vec{w}) &= \int_{(a,b)\times(c,d)} \mathrm{d}x_1\,\mathrm{d}x_2 (\partial_1 w_1(x_1,x_2) + \partial_2 w_2(x_1,x_2)) \\
&= \int_c^d \mathrm{d}x_2 \int_a^b \mathrm{d}x_1\,\partial_1 w_1(x_1,x_2) + \int_a^b \mathrm{d}x_1 \int_c^d \mathrm{d}x_2\,\partial_2 w_2(x_1,x_2) \\
&= \int_c^d \mathrm{d}x_2 (w_1(b,x_2) - w_1(a,x_2)) + \int_a^b \mathrm{d}x_1 (w_2(x_1,d) - w_2(x_1,c)) \\
&= \int_c^d (1,0)\cdot\vec{w}(b,x_2)\mathrm{d}x_2 + \int_c^d (-1,0)\cdot\vec{w}(a,x_2)\mathrm{d}x_2 \\
&\quad + \int_a^b (0,1)\cdot\vec{w}(x_1,d)\mathrm{d}x_2 + \int_a^b (0,-1)\cdot\vec{w}(x_1,c)\mathrm{d}x_2
\end{aligned}$$

The last four integrals can be identified as the integral of $\vec{n}\cdot\vec{w}$ on the four sides of the rectangle. Therefore we have $\int_\Omega \operatorname{div}(\vec{w}) = \int_{\partial\Omega} \vec{n}\cdot\vec{w}$ in this case.

Next, we use the change of variable formula of vector calculus to extend the Gauss formula to the case where $\Omega \subseteq \mathbb{R}^2$ is parametrized by the square. More precisely, we assume that there exists a C^2-smooth bijection $\phi : [0,1]\times[0,1] \to \overline{\Omega}$ with positive Jacobian determinant

$$J = \det\begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{pmatrix} > 0\,.$$

According to the change of variable formula, a volume element $\mathrm{d}y_1\mathrm{d}y_2$ at $y\in(0,1)^2$ is mapped to a volume element at $x\in\Omega$ given by

$$\mathrm{d}x_1\mathrm{d}x_2 = J(y)\mathrm{d}y_1\mathrm{d}y_2\,.$$

On the other hand, consider an infinitesimal line element $(\mathrm{d}y_1, 0)$ on the boundary $CD \subseteq \partial[0,1]^2$, see Figure 1. By the chain rule, its image on the side $\phi(C)\phi(D) \subseteq \partial\Omega$ is $(\mathrm{d}x_1,\mathrm{d}x_2) = (\frac{\partial x_1}{\partial y_1}, \frac{\partial x_2}{\partial y_1})\mathrm{d}y_1$. We multiply the length of these line segments by the outer normal vectors of the boundary, then

$$\vec{n}_y \mathrm{d}\sigma_y = (0,1)\mathrm{d}y \qquad \text{and} \qquad \vec{n}_x \mathrm{d}\sigma_x = \Big(-\frac{\partial x_2}{\partial y_1}, \frac{\partial x_1}{\partial y_1}\Big)\mathrm{d}y\,.$$

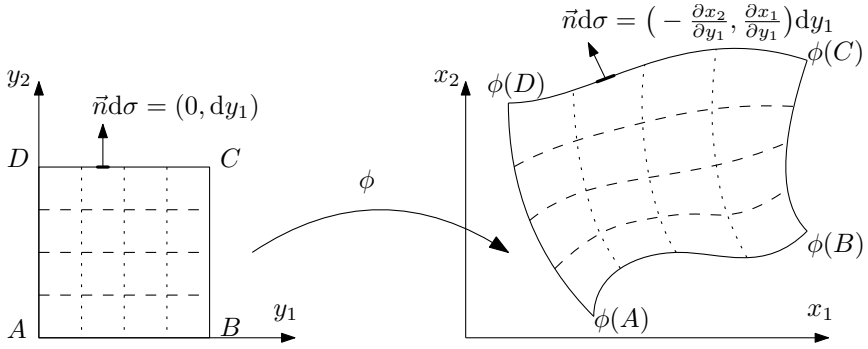


Figure 1 – *A domain parametrized by the square.*

Consider the function $\vec{z}\colon [0,1]^2 \to \mathbb{R}^2$ defined by

$$z_1 = \det\begin{pmatrix} w_1 & \frac{\partial x_1}{\partial y_2} \\ w_2 & \frac{\partial x_2}{\partial y_2} \end{pmatrix} \qquad \text{and} \qquad z_2 = \det\begin{pmatrix} \frac{\partial x_1}{\partial y_1} & w_1 \\ \frac{\partial x_2}{\partial y_1} & w_2 \end{pmatrix}.$$

We leave the reader to prove, using the chain rule and the definition of the determinant, that

$$\mathrm{div}(\vec{w})\mathrm{d}x_1\mathrm{d}x_2 = \mathrm{div}(\vec{z})\mathrm{d}y_1\mathrm{d}y_2 \qquad \text{and} \qquad \vec{w}\cdot\vec{n}_x\mathrm{d}\sigma_x = \vec{z}\cdot\vec{n}_y\mathrm{d}\sigma_y$$

where $\mathrm{div}(\vec{w}) = \frac{\partial w_1}{\partial x_1} + \frac{\partial w_2}{\partial x_2}$ and $\mathrm{div}(\vec{z}) = \frac{\partial z_1}{\partial y_1} + \frac{\partial z_2}{\partial y_2}$ in the first identity, and the second identity is valid on all of the four segments of $\partial(0,1)^2$. Since ϕ is a bijection, we obtain

$$\int_\Omega \mathrm{div}(\vec{w})\mathrm{d}x_1\mathrm{d}x_2 = \int_{(0,1)^2} \mathrm{div}(\vec{z})\mathrm{d}y_1\mathrm{d}y_2 \qquad \text{and} \qquad \int_{\partial\Omega} \vec{w}\cdot\vec{n}_x\mathrm{d}\sigma_x = \int_{\partial(0,1)^2} \vec{z}\cdot\vec{n}_y\mathrm{d}\sigma_y\,.$$

Since we know that the Gauss formula is valid on rectangles, the right hand sides of the above two identities are equal. Therefore so do their left hand sides.

For a domain Ω that cannot be parametrized by the square via a C^2-smooth mapping, in general we can cut the domain into a finite number of sub-domains using smooth curves, so that each sub-domain can be parametrized by the square. Then the Gauss formula on Ω is obtained by summing the Gauss formula on all the sub-domains, and remark that the boundary integrals along the cutting curves cancel out thanks to the opposite directions of the outer normal vectors.

Exercise 4. (Green's function and the Dirichlet problem in the unit ball.) Let $d \geq 3$ and denote by $B = B_1(0)$ the unit ball around the origin in $\mathbb{R}^d$. Assume that f is harmonic in $(1+\epsilon)B$. The goal of this exercise is to reconstruct $f(x)$, where $x \in B$, from the values of f on ∂B.

(1) Solve the above problem for $x = 0$.

(2) Let $x \neq 0$, and denote by $x^* := \frac{x}{|x|^2}$ the inversion of x with respect to ∂B. Prove that for any $y \in \partial B$, we have $|y-x|^2 = |x|^2|y-x^*|^2$.

(3) Denote $G_0(y,x) := |y-x|^{2-d}$. For a fixed $x \in B \setminus \{0\}$, find a function $G(\cdot,x)$ such that $G(\cdot,x) \equiv 0$ on ∂B, and $G(\cdot,x) - G_0(\cdot,x)$ is harmonic inside $(1+\epsilon)B$.

(4) Apply the second Green's identity of Exercise 3 to $v(\cdot) = f(\cdot)$, $u(\cdot) = G(\cdot,x)$, and $\Omega = B \setminus B_\delta(x)$, where δ is a small number. Compute the limit of the integral over $\partial B_\delta(x)$ as $\delta \to 0$.

Solution of Exercise 4.

(1). Let σ be the surface measure on ∂B. The mean value property of harmonic functions gives

$$f(0) = \frac{1}{|\partial B|} \int_{\partial B} f(x) \mathrm{d}\sigma(x) \,.$$

(2). We expand the squared norm $|y - x^*|^2$ as an inner product $(y - x^*) \cdot (y - x^*)$ so that

$$|x|^2 |y - x^*|^2 \;=\; |x|^2 \left(|y|^2 - 2y \cdot \frac{x}{|x|^2} + \frac{1}{|x|^2} \right) \;=\; |x|^2 |y|^2 - 2y \cdot x + 1 \,.$$

But $y \in \partial B$ means that $|y|^2 = 1$. So the right hand side is equal to $|x|^2 - 2y \cdot x + |y|^2 = |x - y|^2$.

(3). Let

$$G(y, x) = |y - x|^{2-d} - |x|^{2-d} |y - x^*|^{2-d}.$$

According to the previous question, $G(y, x) = 0$ for all $y \in \partial B$. Moreover, we have seen in Exercise 2 that $G(y, x) - G_0(y, x) = -|x|^{2-d} |y - x^*|^{2-d}$ is harmonic with respect to y in the domain $\mathbb{R}^d \setminus \{x^*\}$. Since $|x^*| = |x|^{-1} > 1$, this domain contains $(1 + \epsilon)B$ for $\epsilon = |x|^{-1} - 1 > 0$.

(4). Let $G^x(y) = G(y, x)$. Writing the second Green's identity as instructed, we get

$$\int_\Omega (f \Delta G^x - G^x \Delta f) = \int_{\partial \Omega} (f \partial_n G^x - G^x \partial_n f) \tag{2}$$

By Exercise 2, both G^x and $G^x - G_0^x$ are harmonic on the domain $\mathbb{C} \setminus \{x, x^*\}$, which contains Ω. It follows that $\Delta G^x = 0 = \Delta f$ on Ω and thus the left hand side of (2) is zero.

The boundary of Ω has two disjoint components ∂B and $\partial B_\delta(x)$, where the outer normal vector to $\partial \Omega$ on the second component is the opposite of the outer normal vector to $\partial B_\delta(x)$. Hence (2) implies that $0 = \int_{\partial \Omega} \omega = \int_{\partial B} \omega - \int_{\partial B_\delta(x)} \omega$, where $\omega = f \partial_n G^x - G^x \partial_n f$. Since $G^x = 0$ on ∂B, we obtain

$$\int_{\partial B} f \partial_n G^x \;=\; \int_{\partial B_\delta(x)} (f \partial_n G^x - G^x \partial_n f) \tag{3}$$

Since both $G^x - G_0^x$ and f are harmonic in $B \ni x$, the difference $D := f \partial_n (G^x - G_0) - (G^x - G_0^x) \partial_n f$ is bounded in a neighborhood of x. It follows that $|\int_{\partial B_\delta(x)} D| = O(|\partial B_\delta(x)|) \to 0$ as $\delta \to 0$. Thus we can replace G^x by G_0^x on the right hand side of (3) when taking the limit $\delta \to 0$.

Notice that the function G_0^x only depends on the distance $|y - x|$, then it is not hard to see that $G_0^x(y) = \delta^{2-d}$ and $\partial_n G_0^x(y) = (2 - d)\delta^{1-d}$ for all $y \in B_\delta(x)$. Therefore

$$\int_{\partial B_\delta(x)} (f \partial_n G_0^x - G_0^x \partial_n f) \;=\; -\frac{d-2}{\delta^{d-1}} \int_{\partial B_\delta(x)} f \;-\; \frac{1}{\delta^{d-2}} \int_{\partial B_\delta(x)} n \cdot \nabla f$$

Using the mean value property in the first integral and the Gauss theorem in the second, we get

$$\int_{\partial B_\delta(x)} (f \partial_n G_0^x - G_0^x \partial_n f) \;=\; -\frac{d-2}{\delta^{d-1}} |\partial B_\delta(x)| f(x) - \frac{1}{\delta^{d-2}} \int_{B_\delta(x)} \Delta f \;=\; -(d-2)|\partial B| f(x) \,.$$

Then it follows from (3) that $f(x) = -\frac{1}{(d-2)|\partial B|} \int_{\partial B} f \partial_n G^x$, or explicitly,

$$f(x) = \frac{1}{|\partial B|} \int_{\partial B} n_y \cdot \left(\frac{y - x}{|y - x|^d} - \frac{1}{|x|^{d-2}} \frac{y - x^*}{|y - x^*|^d} \right) f(y) \, \mathrm{d}\sigma(y) \,.$$

Topics in Probability — Exercise Sheet 4+5
Solution

About the definition of Green's function. Recall that in the lectures, the Green's function G_Ω of a *bounded* domain $\Omega \subseteq \mathbb{R}^d$ was defined by its analytic propertyies that for all $x \in \Omega$,

(1) $y \mapsto G_\Omega(x,y)$ is harmonic on $\mathbb{R}^d \setminus \{x\}$;

(2) $y \mapsto G_\Omega(x,y)$ is continuous on $\partial\Omega$ and and $G_\Omega(x,y) = 0$ for $y \in \partial\Omega$;

(3) $G_\Omega(x,y) - G_0(x,y)$ is bounded when $y \to x$, where G_0 is the full-space Green's function given by $G_0(x,y) = \varphi(|x-y|)$ where

$$\varphi(r) = \begin{cases} \frac{1}{2\pi}\log(\frac{1}{r}) & d = 2, \\ \frac{1}{(d-2)|\partial B_1|}\frac{1}{r^{d-2}} & d \geq 3. \end{cases}$$

The Green's function was constructed using the solution of the Dirichlet problem on the domain Ω with the boundary condition $f_x(y) = G_0(x,y)$ for all $y \in \partial\Omega$. For this reason we need to assume that the Dirichlet problem on the domain Ω always has a unique solution, that is, every point on $\partial\Omega$ is regular for Ω^c.

One can get rid of this assumption by taking the probablistic interpretation of the Green's function as its definition, that is, $G_\Omega(x,y) = 2\int_0^\infty p^*(t,x,y)\mathrm{d}t$ where $p^*(t,x,y) = p(t,x,y) - \mathbb{E}^x[p(t-\tau, B_\tau, y)\mathbb{1}_{\{\tau<t\}}]$ with $p(t,x,y) = (2\pi t)^{-d/2}\exp(-\frac{|x-y|^2}{2t})$ and τ is the hitting time of Ω^c by the Brownian motion $(B)_{t\geq 0}$. Without the assumption on the regularity of the boundary points of Ω, the function $y \mapsto G_\Omega(x,y)$ is no longer necessarily continuous on $\partial\Omega$, but the following identity remains valid (see Theorem 3.32 of the Brownian motion book by Mörter and Peres):

$$\int_\Omega G_\Omega(x,y)f(y)\mathrm{d}y = 2\mathbb{E}^x\left[\int_0^\tau f(B_t)\mathrm{d}t\right]$$

for any $x \in \Omega$ and bounded and measurable function $f : \Omega \to \mathbb{R}$.

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has the mean value property everywhere on Ω. Moreover, Ω is bounded and g^x is continuous on $\overline{\Omega}$. Thus by the maximum principal, g^x attains its minimum on $\partial\Omega$. But the Green's function is positive inside its domain (as shown by its probabilistic interpretation) and zero on the boundary. This implies that $g^x(y) = G_{\Omega'}(x,y) \geq 0$ for all $y \in \partial\Omega$. It follows that $g^x(y) \geq 0$ for all $y \in \Omega$.

Probabilistic proof. Recall that the Green's function is characterized by the identity that, for any bounded and measurable function $f : \Omega \to \mathbb{R}$ and $x \in \Omega$,

$$\int_\Omega G_\Omega(x,y) f(y) \mathrm{d}y = 2\mathbb{E}^x \left[\int_0^{\tau_\Omega} f(B_t) \mathrm{d}t \right],$$

where $\tau_\Omega = \inf\{t \geq 0 : B_t \notin \Omega\}$ is the exit time from the domain Ω.

If $\Omega \subseteq \Omega'$, then $\tau_\Omega \leq \tau_{\Omega'}$ and thus for all $f \geq 0$, the two sides of the above equation increase when Ω is replaced by Ω' (and f replaced by $f\mathbb{1}_\Omega$). It follows that for all $f \geq 0$,

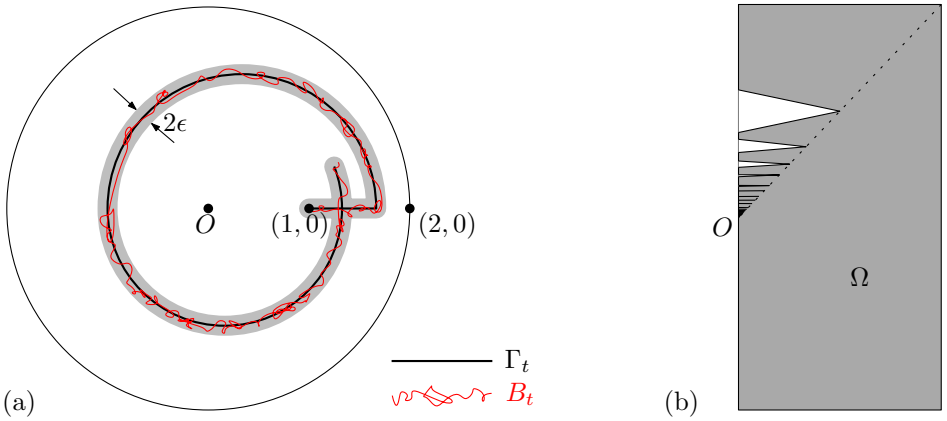
$$\int_\Omega (G_{\Omega'} - G_\Omega)(x,y) f(y) \mathrm{d}y \geq 0\,. \tag{1}$$

Together with the continuity of $G_{\Omega'} - G_\Omega$, this implies that $G_{\Omega'}(x,y) \geq G_\Omega(x,y)$ for all $x, y \in \Omega$.

More precisely, since $G_{\Omega'}$ and G_Ω has the same singularity when $y \to x$, the function $y \mapsto (G_{\Omega'} - G_\Omega)(x,y)$ is continuous on Ω. If $G_\Omega(x_0,y_0) > G_{\Omega'}(x_0,y_0)$ for some $x_0, y_0 \in \Omega$, then we have $G_\Omega(x_0,y) > G_{\Omega'}(x_0,y)$ for all y in some ball $\mathcal{B}$ around y_0. Since $\mathcal{B}$ has non-zero Lebesgue measure, this implies that $\int_{\mathcal{B}} (G_{\Omega'} - G_\Omega)(x_0,y)\mathrm{d}y < 0$. This contradicts (1) because we can take $f = \mathbb{1}_{\mathcal{B}}$.

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According to the above discussion on the topology of Ω^c, it suffices to prove that the probability that $\{B_t : t \in (\tau_k, \tau_{k+1})\}$ contains a closed path separating $\hat{x}$ from ∞ is at least p. By the translation, rotation and scaling invariance of the Brownian motion, the above event has the same probability as the event that a Brownian motion started from $(1, 0)$ draws a closed path separating the origin from ∞ before exiting from the ball of radius 2. This certainly happens with a strictly positive probability: for example, one can use the result of Exercise 6 in Exercise Sheet 1 to show that the Brownian motion approximates any continuous path Γ in $\mathbb{R}^2$ with arbitrary precision with positive probability. That is, $\mathbb{P}^x(\sup_{t\in[0,1]} |B_t - \Gamma_t| \leq \epsilon) > 0$ for any $\epsilon > 0$ and continuous function $\Gamma : [0, 1] \to \mathbb{R}^2$ such that $\Gamma_0 = x$. One can ensure that there is B_t draws a closed path around the origin by choosing Γ and ϵ appropriately, e.g. as in the picture (a) below.



The case of $d \geq 3$. The assumption that Ω the Poincaré coin condition is not sufficient for (2) to hold. Even if we assume in addition that Ω is bounded, one still has the following counter-example: for $h \in \mathbb{R}$ and $\alpha \in (0, \pi/2)$, denote by $\mathtt{C}(h, \alpha)$ the cone based at $(h, h, 0, \ldots, 0)$ and having an opening angle α around the negative x_1-axis. Let $H = (0, 1) \times (-1, 1)^{d-1}$ be the right half of the hypercube centered at the origin, and define $\Omega = H \setminus (\cup_{n\geq 2} \mathtt{C}(\frac{1}{n}, \frac{1}{n^2}))$, see the picture (b) above. One can check that Ω satisfies the Poincaré cone condition. But since the cones $\mathtt{C}(\frac{1}{n}, \frac{1}{n^2})$ removed from H gets sharper and sharper, there cannot be any global constants $c, \sigma > 0$ such that (2) holds for all $x \in \Omega$.

A sufficient condition for (2) is the following global version of Poincaré cone condition: there exists a global constant $\alpha > 0$ such that for all $x \in \partial\Omega$, there is a cone $\mathcal{C}_\alpha(x)$ based at x and of opening angle α such that $\mathcal{C}_\alpha(x) \subseteq \Omega^c$.

If this global Poincaré cone condition is satisfied, then we can adapt the proof for $d = 2$ to show that (2) also holds in this case. More precisely, we define C_k, τ_k, and bound $\mathbb{P}^x(\tau_R < \tau_{\Omega^c})$ in the same way as before, so that the proof boils down to showing that $\mathbb{P}^x(\exists t \in (\tau_k, \tau_{k+1}) : B_t \notin \Omega | \mathcal{F}_{\tau_k}) > p$ for some constant $p > 0$. By the translation, rotation and scaling invariance of the Brownian motion, this probability is bounded from below by the probability that a d-dimensional Brownian motion, started from some point x on the unit sphere, hits the cone based at the origin and of opening angle α around the x_1-axis before exiting the ball of radius 2. Using the same approximation argument as before, this probability is strictly positive for each x on the unit sphere. As the sphere is compact, it is bounded from below by some constant $P > 0$ that only depends on the angle α (and of the dimension d). This completes the proof of (2) under the global Poincaré cone condition.

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$G(\cdot, y)$ and the domain $\Omega \setminus (B_\varepsilon(x) \cup B_\varepsilon(y))$. You may assume that the boundary of Ω is smooth and take for granted the existence of normal derivatives). **Bonus question**: get rid of the smoothness assumption.

Solution of Exercise 4. Let us write $G^x(z) = G_\Omega(z, x)$ for all $x, z \in \Omega$. We assume $x \neq y$ (otherwise there is nothing to prove) and take ϵ small enough so that $B_\epsilon(x)$, $B_\epsilon(y)$ and Ω^c are disjoint.

Appying Green's identity to the domain $\Omega' = \Omega \setminus (B_\epsilon(x) \cup B_\epsilon(y))$ gives

$$\int_{\Omega'} (G^x \Delta G^y - G^y \Delta G^x) = \int_{\partial\Omega'} (G^x \partial_n G^y - G^y \partial_n G^x)$$

By the definition of Green's function, G^x and G^y are harmonic on $\Omega \setminus \{x, y\}$. Thus the left hand side of the above identity vanishes. The surface of integration on the right hand side has three components so that $\int_{\partial\Omega'} = \int_{\partial\Omega} - \int_{\partial B_\epsilon(x)} - \int_{\partial B_\epsilon(y)}$. The integral on $\partial\Omega$ vanishes because $G^x(z) = G^y(z) = 0$ on the boundary of Ω. Therefore we have

$$\int_{\partial B_\epsilon(x)} (G^x \partial_n G^y - G^y \partial_n G^x) + \int_{\partial B_\epsilon(y)} (G^x \partial_n G^y - G^y \partial_n G^x) = 0\,. \tag{3}$$

The definition of the Green's implies that $G^x - G_0^x$ (where $G_0^x(z) = G_0(x, z) = \varphi(|x - z|)$) is harmonic on Ω. On the other hand, since $|x - y| \geq 2\epsilon$, the function G^y is continuous on $B_{2\epsilon}(x)$. It follows that,

$$(G^x \partial_n G^y - G^y \partial_n G^x) - (G_0^x \partial_n G^y - G^y \partial_n G_0^x) \;=\; (G^x - G_0^x) \cdot \partial_n G^y + G^y \cdot \partial_n (G^x - G_0^x)$$

is continuous, and therefore bounded on the closed ball $\overline{B}_\epsilon(x)$. Therefore one can replace G^x by $G_0^x := G_0(\cdot, x)$ in the first integral of (3) with an error that vanishes in the limit $\epsilon \to 0$.

Since the function G_0^x is radially symmetric around x, both $G_0^x(z)$ and $\partial_n G_0^x(z)$ are constant on $\partial B_\epsilon(x)$. So one can take them out of the integral and obtain

$$\int_{\partial B_\epsilon(x)} (G_0^x \partial_n G^y - G^y \partial_n G_0^x) \;=\; \varphi(\epsilon) \int_{\partial B_\epsilon(x)} \partial_n G^y - \varphi'(\epsilon) \int_{\partial B_\epsilon(x)} G^y\,.$$

By Green's identity, the first integral on the right hand side is equal to $\int_{B_\epsilon(x)} \Delta G^y$, which is zero because G^y is harmonic on $B_\epsilon(x)$. Since G^y is continuous at x, the second integral is asymptotically equivalent to $|\partial B_\epsilon(x)| G^y(x) = |\partial B_\epsilon| \cdot G_\Omega(y, x)$. We conclude that

$$\lim_{\epsilon \to 0} \int_{\partial B_\epsilon(x)} (G^x \partial_n G^y - G^y \partial_n G^x) \;=\; \lim_{\epsilon \to 0} -\varphi'(\epsilon) |\partial B_\epsilon| \cdot G_\Omega(y, x)\,.$$

One can check that $\varphi'(\epsilon) = -|\partial B_\epsilon|^{-1}$ for all $d \geq 2$. So the right hand side is equal to $G_\Omega(y, x)$. By symmetry, the second integral in (3) converges to $-G_\Omega(x, y)$ as $\epsilon \to 0$. Hence $G_\Omega(x, y) = G_\Omega(y, x)$.

Bonus question. For this question we will take for granted the continuity of the Green's function with respect to its first variable on the boundary of Ω. That is, we assume that $x \mapsto G(x, y)$ is continuous on $\partial\Omega$ and vanishes on $\partial\Omega$, for every fixed $y \in \Omega$. (This assumption is already implicitly used to cancel the term $\int_{\partial\Omega}$ in the surface integral in the above calculation.)

Consider $x \in \Omega$ and $\epsilon > 0$ small enough so that $B_\epsilon(x) \subseteq \Omega$. Let τ_ϵ be the hitting time of $\partial B_\epsilon(x)$ by the Brownian motion $(B)_{t \geq 0}$. If the Brownian motion starts from x, then B_{τ_ϵ} is distributed uniformly on the sphere $\partial B_\epsilon(x)$. Therefore by strong Markov property,

$$\mathbb{E}^x \left[\int_{\tau_\epsilon}^{\tau} f(B_t) \mathrm{d}t \right] \;=\; \mathbb{E}_x \left[\mathbb{E}_{B_{\tau_\epsilon}} \left[\int_0^{\tau} f(B_t) \mathrm{d}t \right] \right] \;=\; \int_{\partial B_\epsilon(x)} \mathbb{E}_z \left[\int_0^{\tau} f(B_t) \mathrm{d}t \right] \frac{\mathrm{d}\sigma(z)}{|\partial B_\epsilon|}$$

for all bounded measurable function $f : \Omega \to \mathbb{R}$. Assume that f vanishes on $B_{2\epsilon}(x)$, then we can replace τ_ϵ by zero on the left hand side. It follows that

$$\int_\Omega G_\Omega(x,y)f(y)\mathrm{d}y \;=\; \int_{\partial B_\epsilon(x)} \frac{\mathrm{d}\sigma(z)}{|\partial B_\epsilon|} \int_\Omega G_\Omega(z,y)f(y)\mathrm{d}y\,.$$

Since $G_\Omega(z,y)$ is bounded for $(z,y) \in \partial B_\epsilon(x) \times (B_{2\epsilon}(x))^c$, we can exchange the order of the two integrals on the right hand side and deduce that

$$\int_\Omega \left(G_\Omega(x,y) - \int_{\partial B_\epsilon(x)} G_\Omega(z,y) \frac{\mathrm{d}\sigma(z)}{|\partial B_\epsilon|} \right) f(y)\mathrm{d}y \;=\; 0\,.$$

Since f is any bounded measurable function supported on $\Omega \setminus B_{2\epsilon}(x)$ and G_Ω is continuous with respect to its second variable, the above equation implies that $G_\Omega(x,y) = \int_{\partial B_\epsilon(x)} G_\Omega(z,y) \frac{\mathrm{d}\sigma(z)}{|\partial B_\epsilon|}$ whenever $|x-y| > 2\epsilon$. As ϵ is arbitrary, we conclude that $x \mapsto G_\Omega(x,y)$ is harmonic on $\Omega \setminus \{y\}$.

Therefore for each $x \in \Omega$, the function $h^x(y) := G_\Omega(x,y) - G_\Omega(y,x)$ is harmonic on $\Omega \setminus \{x\}$. Moreover, it is bounded when $y \to x$ because both $G_\Omega(x,y)$ and $G_\Omega(y,x)$ have a bounded difference from $G_0(x,y) = G_0(y,x)$. Thus g^x is a harmonic function on Ω. Our assumption on $x \mapsto G(x,y)$ ensures that h^x is continuous and vanishes on $\partial\Omega$. Thus by the maximum principle $h^x(y) = 0$ for all $x, y \in \Omega$.

Alternative. In Theorem 3.30 of the Brownian motion book of Mörter and Peres it is proved for almost every $x, y \in \Omega$ (for the Lebesgue measure) the transition kernel $p^*(t,x,y)$ of the Brownian motion stopped upon hitting Ω^c is symmetric: $p^*(t,x,y) = p^*(t,y,x)$ for all $t > 0$. Since $G_\Omega(x,y) = 2\int_0^\infty p^*(t,x,y)\mathrm{d}t$, we have $G_\Omega(x,y) = G_\Omega(y,x)$ for almost every $x, y \in \Omega$. Then one can extend the equality to every $x, y \in \Omega$ using the continuity of G_Ω with respect to its two variables.

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where we have the second equality because by continuity of the Brownian motion, B_t can visit $\mathcal{R}A$ only after τ_H. Combining this identity with (4), we conclude that

$$\int_A G^+(x,y)\mathrm{d}y \;=\; \int_A G(x,y)\mathrm{d}y - \int_A G(x,\mathcal{R}y)\mathrm{d}y$$

for all $x \in B_1^+(0)$ and measurable set $A \subseteq B_1^+(0)$. Therefore $G^+(x,y) = G(x,y) - G(x,\mathcal{R}y)$.

Analytic approach. For any bounded measurable function $f : B_1(0) \to \mathbb{R}$,

$$F(x) = \int_{B_1(0)} G(x,y)f(y)\mathrm{d}y$$

solves the Poisson problem on $B_1(0)$: F is the unique continuous function on the closure of $B_1(0)$ such that $\Delta F = f$ on $B_1(0)$ and $F = 0$ on $\partial B_1(0)$.

We choose f to satisfy $f(\mathcal{R}x) = -f(x)$ for all $x \in B_1(0)$. Since the Poisson problem is linear and has unique solution, we must have $F(\mathcal{R}x) = -F(x)$. In particular, $F(x) = 0$ for all $x \in H$. It follows that the restriction of F to the closure of $B_1^+(0)$ solves the Poisson problem on $B_1^+(0)$. But

$$F(x) = \int_{B_1^+(0)} G(x,y)f(y)\mathrm{d}y + \int_{B_1^+(0)} G(x,\mathcal{R}y)f(\mathcal{R}y)\mathrm{d}y \;=\; \int_{B_1^+(0)} \left(G(x,y) - G(x,\mathcal{R}y)\right) f(y)\mathrm{d}y\,,$$

for all $x \in B_1^+(0)$ and bounded measurable function f on $B_1^+(0)$. So $G^+(x,y) = G(x,y) - G(x,\mathcal{R}y)$.

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On the other hand, $\tau_{\mathbf{e}_1}$ and τ_H are independent. Using the reflexion principle of 1-dimensional Brownian motion, we can compute their laws as follows

$$\mathbb{P}^x(\tau_H > t) \;=\; \mathbb{P}^0\Big(\sup_{s\in[0,t]} B_s^{(1)} < x_1\Big) \;=\; \mathbb{P}\Big(|\mathcal{N}(0,1)| < \frac{x_1}{\sqrt{t}}\Big) \;\le\; \frac{x_1}{\sqrt{t}}\,,$$
$$\mathbb{P}^x(\tau_{\mathbf{e}_1} < t) \;=\; \mathbb{P}^0\Big(\sup_{s\in[0,t]} B_s^{(1)} > a\Big) \;=\; \mathbb{P}\Big(|\mathcal{N}(0,1)| > \frac{a}{\sqrt{t}}\Big) \;\le\; \exp\Big(-\frac{a^2}{2t}\Big)\,.$$

The first estimate is a consequence of the fact that the density of the standard normal distribution is bounded by $1/2$. The second estimate is classical and can be shown as follows: for all $z > 0$,

$$\big(\mathbb{P}(\mathcal{N}(0,1) > z)\big)^2 \;=\; \iint_{|x|>z,|y|>z} \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}\,\mathrm{d}x\,\mathrm{d}y \;\le\; \frac{1}{2\pi}\int_{\sqrt{2}z}^{\infty} e^{-\frac{r^2}{2}}\cdot\frac{\pi}{2} r\,\mathrm{d}r \;=\; \frac{1}{4}e^{-z^2}\,,$$

so $\mathbb{P}(|\mathcal{N}(0,1)| > z) \le e^{-\frac{z^2}{2}}$. Thanks to the independence between $\tau_{\mathbf{e}_1}$ and τ_H, we have

$$\mathbb{P}^x(\tau_{\mathbf{e}_1} < \tau_H) \;=\; \mathbb{E}^x\Big[\mathbb{P}^x(\tau_{\mathbf{e}_1}(\tau_{\mathbf{e}_1} < t)\big|_{t=\tau_H}\Big] \;\le\; \mathbb{E}^x\Big[\exp\Big(-\frac{a^2}{2\tau_H}\Big)\Big]$$

Using integration by parts, one can express the right hand side in terms of $\mathbb{P}^x(\tau_H > t)$

$$\begin{aligned}\mathbb{E}^x\Big[\exp\Big(-\frac{a^2}{2\tau_H}\Big)\Big] \;&=\; \int_0^\infty e^{-\frac{a^2}{2t}}\cdot\Big(-\frac{\mathrm{d}\mathbb{P}^x(\tau_H > t)}{\mathrm{d}\,t}\Big)\,\mathrm{d}t \\ &=\; \int_0^\infty \frac{a^2}{2t^2} e^{-\frac{a^2}{2t}}\cdot\mathbb{P}^x(\tau_H > t)\mathrm{d}t \;\le\; x_1\int_0^\infty \frac{a^2}{2t^{5/2}} e^{-\frac{a^2}{2t}}\,\mathrm{d}t\end{aligned}$$

With the change of variable $s = \frac{a^2}{2t}$, we see that the integral on the right hand side is equal to $\frac{\sqrt{2}}{a}\int_0^\infty \sqrt{s}e^{-s}\mathrm{d}s = \frac{\sqrt{2\pi}}{2a} < \infty$. That is, $\mathbb{P}^x(\tau_{\mathbf{e}_1} < \tau_H)$ is also bounded by a constant times x_1.

It follows that there exists $C > 0$ such that $\mathbb{P}^x(\tau < \tau_H) \le Cx_1$ for all $x_1 \le 1 - da^2$. With a possible increase of the value of C, the same holds for all $x_1 \in (0,1)$.

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Choose $\epsilon > 0$ small enough so that B_a is outside the ball $B_\epsilon(B_c)$. Let θ be the last time before c that $B_t \notin B_\epsilon(B_c)$. Then $(B_t)_{t\in[\theta,c]}$ is a simple path joining B_c to the circle $\partial B_\epsilon(B_c)$, and it intersects $\partial B_\epsilon(B_c)$ only at B_θ. It is clear that $\Omega := B_\epsilon(B_c) \setminus B_{[\theta,c]}$ is a simply connected domain in $\mathbb{R}^2$. Conditionally on $\mathcal{F}_c$, $(B)_{t\geq c}$ is a Brownian motion started from B_c. We know that almost surely, it hits the boundary of Ω immediately after time c. It can only do so by hitting $B_{[\theta,c]}$ because $\partial B_\epsilon(B_c)$ is at a distance $\epsilon > 0$ from B_c. This proves that $B_{[\theta,c]} \cap B_{(c,b)} \neq \varnothing$ almost surely. In other words, there exists $t_1 \in (a, c]$ and $t_2 \in (c, b)$ such that $B_{t_1} = B_{t_2}$.

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for all $x \in \mathbb{R}^3 \setminus I$. By the triangular inequality, $|x| > R$ implies $|F(x)| \leq \frac{1}{R-1}\int_0^1 y^\alpha \mathrm{d}y$. Therefore $F(x) \to 0$ when $|x| \to \infty$. Finally, when $a \searrow 0$, we have by the monotone convergence theorem

$$F(-a,0,0) \;=\; \int_0^1 \frac{y^\alpha \,\mathrm{d}y}{y+a} \;\longrightarrow\; \int_0^1 \frac{y^\alpha \,\mathrm{d}y}{y} \;=\; \frac{1}{\alpha}\,.$$

(2). The idea is that $F(x) = \infty$ for $x \in I$, therefore for any $M > L$, the level surface C_M, which separates I from the infinity, must "pass through" the origin.

More precisely, consider the semi-circle $\{(r\cos\theta, r\sin\theta, 0) : \theta \in (0,\pi]\}$ where $r \in (0,1)$. By the monotone convergence theorem we have

$$F(r\cos\theta, r\sin\theta, 0) \;=\; \int_0^1 \frac{y^\alpha \,\mathrm{d}y}{\sqrt{r^2+y^2-2ry\cos\theta}} \;\xrightarrow[\theta\to 0]{}\; \int_0^1 \frac{y^\alpha \,\mathrm{d}y}{\sqrt{r^2+y^2-2ry}} \;=\; \infty$$

On the other hand, when $\theta = \pi$, we have $F(-r,0,0) < L < M$. Thus by continuity, there exists a point x_M on the semi-circle such that $F(x_M) = M$. Since $|x_M| = r$ can be taken arbitrarily small, this proves that the origin is in the closure of the C_M.

The fact that the origin is not a regular point of C_M follows from the following lemma by considering the domain $\Omega = \{x \in \mathbb{R}^3 : m < F(x) < M\}$ for some $m < L$, the polar set $P = \{0\}$, and the boundary condition $f(z) = F(z)$ for all $z \in \partial\Omega \setminus \{0\}$ and $f(0) = M$.

Lemma. *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain, and f some (not necessarily continuous) function from $\partial\Omega$ to $\mathbb{R}$. Assume that $F : \overline{\Omega} \to \mathbb{R}$ is a bounded solution to the generalized Dirichlet problem that $\left\{\begin{array}{l}\Delta F(x) = 0 \text{ for all } x \in \Omega \\ F(x) = f(x) \text{ for all } x \in \partial\Omega\end{array}\right\}$ and $F(x)$ is continuous on $\overline{\Omega}\setminus P$, where $P \subseteq \partial\Omega$ is a polar set for the Brownian motion. Then $F(x) = \mathbb{E}^x[f(B_\tau)]$ for all $x \in \overline{\Omega}\setminus P$.*
In particular, if f is continuous but F is discontinuous at some $z \in P$, then z is irregular for Ω.

Proof. We adapt the proof used in the lectures to show that when f is continuous and F is the solution of the Dirichlet problem, we have $F(x) = E^x[f(B_\tau)]$.

Let $\tau_0 = 0$ and for $k \geq 0$, let $\tau_{k+1} = \min\left\{t > \tau_k : |B_t - B_{\tau_k}| = \frac{1}{2}r_k\right\}$ where $r_k = \mathrm{dist}(B_{\tau_k}, \partial\Omega)$. Then $(B_{\tau\wedge\tau_k})_{k\geq 0}$ is a bounded discrete-time martingale, therefore converges almost surely as $k \to \infty$. If $\sup_k \tau_k < \tau$, then we would have both $r_k = 2|B_{\tau_{k+1}} - B_{\tau_k}| \to 0$ and $r_k = \mathrm{dist}(B_{\tau_k}, \partial\Omega) \to \mathrm{dist}(B_{\sup_k \tau_k}, \partial\Omega) > 0$, which is a contradiction. Hence $B_{\tau_k} \to B_\tau$ almost surely as $k \to \infty$. Since the Brownian motion never hits the polar set P, the function F is almost surely continuous at B_τ. It follows that $F(B_{\tau_k}) \to F(B_\tau) = f(B_\tau)$ almost surely as $k \to \infty$.

Conditionally on $\mathcal{F}_{\tau_k}$, the $B_{\tau_{k+1}}$ is distributed uniformly on the sphere $\partial B_{r_k/2}(B_{\tau_k})$. Since F is harmonic in the larger ball $B_{r_k}(B_{\tau_k})$, the mean value property gives $\mathbb{E}^x[F(B_{\tau_{k+1}})|\mathcal{F}_{\tau_k}] = F(B_{\tau_k})$. Therefore $(F(B_{\tau_k}))_{k\geq 0}$ is also a bounded martingale. It converges almost surely and in L^1 to $f(B_\tau)$. This proves that $F(x) = E^x[F(B_{\tau_0})] = E^x[f(B_\tau)]$.

If z is a regular point and f is continuous at z, then $x \mapsto \mathbb{E}^x[f(B_\tau)]$ must be continuous at z. Therefore by the above identity, if F is discontinuous at z, then z is irregular. □

(3). By symmetry $C_M \cap \{x_1 = a\}$ is a circle centered at $(a,0,0)$, let r be its radius. Since the origin is an irregular point for C_M, the region $\{x \in \mathbb{R}^3 : F(x) > M\}$ does not satisfy the Poincaré cone condition at 0. This leads us to consider the regime $r = \epsilon a$, where $\epsilon = \epsilon(a) \to 0$ as $a \to 0$.

Let us estimate $F(a, \epsilon a, 0) = \int_0^1 \frac{y^\alpha \,\mathrm{d}y}{\sqrt{(y-a)^2+\epsilon^2 a^2}}$ in this limit. Fix some $\lambda > 1$. For all $a > 0$ small enough,

$$\frac{y^\alpha \mathbb{1}_{\{y>\lambda a\}}}{\sqrt{(y-a)^2+\epsilon^2 a^2}} \;\leq\; \frac{y^\alpha \mathbb{1}_{\{y>\lambda a\}}}{y-a} \;\leq\; \frac{y^\alpha}{y}\frac{\lambda}{\lambda-1}\,.$$

The right hand side is integrable over $y \in (0,1)$, therefore by the dominated convergence theorem,

$$\int_{\lambda a}^{1} \frac{y^\alpha \,\mathrm{d}y}{\sqrt{(y-a)^2+\epsilon^2 a^2}} \xrightarrow[a\to 0]{} \int_0^1 \frac{y^\alpha \,\mathrm{d}y}{y} \;=\; L\,.$$

On the other hand, for any $\theta > 0$ small enough, we have

$$\int_0^{\lambda a} \frac{y^\alpha \,\mathrm{d}y}{\sqrt{(y-a)^2+\epsilon^2 a^2}} \;=\; \int_0^{\lambda} \frac{a^\alpha x^\alpha \,\mathrm{d}x}{\sqrt{(x-1)^2+\epsilon^2}} \;=\; \left(\int_{-1}^{\lambda-1} - \int_{-\theta}^{\theta}\right) \frac{a^\alpha (x+1)^\alpha \,\mathrm{d}x}{\sqrt{x^2+\epsilon^2}} + \int_{-\theta}^{\theta} \frac{a^\alpha (x+1)^\alpha \,\mathrm{d}x}{\sqrt{x^2+\epsilon^2}}\,.$$

Again by then dominated convergence theorem, the first integral on the right hand side converges to zero when $a \to 0$ and $\epsilon \to 0$. In the second integral, we can bound the numerator by $a^\alpha(1 \pm \theta)^\alpha$ and compute the integral $\int_{-\theta}^{\theta}(x^2+\epsilon^2)^{-1/2}\mathrm{d}x = 2\,\mathrm{arsinh}(\theta/\epsilon)$, where $\mathrm{arsinh}(x) = \log(x + \sqrt{x^2+1})$ is the inverse hyperbolic sine function. When $\epsilon \to 0$, we have $\mathrm{arsinh}(\theta/\epsilon) \sim \log(1/\epsilon)$ independently of θ.

Combining the above estimates, we see that $\lim_{a\to 0} F(a, \epsilon a, 0) = L + 2\lim_{a\to 0} a^\alpha \log(1/\epsilon)$, provided that the limits exist. Take $\lambda > (M-L)/2$ and choose $\epsilon(a) = \exp(-\lambda a^{-\alpha})$ so that $a^\alpha \log(1/\epsilon) = \lambda$. Then we have $\lim_{a\to 0} F(a, \epsilon a, 0) = L + 2\lambda > M$. This implies that, at least on the left of the plan $\{x_1 = a\}$ for some $a > 0$ small enough, C_f is contained inside C_M for $f(x_1) = x_1 \exp(-\lambda x_1^{-\alpha})$.

Now consider the Newton's potential defined by a general measure of density $\rho(y)$ on I. One can check that if

$$\int_0^1 \frac{\rho(y)}{y}\mathrm{d}y < \infty$$

and $\rho(y) > 0$ for all $y > 0$ small enough, then the conclusion of (1) and (2) remains valid. Moreover, if $y \mapsto \rho(y)$ is decreasing in a neighborhood of zero, then in the regime $a \to 0$ and $\epsilon(a) \to 0$, we still have $\lim F(a, \epsilon(a)a, 0) = L + 2\lim \rho(a)\log(1/\epsilon)$, provided that the limits exist. With the same argument as before, one sees that on the left of the plan $\{x_1 = a\}$ for some $a > 0$ small, C_f is contained in C_M for $f(y) = y\exp(-\lambda\rho(y)^{-1})$.

For $k \geq 1$, denote by $\log^{(k)}(x)$ the logarithm iterated k times. Then for any $n \geq 1$, $\gamma > 1$ and δ small enough,

$$\rho(y)^{-1} = \left(\log^{(n)}\left(\frac{1}{y}\right)\right)^\gamma \mathbb{1}_{(0,\delta)}(y) \prod_{k=1}^{n-1} \log^{(k)}\left(\frac{1}{y}\right)\,,$$

defines a density that satisfies the above conditions. For example, when $k = 2$, the corresponding function f is given by

$$f(y) = y^{1+\lambda(-\log y)^\gamma}$$

for $y \in (0,\delta)$. Notice that this function still decays faster than any polynomial when $y \to 0$, while the sharper estimate provided by Wiener's test shows that the origin is a irregular point for C_f with $f(y) = y^\alpha$ with any $\alpha > 1$.