```
> restart;
```

Preliminary: Remark on graph-based argument for positivity.

On multiple occasions in this Maple notebook, we will "check" that a univariate real function f
   is positive on some given interval I by looking at the graph of f on I.
   (This includes also checking inequalities like f(x)>g(x), or checking that some function h(x) does not take real values,
    since the first problem is equivalent to (f-g)(x)>0, and the second problem can be reduced to checking the sign of Im(h(x)).)
These graph-based arguments are not rigorous proofs, but quick sketches which indicate how the rigorous proof show go.

The key remark here is that we only deal with algebraic functions in this notebook,
   and positivity proofs of algebraic functions can be done algorithmically, as follows:

   The (real) zeros of an algebraic function f are always isolated.
   Given the algebraic equation satisfies by f, one can compute algorithmically disjoint intervals that each
      contains exactly one zero of f. (See the documentation of the Maple command RootFinding[Isolate].)
   Then, to prove that f is nonnegative, it suffices to take one value x between each pair of consecutive
      zeros of f, and show that f(x)>0. Again, this can be done algorithmically.

However, spelling out all the steps of these algorithmic proofs will take up too much space and obscure our main objective.
So we choose to adopt the less rigorous, but equally convincing graph-based arguments for checking positivity.

Preliminary: Functional equation satisfied by Z(u,v;t,nu).

%%% We rederive the following results, which is summarized as Equation (15), from our previous paper:
%%% 1. expression of Z(u,v) as a rational function of Z0(u), Z0(v), z10 and u,v,t,nu.
%%% 2. algebraic equation satisfied by Z0(u), z10,z30 and u,v,t,nu.
%%%

Naming conventions:
  Z, Z0,Z1, z10,z11,z20,... are formal variables used to represent the GFs (Generating Functions) of Ising-triangulations.
  They are not assigned any value, and are used as unknowns in the functional equations.
  For Z0,Z1,... we add a letter u or v after the name to distinguish between Z0(u) and Z0(v), Z1(u) and Z1(v), etc.

We start with the two Tutte's equations for our model:

```
> Du_Z  := (Z-Z0v)/u:
  Dv_Z  := (Z-Z0u)/v:
  Du_Z0u:= (Z0u-1)/u:
  Dv_Z0v:= (Z0v-1)/v:

  Tutte1:= u+t*((Du_Z-Z1v)/u + (Du_Z0u+Z1v)*Du_Z - Du_Z0u*Z1v) -
  (nu*Du_Z-Dv_Z)/(nu^2-1);
  Tutte2:= v+t*((Dv_Z-Z1u)/v + (Dv_Z0v+Z1u)*Dv_Z - Dv_Z0v*Z1u) -
  (nu*Dv_Z-Du_Z)/(nu^2-1);
```

$$Tutte1 := u + t\left(\frac{\frac{Z - Z0v}{u} - Z1v}{u} + \frac{\left(\frac{Z0u - 1}{u} + Z1v\right)\ (Z - Z0v)}{u} - \frac{(Z0u - 1)\ Z1v}{u}\right) - \frac{\frac{(Z - Z0v)\ \nu}{u} - \frac{Z - Z0u}{v}}{\nu^2 - 1}$$

$$Tutte2 := v + t\left(\frac{\frac{Z - Z0u}{v} - Z1u}{v} + \frac{\left(\frac{Z0v - 1}{v} + Z1u\right)\ (Z - Z0u)}{v} - \frac{(Z0v - 1)\ Z1u}{v}\right) - \frac{\frac{(Z - Z0u)\ \nu}{v} - \frac{Z - Z0v}{u}}{\nu^2 - 1} \qquad \textbf{(2.1)}$$

By solving the first Tutte's equation for Z, we obtain a rational expression of Z in terms of Z0u,Z0v,Z1v and nu,t,u,v.

```
> Z_rational_with_Z1v := solve(Tutte1,Z):
  Z0v + (u*(Z0v-Z0u) + (nu^2-1)*v*(u^2-t*Z0u*Z1v)) / (nu*v-u-(nu^2
  -1)*t*v*(Z0u/u+Z1v));  # a human-readable expression of
  Z_rational
  is(% = Z_rational_with_Z1v);
  indets(Z_rational_with_Z1v);
```

$$Z0v + \frac{u\ (Z0v - Z0u) + \left(\nu^2 - 1\right)\ v\ \left(-t\ Z0u\ Z1v + u^2\right)}{\nu\ v - u - \left(\nu^2 - 1\right)\ t\ v\ \left(\frac{Z0u}{u} + Z1v\right)}$$

$$true$$

$$\{Z0u, Z0v, Z1v, \nu, t, u, v\} \qquad \textbf{(2.2)}$$

The following function extracts formally the coefficient of [u^k] or [v^k]
   from a polynomial expression containing the variables Z, Z0u,Z0v, Z1u,Z1v.
It takes into account the naming convention at the beginning of this worksheet.
The result is correct for small values of k.
The indicator for correct is that the result does not contain the variables RES, RES0 and RES1.

```
> extract_coeff := proc(expr, u_or_v::name, k::integer)
     local expansion, coef, RES,RES0,RES1;
     if is(u_or_v=v) then
        expansion := subs({
           Z  = Z0u + Z1u*v + Z2u*v^2 + Z3u*v^3 + RES*v^4,
           Z0v=   1 + z10*v + z20*v^2 + z30*v^3 +RES0*v^4,
           Z1v= z10 + z11*v + RES1*v^2
                        }, expr);
        coef := coeff(expand(expansion), v, k);
     else
        expansion := subs({
           Z  = Z0v + Z1v*u + Z2v*u^2 + Z3v*u^3 + RES*u^4,
           Z0u=   1 + z10*u + z20*u^2 + z30*u^3 +RES0*u^4,
           Z1u= z10 + z11*u + RES1*u^2
                        }, expr);
        coef := coeff(expand(expansion), u, k);
     end if;
     return coef;
  end proc:
```

We extract the coefficients of v^0 and v^1 from the two Tutte's equations.
This gives a system of three algebraic equations relating Z0u, Z1u and Z2u :

```
> Tutte1_v0 := extract_coeff(Tutte1,v,0);
  Tutte1_v1 := extract_coeff(Tutte1,v,1);
  Tutte2_v0 := extract_coeff(Tutte2,v,0);

  indets(Tutte1_v0);
  indets(Tutte1_v1);
  indets(Tutte2_v0);
```

$$Tutte1_v0 := u + \frac{t\ Z0u^2}{u^2} - \frac{t\ Z0u}{u^2} - \frac{t\ z10}{u} + \frac{\nu}{\left(\nu^2-1\right)\ u} - \frac{\nu\ Z0u}{\left(\nu^2-1\right)\ u} + \frac{Z1u}{\nu^2-1}$$

$$Tutte1_v1 := \frac{Z2u}{\nu^2-1} + \frac{t\ z10\ Z1u}{u} + \frac{t\ Z0u\ Z1u}{u^2} - \frac{t\ Z0u\ z10}{u^2}$$

$$-\frac{\nu\ Z1u}{\left(\nu^2-1\right)\ u}+\frac{\nu\ z10}{\left(\nu^2-1\right)\ u}-\frac{t\ z11}{u}-\frac{t\ z10^2}{u}$$

$$Tutte2_v0 := -\frac{\nu\ Z1u}{\nu^2-1}+\frac{Z0u}{\left(\nu^2-1\right)\ u}-\frac{1}{\left(\nu^2-1\right)\ u}+Z1u^2\ t+Z2u\ t$$

$$\{Z0u, Z1u, \nu, t, u, z10\}$$

$$\{Z0u, Z1u, Z2u, \nu, t, u, z10, z11\}$$

$$\{Z0u, Z1u, Z2u, \nu, t, u\} \qquad \textbf{(2.3)}$$

By eliminating Z2u and isolating Z1u in the above system of three equations,
   we obtain the rational expression of Z1u in terms of Z0,z10 and u, t,nu,
   as well as an algebraic equation for Z0u with coefficients depending on z10,z11 and u,t,nu.

```
> eliminate({Tutte1_v0,Tutte1_v1,Tutte2_v0},{Z1u,Z2u}):
  Z1u_rational := subs(%[1],Z1u);
  eq_Z0u_with_z11 := %%[2,1];
  indets(Z1u_rational);
  indets(eq_Z0u_with_z11);
```

$$Z1u_rational := -\frac{1}{u^2}\left(Z0u^2\ \nu^2\ t-\nu^2\ t\ u\ z10+\nu^2\ u^3-Z0u\ \nu^2\ t\right.$$

$$\left.-t\ Z0u^2-\nu\ Z0u\ u+t\ z10\ u-u^3+t\ Z0u+\nu\ u\right)$$

$$eq_Z0u_with_z11 := Z0u^4\ \nu^4\ t^3-Z0u^2\ \nu^4\ t^3\ u\ z10+2\ Z0u^2\ \nu^4\ t^2\ u^3$$
$$-\nu^4\ t^2\ u^4\ z10+\nu^4\ t\ u^6-Z0u^3\ \nu^4\ t^3-Z0u\ \nu^4\ t^2\ u^3-2\ Z0u^4\ \nu^2\ t^3$$
$$-2\ Z0u^3\ \nu^3\ t^2\ u+2\ Z0u^2\ \nu^2\ t^3\ u\ z10-4\ Z0u^2\ \nu^2\ t^2\ u^3$$
$$+Z0u\ \nu^3\ t^2\ u^2\ z10-2\ Z0u\ \nu^3\ t\ u^4+2\ \nu^2\ t^2\ u^4\ z10$$
$$+\nu^2\ t^2\ u^3\ z10^2-2\ \nu^2\ t\ u^6+2\ Z0u^3\ \nu^2\ t^3+2\ Z0u^2\ \nu^3\ t^2\ u$$
$$+2\ Z0u\ \nu^2\ t^2\ u^3+Z0u\ \nu^2\ t^2\ u^2\ z10+\nu^3\ t\ u^4+\nu^2\ t^2\ u^3\ z11$$
$$+Z0u^4\ t^3+2\ Z0u^3\ \nu\ t^2\ u+Z0u^2\ \nu^2\ t\ u^2-Z0u^2\ t^3\ u\ z10$$
$$+2\ Z0u^2\ t^2\ u^3-Z0u\ \nu\ t^2\ u^2\ z10+2\ Z0u\ \nu\ t\ u^4-t^2\ u^4\ z10$$
$$-t^2\ u^3\ z10^2+t\ u^6-Z0u^3\ t^3-2\ Z0u^2\ \nu\ t^2\ u+Z0u^2\ \nu\ t\ u^2$$
$$-Z0u\ \nu^2\ t\ u^2-Z0u\ t^2\ u^3-Z0u\ t^2\ u^2\ z10-\nu\ t\ u^4-2\ \nu\ t\ u^3\ z10$$
$$+\nu\ u^5-t^2\ u^3\ z11-Z0u\ \nu\ t\ u^2-Z0u\ u^3+u^3$$

$$\{Z0u, \nu, t, u, z10\}$$

$$\{Z0u, \nu, t, u, z10, z11\} \qquad \textbf{(2.4)}$$

Using Z1u_rational, we can replace the term Z1v in Z_rational by its rational expression in terms of Z1v,z10 and v,t,nu.

This gives a rational expression of Z in terms of Z0u,Z0v,z10 and u, v,t,nu.
(Second half of Equation (15) in the paper)

```
> Z_rational := simplify(subs(Z1v = subs({Z0u=Z0v,u=v},
  Z1u_rational), Z_rational_with_Z1v));
  indets(Z_rational);
```

$$Z_rational := \left(\left(-v^2\ \nu^2+v^2\right)\ u^3+Z0u\ u^2\ v+\left((\nu-1)^2\ (\nu+1)^2\ (Z0u+Z0v)\ \left(-Z0v^2+z10\ v+Z0v\right)\ t^2-\left(v^2\ \nu^2+(-Z0v+1)\ \nu-v^2\right)\ (Z0u+Z0v)\ (\nu-1)\ v\ (\nu+1)\ t-v^2\ \nu\ Z0v\right)\ u+t\ v^2\ Z0u\ Z0v\ (\nu-1)\ (\nu+1)\right)\Big/\left(v\ u^2+\left((\nu-1)^2\ (\nu+1)^2\ (-Z0v^2+z10\ v+Z0v)\ t^2-\left(v^2\ \nu^2+(-Z0v+1)\ \nu-v^2\right)\ (\nu-1)\ v\ (\nu+1)\ t-v^2\ \nu\right)\ u+t\ v^2\ Z0u\ (\nu-1)\ (\nu+1)\right)$$

$$\{Z0u, Z0v, \nu, t, u, v, z10\} \qquad \textbf{(2.5)}$$

The equation eq_Z0u obtained above is not yet a closed functional equation (with one catalytic variable)
   which determines Z0u, since the equation still depends on z11.
We extract the first coefficient in u from the equation eq_Z0u to express z11 in terms of z10, z30 and t,nu.

```
> extract_coeff(eq_Z0u_with_z11,u,0), extract_coeff
  (eq_Z0u_with_z11,u,1);  # The first two coefficients in u are
  trivial.
  eq_z20 := extract_coeff(eq_Z0u_with_z11,u,2);
  eq_z11 := extract_coeff(eq_Z0u_with_z11,u,3);
  indets(eq_z20);
  indets(eq_z11);  # The next two coefficients relates z11 to z10,
  z20 and z30.
```

$$0, 0$$

$$eq_z20 := \nu^4\ t^3\ z10^2+\nu^4\ t^3\ z20-2\ \nu^2\ t^3\ z10^2-\nu^3\ t^2\ z10-2\ \nu^2\ t^3\ z20+\nu^2\ t^2\ z10+t^3\ z10^2+\nu\ t^2\ z10+t^3\ z20-t^2\ z10$$

$$eq_z11 := 2\ \nu^4\ t^3\ z10^3+4\ \nu^4\ t^3\ z10\ z20+\nu^4\ t^3\ z30-4\ \nu^2\ t^3\ z10^3-3\ \nu^3\ t^2\ z10^2-8\ \nu^2\ t^3\ z10\ z20+\nu^4\ t^2-2\ \nu^3\ t^2\ z20-2\ \nu^2\ t^3\ z30+2\ \nu^2\ t^2\ z10^2+2\ t^3\ z10^3+\nu^2\ t^2\ z11+3\ \nu\ t^2\ z10^2+4\ t^3\ z10\ z20-2\ \nu^2\ t^2+\nu^2\ t\ z10+2\ \nu\ t^2\ z20+t^3\ z30-2\ t^2\ z10^2-\nu\ t\ z10-t^2\ z11+t^2$$

$$\{\nu, t, z10, z20\}$$

$$\{\nu, t, z10, z11, z20, z30\} \qquad \textbf{(2.6)}$$

```
> z20_rational := solve(eq_z20,z20);
  z11_rational := solve(subs(z20 = z20_rational, eq_z11), z11); #
```

We elimiate z20 in these two equations, and solve for z11.

$$z20_rational := -\frac{(\nu\ t\ z10 + t\ z10 - 1)\ z10}{t\ (\nu+1)}$$

$$z11_rational := \frac{1}{t\ (\nu+1)}\left(2\ \nu^3\ t^2\ z10^3 + 2\ \nu^2\ t^2\ z10^3 - \nu^3\ t^2\ z30 - 2\ t^2\ z10^3\ \nu - \nu^2\ t^2\ z30 - 3\ \nu^2\ t\ z10^2 - 2\ t^2\ z10^3 - \nu^3\ t + t^2\ z30\ \nu - \nu\ t\ z10^2 - \nu^2\ t + t^2\ z30 + 2\ t\ z10^2 + t\ \nu + \nu\ z10 + t\right) \quad (2.7)$$

By substituting z10 with the above expression,
  we obtain a algebraic equation with one catalytic variable for Z0u.
(First half of Equation (15) in the paper)

```
> eq_Z0u := simplify(subs(z11=z11_rational, eq_Z0u_with_z11));
  indets(eq_Z0u);
```

$$eq_Z0u := 2\ (\nu+1)^2\left(\left(z10^3 - \frac{z30}{2}\right)\ u^3 - \frac{u\ Z0u^2\ z10}{2} + \frac{Z0u^3\ (Z0u-1)}{2}\right)\ (\nu-1)^2\ t^3 - (\nu+1)\ u\ (\nu-1)\ \left(\left(\nu^2\ z10 - z10\right)\ u^3 + \left(\left(-2\ Z0u^2 + Z0u + 1\right)\ \nu^2 + 3\ z10^2\ \nu + 2\ Z0u^2 - 3\ z10^2 - Z0u - 1\right)\ u^2 - Z0u\ z10\ (\nu+1)\ u + 2\ \nu\ Z0u^2\ (Z0u-1)\right)\ t^2 + u^2\ \left(\left(\nu^4 - 2\ \nu^2 + 1\right)\ u^4 + \left(\left(-2\ Z0u + 1\right)\ \nu^3 + (2\ Z0u - 1)\ \nu\right)\ u^2 + \nu\ z10\ (\nu-3)\ u + \nu\ Z0u\ (Z0u-1)\ (\nu+1)\right)\ t + u^3\ \left(\nu\ u^2 - Z0u + 1\right)$$

$$\{Z0u, \nu, t, u, z10, z30\} \quad (2.8)$$

2. Rational parametrization of Z(u,v;t,nu) and its specialization to the critical line.

Objects defined in this section:

T_S,U_S, Z_S,Z0_S, z10_S,z30_S, ... :
  RPs (rational parametrizations) of the generating functions before restriction to the critical line.
  They depend on the parameters H,K,S,nu, and correspond to the rational functions named with a \hat in the paper.

nu_HT, T_HT   ("_HT" meaning "high temperature")
nu_LT, T_LT   ("_LT" meaning "low temperature") :

RP of the critical line t=t_c(nu),

nu_c and R1,Rc,Rinf:
The critical value of nu and the values of the parameter R that correspond to nu=1, nu=nu_c, and nu=infinity, respectively.

U_HT, Z_HT, Hc_HT, x_HT, ...
U_LT, Z_LT, Hc_LT, x_LT, ...
U_cT, Z_cT, Hc_cT, x_cT .... ("_cT" meaning "critical temperautre")
:
Parametrizations of the generating functions (or critical points) after restriction to the critical line.
They depend on the parameters H,K,R, and correspond to the functions named with a \check in the paper.
We have the relations ?_cT = subs(R=Rc, ?_HT) = subs(R=Rc, ?_LT), where ?=U,Z,Hc,x, ...

2.1 Rational parametrization of Z0(u;t,nu).

%%% We use previously known RPs of the functions z10(t) and z30(t) to replace the
%%% unknowns z10,z30 and t in the equation eq_Z0u by a single variable S.
%%% This will leave us an algebraic equation (with no catalytic variable) eq_Z0_S satisfied by Z0u,U,S,nu, where U=t*u.
%%%

The previously known RPs of z10(t) and z30(t) have the form:

$$t^2 = T_S$$
$$t^3 z10 = z10_S$$
$$t^9 z30 = z30_S,$$

where

```
> T_S := (nu-S)*(S+nu-2)*(4*S^3-S^2-2*S+nu^2-2*nu) / (32*(nu^2
  -1)^3*S^2);
  z10_S := (nu-S)^2*(S+nu-2)*(3*S^3-nu*S^2-nu*S+nu^2-2*nu) /
  (64*(nu^2-1)^4*S^2);
  z30_S := (nu-S)^5*(S+nu-2)^5*(160*S^10 - 128*S^9 - 16*(2*nu^2
  -4*nu+3)*S^8 + 32*(2*nu^2-4* nu+3)*S^7 - 7*(16*nu^2-32*nu+27)*
  S^6 - 2*(32*nu^2-64*nu+57)*S^5 + (32*nu^4-128*nu^3+183*nu^2
  -110*nu+20)*S^4 - 4*(7*nu^2-14*nu-2)*S^3 + nu*(nu-2)*(9*nu^2
  -18*nu-20)*S^2 + 14* nu^2*(nu-2)^2*S - 3*nu^3*(nu-2)^3) /
  (2^22*(nu^2-1)^12*S^8);
```

$$T_S := \frac{(\nu - S)\ (S + \nu - 2)\ \left(4\ S^3 - S^2 + \nu^2 - 2\ S - 2\ \nu\right)}{32\ \left(\nu^2 - 1\right)^3\ S^2}$$

$$z10_S := \frac{(\nu - S)^2\ (S + \nu - 2)\ \left(3\ S^3 - \nu\ S^2 - \nu\ S + \nu^2 - 2\ \nu\right)}{64\ \left(\nu^2 - 1\right)^4\ S^2}$$

(3.1.1)

$$z30_S := \frac{1}{4194304\,\left(\nu^2-1\right)^{12}\,S^8}\left(\left(\nu-S\right)^5\,\left(S+\nu-2\right)^5\,\left(160\,S^{10}-128\,S^9-16\,\left(2\,\nu^2-4\,\nu+3\right)\,S^8+32\,\left(2\,\nu^2-4\,\nu+3\right)\,S^7-7\,\left(16\,\nu^2-32\,\nu+27\right)\,S^6-2\,\left(32\,\nu^2-64\,\nu+57\right)\,S^5+\left(32\,\nu^4-128\,\nu^3+183\,\nu^2-110\,\nu+20\right)\,S^4-4\,\left(7\,\nu^2-14\,\nu-2\right)\,S^3+\nu\,\left(\nu-2\right)\,\left(9\,\nu^2-18\,\nu-20\right)\,S^2+14\,\nu^2\,\left(\nu-2\right)^2\,S-3\,\nu^3\,\left(\nu-2\right)^3\right)\right) \quad \textbf{(3.1.1)}$$

We replace z10,z30 and t by their parametrizations (as well as replacing u by U=t*u)
in the equation eq_Z0u with one catalytic variable for Z0u.

```
> subs({u=U/t, z10=z10_S/t^3, z30=z30_S/t^9}, eq_Z0u):
  eq_Z0_S_extra_factors := subs(t=sqrt(T_S), numer(factor(%))):
    # Displaying the explicit expression is not very helpful.
```

The resulting equation factorizes into four factors.
Only one of them depends on the variables U and Z0u.
We retain this factor as the equation eq_Z0_S:

```
> factors( numer(factor(eq_Z0_S_extra_factors)) )[2,..,1]:  #
  This gives the list of factors of  eq_Z0_S_extra_factors.
  select( f->evalb({U,Z0u} subset indets(f)), %):           #
  This selects the factors that depend non-trivially on U or
  Z0u.
  nops(%);                                                  #
  There is only one such factor.
  eq_Z0_S := %%[1]:    # Again, the explicit expression is not
  very helpful.
```

$$1 \quad \textbf{(3.1.2)}$$

When viewed as an equation with respect to the variables U and Z0u,
eq_Z0_S has degree 6, and defines a plane curve of genus zero :

```
> indets(eq_Z0_S);
  degree(eq_Z0_S,{Z0u,U});
  algcurves[genus](eq_Z0_S, U, Z0u);
```

$$\{S, U, Z0u, \nu\}$$

$$6$$

$$0 \quad \textbf{(3.1.3)}$$

%%%
%%% The fact that eq_Z0_u defines a plane curve of genus zero implies that
%%% Z0u should have a RP of the form Z0u = Z0_S(H;S,nu) and U = U_S(H;S,nu).
%%% Let us compute this RP.
%%%

In theory, the RP can be computed algorithmically using the command "algcurves[parametrization]".
In practice, the direct computation takes too much time:

```
> #algcurves[parametrization](eq_Z0_S,U,Z0u,H);  # This takes
  too much time.
```

Instead, we use the interpolation method outlined in Remark 9 of the paper to obtain a RP of Z0u.

Computations in this part is time-consumming (mainly due to the first block, which takes about 15 min on the authors' laptops).

%%% Reader can skip the execution this part, and pick up the computation with
%%% the recorded result of U_S and Z0_S, which is RP of Z0(u;t, nu) given in Lemma 8 of the paper.
%%%

We start by computing RPs of Z0u when nu takes the fixed values 2,3,...,10. (According to experiment, 10 is the smallest number which gives the correct result at the end.)

```
> N := 10:
  t_0 := time():
  for n in $2..N do
     t_1 := time();
     RP[n] := algcurves[parametrization]( subs(nu=n,eq_Z0_S) ,
  U,Z0u,H);
     RP[n] := map(factor, RP[n]);
     printf("RP for Z0u at nu=%a computed in %g s\n",n, time()
  -t_1)
  end do:
  printf("total: %g s\n", time()-t_0):
RP for Z0u at nu=2 computed in 48.276 s
RP for Z0u at nu=3 computed in 70.005 s
RP for Z0u at nu=4 computed in 108.986 s
RP for Z0u at nu=5 computed in 95.601 s
RP for Z0u at nu=6 computed in 79.748 s
RP for Z0u at nu=7 computed in 99.436 s
RP for Z0u at nu=8 computed in 94.118 s
RP for Z0u at nu=9 computed in 88.634 s
RP for Z0u at nu=10 computed in 99.595 s
total: 784.401 s
```

For each integer n=2,3,...,20, the variable RP[n] contains a pair of rational functions which is a RP of Z0(u) when nu=n. We check that all these RPs have the following common properties:

1. the value H=infinity of the parameter is mapped to (U,Z0u)=(0,1) by the pair of rational functions.
2. The two rational functions have only one (common) pole (without counting multiplicity).

```
> degree( mul(sqrfree( denom(RP[2][1])*denom(RP[2][2]) )[2,..,
```

```
  1] ), H);
  # The above line computes the total number of poles of the
  rational function RP[2][1] and RP[2][2].
  # The command "sqrfree" is used to eliminate any
  multiplicity in the zeros of the denominators.

  "At H=infinity,   U= ",   seq( limit(RP[n][1], H=infinity) ,
  n=2..N);
  "At H=infinity, Z0u= ",   seq( limit(RP[n][2], H=infinity) ,
  n=2..N);
  "   Number of poles= ",   seq( degree( mul(sqrfree( denom(RP
  [n][1])*denom(RP[n][2]) )[2,..,1] ), H) , n=2..N);
```

$$1$$

"At H=infinity, U= ", 0, 0, 0, 0, 0, 0, 0, 0, 0

"At H=infinity, Z0u= ", 1, 1, 1, 1, 1, 1, 1, 1, 1

"   Number of poles= ", 1, 1, 1, 1, 1, 1, 1, 1, 1 **(3.1.1.1)**

We notice that the point (U,Z0u)=(0,1) is not a multiple point on the algebraic curve defined by eq_Z0_S.

```
> simplify( subs({U=0,Z0u=1}, eq_Z0_S) );
  simplify( subs({U=0,Z0u=1}, diff(eq_Z0_S,Z0u)) );
```

$$0$$

$$\frac{(S+\nu-2)^4\ (-\nu+S)^4\ (4\ S^3-S^2+\nu^2-2\ S-2\ \nu)^4}{256} \qquad \textbf{(3.1.1.2)}$$

Thus H=infinity must be the value parametrizing the origin in the branch of the generating function Z0u(U) that we are interested in.
So it makes sense to map it to H=0 at the next change of variable.

On the other hand, let us map the unique pole of each RP[n] to infinity, so that the two rational functions become polynomials after the change.
This suggests the Möbius change of variable H -> H_pole[n]-1/H, where H_pole[n] is the unique pole of RP[n].

```
> for n in $2..N do
     H_pole[n] := {solve(denom(RP[n][1])*denom(RP[n][2]), H)}
  [1];
     RP_poly[n]:= subs(H = H_pole[n]-1/H , RP[n])
  end do:
```

After the change of variable, each rational parametrization RP_poly[n] contains a pair of polynomials.
We check that they have the same degree (4 and 6 respectively) for all the different values of nu=n:

```
> "Degree of   U(H) = ", seq( degree(RP_poly[n][1], H) , n=2..
  N);
```

```
  "Degree of Z0u(H) = ", seq( degree(RP_poly[n][2], H) , n=2..
  N)
```

```
      "Degree of   U(H) = ", 4, 4, 4, 4, 4, 4, 4, 4, 4
      "Degree of Z0u(H) = ", 6, 6, 6, 6, 6, 6, 6, 6, 6
```

(3.1.1.3)

Finally, we rescale the parameter H by a value H_scale[n] selected by hand
   which makes the coefficients of the above two polynomials look as simple as possible.

```
> for n in $2..N do
     H_scale[n] := -8*S*coeff(RP_poly[n][1], H,4) / coeff
  (RP_poly[n][1], H,3);
     RP_normal[n] := subs(H = H/H_scale[n] , RP_poly[n]);
     RP_normal[n] := map(collect, RP_normal[n], H, factor);

     print( RP_normal[n][1] )
  end do:
```

$$-\frac{H^4}{2304}+\frac{H^3\ S}{288}-\frac{S\ (S+1)\ H^2}{144}+\left(\frac{1}{36}\ S^2-\frac{1}{144}\ S-\frac{1}{72}\right)\ H$$

$$-\frac{H^4}{16384}+\frac{H^3\ S}{2048}-\frac{S\ (S+1)\ H^2}{1024}+\frac{(S+1)\ \left(4\ S^2-5\ S+3\right)\ H}{1024\ S}$$

$$-\frac{H^4}{57600}+\frac{H^3\ S}{7200}-\frac{S\ (S+1)\ H^2}{3600}+\frac{\left(4\ S^3-S^2-2\ S+8\right)\ H}{3600\ S}$$

$$-\frac{H^4}{147456}+\frac{H^3\ S}{18432}-\frac{S\ (S+1)\ H^2}{9216}+\frac{\left(4\ S^3-S^2-2\ S+15\right)\ H}{9216\ S}$$

$$-\frac{H^4}{313600}+\frac{H^3\ S}{39200}-\frac{S\ (S+1)\ H^2}{19600}+\frac{\left(4\ S^3-S^2-2\ S+24\right)\ H}{19600\ S}$$

$$-\frac{H^4}{589824}+\frac{H^3\ S}{73728}-\frac{S\ (S+1)\ H^2}{36864}+\frac{\left(4\ S^3-S^2-2\ S+35\right)\ H}{36864\ S}$$

$$-\frac{H^4}{1016064}+\frac{H^3\ S}{127008}-\frac{S\ (S+1)\ H^2}{63504}+\frac{\left(4\ S^3-S^2-2\ S+48\right)\ H}{63504\ S}$$

$$-\frac{H^4}{1638400}+\frac{H^3\ S}{204800}-\frac{S\ (S+1)\ H^2}{102400}+\frac{\left(4\ S^3-S^2-2\ S+63\right)\ H}{102400\ S}$$

$$-\frac{H^4}{2509056}+\frac{H^3\ S}{313632}-\frac{S\ (S+1)\ H^2}{156816}+\frac{\left(4\ S^3-S^2-2\ S+80\right)\ H}{156816\ S}$$

(3.1.1.4)

Now the coefficients of the polynomials (in H) RP_normal[n][1] and RP_normal[n][2] look simple enough
  so that it is not hard to guess rational functions nu that interpolate between the special values n=2,3,...,20.
This can also be done more systematically using the command "CurveFitting[RationalInterpolation]".

```
> U_S  := CurveFitting[RationalInterpolation]( [seq( [n,
```

```
RP_normal[n][1]], n=2..N)] ,nu):
Z0_S := CurveFitting[RationalInterpolation]( [seq( [n,
RP_normal[n][2]], n=2..N)] ,nu):

H_renorm := subs({H=0,nu=2,S=2}, factor(U_S/H) ):   # For
the RP given in the paper,  H_renorm = 1/6. We renormlaize
the parameter H to make it so.
U_S  := factor( subs(H=H/(6*H_renorm), U_S) );      #
Without this step of renormalization to prevent different
versions of Maple may produce
Z0_S := factor( subs(H=H/(6*H_renorm), Z0_S) );     #
parametrizations whose parameter differ by a constant from
each other.
```

The result of the above computations is registered below (which is Eq (17)-(18) of Lemma 8 in the paper).
One can pick up the computation from here and check directly the correctness of this RP of Z0(u;t,nu),
   without entering into the details of how this RP was obtained.

```
> U_S := H*( 2*(4*S^3-S^2-2*S +nu^2-2*nu) -4*(S+1)*S^2*H +4*S^2*
  H^2 -S*H^3 ) / ( 16*(1-nu^2)^2*S );
  Z0_S:= U_S/T_S* ( (S-nu)*(S+nu-2) +2*(S-nu)*S*H -2*S^2*H^2 +S*
  H^3 ) / ( 4*(1-nu^2)*S*H );
```

$$U_S := \frac{1}{16\ \left(-\nu^2+1\right)^2\ S}\left(H\ \left(8\ S^3-2\ S^2+2\ \nu^2-4\ S-4\ \nu-4\ (S+1)\ S^2\ H+4\ S^2\ H^2-S\ H^3\right)\right)$$

$$Z0_S := \left(\left(8\ S^3-2\ S^2+2\ \nu^2-4\ S-4\ \nu-4\ (S+1)\ S^2\ H+4\ S^2\ H^2-S\ H^3\right)\ \left(\nu^2-1\right)^3\ \left((-\nu+S)\ (S+\nu-2)+2\ (-\nu+S)\ S\ H-2\ S^2\ H^2+S\ H^3\right)\right)\Big/\left(2\ \left(-\nu^2+1\right)^3\ (\nu-S)\ (S+\nu-2)\ \left(4\ S^3-S^2+\nu^2-2\ S-2\ \nu\right)\right) \quad \textbf{(3.1.4)}$$

By construction, the above RP parametrizes the point (u,Z0u)=(0,1) by H=0, and U_S=infinity when H=infinity.
We check that it satisfies the original equation eq_Z0_S.

```
> simplify( subs({U=U_S,Z0u=Z0_S}, eq_Z0_S) );
```

$$0 \quad \textbf{(3.1.5)}$$

2.2 Rational parametrization of Z(u,v;t,nu).

%%% The RP of Z(u,v;t,nu) is of the form t*u=U_S, t*v=subs(H=K, U_S) and Z=Z_S,

%%% where U_S is given in the RP of Z0(u;t,nu),
%%% and Z_S can be obtained by plugging the known RPs of Z0u,Z0v and z10 into Z_rational.
%%%

```
> subs({u=U_S/t, v=subs(H=K,U_S)/t, z10=z10_S/t^3, Z0u=Z0_S,
  Z0v=subs(H=K,Z0_S)}, Z_rational):
  Z_S := factor(subs(t=sqrt(T_S), collect(%,t) )):      # The
  substitution is done in two steps to help Maple dealing with
  the square root.
```

We save the useful RPs in text file, which can be viewed and used without Maple.

```
> save z10_S, T_S,U_S,Z_S, "Ising-IPT_2--RP_any_t";
```

2.3 Specialization of Z(u,v;nu,t) to the critical line.

Rational parametrization of the critical line t=t_c(nu).

%%% We compute a piecewise rational parametrization of the function t_c(nu).
%%% It will have the form
%%% nu=nu_HT, t_c^2=T_HT when 1<nu<=nu_c,
%%% and nu=nu_LT, t_c^2=T_LT when nu>=nu_c,
%%% where nu_HT,nu_LT,T_HT,T_LT are rational functions of the parameter R.
%%%

We start by computing the value S_c(nu) which parametrizes t_c (nu) in the RP t^2=T_S and t^3*z10(t) = z10_S.
For this we apply the method of [Appendix B, ??].

The first step is to identify the value of S0(nu) that parametrizes the origin, where t^2=0 and t*z10(t)=0.

```
> solve({T_S = 0, z10_S/T_S = 0}, S);
  S0 := subs(%, S);
```

$$\{S = \nu\}$$

$$S0 := \nu \qquad \textbf{(3.3.1.1)}$$

Next, we check the sign of the derivative of T_S (with respect to S) at S0. It is negative.

```
> dT_S := factor(diff(T_S, S));
  signum(subs(S=S0,dT_S)) assuming nu>1;
```

$$dT_S := -\frac{\left(3\ S^2-\nu^2+2\ \nu\right)\ \left(2\ S^3-3\ S^2-\nu^2+2\ \nu\right)}{16\ (\nu-1)^3\ (\nu+1)^3\ S^3}$$

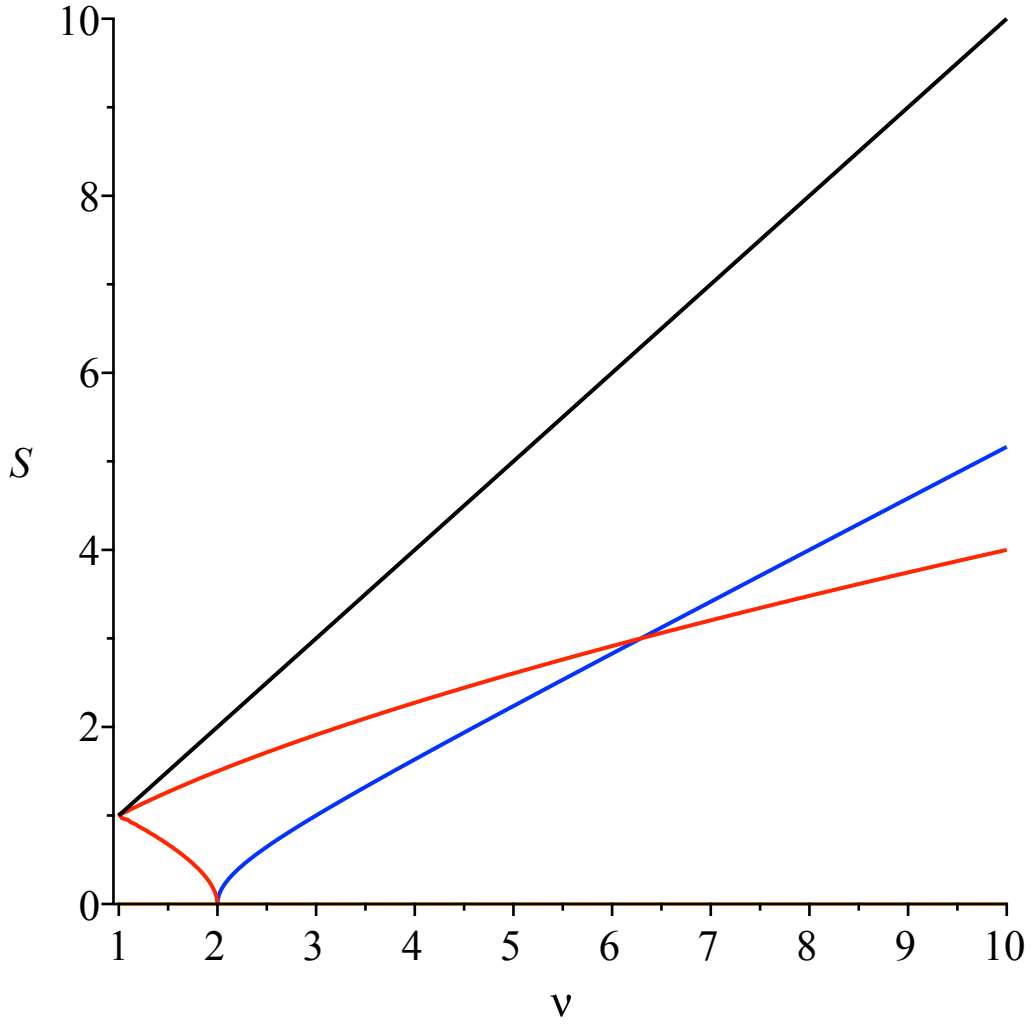
$$-1 \qquad \textbf{(3.3.1.2)}$$

Then, the method of [Appendix B, ??] prescribes that S_c(nu) is

the largest (real) value of S below S0
   which is either a zero of the derivative dT_S, or a pole of z10_S.
We compute the equations satisfied by these zeros and poles, and plot them to find the largest one below S0.

```
> factors( numer(dT_S)*denom(z10_S) )[2,..,1]:
      # This gives the list of factors appearing in the numerator of dT_S or the denominator of z10_S.
  eq_Sc := sort(select( f->evalb(S in indets(f)), %), key=degree);  # This selects the factors that depend non-trivially on S, and sorts the result by degree.
  plots[display](
     plots[implicitplot](eq_Sc[1], nu=1..10, S=0..10, numpoints = 10000, color = "Orange"),
     plots[implicitplot](eq_Sc[2], nu=1..10, S=0..10, numpoints = 10000, color = "Blue"),
     plots[implicitplot](eq_Sc[3], nu=1..10, S=0..10, numpoints = 10000, color = "Red"),
     plot(S0, nu=1..10, color = "Black")
  );
```

$$eq_Sc := \left[S,\ S^2-\frac{1}{3}\ \nu^2+\frac{2}{3}\ \nu,\ S^3-\frac{3}{2}\ S^2-\frac{1}{2}\ \nu^2+\nu\right]$$



From the plot, we see that S_c(nu) satisfies two different algebraic equations
for small values of nu (high temperature) and for large values of nu (low temperature).
The crossover happens as the point (nu_c,S_c), where the two equations has a common solution.

```
> eq_Sc_HT := eq_Sc[3];
  eq_Sc_LT := eq_Sc[2];
  solve({eq_Sc_HT=0, eq_Sc_LT=0, nu>1,S>0});
  nu_cT,S_cT := op( subs(%,[nu,S]) );
```

$$eq_Sc_HT := S^3 - \frac{3}{2}\ S^2 - \frac{1}{2}\ \nu^2 + \nu$$

$$eq_Sc_LT := S^2 - \frac{1}{3}\ \nu^2 + \frac{2}{3}\ \nu$$

$$\{S = 3,\ \nu = 1 + 2\ \sqrt{7}\}$$

$$nu_cT,\ S_cT := 1 + 2\ \sqrt{7},\ 3 \qquad \textbf{(3.3.1.3)}$$

Now let us show that S_c(nu) has a RP at both high temperatures and low temperatures.

At low temperatures, S_c(nu) lives on a conic (plane curve defined by a quadratic equation).
A conic always has a RP, which can be obtained easily using as parameter
   the slope R of a secant line passing through any fixed point (e.g. the origin).

```
> degree(eq_Sc_LT);
  solve({eq_Sc_LT = 0, R = nu/S}, {S, nu});
  nu_lt,S_lt := op( subs(%,[nu,S]) );
```

$$2$$

$$\left\{S = \frac{2\ R}{R^2 - 3},\ \nu = \frac{2\ R^2}{R^2 - 3}\right\}$$

$$nu_lt, S_lt := \frac{2\ R^2}{R^2 - 3}, \frac{2\ R}{R^2 - 3} \qquad \textbf{(3.3.1.4)}$$

At high temperatures, S_c(nu) lives on a cubic curve. Moreover, the curve has a double point at (S,nu)=(1,1).
A such curve also always has a RP, which can be obtained using as parameter
   the slope R of a secant line passing through the double point (1,1).

```
> degree(eq_Sc_HT);
  solve({eq_Sc_HT=0, diff(eq_Sc_HT, S)=0, diff(eq_Sc_HT, nu)=
  0}, {S, nu});
  solve({eq_Sc_HT=0, R = (nu-1)/(S-1)}, {S, nu});
  nu_HT,S_HT := op(factor( subs(%,[nu,S]) ));
```

$$3$$

$$\{S = 1,\ \nu = 1\}$$

$$\left\{S = \frac{R^2}{2} - \frac{1}{2},\ \nu = \frac{1}{2}\ R^3 - \frac{3}{2}\ R + 1\right\}$$

$$nu_HT, S_HT := \frac{(R+2)\ (R-1)^2}{2}, \frac{(R-1)\ (R+1)}{2} \qquad \textbf{(3.3.1.5)}$$

We compute the two values of R corresponding to nu=nu_c and nu=infinity
  in the low temperature parametrization.

```
> solve({nu_lt = nu_cT, S_lt = S_cT}, explicit = true);
  solve({1/nu_lt = 0,  1/S_lt = 0}, explicit = true);
  R_c_lt, r_inf := subs(%%,R), subs(%[1],R);
```

$$\left\{R = \frac{1}{3} + \frac{2\ \sqrt{7}}{3}\right\}$$

$$\{R = \sqrt{3}\}, \{R = -\sqrt{3}\}$$

$$R_c_lt, r_inf := \frac{1}{3} + \frac{2\sqrt{7}}{3}, \sqrt{3} \qquad \textbf{(3.3.1.6)}$$

And the two values of R corresponding to nu=1 and nu=nu_c
  in the high temperature parametrization.

```
> solve( {nu_HT = 1, S_HT=1}, explicit=true );


  # Strangly, the command "solve" is unable to find the
  solution of the system :
  #    solve({nu_HT = nu_cT, S_HT = S_cT});
  # The core of the problem can be seen in the following
  example:
  #    solve({x^2=2, x^3=2*sqrt(2) },[x]);
  # For this reason we use the following alternative to solve
  {nu_HT=nu_cT, S_HT=S_cT}.
  {solve(nu_HT = nu_cT)} intersect {solve(S_HT = S_cT)};

  R_1, R_c_HT := subs(%%[1],R), %[1];
```

$$\{R = \sqrt{3}\}, \{R = -\sqrt{3}\}$$

$$\{\sqrt{7}\}$$

$$R_1, R_c_HT := \sqrt{3}, \sqrt{7} \qquad \textbf{(3.3.1.7)}$$

We apply a Möbius transform to the parameter R of low
temperature parametrization,
   so that the parametrizations at low temperature and at high
temperatures
   are supported by two consecutive intervals of R.

```
> R_transform := 9/(2*R-1);
  nu_LT, S_LT := op(factor( subs(R=R_transform, [nu_lt,S_lt])
  ));
  R_c_LT, R_inf := factor(solve(R_transform = R_c_lt)), factor
  (solve(R_transform = r_inf));
  is(R_c_LT=R_c_HT);

  R1,Rc,Rinf := R_1, R_c_LT, R_inf;
```

$$R_transform := \frac{9}{2\,R - 1}$$

$$nu_LT, S_LT := -\frac{27}{2\,R^2 - 2\,R - 13}, -\frac{3\,(2\,R - 1)}{2\,R^2 - 2\,R - 13}$$

$$R_c_LT, R_inf := \sqrt{7}, \frac{1}{2} + \frac{3\sqrt{3}}{2}$$

$$true$$

$$R1, Rc, Rinf := \sqrt{3}, \sqrt{7}, \frac{1}{2} + \frac{3\sqrt{3}}{2} \qquad \textbf{(3.3.1.8)}$$

Finally, we obtain a piecewise parametrization of t_c(nu) by plugging nu=nu_?T and S=S_?T into T_S (where ? = H,L)

```
> T_HT := factor(subs({nu=nu_HT, S=S_HT}, T_S));
  T_LT := factor(subs({nu=nu_LT, S=S_LT}, T_S));
```

$$T_HT := \frac{3\,R^2 - 1}{2\,\left(R^3 - 3\,R + 4\right)^3\,R^3}$$

$$T_LT := \frac{\left(2\,R^2 - 2\,R - 13\right)^3\,\left(R+1\right)^2\,\left(2\,R^2 + 10\,R - 19\right)}{128\,\left(R+4\right)^3\,\left(R^2 - R + 7\right)^3\,\left(-5 + R\right)} \qquad \textbf{(3.3.1.9)}$$

We check that t_c(nu) is continuous at nu_cT, and record its value there.

```
> simplify(subs(R=Rc, T_HT-T_LT ));
  T_cT := simplify( subs(R=Rc,T_HT) );
```

$$0$$

$$T_cT := \frac{5\sqrt{7}}{1568\,\left(\sqrt{7}+1\right)^3} \qquad \textbf{(3.3.1.10)}$$

Rational parametrization of Z(u,v;t,nu) on the critical line.

```
%%% On the critical line, Z(u,v;nu) has a piecewise RP of the
form
%%%            nu = nu_?T
%%%    t_c(nu)*u = U_?T
%%%    t_c(nu)*v = subs(H=K, U_?T)
%%%    Z(u,v;nu) = Z_?T
%%% where ?=H when 1<nu<=nu_c and ?=L when nu>=nu_c
%%%
```

The RP is obtained by plugging nu=nu_?T and S=S_?T into U_S and Z_S.

```
> U_HT := factor(subs({nu=nu_HT,S=S_HT}, U_S));
  U_LT := factor(subs({nu=nu_LT,S=S_LT}, U_S));
  Z_HT := factor(subs({nu=nu_HT,S=S_HT}, Z_S)):    # The
  explicit expression is too long to be helpful.
  Z_LT := factor(subs({nu=nu_LT,S=S_LT}, Z_S)):
```

$$U_HT := -\frac{H\,\left(H\,R^4 - 2\,H^2\,R^2 - 3\,R^4 + H^3 + 2\,H^2 + 10\,R^2 - H - 3\right)}{\left(R^2-3\right)^2\,\left(R^3 - 3\,R + 4\right)^2\,R^2}$$

$$U_LT := -\left(H\,\left(4\,H^3\,R^4 - 8\,H^3\,R^3 - 48\,H^3\,R^2 + 48\,H^2\,R^3 + 52\,H^3\,R - 72\,H^2\,R^2 - 48\,H\,R^3 + 16\,R^4 + 169\,H^3\right.\right. \qquad \textbf{(3.3.2.1)}$$

$$-288\ H^2\ R+216\ H\ R^2+16\ R^3+156\ H^2+144\ H\ R-552\ R^2-120\ H+208\ R+760)\ (2\ R^2-2\ R-13)^2)\Big/(256\ (R+4)^2\ (-5+R)^2\ (R^2-R+7)^2)$$

We check that the high and low temperature RPs coincide when R= Rc, and record their common value.

```
> simplify( subs(R=Rc, U_HT-U_LT) );
  simplify( subs(R=Rc, Z_HT-Z_LT) );
  U_cT := simplify( subs(R=Rc, U_HT) );
  Z_cT := simplify( subs(R=Rc, Z_HT) );
```

$$0$$

$$0$$

$$U_cT := -\frac{H\ (H^3-12\ H^2+48\ H-80)}{1792\ (\sqrt{7}+1)^2}$$

$$Z_cT := \Big(-8\ (H^4+K^4-12\ H^3-12\ K^3+48\ H^2+48\ K^2-80\ H-80\ K)\ \Big(H^3+\Big(-\frac{1}{2}\ K^2+3\ K-12\Big)\ H^2+(3\ K^2-18\ K+48)\ H+K^3-12\ K^2+48\ K-80\Big)\ \sqrt{7}+2\ H^9+(-K^2+6\ K-60)\ H^8+(24\ K^2-144\ K+776)\ H^7+(2\ K^3-256\ K^2+1488\ K-5712)\ H^6+(-36\ K^3+1600\ K^2-8736\ K+26496)\ H^5+(-4\ K^4+296\ K^3-6480\ K^2+32096\ K-80384)\ H^4+(2\ K^6-36\ K^5+296\ K^4-2256\ K^3+19136\ K^2-77056\ K+159744)\ H^3+(-K^8+24\ K^7-256\ K^6+1600\ K^5-6480\ K^4+19136\ K^3-53504\ K^2+129792\ K-205824)\ H^2+(6\ K^8-144\ K^7+1488\ K^6-8736\ K^5+32096\ K^4-77056\ K^3+129792\ K^2-169984\ K+174080)\ H+2\ (K^3-6\ K^2+4\ K-8)\ (K^3-12\ K^2+48\ K-80)^2\Big)\Big/(1280\ H^3+(-640\ K^2+3840\ K-15360)\ H^2+(3840\ K^2-23040\ K+61440)\ H+1280\ K^3-15360\ K^2+61440\ K-102400)$$ **(3.3.2.2)**

2.4 Domain of convergence of Z(u,v;nu) and its parametrization.

Definition and parametrization of u_c(nu).

%%% By definition, Hc(R) is the smallest positive zero of the derivative d(U_R)/dH,
%%% u_c(nu) is parametrized by replacing H with Hc(R) in t_c (nu)*u=U_R(H).
%%% We compute Hc(R) at high temperatures and at low temperatures directly from its definition.
%%% The result is of the form Hc(R)=Hc_HT for R1<R<=Rc and Hc (R)=Hc_LT for Rc<=R<Rinf .
%%%

```
> dU_HT := factor(diff(U_HT,H));
  dU_LT := factor(diff(U_LT,H));
```

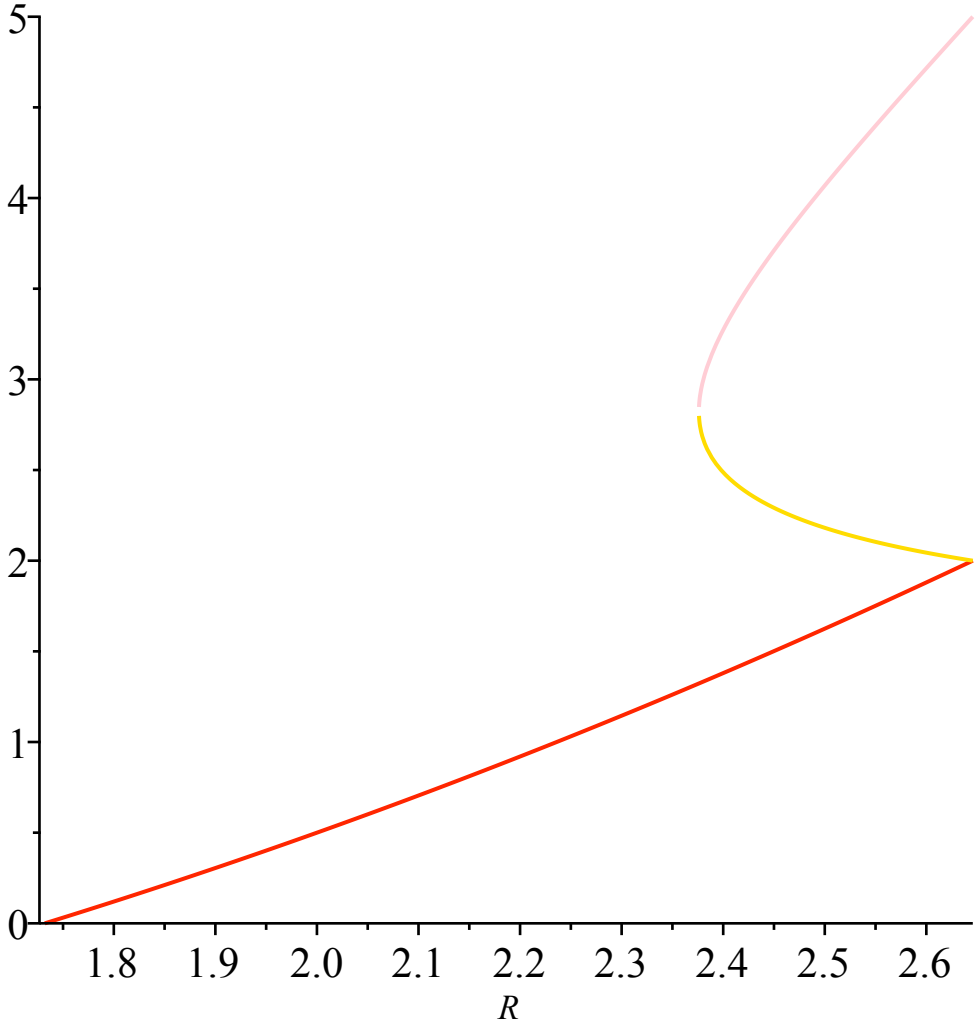
$$dU_HT := -\frac{(-2\ H\ R^2+2\ H^2+3\ R^2-1)\ (-R^2+2\ H+3)}{(R^2-3)^2\ (R^3-3\ R+4)^2\ R^2}$$

$$dU_LT := -\Big((2\ R^2-2\ R-13)^2\ (2\ H\ R^2-2\ H\ R+2\ R^2-13\ H+10\ R-19)\ (2\ H^2\ R^2-2\ H^2\ R-2\ H\ R^2-13\ H^2+8\ H\ R+2\ R^2+10\ H-8\ R-10)\Big)\Big/\Big(64\ (R+4)^2\ (-5+R)^2\ (R^2-R+7)^2\Big) \quad \textbf{(3.4.1.1)}$$

We select the smallest positive solution of dU_HT=0 to be Hc_HT.

```
> Hc_HT_list := [solve(dU_HT=0, H)];
  plot(Hc_HT_list, R=R1..Rc, color = ["Red", "Pink", "Gold"]);
  Hc_HT := Hc_HT_list[1];
```

$$Hc_HT_list := \left[\frac{R^2}{2}-\frac{3}{2},\ \frac{R^2}{2}+\frac{\sqrt{R^4-6\ R^2+2}}{2},\ \frac{R^2}{2}-\frac{\sqrt{R^4-6\ R^2+2}}{2}\right]$$

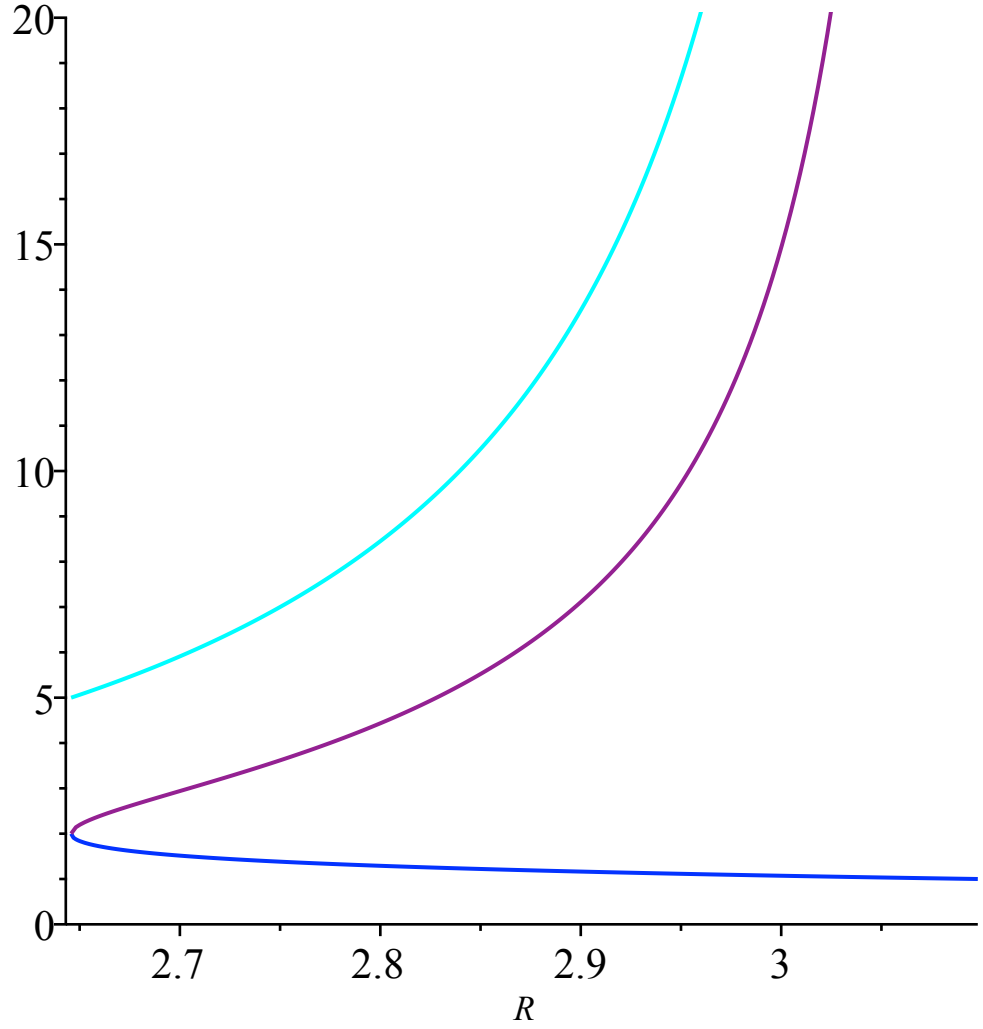


$$Hc_HT := \frac{R^2}{2} - \frac{3}{2} \qquad \textbf{(3.4.1.2)}$$

Similarly, Hc_LT is the smallest positive solution of dU_LT=0.

```
> Hc_LT_list := [solve(dU_LT, H)];
  plot(Hc_LT_list, R=Rc..Rinf, color = ["Cyan", "Blue",
  "Purple"],view=[Rc..Rinf, 0..20]);
  Hc_LT := Hc_LT_list[2];
```

$$Hc_LT_list := \left[-\frac{2\ R^2 + 10\ R - 19}{2\ R^2 - 2\ R - 13},\ \frac{R^2 + \sqrt{-3\ R^4 + 12\ R^3 + 36\ R^2 - 84\ R - 105} - 4\ R - 5}{2\ R^2 - 2\ R - 13},\ -\frac{-R^2 + \sqrt{-3\ R^4 + 12\ R^3 + 36\ R^2 - 84\ R - 105} + 4\ R + 5}{2\ R^2 - 2\ R - 13} \right]$$



$$Hc_LT := \frac{R^2 + \sqrt{-3\ R^4 + 12\ R^3 + 36\ R^2 - 84\ R - 105}\ - 4\ R - 5}{2\ R^2 - 2\ R - 13} \quad \textbf{(3.4.1.3)}$$

Proof of Lemma 10.

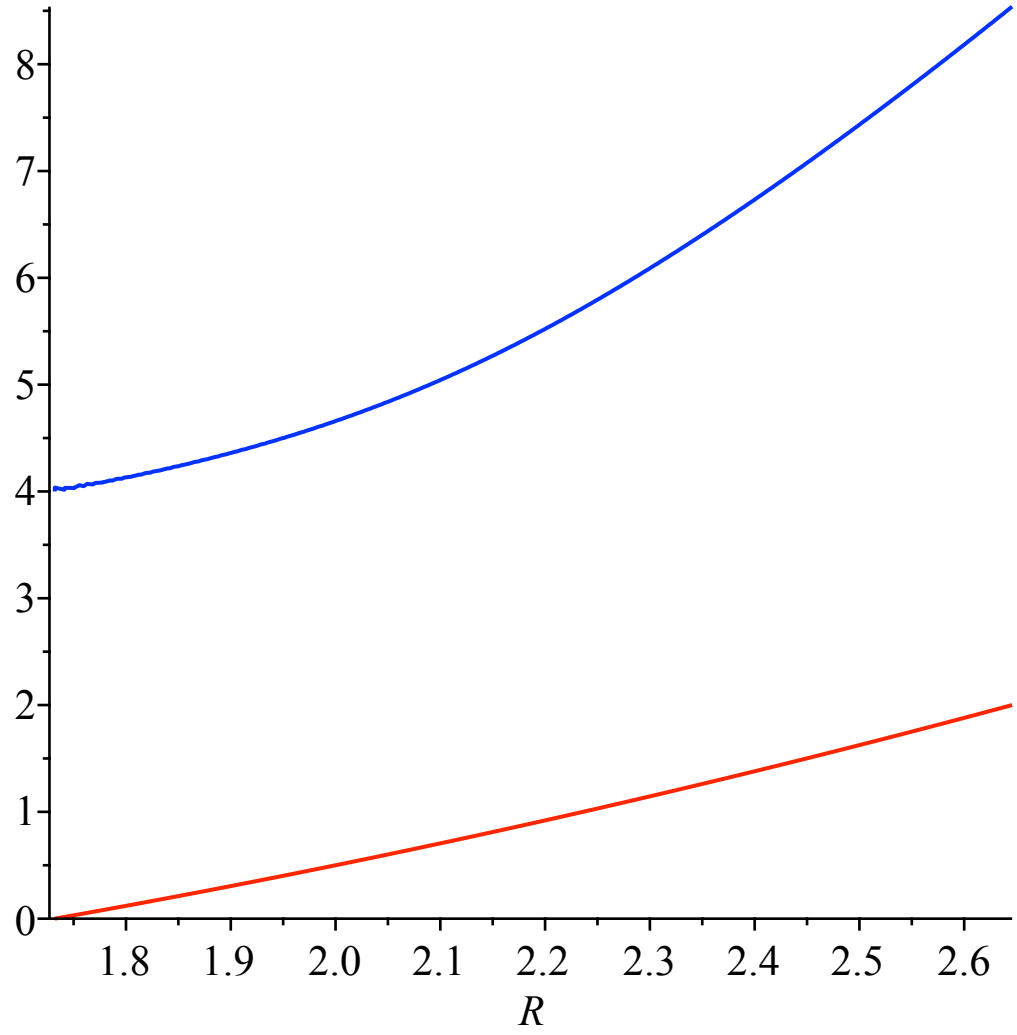
%%% The proof of Lemma 10 requires checking the following two statements explicitly:
%%%
%%% 1. the derivative of U_R is always positive at H=0.
%%%

```
> signum(subs(H=0,dU_HT)) assuming R1<R and R<=Rc;
  signum(subs(H=0,dU_LT)) assuming Rc<=R and R<Rinf;
```

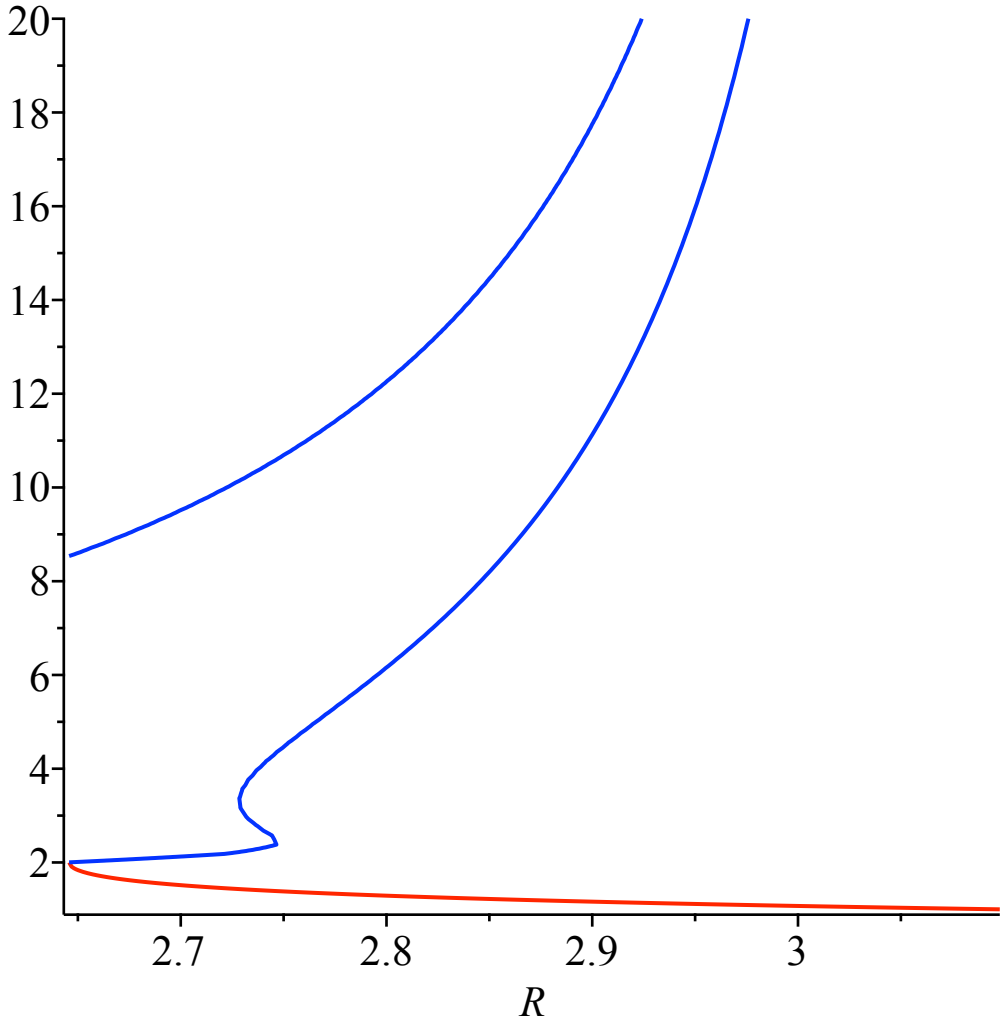
$$1$$
$$1 \quad \textbf{(3.4.2.1)}$$

%%%
%%% 2. the rational function Z_R(H,H) has no pole in the interval [0,Hc(R)].
%%%

```
> ZHH_HT := factor(subs(K=H,Z_HT)):
  plots[display](
     plot(Hc_HT, R=R1..Rc, color = "Red"),
     plots[implicitplot](denom(ZHH_HT)=0, R=R1..Rc, H=0..20,
  numpoints = 10000, color = "Blue")
  );
  # The blue curve is always above the red curve.
```




```
> ZHH_LT := factor(subs(K=H,Z_LT)):
  plots[display](
     plot(Hc_LT, R=Rc..Rinf, color = "Red"),
     plots[implicitplot](denom(ZHH_LT)=0, R=Rc..Rinf, H=0..20,
  numpoints = 10000, color = "Blue")
  );
  # The blue curves are always above the red curve.
```



x_R(H) is the function U_R(H) renormalized by its value at Hc(R), so that x_R(Hc(R))=1 .

```
> x_HT := factor( U_HT/subs(H=Hc_HT, U_HT) );
  x_LT := simplify( U_LT/subs(H=Hc_LT, U_LT) );
```

$$x_HT := \frac{16\ H\ (H\ R^4 - 2\ H^2\ R^2 - 3\ R^4 + H^3 + 2\ H^2 + 10\ R^2 - H - 3)}{(R^2 - 3)^2\ (R^2 - 4\ R + 1)\ (R^2 + 4\ R + 1)}$$

$$x_LT := \left(2\ \left(R^2 - R - \frac{13}{2}\right)^2\ H\ \left(\left(H^3 + 4\right)\ R^4 + \left(-2\ H^3 + 12\ H^2 - 12\ H + 4\right)\ R^3 + \left(-12\ H^3 - 18\ H^2 + 54\ H - 138\right)\ R^2 + \left(13\ H^3 - 72\ H^2 + 36\ H + 52\right)\ R + \frac{169\ H^3}{4} + 39\ H^2 - 30\ H + 190\right)\right) \Big/ \left(3\ \left(R^2 + \sqrt{-3\ R^4 + 12\ R^3 + 36\ R^2 - 84\ R - 105} - 4\ R - 5\right)\ (R + 1)\ (-5\right. \quad \textbf{(3.4.1)}$$

$$+R)\left(\left(-\frac{R}{2}+\frac{1}{4}\right)\sqrt{-3\,R^4+12\,R^3+36\,R^2-84\,R-105}+R^4+\frac{5\,R^3}{2}-\frac{69\,R^2}{4}-\frac{31\,R}{2}+\frac{229}{4}\right)\Bigg)$$

We record the special case of the critical temperature (i.e. when R=Rc).

```
> Hc_cT:= simplify( subs(R=Rc, Hc_HT) );
  x_cT := simplify( subs(R=Rc, x_HT ) );
```

$$Hc_cT := 2$$

$$x_cT := -\frac{H\,(H^3-12\,H^2+48\,H-80)}{48} \qquad \textbf{(3.4.2)}$$

We save the useful RPs in text file, which can be viewed and used without Maple.

```
> save R1,Rc,Rinf,
       nu_HT,S_HT,T_HT, U_HT,Z_HT, Hc_HT,x_HT,
       nu_LT,S_LT,T_LT, U_LT,Z_LT, Hc_LT,x_LT,
       nu_cT,S_cT,T_cT, U_cT,Z_cT, Hc_cT,x_cT,
       "Ising-IPT_2--RP_ciritcal_line"
```

3. Dominant singularity structure of Z(u,v; nu).

%%% The computations in this section do not define new objects. They consist of
%%%   checking the various claims made in the proofs of Lemmas 13, 16 and 18.
%%%

3.1 The conformal bijection x_R from H_epsilon (R) to Delta_epsilon.

Proof of Lemma 13.

%%% The proof of Lemma 13 requires checking the following four statements explicitly:
%%%
%%% 1. Hc(R) is a simple zero of the derivative of x_R(H) for R in (R1,Rinf) except at R=Rc, where it is a double zero.
%%%   We check this at the high temperatures, at the low temperatures, and at the critical temperature separately.
%%%

By construction, Hc_HT is a zero of the derivative of x_HT.

We check that it is a simple zero for R in (R1,Rc).

```
> factor( subs(H=Hc_HT, diff(x_HT,H,H)) );   # The second
  derivative of x_HT at Hc_HT ...
  solve( {%=0, R1<R, R<Rc}, [R] );           # ... does not
  vanishes for R in (R1,Rc).
```

$$-\frac{16\,(R^2-7)\,(R^2+1)}{(R^2-3)^2\,(R^2-4\,R+1)\,(R^2+4\,R+1)}$$

$$[\,] \qquad \textbf{(4.1.1.1)}$$

Similarly, we check that Hc_LT is a simple zero of the derivative of x_LT for R in (Rc,Rinf).

```
> evala( subs(H=Hc_LT, diff(x_LT,H,H)) );    # The second
  derivative of x_LT at Hc_LT ...
  solve( {%=0, Rc<R, R<Rinf}, [R] );         # ... does not
  vanishes for R in (Rc,Rinf).
```

$$\left(2\,\left(4\,R^8+4\,\sqrt{-3\,R^4+12\,R^3+36\,R^2-84\,R-105}\;R^6+24\,R^7\right.\right.$$
$$+16\,\sqrt{-3\,R^4+12\,R^3+36\,R^2-84\,R-105}\;R^5-64\,R^6$$
$$-80\,\sqrt{-3\,R^4+12\,R^3+36\,R^2-84\,R-105}\;R^4-580\,R^5$$
$$-244\,\sqrt{-3\,R^4+12\,R^3+36\,R^2-84\,R-105}\;R^3+627\,R^4$$
$$+631\,\sqrt{-3\,R^4+12\,R^3+36\,R^2-84\,R-105}\;R^2+4468\,R^3$$
$$+862\,\sqrt{-3\,R^4+12\,R^3+36\,R^2-84\,R-105}\;R-4688\,R^2$$
$$\left.-1784\,\sqrt{-3\,R^4+12\,R^3+36\,R^2-84\,R-105}-11088\,R\right.$$
$$\left.+14441\right)\Big)\Big/\left(\left(4\,R^6+24\,R^5-60\,R^4-388\,R^3+537\,R^2\right.\right.$$
$$\left.\left.+1416\,R-2021\right)\,(R+1)\,(-5+R)\right)$$

$$[\,] \qquad \textbf{(4.1.1.2)}$$

And Hc_cT is a double zero of x_cT:

```
> subs(H=Hc_cT, diff(x_cT,H,H) );    # The second derivative
  of x_cT at Hc_cT vanishes,
  subs(H=Hc_cT, diff(x_cT,H,H,H) );  # while the third
  derivative does not vanish.
```

$$0$$

$$\frac{1}{2} \qquad \textbf{(4.1.1.3)}$$

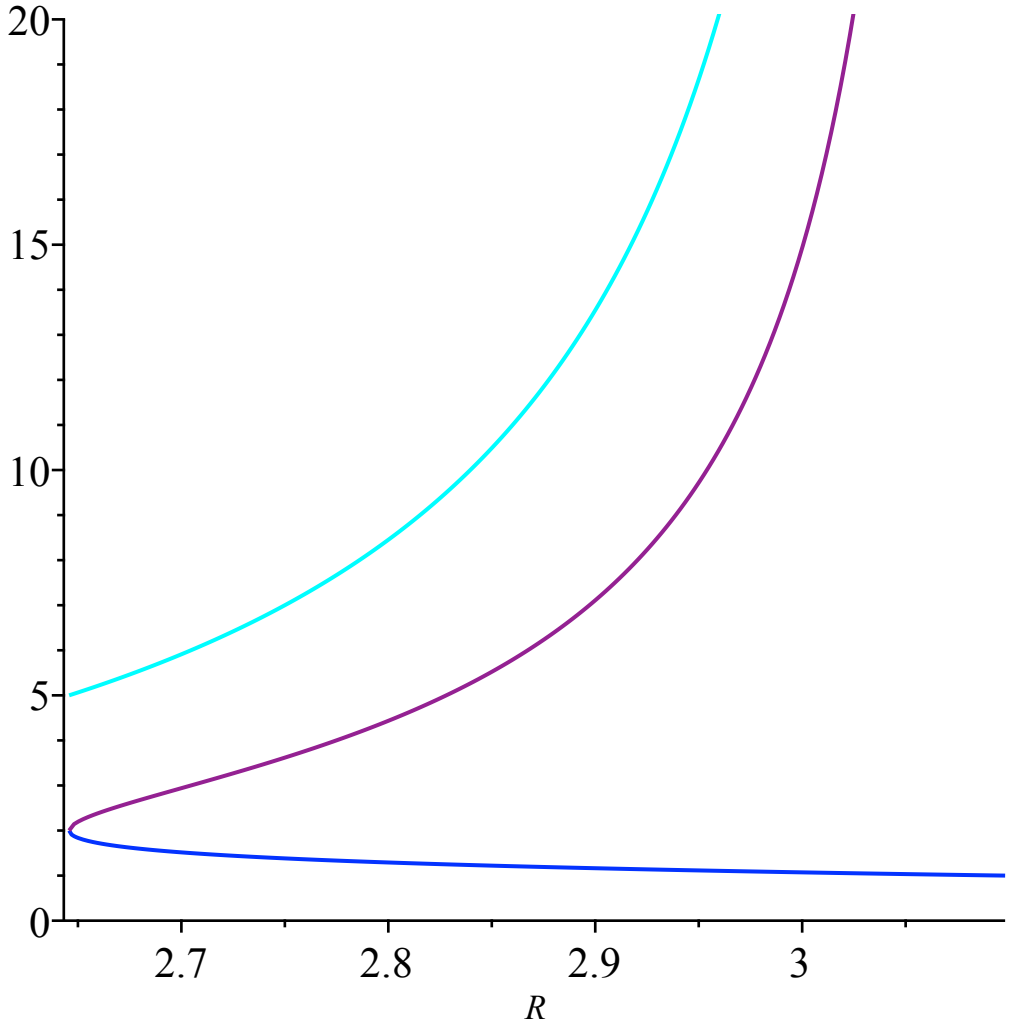
%%%

%%% 2. For all R in [Rc,Rinf), the three zeros of U_LT are all on the positive real line.

%%%

This can be seen from a graph plotted when computing the parametrization Hc_LT. Recall that graph:

```
> plot(Hc_LT_list, R=Rc..Rinf, color = ["Cyan", "Blue",
  "Purple"],view=[Rc..Rinf, 0..20]);
```




```
%%%
%%% 3. The function chi(H,R) and its partial derivative with
respect to H
%%%    have only one common zero (H,R) satisfying R1<R<=Rc.
%%%    And at this point, the parametrization x(H,R) has a
modulus different from 1.
%%%
> chi := factor( diff(x_HT,H)/(H-Hc_HT) );
  solve([chi=0, diff(chi,H)=0, R1<R,R<=Rc],{H,R});
  simplify( subs(%,x_HT) );
  evalf(%)
```

$$\chi := \frac{32\ \left(-2\ H\ R^2+2\ H^2+3\ R^2-1\right)}{\left(R^2-3\right)^2\ \left(R^2-4\ R+1\right)\ \left(R^2+4\ R+1\right)}$$

$$\left\{H=\frac{3}{2}+\frac{\sqrt{7}}{2},\ R=\sqrt{3+\sqrt{7}}\right\}$$

$$\frac{72\sqrt{7}}{413}+\frac{188}{413}$$

$$0.9164505916 \qquad \textbf{(4.1.1.4)}$$

%%%

%%% 4. The system of algebraic equations (37) has no solutions when the variables r,s,R are real.

%%%

The system of equations (37):

```
> eq_chi_complex := [ chi=0,
          x_HT=(s+I)/(s-I),
          diff(x_HT,R) - diff(x_HT,H)*diff(chi,R)/diff(chi,H)
  = I*r*(s+I)/(s-I) ];
```

$$eq_chi_complex := \left[\frac{32\left(-2\,H\,R^2+2\,H^2+3\,R^2-1\right)}{\left(R^2-3\right)^2\left(R^2-4\,R+1\right)\left(R^2+4\,R+1\right)}\right.$$

$$=0,$$

$$\frac{16\,H\left(H\,R^4-2\,H^2\,R^2-3\,R^4+H^3+2\,H^2+10\,R^2-H-3\right)}{\left(R^2-3\right)^2\left(R^2-4\,R+1\right)\left(R^2+4\,R+1\right)}$$

$$=\frac{s+\mathrm{I}}{s-\mathrm{I}},\ \frac{16\,H\left(4\,H\,R^3-4\,H^2\,R-12\,R^3+20\,R\right)}{\left(R^2-3\right)^2\left(R^2-4\,R+1\right)\left(R^2+4\,R+1\right)}$$

$$-\left(64\,H\left(H\,R^4-2\,H^2\,R^2-3\,R^4+H^3+2\,H^2+10\,R^2-H\right.\right.$$

$$\left.\left.-3\right)\,R\right)\Big/\left(\left(R^2-3\right)^3\left(R^2-4\,R+1\right)\left(R^2+4\,R+1\right)\right)$$

$$-\left(16\,H\left(H\,R^4-2\,H^2\,R^2-3\,R^4+H^3+2\,H^2+10\,R^2-H-3\right)\,(2\,R-4)\right)\Big/\left(\left(R^2-\right.\right.$$

$$+1)\Big)-\big(16\ H\ \big(H\ R^4-2\ H^2\ R^2-3\ R^4+H^3+2\ H^2$$

$$+10\ R^2-H-3\big)\ (2\ R+4)\big)\Big/\Big(\big(R^2-3\big)^2\ (R^2-4\ R$$

$$+1)\ \big(R^2+4\ R+1\big)^2\Big)$$

$$-\frac{1}{32\ (-2\ R^2+4\ H)}\Bigg(\Bigg(\big(16\ \big(H\ R^4-2\ H^2\ R^2-3\ R^4+H^3$$

$$+2\ H^2+10\ R^2-H-3\big)\big)\Big/\Big(\big(R^2-3\big)^2\ (R^2-4\ R+1)\ (R^2$$

$$+4\ R+1\big)\Big)+\frac{16\ H\ \big(R^4-4\ H\ R^2+3\ H^2+4\ H-1\big)}{\big(R^2-3\big)^2\ (R^2-4\ R+1)\ (R^2+4\ R+1)}\Bigg)$$

$$\Bigg(\frac{32\ (-4\ H\ R+6\ R)}{\big(R^2-3\big)^2\ (R^2-4\ R+1)\ (R^2+4\ R+1)}$$

$$-\frac{128\ (-2\ H\ R^2+2\ H^2+3\ R^2-1)\ R}{\big(R^2-3\big)^3\ (R^2-4\ R+1)\ (R^2+4\ R+1)}$$

$$-\frac{32\ (-2\ H\ R^2+2\ H^2+3\ R^2-1)\ (2\ R-4)}{\big(R^2-3\big)^2\ (R^2-4\ R+1)^2\ (R^2+4\ R+1)}$$

$$-\frac{32\ (-2\ H\ R^2+2\ H^2+3\ R^2-1)\ (2\ R+4)}{\big(R^2-3\big)^2\ (R^2-4\ R+1)\ (R^2+4\ R+1)^2}\Bigg)\ \big(R^2-3\big)^2\ (R^2$$

$$-4\ R+1)\ (R^2+4\ R+1)\Bigg)=\frac{\mathrm{I}\ r\ (s+\mathrm{I})}{s-\mathrm{I}}\Bigg]$$

We eliminate the complex variable H, and take the real and imaginary parts of the remaining equations.

```
> eliminate( {chi=0, x_HT=(s+I)/(s-I), diff(x_HT,R) - diff
  (x_HT,H)*diff(chi,R)/diff(chi,H) = I*r*(s+I)/(s-I)}, H )[2]:
  eq_chi_real := [ op( map(RealDomain[Re],%) ), op( map
  (RealDomain[Im],%) ) ];
```

$$eq_chi_real := \big[\big(R^4-6\ R^2+2\big)\ \big(R^{28}\ r^2\ s^2-R^{28}\ r^2 \qquad \textbf{(4.1.1.6)}$$

$$-74\ R^{26}\ r^2\ s^2+74\ R^{26}\ r^2+2323\ R^{24}\ r^2\ s^2-2323\ R^{24}\ r^2$$

$$-40484\ R^{22}\ r^2\ s^2+40484\ R^{22}\ r^2+432121\ R^{20}\ r^2\ s^2$$

$$-432121\ R^{20}\ r^2-4608\ R^{20}\ s^2-2956214\ R^{18}\ r^2\ s^2$$

$$
\begin{aligned}
&+ 4608\ R^{20} + 2956214\ R^{18}\ r^2 + 105984\ R^{18}\ s^2 \\
&+ 13240099\ R^{16}\ r^2\ s^2 - 105984\ R^{18} - 13240099\ R^{16}\ r^2 \\
&- 919552\ R^{16}\ s^2 - 38844920\ R^{14}\ r^2\ s^2 + 919552\ R^{16} \\
&+ 38844920\ R^{14}\ r^2 + 3348480\ R^{14}\ s^2 + 72826651\ R^{12}\ r^2\ s^2 \\
&- 3348480\ R^{14} - 72826651\ R^{12}\ r^2 - 1924096\ R^{12}\ s^2 \\
&- 81757206\ R^{10}\ r^2\ s^2 + 1924096\ R^{12} + 81757206\ R^{10}\ r^2 \\
&- 11426816\ R^{10}\ s^2 + 47436273\ R^8\ r^2\ s^2 + 11426816\ R^{10} \\
&- 47436273\ R^8\ r^2 - 4876288\ R^8\ s^2 - 9912996\ R^6\ r^2\ s^2 \\
&+ 4876288\ R^8 + 9912996\ R^6\ r^2 - 4425728\ R^6\ s^2 \\
&+ 943083\ R^4\ r^2\ s^2 + 4425728\ R^6 - 943083\ R^4\ r^2 \\
&- 2892288\ R^4\ s^2 - 42282\ R^2\ r^2\ s^2 + 2892288\ R^4 \\
&+ 42282\ R^2\ r^2 + 1781248\ R^2\ s^2 + 729\ r^2\ s^2 - 1781248\ R^2 \\
&- 729\ r^2),\ R^{16}\ s^2 - R^{16} - 24\ R^{14}\ s^2 + 56\ R^{14} + 196\ R^{12}\ s^2 \\
&- 980\ R^{12} - 504\ R^{10}\ s^2 + 6392\ R^{10} - 738\ R^8\ s^2 - 17006\ R^8 \\
&+ 2808\ R^6\ s^2 + 16168\ R^6 + 4340\ R^4\ s^2 - 4132\ R^4 \\
&+ 1176\ R^2\ s^2 - 3288\ R^2 - 343\ s^2 + 487,\ (R^4 - 6\ R^2 \\
&+ 2)\ (2\ R^{28}\ r^2\ s - 148\ R^{26}\ r^2\ s + 32\ R^{25}\ r\ s^2 \\
&+ 4646\ R^{24}\ r^2\ s + 32\ R^{25}\ r - 1376\ R^{23}\ r\ s^2 \\
&- 80968\ R^{22}\ r^2\ s - 1376\ R^{23}\ r + 21664\ R^{21}\ r\ s^2 \\
&+ 864242\ R^{20}\ r^2\ s + 21664\ R^{21}\ r - 142112\ R^{19}\ r\ s^2 \\
&+ 9216\ R^{20}\ s - 5912428\ R^{18}\ r^2\ s - 142112\ R^{19}\ r \\
&+ 126976\ R^{17}\ r\ s^2 - 211968\ R^{18}\ s + 26480198\ R^{16}\ r^2\ s \\
&+ 126976\ R^{17}\ r + 3523648\ R^{15}\ r\ s^2 + 1839104\ R^{16}\ s \\
&- 77689840\ R^{14}\ r^2\ s + 3523648\ R^{15}\ r - 20675264\ R^{13}\ r\ s^2 \\
&- 6696960\ R^{14}\ s + 145653302\ R^{12}\ r^2\ s - 20675264\ R^{13}\ r \\
&+ 49390528\ R^{11}\ r\ s^2 + 3848192\ R^{12}\ s \\
&- 163514412\ R^{10}\ r^2\ s + 49390528\ R^{11}\ r \\
&- 51853280\ R^9\ r\ s^2 + 22853632\ R^{10}\ s + 94872546\ R^8\ r^2\ s \\
&- 51853280\ R^9\ r + 19477280\ R^7\ r\ s^2 + 9752576\ R^8\ s \\
&- 19825992\ R^6\ r^2\ s + 19477280\ R^7\ r - 5752800\ R^5\ r\ s^2 \\
&+ 8851456\ R^6\ s + 1886166\ R^4\ r^2\ s - 5752800\ R^5\ r \\
&+ 595296\ R^3\ r\ s^2 + 5784576\ R^4\ s - 84564\ R^2\ r^2\ s
\end{aligned}
$$

$$+595296\ R^3\ r-19008\ R\ r\ s^2-3562496\ R^2\ s+1458\ r^2\ s$$
$$-19008\ R\ r),\ 2\ R^{16}\ s-80\ R^{14}\ s+1176\ R^{12}\ s$$
$$-9200\ R^{10}\ s+40268\ R^8\ s-87344\ R^6\ s+67992\ R^4\ s$$
$$-13968\ R^2\ s+1154\ s]$$

The resulting system has no real solutions.

```
> RootFinding[HasRealRoots]( eq_chi_real );
```

$$false \qquad \textbf{(4.1.1.7)}$$

3.2 Holomorphicity of Z_R on H_epsilon(R)* H_epsilon(R).

Proof of Lemma 16.

%%% The proof of Lemma 16 requires checking the following two statements explicitly:
%%%
%%% 1. When (H,K)=(Hc(R),Hc(R)), the denominator of Z_R(H,K) vanishes for all R in (R1,Rc],
%%% and does not vanishes for all R in (Rc,Rinf).
%%%

```
> N_HT,D_HT := numer(Z_HT), denom(Z_HT):
  N_LT,D_LT := numer(Z_LT), denom(Z_LT):

  simplify( subs({H=Hc_HT,K=Hc_HT}, D_HT) );
  simplify( subs({H=Hc_LT,K=Hc_LT}, D_LT) );
  solve( {%=0, Rc<R, R<Rinf},[R] );
```

$$0$$

$$\frac{1}{(2\ R^2-2\ R-13)^2}\left(6144\ \left(\left(-\frac{1}{6}\ R^4-\frac{11}{12}\ R^3+\frac{25}{8}\ R^2+\frac{73}{12}\ R\right.\right.\right.$$
$$\left.\left.-\frac{271}{24}\right)\ \sqrt{-3\ R^4+12\ R^3+36\ R^2-84\ R-105}+\left(R^4+\frac{1}{4}\ R^3\right.\right.$$
$$\left.\left.-\frac{105}{8}\ R^2-\frac{11}{4}\ R+\frac{401}{8}\right)\ (R^2-7)\right)\ (-5+R)^3\ \left(R^2+5\ R\right.$$
$$\left.-\frac{19}{2}\right)\ (R+1)^3\Bigg)$$

$$[\,] \qquad \textbf{(4.2.1.1)}$$

%%%
%%% 2. The system of algebraic equations (43) does not have any solutions.
%%% We check this fact in the low and high temperature regimes separately.
%%%

We define the list of equations in the system (43).

```
> cross_prod_HT := diff(N_HT,H)*diff(D_HT,K)-diff(D_HT,H)*diff
  (N_HT,K):
  nonzero_HT    := numer(factor( U_HT/H * subs(H=K,U_HT)/K *
  dU_HT ));

  eq_pole_HT    := { N_HT, D_HT, cross_prod_HT, X*nonzero_HT-1
  }:
```

$$nonzero_HT := -(H\,R^4 - 2\,H^2\,R^2 - 3\,R^4 + H^3 + 2\,H^2 + 10\,R^2 - H - 3)\,(K\,R^4 - 2\,K^2\,R^2 - 3\,R^4 + K^3 + 2\,K^2 + 10\,R^2 - K - 3)\,(-2\,H\,R^2 + 2\,H^2 + 3\,R^2 - 1)\,(-R^2 + 2\,H + 3) \quad \textbf{(4.2.1.2)}$$

The following Maple function take a polynomial with rational coefficients as input,
   factorizes it over the field of rational numbers, and removes all factors which
   (1) only depend on the variable R, and
   (2) have no root in the interval (R1,Rc].
These factors are irrelavant since they do not vanish.
We apply this function to all the equations in eq_pole_HT.

```
> relevant_factors_HT := proc(poly::polynom(rational))
     local is_irrelev;
     is_irrelev := p -> indets(p)={R} and solve([p, R1<R, R<=
  Rc])=NULL;
                  # This function tests whether a factor
  satisfies the two conditions given in the above comment.
     return mul( remove(is_irrelev, factors(poly)[2,..,1]) )
  end proc:


  eq_pole_reduced_HT := map(relevant_factors_HT, eq_pole_HT):
```

For a list of polynomials eq_sys involving the variable R,
   "Groebner[UnivariatePolynomial](X,eq_sys)" computes the univariate polynomial P(X)
   of least degree in the polynomial ideal generated by the polynomials in eq_sys.
   (It returns 0 if no such polynomial exists.)
In particular, if P(X) is not 0, then the coordinate X of any common zero of
   the polynomials in the list eq_sys must be a zero of P(X).
Thus, if P(X) does not have any real root, then the original system eq_sys
   does not have any solution with a real value of X.

We check that eq_pole_reduced_HT has no solution with real value of R using the above fact:

```
> factor(  Groebner[UnivariatePolynomial](R,
```

```
  eq_pole_reduced_HT)  );
```

$$(R^2+1)^3 \qquad \textbf{(4.2.1.3)}$$

We repeat the same computations for in the low temperature regime:

```
> nonzero_LT    := numer(factor( U_LT/H * subs(H=K,U_LT)/K *
  dU_LT )):
  cross_prod_LT := diff(N_LT,H)*diff(D_LT,K)-diff(D_LT,H)*diff
  (N_LT,K):
  eq_pole_LT    := { N_LT, D_LT, cross_prod_LT, X*nonzero_LT-1
  }:

  relevant_factors_LT := proc(poly::polynom(rational))
     local is_irrelev;
     is_irrelev := p -> indets(p)={R} and solve([p, Rc<=R,
  R<Rinf])=NULL;
     return mul( remove(is_irrelev, factors(poly)[2,..,1]) )
  end proc:

  map(relevant_factors_LT, eq_pole_LT):
  factor(Groebner[UnivariatePolynomial](R, %));
```

$$1 \qquad \textbf{(4.2.1.4)}$$

Proof of Lemma 18.

%%% The proof of Lemma 18 requires checking the following statement explicitly:
%%% For all R in (R1,Rc), the partial derivative of D_HT with respect to H (or K)
%%% does not vanish at (H,K)=(Hc(R),Hc(R)), where D_HT is the denominator of Z_HT.
%%%

```
> factor( subs({H=Hc_HT,K=Hc_HT}, diff(D_HT,H)) );
  solve({%=0, R1<R, R<Rc},[R]);
```

$$\frac{(R^6-17\ R^4+75\ R^2+29)\ (3\ R^2-1)\ (R^2-3)^3}{2}$$

$$[\,] \qquad \textbf{(4.2.2.1)}$$

4. Asymptotic expansions of Z(u,v;nu) at its dominant singularity

%%% First, we check the vanishing of the first partial derivatives of Z_R claimed by Equation (48) in the paper.
%%%

We check that when K=Hc(R), the first partial derivative of Z_R with respect to K vanishes for all R in (R1,Rinf) ...

```
> simplify( subs(K=Hc_HT, diff(Z_HT,K)) );
  simplify( subs(K=Hc_LT, diff(Z_LT,K)) );
```

$$0$$

$$0 \qquad (5.1)$$

... and the second derivative with respect to K vanishes when R=Rc.

```
> simplify( subs(K=Hc_cT, diff(Z_cT,K,K)) );
```

$$0 \qquad (5.2)$$

Explicit parametrizations of b(nu).

%%% We compute the explicit parametrization of b(nu) using its expression in Lemma 20 and 21.
%%% It will be named b_HT, b_LT and b_cT at high, low and critical temperatures, respectively.
%%%

The following functions compute the coefficients x_n and Z_{m,n} defined at the end of Lemma 20.

```
> xn_HT := n -> coeftayl( subs(H=Hc_HT-h, 1-x_HT), h=0, n ):
  xn_LT := n -> coeftayl( subs(H=Hc_LT-h, 1-x_LT), h=0, n ):
  xn_cT := n -> coeftayl( subs(H=Hc_cT-h, 1-x_cT), h=0, n ):

  Zmn_LT := (m,n) -> coeftayl( subs({H=Hc_LT-h,K=Hc_LT-k},
  Z_LT), [h,k]=[0,0], [m,n] ):
  #  Zmn_LT does not have counterparts in the high and critical
  temperature regimes,
  #  since the rational function Z_HT has a pole at (H,K)=
  (Hc_HT,Hc_HT), thus is not analytic there.
```

Using these functions we can compute (the parametrizations of) the three nonzero coefficients defined in Lemma 20,
   which by Lemma 21 are exactly the values of b(nu) at the high, low and critical temperautres (up to a sign).

```
> xn_LT(3)/xn_LT(2) :
  (Zmn_LT(3,3) - 2*%*Zmn_LT(2,3) + %^2 *Zmn_LT(2,2)) / xn_LT(2)
  ^3 :
  b_LT := simplify( evala(%) );
```

$$b_LT := \left(423\ (R+1)\ (-5+R)\ \left(R^2-R-\frac{13}{2}\right)\ \left(\right.\right. \qquad (5.1.1)$$

$$-\frac{1616334570337930567995\ R^{15}}{96256}-\frac{5612506220887268634153\ R^{14}}{192512}$$

$$+\frac{8260340363041606881859\ R^{13}}{96256}$$

$$+\frac{15708688712034376691701\ R^{12}}{192512}$$
$$-\frac{10041539337339175595878 9\ R^{11}}{288768}$$
$$-\frac{4946512207301927489591 9\ R^{10}}{577536}$$
$$+\frac{2989402169429624632785 89\ R^{9}}{288768}$$
$$-\frac{7385660786147208257848 3\ R^{8}}{192512}$$
$$-\frac{1937347321134736630380 97\ R^{7}}{96256}$$
$$-\frac{6602088051343984213477 27\ R}{288768}+\left(-\frac{1685181008580080781}{3008}\ R^{15}\right.$$
$$-\frac{132501456705305652657}{192512}\ R^{14}+\frac{563322747019397505159}{192512}\ R^{13}$$
$$+\frac{626798344728719979895}{385024}\ R^{12}-\frac{10236149405845842348481}{866304}\ R^{11}$$
$$+\frac{1473527210138972626171}{1732608}\ R^{10}+\frac{58868066516667181263421}{1732608}\ R^{9}$$
$$-\frac{25734077212432066249451}{1155072}\ R^{8}-\frac{873684044921032273960 9}{144384}\ R^{7}$$
$$-\frac{12101993467794438374790 5}{1732608}\ R-\frac{92}{423}\ R^{38}-\frac{2852}{423}\ R^{37}$$
$$-\frac{26357}{423}\ R^{36}+\frac{18668}{141}\ R^{35}+\frac{3070445}{564}\ R^{34}+\frac{3431827}{188}\ R^{33}$$
$$-\frac{67780583}{376}\ R^{32}-\frac{57147310}{47}\ R^{31}-\frac{2676296714341}{3384}\ R^{27}$$
$$+\frac{15769886501547413389}{96256}\ R^{16}+\frac{6263560457}{2256}\ R^{30}$$
$$-\frac{30867416137}{13536}\ R^{28}-\frac{1284756581795}{1504}\ R^{26}+\frac{64739930841063}{3008}\ R^{24}$$
$$-\frac{1929546938442555}{6016}\ R^{22}+\frac{61719171050372443}{18048}\ R^{20}$$
$$-\frac{3954346664966375023}{144384}\ R^{18}+\frac{1264104763686793550424 25}{1732608}\ R^{2}$$

$$+\frac{11897780076554616523357}{288768}R^3-\frac{4974608841443913620468 3}{385024}R^4$$
$$+\frac{15322053620572464578161}{192512}R^6+\frac{8238213382071457622081}{192512}R^5$$
$$+\frac{259573382947}{6768}R^{29}+\frac{18015903051181}{1504}R^{25}$$
$$-\frac{53665036251399}{376}R^{23}+\frac{8468810470428073}{6016}R^{21}$$
$$-\frac{53830560705990743}{4512}R^{19}+\frac{4245588054767067695}{48128}R^{17}$$
$$+\frac{65036112719542549478561}{3465216}\Bigg)$$
$$\sqrt{-3\,R^4+12\,R^3+36\,R^2-84\,R-105}+R^{40}+29\,R^{39}$$
$$+\frac{71561\,R^{38}}{282}-\frac{80453\,R^{37}}{141}-\frac{12224855\,R^{36}}{564}-\frac{13692547\,R^{35}}{188}$$
$$+\frac{262679573\,R^{34}}{376}+\frac{916131157\,R^{33}}{188}-\frac{3780545235\,R^{32}}{376}$$
$$-\frac{7288404213\,R^{31}}{47}-\frac{226363800825721\,R^{27}}{4512}$$
$$+\frac{333362491364912429181\,R^{16}}{48128}-\frac{9970617887\,R^{30}}{376}$$
$$+\frac{19958509587659\,R^{28}}{4512}-\frac{327333276014187\,R^{26}}{3008}$$
$$+\frac{20276635061946531\,R^{24}}{12032}-\frac{465002436295616901\,R^{22}}{24064}$$
$$+\frac{8341252136236119815\,R^{20}}{48128}-\frac{1189569277955695590919\,R^{18}}{96256}$$
$$+\frac{1563408253446404552569741\,R^2}{577536}$$
$$+\frac{6083418254284942019 9017\,R^3}{96256}$$
$$-\frac{7403581498338001503 12645\,R^4}{192512}$$
$$+\frac{3811688542555858521 11317\,R^6}{192512}$$

$$+\frac{1815673447790660555222297\ R^5}{96256}+\frac{457951365448\ R^{29}}{141}$$
$$+\frac{916889437893273\ R^{25}}{1504}-\frac{73548771030884637\ R^{23}}{12032}$$
$$+\frac{633909396204100305\ R^{21}}{12032}-\frac{193981685073437776 15\ R^{19}}{48128}$$
$$+\frac{133314222613575679809\ R^{17}}{48128}+\frac{3418128752063667383954 65}{577536}\Bigg)\Bigg)$$
$$\Bigg/\left(8788\ \left(R^2-7\right)^2\ \left(R^4+2\ R^3-12\ R^2-17\ R+\frac{227}{4}\right)^5\ \left(R^6\right.\right.$$
$$\left.+3\ R^5-\frac{441}{26}\ R^4-\frac{634}{13}\ R^3+\frac{1722}{13}\ R^2+\frac{2403}{13}\ R-\frac{11077}{26}\right)^3\ \left(R^2$$
$$\left.\left.+5\ R-\frac{19}{2}\right)\right)$$

```
> factor(diff( subs(K=Hc_HT, Z_HT), H)):
  b_HT_raw := subs(H=Hc_HT, %) / xn_HT(2)^(1/2) :          #
  Maple has some difficulty simplifying the square roots in this
  expression.
  b_HT := sqrt( (1+R^2)*(7-R^2)^3*(14*R^2-1-R^4)^5 )/( sqrt(2)*
  (3*R^2-1)*(29+75*R^2-17*R^4+R^6)^2 );
  simplify( b_HT - b_HT_raw ) assuming R1<R and R<Rc;      #
  Instead, let us use it to check that the expression given in
  the paper is correct.
```

$$b_HT := \frac{\sqrt{\left(R^2+1\right)\ \left(-R^2+7\right)^3\ \left(-R^4+14\ R^2-1\right)^5}\ \sqrt{2}}{2\ \left(3\ R^2-1\right)\ \left(R^6-17\ R^4+75\ R^2+29\right)^2}$$

$$0 \qquad \textbf{(5.1.2)}$$

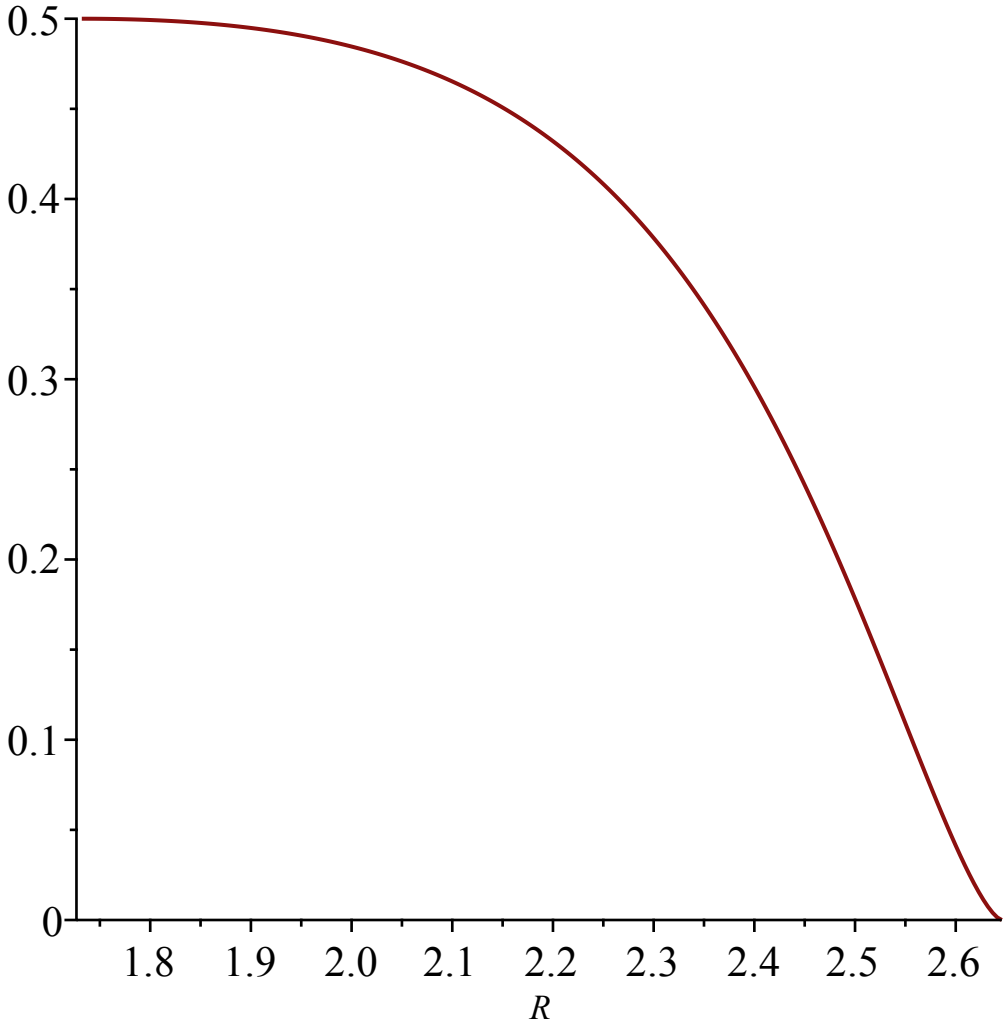
```
> factor( subs(K=H, diff(Z_cT,H,K)) / (Hc_cT-H)^3 ):
  - 4/13* subs(H=Hc_cT, %) / xn_cT(3)^(5/3):
  b_cT := simplify(%);
```

$$b_cT := -\frac{27\ 12^{2\,|\,3}}{80} \qquad \textbf{(5.1.3)}$$

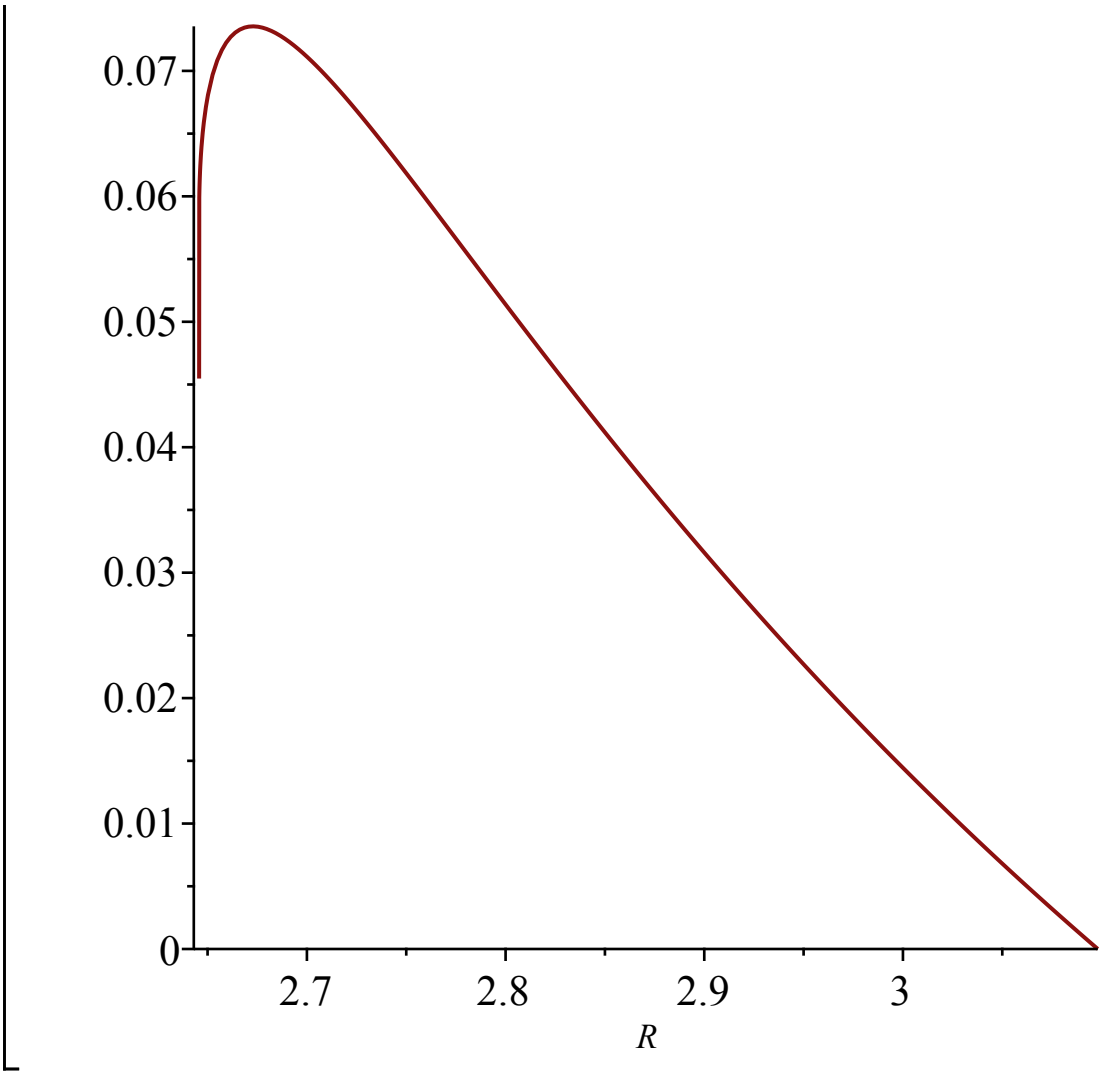
Proof of Lemma 20

We check that b_HT and b_LT are indeed nonzero, as claimed in Lemma 20.

```
> plot(b_HT, R=R1..Rc);
```




```
> module()  Digits := 30;
  print( plot( factor(b_LT *(R-Rc)^2) ,R=Rc..Rinf) );
  end module:
```



Explicit parametrizations of A(u,nu).

```
%%% We compute the explicit parametrization of A(u;nu) using its
expression in Corollary 26.
%%% It will be named  A_HT, A_LT and A_cT at high, low and critical
temperatures, respectively.
%%%
```

The following functions compute the coefficients Z_n defined at the end of Corollary 26.

```
> Zn_HT := n -> coeftayl( subs(K=Hc_HT-k,   Z_HT), k=0, n ):
  Zn_LT := n -> coeftayl( subs(K=Hc_LT-k,   Z_LT), k=0, n ):
  Zn_cT := n -> coeftayl( subs(K=Hc_cT-k,   Z_cT), k=0, n ):
```

Computation of the parametrizations of A(u) given in Corollary 26.

```
> (Zn_LT(3) - xn_LT(3)/xn_LT(2) *Zn_LT(2) ) / xn_LT(2)^(3/2) :
  A_LT := simplify(evala(%)) assuming Rc<R and R<Rinf;
```

$$A_LT := \left(9\left(\left(\left(\left(\frac{1}{3}H^3-\frac{13}{6}H^2+6\right)R^8+\left(H^3-\frac{23}{3}H^2-\frac{68}{3}H+60\right)R^7+\left(-\frac{53}{6}H^3+69H^2-\frac{278}{3}H\right)R^6+\left(472H+\frac{1147}{6}H^2-\frac{85}{3}H^3-1266\right)R^5+\left(\frac{4334}{3}H+\frac{433}{4}H^3-\frac{20657}{24}H^2-\frac{1425}{2}\right)R^4+\left(10584-\frac{12883}{3}H-1222H^2+\frac{2311}{12}H^3\right)R^3+\left(-1785-\frac{13081}{24}H^3+\frac{48679}{12}H^2-\frac{8101}{2}H\right)R^2+\left(-31242-\frac{1443}{4}H^3+\frac{13451}{6}H^2+\frac{40177}{3}H\right)R-\frac{31501\,H}{6}+\frac{23491\,H^3}{24}-\frac{51467\,H^2}{8}+\frac{54567}{2}\right)\sqrt{-3R^4+12R^3+36R^2-84R-105}+\left(H^4-\frac{1}{3}H^3-\frac{13}{6}H^2-\frac{4}{3}H-6\right)R^{10}+\left(4H^4+\frac{41}{3}H^3-\frac{80}{3}H^2-\frac{40}{3}H-24\right)R^9+\left(-39H^4+\frac{133}{2}H^3+45H^2-92H+288\right)R^8+\left(-126H^4-426H^3+\frac{1997}{2}H^2-164H+1026\right)R^7+\left(\frac{1257}{2}H^4-\frac{5639}{4}H^3-\frac{3067}{8}H^2+2469H-\frac{9783}{2}\right)R^6+\left(3812H+1308H^4-\frac{50247}{4}H^2+\frac{21475}{4}H^3-12195\right)R^5+\left(-24993H-4719H^4\right.\right.\right.$$

$$+\frac{71069}{8}\,H^3+\frac{29371}{4}\,H^2+40599\Big)\,R^4+\Big(44100+4204\,H$$

$$+56731\,H^2-\frac{11181}{2}\,H^4-\frac{57561}{2}\,H^3\Big)\,R^3+\Big(-\frac{285219}{2}-13510\,H^3$$

$$-\frac{401229}{8}\,H^2+75616\,H+\frac{261393}{16}\,H^4\Big)\,R^2+\Big(-18267+\frac{650321}{12}\,H^3$$

$$-\frac{840329}{12}\,H^2-\frac{328264}{3}\,H+\frac{33553}{4}\,H^4\Big)\,R+121260+\frac{166112\,H}{3}$$

$$-\frac{387569\,H^3}{24}+\frac{849037\,H^2}{12}-\frac{341549\,H^4}{16}\Big)$$

$$\Big(\Big(\big(-4\,R^6-16\,R^5+80\,R^4+244\,R^3-631\,R^2-862\,R+1784\big)\,\sqrt{-3\,R^4+12\,R^3+3}$$

$$-\frac{2063}{4}\Big)\,\big(R^2-7\big)\Big)\Big/\big(4\,R^8+8\,R^7-176\,R^6-268\,R^5+2389\,R^4$$

$$+1208\,R^3-10370\,R^2+1004\,R+10105\big)^{1/2}\Big)\qquad(-5+R)^2\,(R$$

$$+1)^2\,\Big(\Big(-\frac{192437929521}{256}+\frac{1034188399}{16}\,R^8-\frac{7408695715}{64}\,R^7$$

$$+\frac{31391704211}{64}\,R^5+\frac{105609670099}{256}\,R^4-\frac{78667891963}{64}\,R^3$$

$$-\frac{341782941}{64}\,R^2+\frac{86210145431}{64}\,R-\frac{29}{2}\,R^{18}+742\,R^{17}+2414\,R^{16}$$

$$-\frac{66509}{4}\,R^{15}-\frac{574889}{8}\,R^{14}+\frac{926215}{4}\,R^{13}+\frac{8945359}{8}\,R^{12}$$

$$-\frac{37648405}{16}\,R^{11}-\frac{342089423}{32}\,R^{10}+\frac{75395453}{4}\,R^9-15\,R^{19}-R^{20}$$

$$-\frac{29991620515}{128}\,R^6\Big)\,\sqrt{-3\,R^4+12\,R^3+36\,R^2-84\,R-105}+\big(R^2$$

$$
\begin{aligned}
&-7)\left(R^{20}+15\,R^{19}+\frac{73}{2}\,R^{18}-552\,R^{17}-\frac{6417}{2}\,R^{16}+\frac{28713}{4}\,R^{15}\right.\\
&+\frac{673437}{8}\,R^{14}-\frac{94887}{4}\,R^{13}-1257045\,R^{12}-\frac{4175203}{16}\,R^{11}\\
&+\frac{396751083}{32}\,R^{10}+\frac{8968841}{8}\,R^{9}-\frac{2642798409}{32}\,R^{8}\\
&+\frac{1686770559}{64}\,R^{7}+\frac{44662405119}{128}\,R^{6}-\frac{18678871767}{64}\,R^{5}\\
&-\frac{202133454057}{256}\,R^{4}+\frac{73810328763}{64}\,R^{3}+\frac{15510690869}{32}\,R^{2}\\
&\left.-\frac{105876868215}{64}\,R+\frac{207692405939}{256}\right)\Big)\left(R^{2}-R-\frac{13}{2}\right)\Big)\Big/\\
&\left(64\left(R^{2}+5\,R-\frac{19}{2}\right)\left(-\frac{1}{4}\left(3\left((H+2)\,R^{2}+(-H+4)\,R\right.\right.\right.\right.\\
&\left.\left.\left.\left.-\frac{13\,H}{2}-16\right)\left(R^{2}-7\right)\sqrt{-3\,R^{4}+12\,R^{3}+36\,R^{2}-84\,R-105}\right)\right.\right.\\
&\left.\left.+\left(H^{2}+\frac{3}{4}\,H-\frac{3}{2}\right)R^{6}+\left(\frac{27}{4}\,H-3\,H^{2}\right)R^{5}+\left(-\frac{261}{8}\,H-\frac{33}{2}\,H^{2}\right.\right.\right.\\
&\left.\left.\left.+\frac{117}{2}\right)R^{4}+\left(38\,H^{2}-75\,H-39\right)R^{3}+\left(-\frac{1539}{4}+\frac{429}{4}\,H^{2}\right.\right.\right.\\
&\left.\left.\left.+\frac{1917}{8}\,H\right)R^{2}+\left(180-\frac{507}{4}\,H^{2}+\frac{963}{4}\,H\right)R-\frac{1599\,H}{4}-\frac{2197\,H^{2}}{8}\right.\right.\\
&\left.\left.+\frac{1875}{4}\right)^{2}\left(\frac{\sqrt{-3\,R^{4}+12\,R^{3}+36\,R^{2}-84\,R-105}}{2}+\left(H\right.\right.\right.\\
&\left.\left.\left.-\frac{1}{2}\right)R^{2}+(-H+2)\,R-\frac{13\,H}{2}+\frac{5}{2}\right)^{2}\left(R^{2}-7\right)\left(R^{4}+2\,R^{3}\right.\\
&\left.\left.-12\,R^{2}-17\,R+\frac{227}{4}\right)^{3}\left(R^{6}+6\,R^{5}-15\,R^{4}-97\,R^{3}+\frac{537}{4}\,R^{2}\right.\right.\\
&\left.\left.+354\,R-\frac{2021}{4}\right)\right)
\end{aligned}
$$

```
> (Zn_HT(3) - xn_HT(3)/xn_HT(2) *Zn_HT(2) ) / xn_HT(2)^(3/2) :
```

```
  A_HT := simplify(%) assuming R1<R and R<Rc;
```

$$A_HT := -\left(\left(-R^2+4\,R-1\right)^{5/2}\left(R^2-3\right)^2\sqrt{2}\left(\left(\frac{7}{4}\,H^2-5\,H-1\right)R^8+\left(-3\,H^3-17\,H^2+56\,H+28\right)R^6+\left(H^4+35\,H^3-\frac{3}{2}\,H^2-90\,H-198\right)R^4+\left(-14\,H^4-33\,H^3-9\,H^2+80\,H+28\right)R^2+H^4-7\,H^3-\frac{105\,H^2}{4}-25\,H-1\right)\left(R^2+4\,R+1\right)^{5/2}\right)\Big/\left(2\left(-R^4+6\,R^2+7\right)^{3/2}\left(3\,R^2-1\right)\left(H\,R^4-2\,R^4-14\,H\,R^2+8\,H^2+28\,R^2+H-2\right)^2\left(R^2+1\right)\left(-R^2+2\,H+3\right)^2\right) \quad \textbf{(5.3.2)}$$

```
> (Zn_cT(4) - xn_cT(4)/xn_cT(3) *Zn_cT(3)) / xn_cT(3)^(4/3) :
  A_cT := simplify(%);
```

$$A_cT := \frac{9\left(3\,H^2-16\,H+36\right)12^{1/3}}{20\left(H^2-6\,H+12\right)^2} \quad \textbf{(5.3.3)}$$

We save the useful RPs in text file, which can be viewed and used without Maple.

```
> save nu_cT, R1,Rc,Rinf,
       b_HT, b_LT, b_cT,
       A_HT, A_LT, A_cT, "Ising-IPT_2--RP_b_A"
```

6.Peeling processes and perimeter processes

We compute the RPs of the generating functions Z0(u) and Z1(u), at the high, low and critical temperatures, respectively.

```
> Z0_HT := simplify(subs(K=0, Z_HT)):
  Z0_LT := simplify(subs(K=0, Z_LT)):
  Z0_cT := simplify(subs(K=0, Z_cT)):

  Z1_HT := simplify(subs(K=0, diff(Z_HT,K)/ diff(subs(H=K,U_HT),K)
  )):  # More precisely, this is the parametrization of Z1(u)/t,
  since U_HT is the parametrization of t*u.
  Z1_LT := simplify(subs(K=0, diff(Z_LT,K)/ diff(subs(H=K,U_LT),K)
  )):
  Z1_cT := simplify(subs(K=0, diff(Z_cT,K)/ diff(subs(H=K,U_cT),K)
  )):
```

We compute corresponding parametrizations of the order parameter 0 (nu) defined in Proposition 5.

```
> (nu_HT+1)*T_HT*( Z0_HT/U_HT - diff(Z0_HT,H)/diff(U_HT,H) - U_HT*
  diff(Z1_HT,H)/diff(U_HT,H) ):
  Order_HT := simplify( subs(H=Hc_HT, factor(%)) );

  (nu_LT+1)*T_LT*( Z0_LT/U_LT - diff(Z0_LT,H)/diff(U_LT,H) - U_LT*
  diff(Z1_LT,H)/diff(U_LT,H) ):
  Order_LT := simplify( subs(H=Hc_LT, factor(%)) );

  (nu_cT+1)*T_cT*( Z0_cT/U_cT - diff(Z0_cT,H)/diff(U_cT,H) - U_cT*
  diff(Z1_cT,H)/diff(U_cT,H) ):
  Order_cT := simplify( subs(H=Hc_cT, factor(%)) );
```

$$Order_HT := 0$$

$$Order_LT := -\left(3\ (R+1)\ (-5+R)\ \left(-\left(R-\frac{1}{2}\right)\ \left(R^6-\frac{13}{6}\ R^5-\frac{259}{12}\ R^4+\frac{100}{3}\ R^3+\frac{1693}{12}\ R^2-\frac{701}{6}\ R-\frac{1723}{6}\right)\sqrt{-3\ R^4+12\ R^3+36\ R^2-84\ R-105}+(R+4)\ \left(R^8+R^7-\frac{109}{3}\ R^6-\frac{15}{2}\ R^5+\frac{3219}{8}\ R^4+\frac{55}{12}\ R^3-\frac{13961}{8}\ R^2+\frac{47}{4}\ R+\frac{29777}{12}\right)\ (R-2)\right)\right)\Bigg/\left(2\ (R^2-R+7)\ \left(R^2+2\ R+\frac{\sqrt{-3\ R^4+12\ R^3+36\ R^2-84\ R-105}}{3}-8\right)\ \left(\left(-\frac{R}{2}+\frac{1}{4}\right)\sqrt{-3\ R^4+12\ R^3+36\ R^2-84\ R-105}+R^4+\frac{5\ R^3}{2}-\frac{69\ R^2}{4}-\frac{31\ R}{2}+\frac{229}{4}\right)\ \left(R^2-R-\frac{13}{2}\right)\ \left(R^2+\sqrt{-3\ R^4+12\ R^3+36\ R^2-84\ R-105}-4\ R-5\right)\right)$$

$$Order_cT := \frac{\sqrt{7}}{14} \qquad \textbf{(6.1)}$$

We check that the order parameter is right-continuous at nu=nu_cT (corresponding to R=Rc),
and that it has a finite limit when nu->infinity, and compute the ratio between the two limits.

```
> simplify( limit(Order_LT, R=Rc)-Order_cT );
  Order_inf := limit(Order_LT, R=Rinf);
  Order_inf / Order_cT;
```

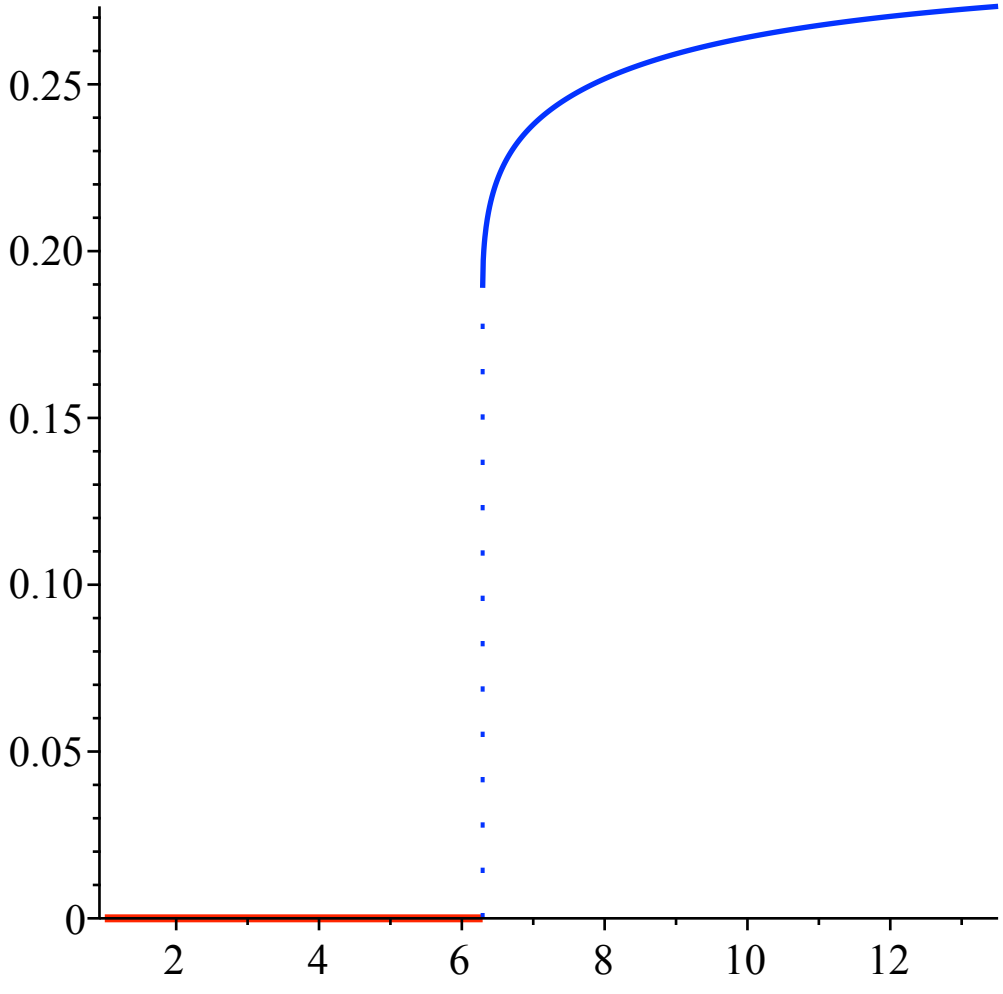
$$0$$

$$Order_inf := \frac{\sqrt{3}}{6}$$

$$\frac{\sqrt{7}\ \sqrt{3}}{3} \qquad \textbf{(6.2)}$$

The plot suggests that the order parameter is strictly increasing when nu>nu_cT.
(For the methods of a rigorous proof, see the first preliminary in this Maple notebook.)

```
> plots[display](
  [ plot([nu_HT, Order_HT, R=R1..Rc],       color='red' ,
  thickness=3),
    plot([nu_LT, Order_LT, R=Rc..Rinf-0.2], color='blue',
  thickness=2) ],
    plot([nu_cT, Ord,    Ord=0..Order_cT ], color='blue',
  linestyle=spacedot));
```



The following functions take the parametrization p_HT or p_LT of a function p(nu),

and compute the the parametrization of its derivative p'(nu) with respect to nu.

```
> d_HT := p_HT -> diff(p_HT,R)/diff(nu_HT,R);
  d_LT := p_LT -> diff(p_LT,R)/diff(nu_LT,R);
```

$$d_HT := p_HT \to \frac{\frac{\partial}{\partial R}\, p_HT}{\frac{\partial}{\partial R}\, nu_HT}$$

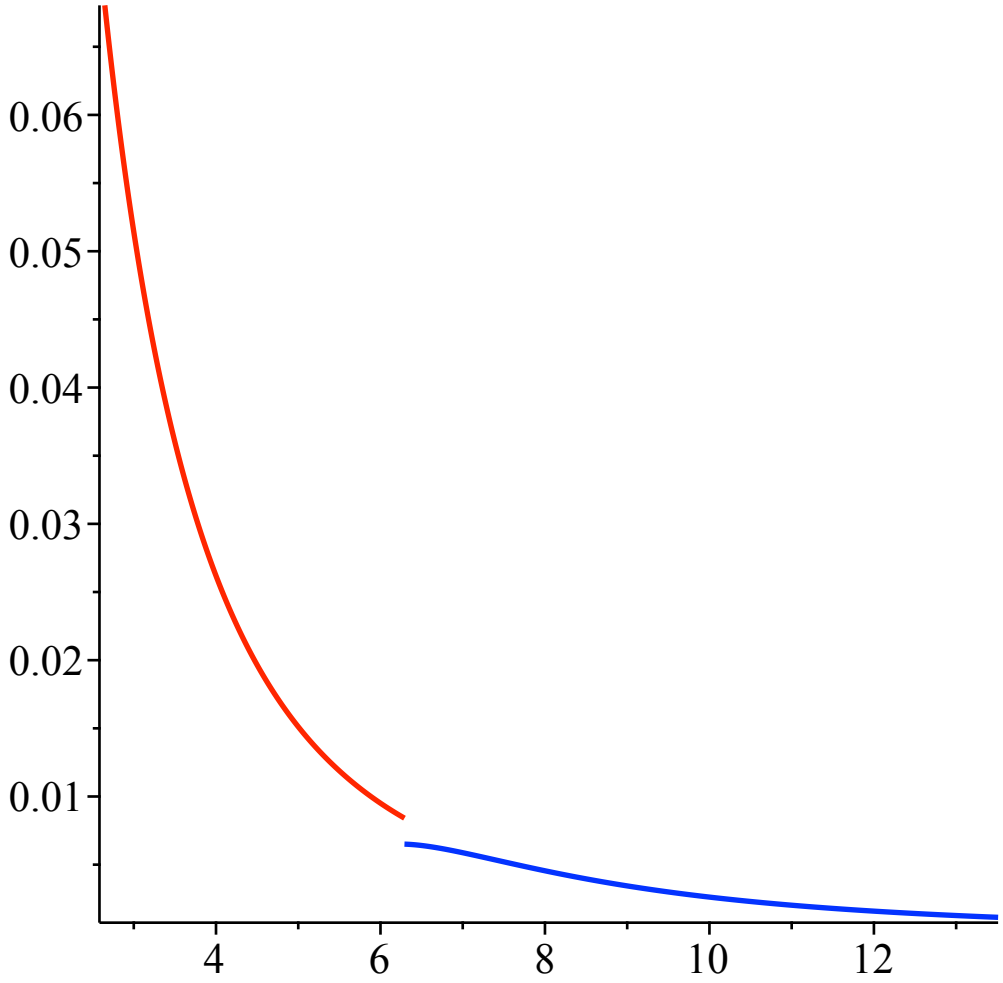
$$d_LT := p_LT \to \frac{\frac{\partial}{\partial R}\, p_LT}{\frac{\partial}{\partial R}\, nu_LT} \qquad \textbf{(6.3)}$$

The free energy per volume Fvol(nu) = -log(t_c(nu)) has a third order discontinuity at nu_c.

```
> Fvol_HT := simplify( -log(T_HT)/2 ):
  Fvol_LT := simplify( -log(T_LT)/2 ):
  evala( subs(R=Rc, (d_HT@@2)(Fvol_HT) - (d_LT@@2)(Fvol_LT)) );
  evala( subs(R=Rc, (d_HT@@3)(Fvol_HT) - (d_LT@@3)(Fvol_LT)) );
  plots[display](
  [ plot([nu_HT, (d_HT@@3)( Fvol_HT ), R=R1+0.4..Rc  ], color='red'
  , thickness=2),
    plot([nu_LT, (d_LT@@3)( Fvol_LT ), R=Rc..Rinf-0.2], color=
  'blue', thickness=2) ]);
```

$$0$$

$$\frac{7\sqrt{7}}{9720}$$



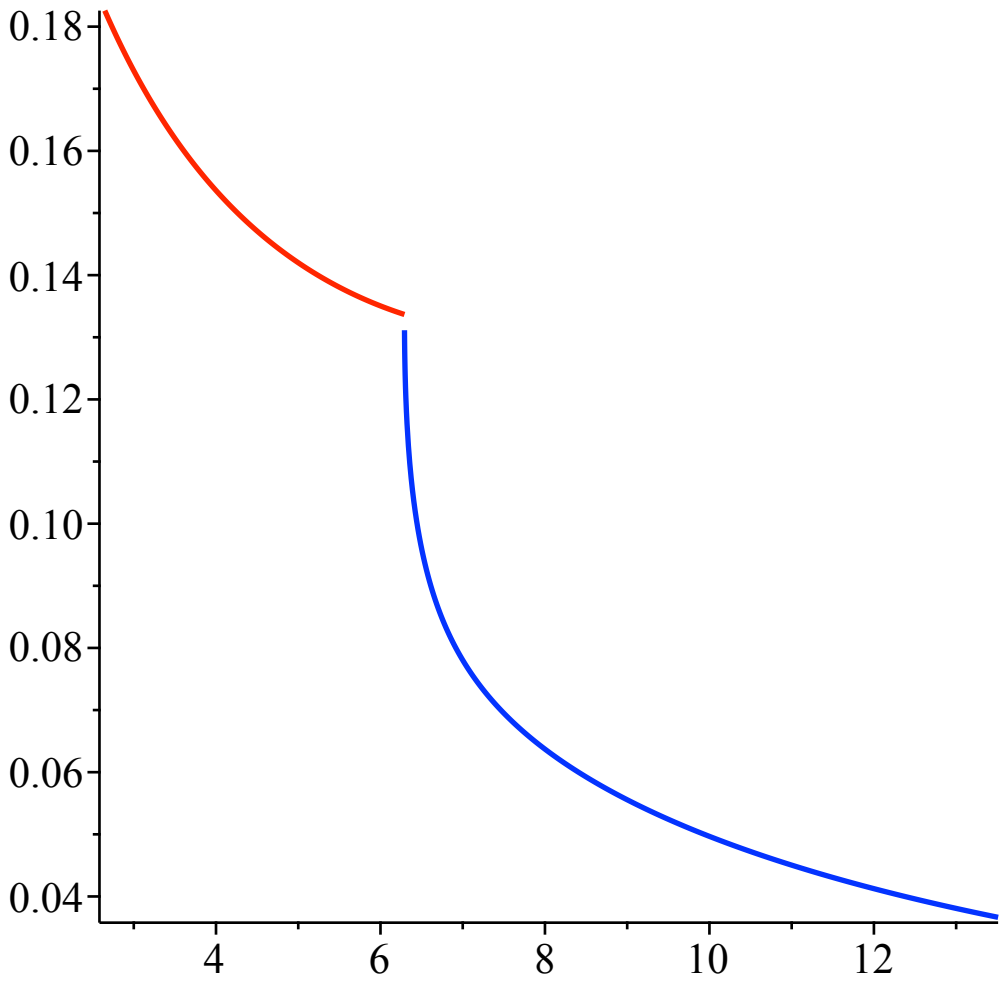
The free energy per perimeter Fper(nu) = -log(u_c(nu)) has a second order discontinuity at nu_c.

```
> Fper_HT := simplify( -log(subs(H=Hc_HT, U_HT)) - Fvol_HT ):
  Fper_LT := simplify( -log(subs(H=Hc_LT, U_LT)) - Fvol_LT ):
  evala( limit( (d_HT@@1)(Fper_HT) - (d_LT@@1)(Fper_LT), R=Rc ) );
  evala( limit( (d_HT@@2)(Fper_HT), R=Rc ) );
  limit( simplify((d_LT@@2)(Fper_LT)), R=Rc, right);
  plots[display](
  [ plot([nu_HT, (d_HT@@1)( Fper_HT ), R=R1+0.4..Rc  ], color='red'
  , thickness=2),
    plot([nu_LT, (d_LT@@1)( Fper_LT ), R=Rc..Rinf-0.2], color=
  'blue', thickness=2) ]);
```

$$0$$

$$-\frac{857}{37800}+\frac{\sqrt{7}}{144}$$

$$-\infty$$



The drift of the perimeter process (X,Y) of the infinite Ising-triangulation of law P_{infty,infty} is given by
E_infty[X_1] = t_c*(2*Z0(u_c)/u_c + Z1(u_c) - Z0'(u_c) - nu*u_c*Z1'(u_c)) when nu<=nu_c
and E_infty[X_1] = -infty when nu>nu_c
In the high temperature case, its parametrization is given by

```
> T_HT*( 2*Z0_HT/U_HT + Z1_HT - diff(Z0_HT,H)/diff(U_HT,H) - nu_HT*
  U_HT*diff(Z1_HT,H)/diff(U_HT,H) ):
  EX1_HT := factor(subs(H=Hc_HT, factor(%)));
```

$$EX1_HT := \frac{R^6 - 17\ R^4 + 75\ R^2 + 29}{16\ \left(R^3 - 3\ R + 4\right)\ \left(R^2 + 1\right)} \qquad (6.4)$$

We compute the limit of E_infty[X_1] when nu->0, given in the remarks on "Pure gravity-like behavior".

```
> subs(R=R1,EX1_HT)
```

$$\frac{1}{2} \qquad (6.5)$$

The probabilities of various peeling events in the low temperature regime:

```
> P_C_plus := evala( T_LT /subs(H=Hc_LT,U_LT) ):
  P_L_plus := evala( T_LT* subs(H=Hc_LT, Z1_LT) ):
  P_R_plus := evala( T_LT* subs(H=Hc_LT, (Z0_LT-1)/U_LT ) ):

  P_C_minus:= evala( nu_LT* P_C_plus ):
  P_L_minus:= evala( nu_LT* P_R_plus ):
  P_R_minus:= evala( nu_LT* P_L_plus ):

  a0_LT := evala( subs(H=0,A_LT) ):
  a1_LT := evala( subs(H=0, evala( (A_LT-a0_LT)/U_LT )) ):   #
  parametrizes a1/t.

  P_Rinfinity_plus := evala( T_LT*a0_LT/b_LT * subs(H=Hc_LT, (A_LT-
  a0_LT)/U_LT ) ):
  P_Linfinity_plus := evala( T_LT*a1_LT/b_LT * subs(H=Hc_LT,  A_LT-
  a0_LT) ):

  P_Rinfinity_minus:= evala( nu_LT* P_Linfinity_plus  ):
  P_Linfinity_minus:= evala( nu_LT* P_Rinfinity_plus ):
```

We compute their limits when nu->infinity. Notice that since the probabilities whose names end with "_plus" can all be
  written as (a probability ending with _minus) divided by nu, their limits when nu->infinity are necessarily zero.
We compute the limits of probabilities of the remaining peeling events, and check that the limits sum to 1:

```
> evala([ limit(P_C_minus, R=Rinf),
          limit(P_L_minus, R=Rinf),
          limit(P_R_minus, R=Rinf),
          limit( P_Linfinity_minus ,R=Rinf),
          limit( P_Rinfinity_minus ,R=Rinf)  ]);
  add(%);
```

$$\left[\frac{\sqrt{3}}{3}, \frac{1}{2} - \frac{\sqrt{3}}{6}, \frac{1}{2} - \frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{12}, 0\right]$$

$$1 \qquad (6.6)$$