ON THE CONNECTEDNESS OF p-ADIC PERIOD DOMAINS.

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Abstract. We prove that all p-adic period domains (and their non-minuscule analogues) are geometrically connected. This answers a question of Hartl [Har13] and has consequences to the geometry of Shimura and local Shimura varieties.

Contents

1. Introduction

In the study of Shimura varieties and p-adic Hodge theory, p-adic period domains and their geometric properties are recurring themes. These period domains arise as p-adic analytic open subsets of flag varieties attached to reductive groups. These open subsets arise as the open image of the Grothendieck–Messing period morphism, which stems from the theory of p-divisible groups. The first appearance of p-adic period domains in the literature is due to Drinfeld [Dri76], who introduced the Drinfeld upper half-space Ω_n. This was later complemented by Gross–Hopkins [HG94], who treated the period morphism for the Lubin–Tate tower. However, the first rigorous definition of p-adic period domains in terms of weakly admissible and admissible loci was given in the seminal book of Rapoport–Zink [RZ96], which initiated their systematic study. Since then, additional significant contributions to their study include the works of Hartl [Har08], Rapoport–Viehmann [RV14], Scholze–Weinstein [SW13, SW20], Chen–Fargues–Shen [CFS21] among others. We refer to the book of Dat–Orlik–Rapoport [DOR10] for a detailed introduction to the subject replete with examples.

The purpose of this article is to prove that p-adic period domains are geometrically connected. Our result confirms in complete generality a conjecture of Hartl, see [Har13, Conjecture 6.5]. It is also a key ingredient in understanding p-adic uniformization of Newton strata on Shimura varieties, compare also with the work of the first author and Lim–Xu [GLX22]. All of our work is done within Scholze's framework of diamonds [Sch17] which allows us to formulate and prove a more general statement. Namely, we prove that the b-admissible loci of the B^+_{dR}-affine Grassmannians (not to be confused with the μ-admissible loci in Witt flag varieties!) are geometrically connected.

1.1. **The main theorem.** Let us formulate precisely our main result. We consider a p-adic shtuka datum (G, b, μ) in the sense of Rapoport–Viehmann [RV14, Definition 5.1] but dropping the minuscule assumption on μ, compare with [SW20, Definition 23.1.1]. This consists of a reductive group G over $\mathbb{Q}_p$, an element b of the Kottwitz set $B(G) = G(\breve{\mathbb{Q}}_p)/\mathrm{ad}_\varphi(G(\breve{\mathbb{Q}}_p))$ in the sense of Kottwitz [Kot85, Kot97], and a geometric conjugacy class of (not necessarily minuscule) cocharacters $\mu \in \mathrm{Hom}(\mathbb{G}_m, G_{\bar{\mathbb{Q}}_p})/\mathrm{ad}(G(\bar{\mathbb{Q}}_p))$, such that $b \in B(G,\mu)$ as in [Kot97, §6]. Let E over $\mathbb{Q}_p$ be the reflex field of μ, i.e., the finite field extension over which the conjugacy class of μ is defined. We let $\mathbb{C}_p$ be a completed algebraic closure of $\mathbb{Q}_p$, $\breve{E} \subset \mathbb{C}_p$ the completion of the maximal unramified extension of E, and Γ denote the absolute Galois group of $\mathbb{Q}_p$.

Given $b \in B(G)$ and a characteristic p perfectoid space S, one can construct a G-bundle $\mathcal{E}_b$ over the relative Fargues–Fontaine curve $X_{\mathrm{FF},S}$ functorially in S. Attached to (G,μ), we have the spatial diamond $\mathrm{Gr}_{G,\mu}$ over $\mathrm{Spd}\,\breve{E}$ that parametrizes B^+_{dR}-lattices with G-structure that are bounded by μ in the Bruhat order [SW20, §§19-22]. Moreover, using Beauville–Laszlo glueing one can identify $\mathrm{Gr}_{G,\mu}$ with the moduli space of G-bundle modifications of $\mathcal{E}_b$

$$\mathrm{Gr}_{G,\mu}(S) = \{(\mathcal{E}, f) \mid f : \mathcal{E} \dashrightarrow \mathcal{E}_b,\ \mathrm{rel}(f) \leq \mu\}/\cong$$

whose relative position is bounded by μ (see [FS21, §III.3] for the case $b = 1$). This gives a Beauville–Laszlo uniformization map:

$$\mathcal{BL}_b : \mathrm{Gr}_{G,\mu} \to \mathrm{Bun}_G,\ (\mathcal{E}, f) \mapsto \mathcal{E},$$

that is analogous to that of [FS21, Proposition III.3.1]. Here, Bun_G denotes the small v-stack of G-bundles on the Fargues–Fontaine curve as in the book of Fargues–Scholze [FS21, §III]. Let Bun^1_G denote the sub-v-stack of Bun_G of those G-bundles that are fiberwise trivial [FS21, §III.2.3]. By [SW20, Corollary 22.5.1, Proposition 24.1.2], the b-admissible locus,[1] $\mathrm{Gr}^b_{G,\mu} := \mathcal{BL}_b^{-1}(\mathrm{Bun}^1_G)$, is non-empty and open in $\mathrm{Gr}_{G,\mu}$. Our main theorem is the following:

Theorem 1.1. *The map $\mathrm{Gr}^b_{G,\mu} \to \mathrm{Spd}\,\breve{E}$ has connected geometric fibers. Moreover, $\mathrm{Gr}^b_{G,\mu} \subset \mathrm{Gr}_{G,\mu}$ remains a dense open after base change along geometric points $\mathrm{Spd}(C, C^+) \to \mathrm{Spd}\,\breve{E}$.*

Remark 1.2. When μ is minuscule and G is quasi-split we have an identification $\mathrm{Gr}_{G,\mu} = (G/P_\mu)^\diamondsuit$, where P_μ is the parabolic subgroup of G defined as the $\mathbb{G}_m$-attractor of $-\mu$. In this case, $\mathrm{Gr}_{G,\mu}$ is (the diamond attached to) a classical flag variety (see [AGLR22, §2.2] for a discussion of the diamond functor). Moreover, we also have a formula:

$$\mathrm{Gr}^b_{G,\mu} = \pi_{\mathrm{GM}}(\mathcal{M}^\diamondsuit_{(G,b,\mu)})$$

where $\mathcal{M}_{(G,b,\mu)}$ is the local Shimura variety attached to (G,b,μ) and π_{GM} is the Grothendieck–Messing period morphism, compare with [RV14, Section 5.2] and [SW20, Definition 24.1.3]. By [Sch17, Lemma 15.6], $\mathrm{Gr}^b_{G,\mu}$ is the diamond associated with a unique open subset $\mathcal{F}(G,b,\mu)^a$ of the rigid-analytic space attached to G/P_μ. This open subset is the p-adic period domain associated to (G,b,μ), and $\mathrm{Gr}^b_{G,\mu} = \mathcal{F}(G,b,\mu)^{a,\diamondsuit}$. In particular, Theorem 1.1 shows that $\mathcal{F}(G,b,\mu)^a$ has connected geometric fibers.

Let us put Theorem 1.1 in context. In [Kis17], Kisin uses in an essential way the connected components of affine Deligne–Lusztig varieties (ADLVs) to study the Langlands–Rapoport conjecture for integral models of Shimura varieties, see [LR87]. On the other hand, Chen [Che14] uses the connected components of ADLVs to derive her main results on connected components of local Shimura varieties (LSVs). These two works motivated Chen–Kisin–Viehmann [CKV15] to compute the connected components of ADLVs at hyperspecial parahoric level building on previous work of Viehmann [Vie08]. Since then, several authors have pushed the strategy of [CKV15] to compute connected components of ADLVs deriving as corollaries results on the geometry of integral models of Shimura varieties (see the following results of Nie [Nie18, Theorem 1.1], He–Zhou [HZ20, Theorem 0.1], Hamacher [Ham20, Theorem 1.1(3)], Nie [Nie23, Theorem 0.2]).

[1] We warn the reader that in the literature $\mathrm{Gr}^b_{G,\mu}$ often stands for $\mathcal{BL}_1^{-1}(\mathrm{Bun}^b_G)$ instead.

Our result is an essential stepping stone in finishing the computation of connected components of mixed characteristic closed ADLVs in full generality, i.e., for all $(\mathcal{I}, \mu, b)$ with $\mathcal{I}$ being an Iwahori group $\mathbb{Z}_p$-model of G in the sense of Bruhat–Tits [BT84]. For this reason, it carries decisive implications to the geometry of integral models of Shimura varieties and local Shimura varieties. In order to explain this, we still need to fix some notation. Let φ denote the canonical lift of arithmetic Frobenius to $\breve{\mathbb{Z}}_p$. Let $\mathrm{Adm}(\mu) \subset \mathcal{I}(\breve{\mathbb{Z}}_p)\backslash G(\breve{\mathbb{Q}}_p)/\mathcal{I}(\breve{\mathbb{Z}}_p)$ denote the μ-admissible set of Kottwitz–Rapoport [KR00] (again, this bears no relation to b-admissibility). We consider the closed affine Deligne–Lusztig variety attached to $(\mathcal{I}, b, \mu)$, given by the formula

$$X_{\mathcal{I},\mu}(b) = \{g\mathcal{I}(\breve{\mathbb{Z}}_p) \mid g^{-1}b\varphi(g) \in \mathcal{I}(\breve{\mathbb{Z}}_p)\,\mathrm{Adm}(\mu)\mathcal{I}(\breve{\mathbb{Z}}_p)\}. \tag{1.1}$$

It admits the structure of a perfect scheme locally perfectly of finite presentation by the representability result of Bhatt–Scholze [BS17] on the Witt flag varieties defined by Zhu [Zhu17]. Let $\kappa_G : G(\breve{\mathbb{Q}}_p) \to \pi_1(G)_I$ denote the Kottwitz map in the sense of [Kot97, 7.4]. The map κ_G induces a map $\omega_G : \pi_0(X_{\mathcal{I},\mu}(b)) \to \pi_1(G)_I$ that factors through a unique coset $c_{b,\mu}\pi_1(G)_I^{\varphi} \in \pi_1(G)_I/\pi_1(G)_I^{\varphi}$. The following is a consequence of our main theorem.

Corollary 1.3. *The Kottwitz map induces a bijection*

$$\omega_G : \pi_0(X_{\mathcal{I},\mu}(b)) \xrightarrow{\simeq} c_{b,\mu}\pi_1(G)_I^{\varphi}, \tag{1.2}$$

whenever (b, μ) is Hodge–Newton irreducible.

Indeed, in work of the first author with Lim–Xu [GLX22], it is explained how to deduce Corollary 1.3 from Theorem 1.1. This work, together with [GLX22], finishes the problem of computing the connected components of closed ADLVs in mixed characteristic.

1.2. **Sketch of the proof.** Let us briefly explain the proof of Theorem 1.1 in the case where G is quasi-split. Fix a Borel $B \subseteq G$. When b is basic, Theorem 1.1 can be proved directly, and it is an unpublished result of Hansen–Weinstein. Suppose that b is not basic and let $P \subseteq G$ be the parabolic subgroup generated by B and the centralizer of ν_b.

To prove connectedness, we may and do replace $\mathrm{Gr}_{G,\mu}$ by its dense open subset $L^+P \cdot \xi^\mu$. Now, by Beauville–Laszlo gluing, $L^+P \cdot \xi^\mu$ gets identified with the space of modifications of $\mathcal{E}_p$, where $\mathcal{E}_p$ is the Harder–Narasimhan P-reduction of $\mathcal{E}_b$. Moreover, on this open subset we have a factorization:

$$\mathcal{BL}_b : L^+P \cdot \xi^\mu \to \mathrm{Bun}_P \xrightarrow{p} \mathrm{Bun}_G \tag{1.3}$$

with the first map being the analogous Beauville-Laszlo map $\mathcal{BL}_p$ for the P-torsor $\mathcal{E}_p$. Recall the following general fact. Let X be a connected locally spatial diamond that is smooth and partially proper over $\mathrm{Spa}\,\mathbb{C}_p$. Suppose that we have an open immersion $j : U \to X$ and a complementary closed immersion $i : Z \to X$. For U to be connected, it suffices that $\dim(Z) < \dim(X)$ by [Han21, Corollary 4.11]. In our case, roughly $X = L^+P \cdot \xi^\mu$ and $U = L^+P \cdot \xi^\mu \cap \mathrm{Gr}^b_{G,\mu}$ and we have left to show that the dimension of $X \setminus U$ drops, i.e., $\dim(X \setminus U) < \dim(X)$. An important observation is that the non-empty fibers of $\mathcal{BL}_p$ are torsors under the group of unipotent filtered automorphisms of $\mathcal{E}_b$, see Lemma 3.3 for a precise statement. In particular, they all have the same dimension. Also, $\mathcal{BL}_p$ factors through one connected component $\mathrm{Bun}_P^\kappa \subseteq \mathrm{Bun}_P$ determined by $\mu - \nu_b$.

Let $Y = \mathrm{Bun}_P^\kappa \setminus p^{-1}(\mathrm{Bun}_G^1)$, with p as in 1.3. The second key point is that $\dim(Y) < \dim(\mathrm{Bun}_P^\kappa)$. To prove this, we study the following diagram,

$$\begin{tikzcd} \mathrm{Bun}_P^m \arrow[r] \arrow[d] & \mathrm{Bun}_P \arrow[r] \arrow[d] & \mathrm{Bun}_G \\ \mathrm{Bun}_M^m \arrow[r] & \mathrm{Bun}_M & \end{tikzcd} \tag{1.4}$$

where M is the Levi quotient of P, $m \in B(M)$ and the square is Cartesian. When m is basic and ν_m is anti-dominant as a coweight of G, $\mathrm{Bun}_P^m \to \mathrm{Bun}_G$ is smooth of a dimension that can be made explicit. In our situation of interest, the assumption $b \in B(G,\mu)$ yields the nonpositivity condition $\mu^\diamond - \nu_b \in \mathbb{Q}_{\geq 0}\Phi_G^+$ and the relevant basic element $m \in B(M)$ satisfies that ν_m (which is related to $\nu_b - \mu^\diamond$) is in $\mathbb{Q}_{\leq 0}\Phi_G^+$. Therefore, we perform an inductive argument reducing the non-positive to the anti-dominant case via a sequence of carefully chosen parabolics and their Levis.

1.3. **Organization of the paper.** We now explain the organization of this article. We start § with some cohomological considerations to formally deal with dimension. Then, we make some preparations explaining the combinatorics involving the induction process that reduces the non-positive case to the anti-dominant case. Afterwards, we bound dimensions of Newton strata that arise from the diagram 1.4. Finally, § is dedicated to proving Theorem 1.1.

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2. Bounding dimensions of Newton strata.

2.1. **Dimension for stacky maps.** In the following sections we bound the dimensions of certain Artin v-stacks. Since we do not intend to develop foundations, we will work with an ad-hoc notion of dimension. Let $f : X \to Y$ be a *fine* morphism of Artin v-stacks in the sense of [GHW22, Definition 1.3] and let $n \in \mathbb{N}$. Let $S \to Y$ be a map with S a spatial diamond, let $f_S : X_S \to S$ denote the base change, and let $\mathcal{F} \in D^{\leq 0}_{\text{ét}}(X_S, \mathbb{F}_\ell)$.

Definition 2.1. We say that the *ℓ-cohomological dimension of f is bounded by n*, which we abbreviate as $\dim_\ell(f) \leq n$ if, for all $S \to Y$ and $\mathcal{F}$ as above,

$$f_{S,!}\mathcal{F} \in D^{\leq 2n}_{\text{ét}}(S, \mathbb{F}_\ell), \tag{2.1}$$

and we write $\dim_\ell(X) \leq n$ when $Y = *$.

Convention 2.2. From now on we will only consider maps of Artin v-stacks that are fine and we will not include this adjective in our statements.

Actually, the stacky morphisms used in this article are all obtained as compositions of smooth maps and locally closed immersions which are all fine morphisms.

Lemma 2.3. *Let $f : X \to Y$ and $g : Y \to Z$ be map of Artin v-stacks such that $\dim_\ell(f) \leq n$ and $\dim_\ell(g) \leq m$. Then $\dim_\ell(g \circ f) \leq m + n$.*

Proof. Let $S \to Z$ be a map and denote by X_S and Y_S the base changes. Let $\mathcal{F} \in D^{\leq 0}_{\text{ét}}(X_S, \mathbb{F}_\ell)$. Observe that $f_{S,!}\mathcal{F}[2n] \in D^{\leq 0}_{\text{ét}}(Y_S, \mathbb{F}_\ell)$, which implies that $g_{!,S} f_{S,!}\mathcal{F}[2n] \in D^{\leq 2m}_{\text{ét}}(S, \mathbb{F}_\ell)$. It follows that $\dim_\ell(g \circ f) \leq n + m$. □

Lemma 2.4. *Let $f : X \to Y$ be a map of Artin v-stacks. Suppose that for any $s : \mathrm{Spa}(C, C^+) \to Y$ the fibers satisfy $\dim_\ell(X_s) \leq n$. Then $\dim_\ell(f) \leq n$.*

Proof. This follows from [Sch17, Theorem 1.9.(2)], [GHW22, Theorem 1.4.(4)], since $\mathcal{F} \in D^{\leq 2n}_{\text{ét}}(S, \mathbb{F}_\ell)$ can be checked on geometric point. $\square$

The next lemma carries some weight in our paper and gives a cancelation property for ℓ-dimension in the presence of a smooth cover.

Lemma 2.5. *Let $f : X \to Y$ be a surjective ℓ-cohomologically smooth map of Artin v-stacks with constant ℓ-dimension d. Let $g : Y \to Z$ be a map of Artin v-stacks. Then $\dim_\ell(g) \leq n$ if and only if $\dim_\ell(g \circ f) \leq n + d$.*

Proof. Assume first that $\dim_\ell(g) \leq n$. In order to bound $\dim_\ell(g \circ f)$, it suffices by Lemma 2.3 to prove $\dim_\ell(f) \leq d$. It suffices to prove that $\mathrm{RHom}(f_{S,!}\mathcal{F}, \mathcal{G}) = 0$ for every map $S \to Y$, every object $\mathcal{G} \in D^{\geq 2d+1}_{\text{ét}}(S, \mathbb{F}_\ell)$ and every object $\mathcal{F} \in D^{\leq 0}_{\text{ét}}(X, \mathbb{F}_\ell)$. By adjunction, we may prove $\mathrm{RHom}(\mathcal{F}, f^!_S\mathcal{G}) = 0$ instead. Now, by ℓ-cohomological smoothness $f^!\mathcal{G} = f^*\mathcal{G} \otimes f^!\mathbb{F}_\ell$ and $f^!\mathbb{F}_\ell$ is an invertible object in $D_{\text{ét}}(X, \mathbb{F}_\ell)$ concentrated in degree $-2d$. In particular, $f^!_S\mathcal{G} \in D^{\geq 1}_{\text{ét}}(X_S, \mathbb{F}_\ell)$ while $\mathcal{F} \in D^{\leq 0}_{\text{ét}}(X_S, \mathbb{F}_\ell)$, so the required vanishing follows by the corresponding property for the natural t-structure.

For the converse, we have to show that $\dim_\ell(g) \leq n$. Let $S \to Z$ a map with S a spatial diamond, let $\mathcal{F} \in D^{\leq 0}_{\text{ét}}(Y_S, \mathbb{F}_\ell)$ and let $\mathcal{G} \in D^{\geq 2n+1}_{\text{ét}}(S, \mathbb{F}_\ell)$. As above, it suffices to prove:

$$\mathrm{RHom}(\mathcal{F}, g^!_S\mathcal{G}) = 0 \tag{2.2}$$

In other words, we wish to prove that $g^!_S\mathcal{G} \in D^{\geq 1}_{\text{ét}}(Y_S, \mathbb{F}_\ell)$, for all $\mathcal{G} \in D^{\geq 2n+1}_{\text{ét}}(S, \mathbb{F}_\ell)$. This can be verified on geometric points so we may show

$$f^*_S g^!_S\mathcal{G} \in D^{\geq 1}_{\text{ét}}(X_S, \mathbb{F}_\ell) \tag{2.3}$$

instead, since f_S is surjective. By smoothness, $f^!_S\mathbb{F}_\ell \in D^{-2d}_{\text{ét}}(X_S, \mathbb{F}_\ell)$ is an invertible object and $f^*_S g^!_S\mathcal{G} = f^!_S g^!_S\mathcal{G} \otimes (f^!_S\mathbb{F}_\ell)^{-1}$. On the other hand, it follows from the bound $\dim_\ell(g \circ f) \leq n + d$ that $f^!_S g^!_S\mathcal{G} \in D^{\geq 1-2d}_{\text{ét}}(X_S, \mathbb{F}_\ell)$ by testing RHom against the natural t-structure and passing to adjoints. In particular, we can verify that (2.3) holds. $\square$

Lemma 2.6. *Let $f : X \to Y$ be a map of Artin v-stacks. Let $i : Z \to X$ be a closed immersion and let $j : U \to X$ denote the complementary open immersion. Suppose that $\dim_\ell(i \circ f) \leq n$ and that $\dim_\ell(j \circ f) \leq n$, then $\dim_\ell(f) \leq n$. Conversely if $\dim_\ell(f) \leq n$ then $\dim_\ell(i \circ f) \leq n$ and $\dim_\ell(j \circ f) \leq n$.*

Proof. Notice that the fibers of j and i are 0-dimensional. By Lemma 2.3, the second claim follows. For the first claim, let $\mathcal{F} \in D^{\leq 0}_{\text{ét}}(X, \mathbb{F}_\ell)$, and consider the following distinguished triangle

$$f_! j_! j^*\mathcal{F} \to f_!\mathcal{F} \to f_! i_* i^*\mathcal{F} \tag{2.4}$$

in the derived category. We may pass to geometric fibers, where one of the terms vanish. $\square$

2.2. **Averages of coweights.** Let G be a quasi-split reductive group over $\mathbb{Q}_p$ and let $T \subset B \subset G$ be a pair consisting of a maximal torus that is maximally $\mathbb{Q}_p$-split and a Borel both defined over $\mathbb{Q}_p$. Let Φ_G be the absolute root system of G with respect to T and Δ_G the basis of positive simple absolute roots with respect to B. We let $X_*(T)$ denote the set of geometric cocharacters and denote by $X_*(T)_\mathbb{Q}$ the resulting rational vector space. We use the symbol M to denote a standard Levi of G defined over $\mathbb{Q}_p$, and by Δ_M the induced base of positive simple roots.

Definition 2.7. We say that $\nu \in X_*(T)_\mathbb{Q}$ is *M-dominant* (resp. *M-central*) if $\langle \alpha, \nu \rangle \geq 0$ (resp. $\langle \alpha, \nu \rangle = 0$) for all $\alpha \in \Delta_M$ and denote by $X_*(T)^{+_M}_\mathbb{Q}$ the convex set of M-dominant vectors in $X_*(T)_\mathbb{Q}$.

Following Schremmer [Sch22], we now define the so called M*-average* of ν:

$$\mathrm{av}_M(\nu) = \frac{1}{|W_M|} \sum_{w \in W_M} w\nu \tag{2.5}$$

where W_M denotes the absolute Weyl group of M.

Lemma 2.8. *The* M*-average* $\mathrm{av}_M(\nu)$ *is the unique* M*-central* $\mu \in X_*(T)_{\mathbb{Q}}$ *whose difference* $\mu - \nu$ *lies in the* $\mathbb{Q}$*-vector space spanned by the* M*-coroots* $\Delta_M^\vee$*.*

Proof. Notice that $\mathrm{av}_M(\nu)$ is W_M-invariant by definition. Also, a vector is W_M-invariant if and only if it is M-central. To see that the diference is spanned by $\Delta_M^\vee$, it is enough to check that the M-coroots evaluate to 0 under av_M. This is clear by summing left $\{1, s_\alpha\}$-cosets in W_M first, since $s_\alpha(\alpha^\vee) = -\alpha^\vee$. □

Thinking in terms of fundamental weights reveals that $2\rho_G - 2\rho_M$ pairs to 0 with every $\alpha^\vee \in \Delta_M^\vee$. Thus, it follows that $\langle 2\rho_G - 2\rho_M, \nu\rangle = \langle 2\rho_G - 2\rho_M, \mathrm{av}_M(\nu)\rangle$. We study how averaging interacts with the notion of non-positivity presented below.

Definition 2.9. We say that $\nu \in X_*(T)_{\mathbb{Q}}$ is *non-positive* (resp. *non-negative*) if it belongs to the convex hull of $X_*(Z_G)_{\mathbb{Q}}$ and $\mathbb{Q}_{\leq 0}\alpha^\vee$ (resp. $\mathbb{Q}_{\geq 0}\alpha^\vee$), where Z_G is the center of G and α runs over Δ_G. The convex set of non-positive vectors is denoted by $X_*(T)_{\mathbb{Q}}^{\leq 0}$.

In our definition above, ν is non-positive if and only if the inequality $\nu_{\mathrm{ad}} \leq 0$ holds in the rational Bruhat order of $X_*(T_{\mathrm{ad}})_{\mathbb{Q}}$. Here T_{ad} denotes the image of T in the adjoint group G_{ad} of G. An anti-dominant vector is necessarily non-positive, but for most groups the converse doesn't hold. In the following, we note that averaging preserves non-negativity, compare with [Sch22, Lemma 3.1].

Proposition 2.10. *The function* av_M *preserves* $X_*(T)_{\mathbb{Q}}^{\leq 0}$*.*

Proof. It suffices to see that it preserves $X_*(Z_G)_{\mathbb{Q}}$ and $\mathbb{Q}_{\leq 0}\alpha^\vee$. This is clear for $X_*(Z_G)_{\mathbb{Q}}$. If $\alpha^\vee \in \Delta_M$ then we already know that $\mathrm{av}_M(\alpha^\vee) = 0$, so it suffices to consider $\mathrm{av}_M(\alpha^\vee)$ for $\alpha \in \Delta_G \setminus \Delta_M$. In this case $w\alpha^\vee$ is a positive coroot for all $w \in W_M$, being a coroot of the unipotent radical of the associated standard parabolic P, and thereby finishing the proof. □

Remark 2.11. If $G = \mathrm{GL}_n$, we may interpret ν as a polygon and its non-positivity as meaning the polygon lies below the straight line connecting its extremities and never crosses it. The vector $\mathrm{av}_M(\nu)$ corresponds to connecting vertices according to a partition of n. In this case, it is visually clear that this partial average polygon lies below the total average polygon, since we started with a non-positive one.

As a corollary, we get the following technical result that is relevant in the next subsection:

Lemma 2.12. *Let* $\nu \in X_*(T)_{\mathbb{Q}}^{\leq 0}$ *be invariant under* Γ *and* M*-central. There is a sequence of standard Levi subgroups* $M = M_0 \subset \cdots \subset M_i \subset \cdots \subset M_k = G$ *defined over* $\mathbb{Q}_p$ *and also of* Γ*-invariant vectors* $\nu = \nu_0, \dots, \nu_i, \dots \nu_k = \mathrm{av}_G(\nu)$ *in* $X_*(T)_{\mathbb{Q}}^{\leq 0}$ *such that the following properties hold*

(1) $\nu_j = \mathrm{av}_{M_j}(\nu_i)$ *for* $j \geq i$.
(2) ν_i *is* M_{i+1}*-anti-dominant.*

Proof. Suppose $\langle \alpha, \nu\rangle \geq 0$ for all $\alpha \in \Delta_G \setminus \Delta_M$. Since $\langle \alpha, \nu\rangle = 0$ for $\alpha \in \Delta_M$ by hypothesis, we also get $\langle \rho_G, \nu\rangle \geq 0$. On the other hand, the convex hull of $X_*(Z_G)_{\mathbb{Q}}$ and $\mathbb{Q}_{\leq 0}\alpha^\vee$ for all $\alpha \in \Delta_G$ pairs non-positively with the strictly dominant weight ρ_G, and it vanishes exactly on G-central elements. Therefore, the only possibility would be $M = G$, in which case $k = 0$ and ν is G-central.

Otherwise, there exists some $\alpha \in \Delta_G \setminus \Delta_M$ such that $\langle \alpha, \nu \rangle < 0$. By Γ-invariance, this holds for its entire Γ-orbit. Now let L be the standard Levi defined over $\mathbb{Q}_p$ with $\Delta_L = \Delta_M \cup \Gamma\alpha$. Clearly ν is L-anti-dominant. Moreover, by Proposition 2.10 $\mathrm{av}_L(\nu)$ is non-positive and L-central, which finishes the proof of the lemma by induction on the cardinality of $\Delta_G \setminus \Delta_M$. □

2.3. **Parabolic Newton strata.** In this subsection, we continue to work under the same assumptions and use similar notations. Pick $b \in B(M)$ and write $\nu_b \in X_*(T)^\Gamma_{\mathbb{Q}}$ for the M-dominant Newton point of b. We warn the reader that we follow the opposite sign convention to [FS21, pages 59 and 90], so that the slope 1 isocrystal $(\breve{\mathbb{Q}}_p, \pi\varphi)$ is sent to the line bundle $\mathcal{O}(1)$ on the Fargues–Fontaine curve: this will lead to sign changes everywhere compared to many of our sources below. We have notions of dominance and positivity for elements of $B(M)$.

Definition 2.13. We say that $b \in B(M)$ is *dominant* (respectively *anti-dominant*) if the M-dominant Newton point ν_b is G-dominant (respectively G-anti-dominant). We say that it is *non-positive* if the M-dominant Newton point ν_b is G-non-positive.

From now on, we assume that $\nu := \nu_b$ is non-positive. Consider a sequence $M = M_0 \subset \cdots \subset M_i \subset \cdots \subset M_k = G$ of standard Levi subgroups defined over $\mathbb{Q}_p$ and of Γ-invariant vectors ν_i as in Lemma 2.12. We inductively choose basic elements $b_i \in B(M_i)$ sharing the same image under the Kottwitz map, i.e., with $\kappa_{M_i}(b_i) = \kappa_{M_i}(b)$. It follows immediately by construction that the Newton point of b_i coincides with ν_i. For $i \leq j$, we still write b_i for its image in $B(M_j)$ under the natural map $B(M_i) \to B(M_j)$. In particular, b_i is non-positive in M_j for all $j \geq i$ and even anti-dominant if $j = i+1$. For simplicity, we also use the shorthand $\rho_i := \rho_{M_i}$ and $\rho_{ij} := \rho_j - \rho_i$. Note that $\langle 2\rho_{ij}, -\nu_i \rangle = \langle 2\rho_{ij}, -\nu_j \rangle$ for all $i \leq j$.

For any $j \geq i$, let $P_{ij} \subset M_{i+1}$ be the standard parabolic whose standard Levi is M_j. We now define the locally closed stratum $\mathrm{Bun}^{b_i}_{P_{ij}}$ by demanding that the square in the following commutative diagram

$$\begin{array}{ccccc} \mathrm{Bun}^{b_i}_{P_{ij}} & \longrightarrow & \mathrm{Bun}_{P_{ij}} & \longrightarrow & \mathrm{Bun}_{M_j} \\ \downarrow & & \downarrow & & \\ \mathrm{Bun}^{b_i}_{M_i} & \longrightarrow & \mathrm{Bun}_{M_i} & & \end{array} \tag{2.6}$$

is Cartesian. The following theorem was shown in [AB21] and further refined in [Ham22].

Theorem 2.14. *The map of Artin v-stacks $\mathrm{Bun}_{P_{ij}} \to \mathrm{Bun}_{M_i}$ is ℓ-cohomologically smooth and has connected geometric fibers. Over the b_i-stratum, it is of relative ℓ-dimension $\langle 2\rho_{ij}, -\nu_i \rangle$.*

Proof. This follows from [Ham22, Proposition 3.16, Proposition 4.7]. □

In the example below, we see concrete examples of these parabolic strata in the case of GL_2 and taking our sign convention into account.

Example 2.15. Let $G = \mathrm{GL}_2$, let $T \subseteq G$ denote the diagonal torus and let $B \subseteq G$ denote the upper diagonal matrices. In our sign convention, we consider $\mathcal{O}(1)$ to have slope 1. Consider the polygons $b_1 = (2, 1)$ and $b_2 = (1, 2)$. For our sign convention, b_1 is G-dominant and corresponds to $\mathcal{O}(2) \oplus \mathcal{O}(1)$ as a T-torsor. On the other hand, b_2 is G-anti-dominant and corresponds to the T-torsor $\mathcal{O}(1) \oplus \mathcal{O}(2)$. In this example, $\mathrm{Bun}^{b_1}_B$ classifies extensions of the form

$$0 \to \mathcal{O}(2) \to \mathcal{E} \to \mathcal{O}(1) \to 0. \tag{2.7}$$

These extensions are trivial and the automorphism group is 1-dimensional. Overall, $\dim(\mathrm{Bun}^{b_1}_B) = -1$ which agrees with the formula $\langle \rho_G, -\nu_{b_1} \rangle$.

Remark 2.16. Suppose $\nu := \nu_0$ is already anti-dominant, then we may take $k = 1$ and $M_k = G$. Then, the b-stratum of Bun_P identifies with the Fargues–Scholze chart $\mathcal{M}_b$ attached to $b \in B(G)$, see [FS21, Example V.3.4] (note the change of sign convention here). If, on the other hand, we worked with dominant coweights, the b-stratum of Bun_P would identify with that of Bun_G by the Harder–Narasimhan filtration.

We want to study the geometry of the natural map $\mathrm{Bun}_P^b \to \mathrm{Bun}_G$ for non-positive basic $b \in B(M)$. We will proceed by induction with the help of our sequence of standard Levis in order to bootstrap a (somewhat weaker) statement from the anti-dominant case. We get the following commutative diagram with Cartesian square

$$\begin{tikzcd} \mathrm{Bun}_{P_{ik}} \arrow[r] \arrow[d] & \mathrm{Bun}_{P_{jk}} \arrow[r] \arrow[d] & \mathrm{Bun}_{M_k} \\ \mathrm{Bun}_{P_{ij}} \arrow[r] \arrow[d] & \mathrm{Bun}_{M_j} & \\ \mathrm{Bun}_{M_i} & & \end{tikzcd} \tag{2.8}$$

with P_{ij} denoting the standard parabolic of M_j with standard Levi equal to M_i. In particular, we get a natural map $\Delta_{ijk} \colon \mathrm{Bun}_{P_{ik}} \to \mathrm{Bun}_{M_i} \times \mathrm{Bun}_{M_j}$, and we define the (b_i, c_j)-strata as the pullback

$$\mathrm{Bun}_{P_{ik}}^{(b_i,c_j)} := \Delta_{ijk}^{-1}(\mathrm{Bun}_{M_i}^{b_i} \times \mathrm{Bun}_{M_j}^{c_j}) \subset \mathrm{Bun}_{P_{ik}} \tag{2.9}$$

for some $b_i \in B(M_i)$ and $c_j \in B(M_j)$.

Proposition 2.17. *Let $b_i \in B(M_i)$ be a sequence of non-positive basic elements with anti-dominant steps. For every $j \geq i$, the b_i-stratum of $\mathrm{Bun}_{P_{ij}}$ contains an open subspace $\mathcal{T}_{ij}$ such that the induced map $f_{b_i} : \mathcal{T}_{ij} \to \mathrm{Bun}_{M_j}$ satisfies the following:*

1. *f_{b_i} factors through the b_j-stratum of Bun_{M_j} and it is ℓ-cohomologically smooth of relative dimension $\langle 2\rho_j - 2\rho_i, -\nu_i \rangle$.*
2. *The dimension of the closed complement drops, i.e., $\dim_\ell(\mathrm{Bun}_{P_{ij}}^{b_i} \setminus \mathcal{T}_{ij}) < \langle 2\rho_j - 2\rho_i, -\nu_i \rangle$.*

Proof. We proceed to construct the open subspace $\mathcal{T}_{ij}$ recursively and show inductively that the natural map towards Bun_{M_j} is ℓ-cohomologically smooth. We do this by induction on $j - i$. If it equals 1, then b_i is M_j-anti-dominant, so the natural map $\mathrm{Bun}_{P_{ij}} \to \mathrm{Bun}_{M_j}$ restricts to an ℓ-cohomologically smooth map over the b_i-stratum by [FS21, Theorem V.3.7]. If we set $\mathcal{T}_i$ as the (b_i, b_j)-stratum of $\mathrm{Bun}_{P_{ij}}$ following (2.9) (here we put $j = k$), then we immediately get the desired properties.

In the general case, we consider $i < j < k$. By the inductive hypothesis, we may and do assume that the result is known for the intermediate pairs (i, j) and (j, k). In fact, we may even assume that b_i is M_j-anti-dominant by passing to the immediate step $j = i + 1$ if necessary. Pulling back the diagram along the b_i- and b_j-strata yields the following commutative diagram with Cartesian squares:

$$\begin{tikzcd} \mathrm{Bun}_{P_{ik}}^{(b_i,b_j)} \arrow[r] \arrow[d] & \mathrm{Bun}_{P_{ik}}^{b_i} \arrow[r] \arrow[d] & \mathrm{Bun}_{P_{ij}}^{b_i} \arrow[d] \\ \mathrm{Bun}_{P_{jk}}^{b_j} \arrow[r] & \mathrm{Bun}_{P_{jk}} \arrow[r] & \mathrm{Bun}_{M_j}. \end{tikzcd} \tag{2.10}$$

By induction, we obtain a map $\mathcal{T}_{jk} \to \mathrm{Bun}_{P_{jk}}$ factoring over the b_j-stratum, i.e., the lower left corner in the above diagram. We now define $\mathcal{T}_{ik} \to \mathrm{Bun}_{P_{ik}}$ as the pullback of this new arrow along the left vertical one in the diagram above, thereby completing it with a new Cartesian square.

$$\begin{tikzcd} \mathcal{T}_{ik} \arrow[r] \arrow[d] & \mathrm{Bun}_{P_{ik}}^{(b_i,b_j)} \arrow[r] \arrow[d] & \mathrm{Bun}_{P_{ik}}^{b_i} \arrow[r] \arrow[d] & \mathrm{Bun}_{P_{ij}}^{b_i} \arrow[d] \\ \mathcal{T}_{jk} \arrow[r] & \mathrm{Bun}_{P_{jk}}^{b_j} \arrow[r] & \mathrm{Bun}_{P_{jk}} \arrow[r] & \mathrm{Bun}_{M_j}. \end{tikzcd} \tag{2.11}$$

By induction, the map $\mathcal{T}_{jk} \to \mathrm{Bun}_{M_k}$ is ℓ-cohomologically smooth and since b_i is M_j-anti-dominant so is also $\mathcal{T}_{ik} \to \mathcal{T}_{jk}$, implying ℓ-cohomological smoothness of the composition. The dimension claim follows since $\mathrm{Bun}_{M_k} \to *$ is ℓ-smooth of dimension 0 by [FS21, Theorem I.4.1.(vii)], while $\mathrm{Bun}_{P_{ik}}^{b_i} \to *$ is ℓ-cohomologically smooth of dimension $\langle 2\rho_{ik}, -\nu_i\rangle$ by Theorem 2.14.

Next, we handle the second claim for the above definition of $\mathcal{T}_{ik}$. Pick $c_j \in B(M_j)$ in the image of $\mathrm{Bun}_{P_{ij}}^{b_i} \to \mathrm{Bun}_{M_j}$ and let μ_j be its M_j-dominant Newton point. We get an ℓ-cohomologically smooth map $\mathrm{Bun}_{P_{ik}}^{(b_i,c_j)} \to \mathrm{Bun}_{P_{jk}}^{c_j}$ of dimension $\langle 2\rho_{ij}, -\nu_i\rangle$. By Theorem 2.14, the map $\mathrm{Bun}_{P_{jk}}^{c_j} \to \mathrm{Bun}_{M_j}$ is ℓ-cohomologically smooth of dimension $\langle 2\rho_{ik}, -\mu_j\rangle$. Since the b_i- and the c_j-strata belong to the same connected component of Bun_{M_j}, we see that $\nu_i - \mu_j$ lies in the $\mathbb{Q}$-span of $\Delta^\vee_{M_j}$ and so it is orthogonal to ρ_{jk}. We conclude that the composition

$$\mathrm{Bun}_{P_{ik}}^{(b_i,c_j)} \to \mathrm{Bun}_{P_{jk}}^{c_j} \to \mathrm{Bun}_{M_j}^{c_j} \tag{2.12}$$

is ℓ-cohomologically smooth of relative dimension $\langle 2\rho_{ik}, -\nu_i\rangle$. The term on the right has strictly negative dimension as soon as $c_j \neq b_j$, so by Lemma 2.3 and Lemma 2.6, it follows that the dimension of the complement of the (b_i, b_j)-stratum of $\mathrm{Bun}_{P_{ik}}$ drops. On the other hand, we have a Cartesian diagram by definition:

$$\begin{tikzcd} \mathrm{Bun}_{P_{ik}}^{(b_i,b_j)} \setminus \mathcal{T}_{ik} \arrow[r] \arrow[d] & \mathrm{Bun}_{P_{ik}}^{(b_i,b_j)} \arrow[r] \arrow[d] & \mathrm{Bun}_{P_{ik}}^{b_i} \arrow[d] \\ \mathrm{Bun}_{P_{jk}}^{b_j} \setminus \mathcal{T}_{jk} \arrow[r] & \mathrm{Bun}_{P_{jk}}^{b_j} \arrow[r] & \mathrm{Bun}_{P_{jk}} \end{tikzcd} \tag{2.13}$$

and we know by induction that the dimension drops along the left bottom horizontal arrow. Since the vertical maps are ℓ-cohomologically smooth of dimension $\langle 2\rho_{ij}, -\nu_i\rangle$, we conclude that dimension also drops along the left upper horizontal arrow. By Lemma 2.6, we may now combine these two dimension drops to reach our desired conclusion.

□

3. $\mathrm{Gr}_{G,\mu}^b$ IS CONNECTED

In contrast with the previous section, we will momentarily not assume that G is quasi-split. Fix C an arbitrary algebraically closed non-archimedean field extension of $\breve{E}$ and consider the Beauville–Laszlo map defined over $\mathrm{Spd}\, C$,

$$\mathcal{BL}_b : \mathrm{Gr}_{G,\mu} \to \mathrm{Bun}_G \times \mathrm{Spd}\, C \to \mathrm{Spd}\, C. \tag{3.1}$$

We fix our sign convention for modifications when $G = \mathbb{G}_m$, and extend the convention by functoriality to all other groups. We consider the inclusion of the ideal sheaf $\mathcal{O}(-1) \subseteq \mathcal{O}$ to be a modification of $\mathcal{O}$ of type $\mu = 1 \in \mathbb{Z} \simeq X_*(\mathbb{G}_m)$. We observe that under our sign convention, $\mathcal{BL}_b$ factors through the unique connected component of Bun_G parametrized by $\kappa_G(b) - \mu^\natural \in \pi_1(G)_\Gamma$, compare with [FS21, Corollary IV.1.23]. We formulate Theorem 1.1 as follows:

Theorem 3.1. *If $b \in B(G, \mu)$, then $\mathrm{Gr}_{G,\mu}^b$ is connected and it is dense in $\mathrm{Gr}_{G,\mu}$.*

Without loss of generality we may assume that G is adjoint (see the proof of [PR24, Proposition 3.1.1]). Moreover, when G is adjoint it follows that if G^* denotes its quasi-split inner form, then G^* is a pure inner form of G. In particular, we have an identification $\mathrm{Bun}_{G^*} \simeq \mathrm{Bun}_G$. This allow us to assume that G is quasi-split, at the expense of proving the more general Theorem 3.2 below.

In order to do this, we will fix additional notation. From now on we assume again that G is quasi-split and we fix $T \subset B \subset G$ as in the previous section. We define an element $\mu^\diamond \in X_*(T)^\Gamma_{\mathbb{Q}}$ given by the formula:

$$\mu^\diamond := \frac{1}{[\Gamma : \Gamma_\mu]} \sum_{\gamma \in \Gamma/\Gamma_\mu} \gamma(\mu), \tag{3.2}$$

where Γ_μ denotes the stabilizer of μ for the Γ-action. Notice that $\langle 2\rho_G, \mu^\diamond\rangle = \langle 2\rho_G, \mu\rangle$, because ρ_G is Γ-invariant.

Let $A_Z(G,\mu) \subset B(G)$ be the set of acceptable elements modulo center, i.e. for which $\mu^\diamond - \nu_b$ is non-negative as in Definition 2.9. This is related to the notion of acceptable elements $A(G,\mu)$ of [RV14, Definition 2.3], in the sense that $A_Z(G,\mu)$ equals the pre-image of $A(G_{\mathrm{ad}}, \mu_{\mathrm{ad}})$ along $B(G) \to B(G_{\mathrm{ad}})$. If $m \in B(M)$, we let m_μ denote the unique basic element in $B(M)$ such that $\kappa_M(m_\mu) = \kappa_M(m) - \mu^\natural$. Let $d = \dim_\ell(\mathrm{Gr}_{G,\mu}) = \langle 2\rho_G, \mu\rangle$ and define the (c,b)-admissible locus $\mathrm{Gr}^{(c,b)}_{G,\mu} := \mathcal{BL}_b^{-1}(\mathrm{Bun}^c_G) \subset \mathrm{Gr}_{G,\mu}$. If $c = 1$, this recovers our usual b-admissible locus.

Theorem 3.2. *If $b \in A_Z(G,\mu)$, then $\mathrm{Gr}^{(b_\mu,b)}_{G,\mu}$ is dense in $\mathrm{Gr}_{G,\mu}$ and connected.*

Proof. To prove that $\mathrm{Gr}^{(b_\mu,b)}_{G,\mu}$ is dense and connected, it suffices to show that $\dim_\ell(\mathrm{Gr}^{(c,b)}_{G,\mu}) < d$ for all $c \in B(G)$ with $c \neq b_\mu$. Since the dimension drops on the complement of the Schubert cell $\mathrm{Gr}^\circ_{G,\mu} \subset \mathrm{Gr}_{G,\mu}$, it suffices to compute the dimension of their intersection $\mathrm{Gr}^{\circ,(c,b)}_{G,\mu}$.

If b is basic, $\mathcal{BL}_b : [\underline{G(\mathbb{Q}_p)} \backslash \mathrm{Gr}^\circ_{G,\mu}] \to \mathrm{Bun}_G$ is smooth of relative dimension d by [FS21, Proposition IV.1.18, Theorem IV.1.19]. In particular, $\dim_\ell(\mathrm{Gr}^{\circ,(g,b)}_{G,\mu}) = d + \dim_\ell(\mathrm{Bun}^g_G)$. Now, b_μ is the unique basic element in the image of $\mathcal{BL}_b$ and for non-basic elements $\dim_\ell(\mathrm{Bun}^g_G) < 0$ by [FS21, IV.1.22]. This finishes the proof in this case.

Suppose now that b is not basic. Let M denote the centralizer of ν_b, and m denote the unique basic element in $B(M)$ lifting b whose Newton point is G-dominant, i.e., $\nu_m = \nu_b$. Now, $\mathrm{Bun}^m_P \simeq \mathrm{Bun}^b_G$ by our choice of m, and we let $\mathcal{E}_p$ denote the unique P-reduction of $\mathcal{E}_b$ on this strata. This is the so-called Harder–Narasimhan reduction of $\mathcal{E}_b$.

The space of P-modifications of $\mathcal{E}_p$ gets identified with $\mathrm{Gr}_P \subset \mathrm{Gr}_G$. We consider $\mathrm{Gr}^\circ_{P,\mu}$, defined as the intersection of $\mathrm{Gr}_{G,\mu}$ with the connected component of Gr_P attached to the G-dominant representative of μ. We can also write this as $\mathrm{Gr}^\circ_{P,\mu} := L^+P \cdot \xi^\mu \cdot L^+P/L^+P$, where $\xi \in B^+_{\mathrm{dR}}$ is a uniformizer and $L^+P = P(B^+_{\mathrm{dR}})$. By our choice of μ, we get an open immersion, $\mathrm{Gr}^\circ_{P,\mu} \subset \mathrm{Gr}^\circ_{G,\mu}$, and a smooth map $\mathrm{Gr}^\circ_{P,\mu} \to \mathrm{Gr}^\circ_{M,\mu}$ of relative dimension $\langle 2\rho_{M,G}, \mu\rangle$, where we set $\rho_{M,G} = \rho_G - \rho_M$. We can also describe $\mathrm{Gr}^\circ_{M,\mu}$ as the quotient $L^+M \cdot \xi^\mu \cdot L^+M/L^+M$ compatibly with the parabolic description and the smooth cover. Moreover, we have commutative diagrams

$$\begin{tikzcd}
\mathrm{Gr}^{m_\mu}_{P,\mu} \arrow[r] \arrow[d] & \mathrm{Gr}^\circ_{P,\mu} \arrow[r] \arrow[d, "\mathcal{BL}_p"] & \mathrm{Gr}^\circ_{G,\mu} \arrow[d, "\mathcal{BL}_b"] & \mathrm{Gr}^\circ_{P,\mu} \arrow[r] \arrow[d, "\mathcal{BL}_p"] & \mathrm{Gr}^\circ_{M,\mu} \arrow[d, "\mathcal{BL}_m"] \\
\mathrm{Bun}^{m_\mu}_P \arrow[r] \arrow[d] & \mathrm{Bun}_P \arrow[r] \arrow[d] & \mathrm{Bun}_G & \mathrm{Bun}_P \arrow[r] & \mathrm{Bun}_M \\
\mathrm{Bun}^{m_\mu}_M \arrow[r] & \mathrm{Bun}_M. & & &
\end{tikzcd} \tag{3.3}$$

with the first row of vertical arrows being Beauville–Laszlo uniformization maps and the top corner $\mathrm{Gr}_{P,\mu}^{m_\mu}$ making the left upper square Cartesian. In particular, the map $\mathrm{Gr}_{P,\mu}^{m_\mu} \to \mathrm{Gr}_{P,\mu}^{\circ}$ is a non-empty open immersion, so the left side has dimension d. Moreover, the map $\mathrm{Gr}_{P,\mu}^{\circ} \to \mathrm{Bun}_M$ is the composition of maps that are either ℓ-cohomologically smooth or pro-étale. Using this and the fact that $m_\mu \in B(M)$ is basic, it follows that the dimension drops on the complement of $\mathrm{Gr}_{P,\mu}^{m_\mu} \to \mathrm{Gr}_{P,\mu}^{\circ}$. We are reduced to showing that for $c \neq b_\mu$ the following inequality holds

$$\dim_\ell(\mathrm{Gr}_{P,\mu}^{m_\mu} \cap \mathrm{Gr}_{G,\mu}^{(c,b)}) < d. \tag{3.4}$$

We claim that for our choice of m, the element $m_\mu \in B(M)$ is non-positive. Recall that by our sign conventions $\kappa_M(m_\mu) = \kappa_M(m) - (\mu^\diamond)^\natural$ in $\pi_1(M)_\Gamma$. This corresponds to the unique connected component in $\pi_0(\mathrm{Bun}_M)$ through which $\mathcal{BL}_m$ factors. It follows that $\nu_{m_\mu} = \mathrm{av}_M(\nu_b - \mu^\diamond)$ since M-central elements in $X_*(T)_{\mathbb{Q}}^\Gamma$ are determined by their image in $\pi_1(M)_{\mathbb{Q}}^\Gamma \simeq \pi_1(M)_{\mathbb{Q},\Gamma}$. Using our assumption that $b \in A_Z(G,\mu)$ and Proposition 2.10 it follows that ν_{m_μ} is non-positive.

An application of Proposition 2.17 shows that

$$\dim_\ell(\mathrm{Bun}_P^{(m_\mu,c)}) < \langle 2\rho_{M,G}, -\nu_{m_\mu}\rangle = \langle 2\rho_{M,G}, \mathrm{av}_M(\mu^\diamond - \nu_b)\rangle. \tag{3.5}$$

By Lemma 3.3 below, the geometric fibers of

$$\mathrm{Gr}_{P,\mu}^{m_\mu} \to \mathrm{Gr}_{M,\mu}^{\circ} \times_{\mathrm{Bun}_M} \mathrm{Bun}_P^{m_\mu} \tag{3.6}$$

have all dimension bounded by $\langle 2\rho_{M,G}, \nu_b\rangle$. Consequently, Lemma 2.4 shows that (3.4) holds. Indeed, $\dim_\ell(\mathrm{Gr}_{P,\mu}^{m_\mu} \cap \mathrm{Gr}_{G,\mu}^{(c,b)})$ is bounded by the dimension of $\mathrm{Gr}_{M,\mu}^{\circ} \times_{\mathrm{Bun}_M} \mathrm{Bun}_P^{(m_\mu,c)}$ plus the dimension of the fibers. The former is strictly smaller than $\langle 2\rho_M, \mu\rangle + \langle 2\rho_{M,G}, \mathrm{av}_M(\mu^\diamond - \nu_b)\rangle$ and the latter is bounded by $\langle 2\rho_{M,G}, \nu_b\rangle$. Moreover, we have equalities

$$\langle 2\rho_{M,G}, \mathrm{av}_M(\mu^\diamond - \nu_b)\rangle = \langle 2\rho_{M,G}, \mu^\diamond - \nu_b\rangle = \langle 2\rho_{M,G}, \mu - \nu_b\rangle, \tag{3.7}$$

from which the bound follows. □

Lemma 3.3. *The geometric fibers of* (3.6) *are either empty or torsors under the kernel of* $\mathrm{Aut}(\mathcal{E}_p) \to \mathrm{Aut}(\mathcal{E}_m)$. *In the former case, their dimension is* $\langle 2\rho_{M,G}, \nu_b\rangle$.

Remark 3.4. The quasi-torsor assertion does not use any special properties of $b \in B(G)$ or of our chosen P-reduction $\mathcal{E}_p \in \mathrm{Bun}_P(C)$. Nevertheless, the precise dimension of this group of automorphisms only holds for the Harder–Narasimhan reduction $\mathcal{E}_p$ of b.

Proof. We begin by observing that the geometric fibers of the Beauville–Laszlo map

$$\mathcal{BL}_{\mathcal{E}_p} : \mathrm{Gr}_P \to \mathrm{Bun}_P \tag{3.8}$$

are torsors on the left for the group $A^{-1}P(B_e)A$ where $A \in P(B_{\mathrm{dR}})$ is the Beauville–Laszlo gluing data for the P-torsor $\mathcal{E}_p$, see [SW20, Theorem 13.5.3.(2)]. Indeed, this is the group of modifications of the form

$$f : \mathcal{E}_p \dashrightarrow \mathcal{E}_p$$

and given a fixed modification $\alpha : \mathcal{E} \dashrightarrow \mathcal{E}_p$, every other modification having $\mathcal{E}$ as its source and $\mathcal{E}_p$ as its target can be obtained by composing α with some f as above. Similarly, the geometric fibers of $\mathrm{Gr}_M \to \mathrm{Bun}_M$ are $A^{-1}M(B_e)A$-torsors. We deduce that the geometric fibers of $\mathrm{Gr}_P \to \mathrm{Bun}_P \times_{\mathrm{Bun}_M} \mathrm{Gr}_M$ are torsors under the group $A^{-1}U(B_e)A$. In other words, the fibers are torsors under the group of modifications $\mathcal{E}_p \dashrightarrow \mathcal{E}_p$ that induce the identity on $\mathcal{E}_m$.

Recall that every $t \in P(B_{\mathrm{dR}})$ has a unique expression $t = u_t \cdot m_t$ with $u \in U(B_{\mathrm{dR}})$ and $m \in M(B_{\mathrm{dR}})$. We claim that if $t \in P(B_{\mathrm{dR}}^+)\xi^\mu P(B_{\mathrm{dR}}^+)$ then $u_t \in U(B_{\mathrm{dR}}^+)$. This follows from the normality of $U(B_{\mathrm{dR}}^+)$ in $P(B_{\mathrm{dR}}^+)$ and from the inclusion $\xi^\mu U(B_{\mathrm{dR}}^+) \subseteq U(B_{\mathrm{dR}}^+)\xi^\mu$, given the fact that μ is dominant. Consequently, if $u \in U(B_{\mathrm{dR}})$ and $x \in \mathrm{Gr}_{P,\mu}^{\circ}$ are such that $u \cdot x \in \mathrm{Gr}_{P,\mu}^{\circ}$, then necessarily $u \in U(B_{\mathrm{dR}}^+)$.

This implies that the non-empty geometric fibers of our map (3.6) form a torsor under the group $U(B_{\mathrm{dR}}^+) \cap A^{-1}U(B_e)A$, the unipotent part of the automorphism group of $(\mathcal{E}_p)$. By [FS21, Proposition III.5.1], its ℓ-dimension equals $\langle 2\rho_{M,G}, \nu_b \rangle$, and we may conclude the same about the non-empty fibers. □

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