PROPAGATION OF COHERENT STATES IN QFT

I - QFT formalism.

h = $\mathbb{C}$-Hilbert = one-particle space

For n bosons: space $\otimes_s^n h$

Fock space $\Gamma_s(h) = \bigoplus_{n=0}^{\infty} \otimes_s^n h = \mathbb{C} \oplus h \oplus h \otimes_s h \oplus \cdots$

$a(u)$ / $a^+(v)$

$a(u)$ = annihilation op. } depend on $0 < \varepsilon \leq 1$
$a^+(v)$ = creation op.

Segal field op. $\phi(u) = \frac{1}{\sqrt{2}}(a(u) + a^+(u))$. Selfadjoint

Weyl op $W(u) = e^{i\phi(u)}$

Coherent states: $W(-i \frac{\sqrt{2}}{\varepsilon} u)\Omega$, $u \in h$

where $\Omega = (1, 0, 0, \dots)$ vacuum

Lifting op. of $A : D(A) \to h$:

$$\Gamma(A)(\otimes^n w) = \otimes^n Aw.$$

Second quantization: $\Gamma(e^{itA}) = e^{\frac{it}{\varepsilon} d\Gamma(A)}$

Wick quantization:

$b : h \to \mathbb{C}$ polynomial = finite L.C. of
$\langle \otimes^q z \mid \tilde{b}_{p,q} (\otimes^p z) \rangle$, $\tilde{b}_{p,q}$ bounded.

$\mathrm{Wick}_\varepsilon(b)$ = closed op. on $\Gamma_s(h)$.

- $\mathrm{Wick}_\varepsilon(b)^\dagger = \mathrm{Wick}_\varepsilon(\bar{b})$
- $e^{\frac{it}{\varepsilon} d\Gamma(A)} \mathrm{Wick}_\varepsilon(b) e^{-\frac{it}{\varepsilon} d\Gamma(A)} = \mathrm{Wick}_\varepsilon(b(e^{-itA} \cdot))$
- $W(-i \frac{\sqrt{2}}{\varepsilon} u)^\dagger \mathrm{Wick}_\varepsilon(b) W(-i \frac{\sqrt{2}}{\varepsilon} u) = \mathrm{Wick}_\varepsilon(b(\cdot + u))$

- Composition

<u>NB</u>: $d\Gamma(A) = \mathrm{Wick}(\langle z \mid Az \rangle)$

II - Main theorem

$\mathcal{C} : h \to h$ complex structure $\left(\begin{array}{l} \mathcal{C} \text{ antilinear} \\ \mathcal{C}^2 = \mathbb{1}_h \\ |\mathcal{C}z| = |z| \end{array} \right)$

$A : D(A) \to h$ selfadjoint, $A \geq m_0 > 0$, $\mathcal{C}A = A\mathcal{C}$

$V = (V^{(j)})_{j \geq 0} \in \Gamma_s(h)$ "nice", $\Gamma(\mathcal{C})V = V$.

Interaction $F_V(z) = \sum_{j=0}^{\infty} \langle \frac{\otimes^j (z + \mathcal{C}z)}{\sqrt{j!}} | V^{(j)} \rangle$

Hamiltonian $H = d\Gamma(A) + \mathrm{Wick}_\varepsilon(F_V)$

Selfadjoint on $\Gamma_s(h)$

$\rightsquigarrow$ propagator $e^{-\frac{it}{\varepsilon}H}$.

Question: $\varphi_0 \in h$ fixed.

Propagation of $e^{-\frac{it}{\varepsilon}H} W\left(\frac{-i\sqrt{2}}{\varepsilon}\varphi_0\right)\Omega$

for ε small?

<u>Theorem 1</u>: There exists $(b_j(t))_{j\geq 0}$ Wick poly. and $T = T(\|\varphi_0\|) > 0$ st $\forall t \in [-T_0, T_0]$, $\forall N \in \mathbb{N}$, $\forall \varepsilon > 0$ small,

$$\Big\| e^{-\frac{it}{\varepsilon}H} W\big(\cdot - i\tfrac{\sqrt{2}}{\varepsilon}\varphi_0\big)\Omega$$

$$- \sum_{j=0}^{N} \varepsilon^{j/2} e^{\frac{i\delta(t)}{\varepsilon}} W\big(\tfrac{-i\sqrt{2}}{\varepsilon}\varphi_t\big) U_2(t,0) \cdot \mathrm{Wick}_1(b_j(t))\Omega$$

$$\leq C(T_0, N)\, \varepsilon^{\frac{N+1}{2}}$$

where

$$\begin{cases} i\dot{\varphi}_t = A\varphi_t + \partial_{\bar{z}} F_V(\varphi_t) \\ \varphi_t|_{t=0} = \varphi_0 \end{cases}$$ Classical field equation

$\delta(t)$ = "classical action"

$U_2(t,0)$ = quadratic propagator

<u>Ref</u>: D. Robert, ~2000: Quantum Mechanics

Z. Ammari, M. Zerzeri, 2014: 1^{st} term

Ammari - Zerzeri - M. : soon!

Example: the $P(\phi)_2$ model.

$\mathfrak{h} = L^2(\mathbb{R}, dk)$, $\mathcal{C}z(k) = \overline{z(-k)}$

$A = \omega$ = mult. by $(m_0^2 + k^2)^{1/2} = \omega(k)$

Polynomial interaction $P = \sum_{j=0}^{2m} \beta_j X^j$, $\beta_{2m} > 0$

Cutoff $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, $g \geq 0$, even

$$\underbrace{V^{(j)}}_{\in L^2_{sym}(\mathbb{R}^j)} = \begin{cases} \beta_0 \|g\|_1 \text{ if } j = 0 \\ \sqrt{j!}\, \beta_j\, \hat{g}(k_1 + \dots + k_j)\, \omega(k_1)^{-1/2} \cdots \omega(k_j)^{-1/2}, & 1 \leq j \leq 2m \\ 0 \text{ if } j > 2m \end{cases}$$

Thm 2: if $\varphi_0 \in D(\omega^{1/2})$ then $T_0 = +\infty$.

(Classical field eq = NLKG ...)

III - "Proof" of Theorem 1

① Idea: L.H.S in thm 1 $= \| \Theta_N(t) - \Theta_N(0) \|$

$$\leq \int_0^{|t|} \| \dot{\Theta}_N(s) \| ds.$$

In $\dot{\Theta}_N(s)$: $F_V(\cdot + \varphi_t)$ appears.

(2) Taylor expansion

$$F_V(\cdot + \varphi_t) = \overbrace{F_V(\varphi_t) + 2\mathrm{Re}\langle z | \partial_{\bar{z}} F_V(\varphi_t)\rangle}^{\text{Canceled by } \dot{\delta}(t)}$$

$$+ \underbrace{F_{V_2(t)}}_{\substack{\text{Quadratic} \\ \rightsquigarrow \text{Propagator } U_2(t,0)}} + F_{R_3(t)}$$

(3) Estimate $\|\dot{\Theta}_N(s)\| \lesssim C(s,N)\, \varepsilon^{\frac{N+1}{2}}$

Key: Number estimates. $\underline{N = d\Gamma(\mathbb{1})}$

$$\left\| \left(\frac{N}{\varepsilon}+1\right)^{-\frac{p}{2}} \mathrm{Wick}(b_{p,q}) \left(\frac{N}{\varepsilon}+1\right)^{-\frac{q}{2}} \right\| \leq C(p,q)\, \varepsilon^{\frac{p+q}{2}} \|\tilde{b}_{p,q}\|$$

$\left(\frac{N}{\varepsilon}+1\right)^{k} \rightsquigarrow S\left(\frac{N}{\varepsilon}+1\right)$, S analytic.