COHERENT STATE DYNAMICS FOR THE SPATIALLY CUTOFF $\mathcal{P}(\Phi)_2$ MODEL

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LAGA

I - General setting

h = $\mathbb{C}$-Hilbert (1 particle).
Phase space (symplectic form $\sigma(u,v) = \mathrm{Im}\langle u|v\rangle$)

$\otimes_s^n h$ for n bosons

Fock space $\Gamma_s(h) = \mathbb{C} \oplus h \oplus (h \otimes_s h) \oplus \cdots$

$\xleftarrow{a_\varepsilon(u)}$ $\quad u, v \in h$

$\xrightarrow{a_\varepsilon^+(v)}$ $\quad 0 < \varepsilon \leq 1$

$\left.\begin{array}{l} a_\varepsilon(u) = \text{annihilation op} \\ a_\varepsilon^+(v) = \text{creation op.} \end{array}\right\}$ Unbounded, Of order $\varepsilon^{1/2}$, CCR $[a_\varepsilon(u), a_\varepsilon^+(v)] = \varepsilon\langle u|v\rangle$

Field op. $\Phi_\varepsilon(u) = \frac{1}{\sqrt{2}}(a_\varepsilon(u) + a_\varepsilon^+(u))$. Selfadjoint

Weyl op $W_\varepsilon(u) = \exp(i\Phi_\varepsilon(u))$

Coherent states $\left\{ W_\varepsilon\left(\frac{-i\sqrt{2}}{\varepsilon}u\right)\Omega, u \in h \right\}$,

$\Omega = (1, 0, 0, \text{—})$ vacuum

Hamiltonian in QFT

- Free part = second quantization

$\forall A : D(A) \to h$ selfadj, $d\Gamma_\varepsilon(A)\big|_{\otimes_s^{m} D(A)} = \varepsilon \sum_{j=1}^{m} (\otimes^{j-1} \mathbb{1}) \otimes A \otimes (\otimes^{m-j} \mathbb{1})$

- Interaction : $b : z \in h \mapsto \sum_{p,q}^{\text{finite}} \langle \otimes^q z | B(\otimes^p z) \rangle$ symbol

(bounded)

$\rightsquigarrow$ $\mathrm{Wick}_\varepsilon(b)$ closed op. on $\Gamma_s(h)$

Nice properties [Ammari - Nier, 2008]

- Adjoint op $\leftrightarrow$ conjugate symbol
- Conjugate op with propag of $d\Gamma_\varepsilon(A)$

$\leftrightarrow$

compose symbol with propag of A
- Composition, finite nb of terms

Prop 0: $W_\varepsilon\left(\frac{-i\sqrt{2}}{\varepsilon} u\right)^{+} \mathrm{Wick}_\varepsilon(b) W_\varepsilon\left(\frac{-i\sqrt{2}}{\varepsilon} u\right) = \mathrm{Wick}_\varepsilon(b(\cdot + u))$

Hamiltonian $H_\varepsilon = d\Gamma_\varepsilon(A) + \mathrm{Wick}_\varepsilon(b)$

II - Main Result

The $P(\Phi)_2$ model [Simon 1974 (book), Dereziński - Gérard 2000]

$h = L^2(\mathbb{R}, dk)$; $\omega(k) = (m_0^2 + k^2)^{1/2}$ dispersion relation $(m_0 > 0)$

Interaction $I = \int_{\mathbb{R}} g(x) \, : P\left(\sqrt{2}\, \phi_\varepsilon\left(\underbrace{\frac{e^{-ikx}}{\sqrt{\omega(k)}}}_{\notin L^2(\mathbb{R}, dk)}\right)\right) : dx$

Wick ordering: all a_ε^+ on the left

$\rightsquigarrow$ limit of...

where * $P = \sum_{j=0}^{2n} \beta_j X^j$, $\beta_j \in \mathbb{R}$, $\beta_{2n} > 0$

* $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, $g \geq 0$, even (spatial cutoff)

Rem: $I = \mathrm{Wick}_\varepsilon(b)$ with b explicit

$H_\varepsilon = d\Gamma_\varepsilon(\omega) + I$ is selfadj [Segal, Høegh-Krohn, Simon ... | 1960s, 1970s]

Aim: for $\varphi_0 \in h$, describe

$$e^{-\frac{it}{\varepsilon} H_\varepsilon} W_\varepsilon\left(\frac{-i\sqrt{2}}{\varepsilon}\varphi_0\right)\Omega \text{ as } \varepsilon \to 0$$

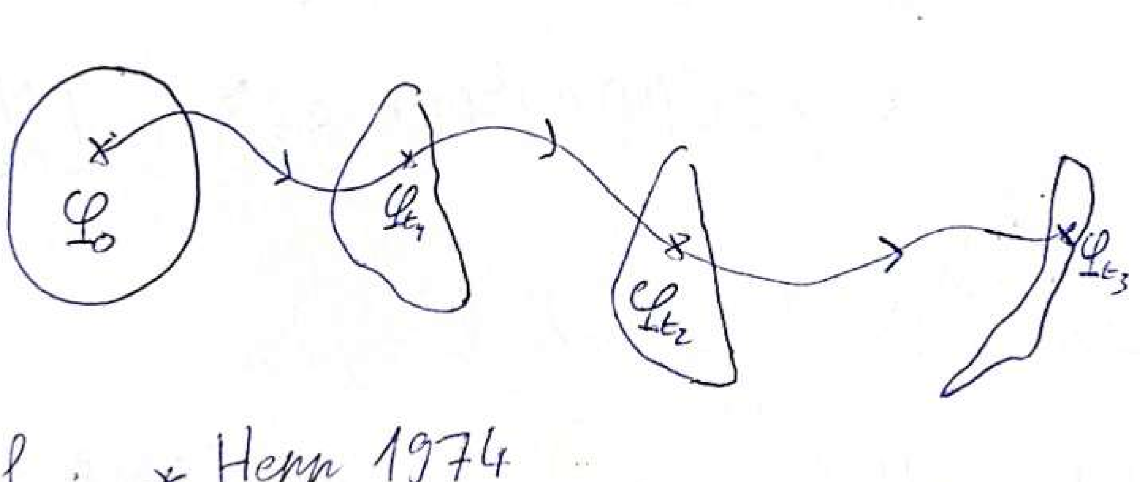
Theorem 1: $\exists (b_j(t))_{j\geq 0}$ polynomial st $\forall N \in \mathbb{N}$, $\forall t \in \mathbb{R}$, we have

$$\Big\| \underbrace{e^{-\frac{it}{\varepsilon}H_\varepsilon} W_\varepsilon\left(\frac{-i\sqrt{2}}{\varepsilon}\varphi_0\right)\Omega}_{\text{initial CS}} - \sum_{j=0}^{N} \varepsilon^{j/2}\, e^{\frac{i\overset{\text{classical action}}{S(t)}}{\varepsilon}} \underbrace{W_\varepsilon\Big(\frac{-i\sqrt{2}}{\varepsilon}\overset{\text{classical trajectory}}{\varphi_t}\Big)\overset{\text{Quadratic propagator}}{U_2(t)}\,\mathrm{Wick}_1(b_j(t))\,\Omega}_{\text{Modified CS}} \Big\|$$

$$\leq \underbrace{C(t,\varphi_0,N)}_{\hookrightarrow\, C^0 \text{ in } t}\, \varepsilon^{\frac{N+1}{2}}$$

for $\varepsilon > 0$ small, where

$$(E)\quad \begin{cases} i\dot{\varphi}_t = \omega\varphi_t + \partial_{\bar z} b(\varphi_t) \\ \varphi_t\big|_{t=0} = \varphi_0 \end{cases} \quad \text{(Classical field equation)}$$



Ref:
* Hepp 1974
* Robert 2000 (QM, Schrödinger, BKW)
* Ammari – Zerzeri 2014 (QFT, 1st term)
  + Malartre 2026 (ArXiv)

COHERENT STATE DYNAMICS FOR $P(\phi)_2$

III - Proof of Theorem 1

1) Write the LHS as $e^{\frac{it}{\varepsilon}H_\varepsilon}\left(\Theta_N(0)-\Theta_N(t)\right)$

Idea: $\|\Theta_N(t)-\Theta_N(0)\| \leqslant \int_0^{|t|}\|\dot{\Theta}_N(s)\|\,ds$

2) Compute $\dot{\Theta}_N(s)$: $b(\cdot+\varphi_t)$ appears (Prop 0)

3) Taylor expansion $b(\cdot+\varphi_t)=b(\varphi_t)+b_{1,t}+\boxed{b_{2,t}}+R_{3,t}$

Quadratic

$\rightsquigarrow$ propagator $U_2(t)$

4) Control the remainder

* Hypercontractivity [Høegh-Krohn, Simon, 1970s]
* Number estimates: $N_\varepsilon = d\Gamma_\varepsilon(\mathbb{1})$ number op.

If $\deg_{(p+q)} b \leq m$ then $\mathrm{Wick}_\varepsilon(b)N^{-\frac{m}{2}}$ is bounded

Rem: Generalization: $\begin{cases} b \text{ poly} \rightsquigarrow b \text{ analytic} \\ \omega \rightsquigarrow A \geq m_0 > 0 \text{ selfadj} \end{cases}$ + assumptions

Q1) Number estimates?

Q2) LWP / GWP of (E)?

IV - The classical field equation

Theorem 2 : for $P(\phi)_2$, (E) is GWP on $h = L^2(\mathbb{R})$

Proof: $\forall s > 0$, let $\mathcal{H}^s_\omega = \mathcal{D}(\omega^s)$.

$$X(t,z) = e^{it\omega} \partial_{\bar{z}} b(e^{-it\omega} z) \quad \begin{pmatrix} \text{Interaction} \\ \text{RPZ} \end{pmatrix}$$

1) $\|X(t,z_1) - X(t,z_2)\|_{\underset{\mathcal{H}^{3/2}_\omega}{2}} \leq C(\|z_1\|_{\underset{\mathcal{H}^{1/2}_\omega}{2}}, \|z_2\|_{\underset{\mathcal{H}^{1/2}_\omega}{2}}) \|z_1 - z_2\|_{\underset{\mathcal{H}^{3/2}_\omega}{2}}$

2)

$\rightsquigarrow$ LWP L^2, LWP $\mathcal{H}^{3/2}_\omega$

3) Energy $h(z) = \langle z | \omega z \rangle + b(z)$
cst along $\mathcal{H}^{3/2}_\omega$ sol $\to$ GWP $\mathcal{H}^{3/2}_\omega$.

4) $\varphi_0^{(n)} \in \mathcal{H}^{3/2}_\omega \xrightarrow[n \to \infty]{L^2} \varphi_0 \in L^2$ gives GWP L^2

Rem: We have a whole GWP theory in $\mathcal{H}^{1/2}_\omega$.

Open Q: $\mathcal{H}^s_\omega$, $\forall s > 0$? ...

① LWP $\mathcal{H}^{1/2}_\omega$: 1) with $\mathcal{H}^{1/2}_\omega$ norms

② $\varphi_0^{(n)} \in \mathcal{H}^{3/2}_\omega \longrightarrow \varphi_0 \in \mathcal{H}^{1/2}_\omega$: h is cst along $\mathcal{H}^{1/2}_\omega$ sol

③ h dominates $\|\cdot\|_{\mathcal{H}^{1/2}_\omega}$ + blowup criterion