Ranges of randomly biased random walks on trees

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Random Networks and Interacting Particle Systems

September 10, 2021

Marked trees

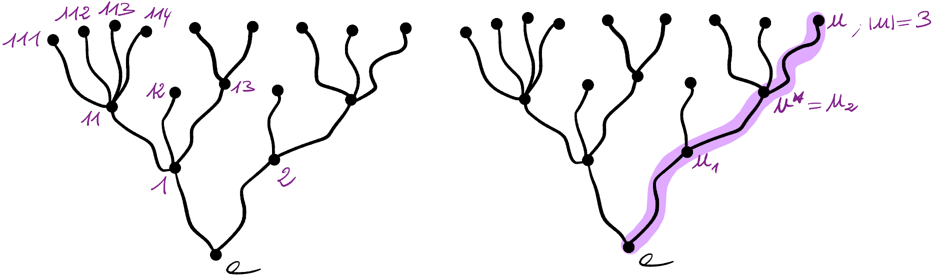
Let

$$\mathscr{U} := \bigcup_{n\in\mathbb{N}} (\mathbb{N}^*)^n.$$

By convention, $(\mathbb{N}^*)^0 = \{e\}$, e is the sequence with length 0. A tree ω is a subset of $\mathscr{U}$ such that

- $e \in \omega$,
- $uv \in \omega \implies u \in \omega$,
- for any $u \in \omega$, there exists $N_u \in \mathbb{N}$ such that $uj \in \omega$ if and only if $j \in \{1, \dots, N_u\}$.

For a tree ω, $(\omega, (A_u; u \in \omega))$ is called a marked tree where $A_u > 0$ for all $u \in \omega$.



Notations: For $u \in \omega$

- $|u| = d_\omega(e, u)$ is the generation of u,
- for all $v \in \omega$, $v \leq u$ if and only if v is an ancestor of u and u_n is the only ancestor of u such that $|u| = n$,
- u^* is the parent of u, the most recent ancestor of u.

Let $V(z)$ be the branching potential:

$$V(z) := -\sum_{u \leq z} \log A_u = -\sum_{i=1}^{|z|} \log A_{z_i}$$

Galton-Watson marked tree $\mathbb{T}$

N with law $(p_k)_{k\in\mathbb{N}}$ on $\mathbb{N}$ such that $\sum_{k\geq 0} kp_k > 1$ and A random variable on $(0,\infty)$.

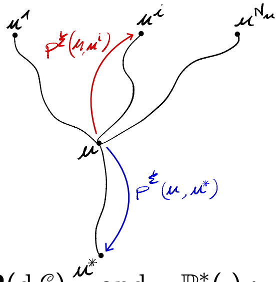
- Generation 0: one individual (e,1).
- Generation $n \geq 1$: generation $n-1$ empty $\Rightarrow$ generation n empty. Otherwise, given generation $n-1$, each individual (u, A_u) of generation $n-1$ gives birth to N_u individuals (u^i, A_{u^i}), $i \in \{1,\dots,N_{u^i}\}$, independently according to (N, A).

$\mathbb{T}$: the genealogical tree of this population with root e.
Random environment: the random marked tree $(\mathbb{T}, A_u; u \in \mathbb{T})$.
$\mathbf{P}$ = law of $(\mathbb{T}, A_u; u \in \mathbb{T})$ and $\mathbf{P}^* := \mathbf{P}(\cdot|\ \text{non-extinction}\ )$.

Random walk on Galton-Watson tree $\mathbb{T}$

Given $\mathscr{E} = (\omega, A(u); u \in \omega)$, $X = (X_m)$ is a random walk under $\{\mathbb{P}_z^{\mathscr{E}}; z \in \omega\}$ on ω with $X_0 = e$ and probability transition:

$$\begin{cases} p^{\mathscr{E}}(u, u^*) = \dfrac{1}{1+\sum_{k=1}^{N_u} A(u^k)} = \dfrac{e^{-V(u)}}{e^{-V(u)}+\sum_{k=1}^{N_u} e^{-V(u^k)}}, \\ \\ p^{\mathscr{E}}(u, u^i) = \dfrac{A(u^i)}{1+\sum_{k=1}^{N_u} A(u^k)} = \dfrac{e^{-V(u^i)}}{e^{-V(u)}+\sum_{k=1}^{N_u} e^{-V(u^k)}}. \end{cases}$$



$$\mathbb{P}(\cdot) := \int_{\mathscr{E}} \mathbb{P}^{\mathscr{E}}(\cdot)\mathbf{P}(\mathrm{d}\mathscr{E}) \quad \text{and} \quad \mathbb{P}^*(\cdot) := \int_{\mathscr{E}} \mathbb{P}^{\mathscr{E}}(\cdot)\mathbf{P}^*(\mathrm{d}\mathscr{E})$$

The slow random walk on Galton-Watson tree $\mathbb{T}$

log-Laplace transform: $\psi(t) := \log \mathbf{E}\left[\sum_{|z|=1} e^{-tV(z)}\right]$.
Boundary case for the branching potential $(V(u))_{u\in\omega}$:

$$\psi(1) = \psi'(1) = 0$$

- X is null recurrent,
- X is slow: $\mathbb{P}^*$- almost surely

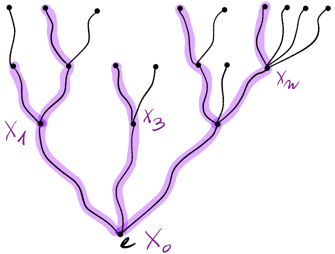
$$\frac{1}{(\log n)^3} \max_{j\leq n} |X_j| \underset{n\to\infty}{\longrightarrow} C_1 > 0.$$

(Faraud, Hu, Shi, 12)

Regular range of the walk

Let $\mathscr{E} = (\omega, A(u); u \in \omega)$ a given environment.

The sub-tree $\{u \in \omega;\ \exists\, j \leq n : X_j = u\}$ of ω:



The regular range up to the time n is defined by:

$$R_n := \#\{u \in \omega;\ \exists\, j \leq n : X_j = u\}.$$

Asymptotic of R_n: in $\mathbb{P}^*$-probability

$$\frac{\log n}{n} R_n \underset{n\to\infty}{\longrightarrow} C_2 > 0.$$

(Andreoletti, Chen, 18)

Extensions of the regular range

Introduce the local time $\mathscr{L}_z^n$ of the vertex z at n:

$$\mathscr{L}_z^n := \sum_{k=1}^{n} \mathbb{1}_{\{X_k = z\}}.$$

We add some constraints both on the trajectories of the walk and of the branching potential: for all $b \in (0,1)$, $k, n \in \mathbb{N}^*$, let $E_n^k \subset \mathbb{R}^k$ and introduce

$$\begin{aligned}\mathscr{R}_n^{(b)} :&= \#\{z \in \omega; \mathscr{L}_z^n \geq n^b \text{ and } (V(z_1), \dots, V(z)) \in E_n^{|z|}\} \\ &= \sum_{z \in \omega} \mathbb{1}_{\{\mathscr{L}_z^n \geq n^b,\, (V(z_1),\dots,V(z)) \in E_n^{|z|}\}}.\end{aligned}$$

and

$$\begin{aligned}\mathscr{R}_n^X :&= \#\{j \leq n; (V(u); e < u \leq X_j) \in E_n^{|X_j|}\} \\ &= \sum_{z \in \omega} \mathscr{L}_z^n \mathbb{1}_{\{(V(z_1),\dots,V(z)) \in E_n^{|z|}\}}.\end{aligned}$$

Define

$$r_n := \mathbf{E}\Big[\sum_{z\in\omega} e^{-V(z)}\mathbb{1}_{\{(V(z_1),\dots,V(z))\in E_n^{|z|}\}}\Big] \in (0,1),$$

and assume technical hypotheses only depending on the branching potential V and $E_n^k, k, n \geq 1$.

Theorem (Andreoletti-K. 21+)

If $(\log n)^\gamma = o(\log r_n)$ *for some* $\gamma \in (0,1)$ *then in* $\mathbb{P}^*$*-probability*

$$\frac{\log^+ \mathscr{R}_n^{(b)} - (1-b)\log n}{\log r_n} \underset{n\to\infty}{\longrightarrow} 1 \quad \textit{and} \quad \frac{\log^+ \mathscr{R}_n^{X} - \log n}{\log r_n} \underset{n\to\infty}{\longrightarrow} 1$$

else in $\mathbb{P}^*$*-probability*

$$\frac{\log^+ \mathscr{R}_n^{(b)}}{\log n} \underset{n\to\infty}{\longrightarrow} 1-b \quad \textit{and} \quad \frac{\log^+ \mathscr{R}_n^{X}}{\log n} \underset{n\to\infty}{\longrightarrow} 1$$

Examples

- Heavy range: $\mathscr{R}_n^{(b)} = \sum_{z\in\omega} \mathbb{1}_{\{\mathscr{L}_z^n \geq n^b\}}$ $b \in (0,1)$

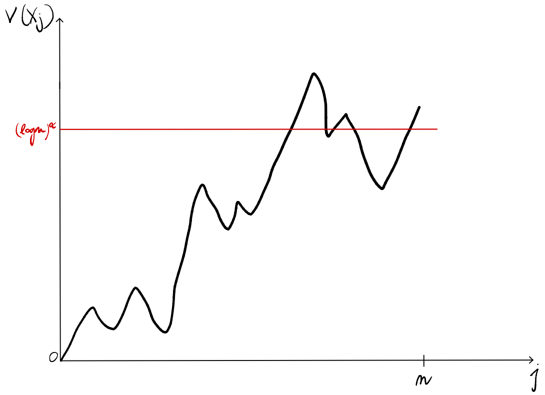
$$\frac{\log^+ \mathscr{R}_n^{(b)}}{\log n} \underset{n\to\infty}{\longrightarrow} 1-b \qquad \text{in } \mathbb{P}^*\text{- probability.}$$

$$\mathscr{R}_n^{(b)} \approx n^{1-b+o(1)} \approx R_n n^{-b+o(1)}$$

- High potential: $\mathscr{R}_n^X = \sum_{j=1}^n \mathbb{1}_{\{V(X_j)\geq(\log n)^\alpha\}}$ with $\alpha \in (1,2)$

$$\frac{\log^+ \mathscr{R}_n^X - \log n}{(\log n)^{\alpha-1}} \underset{n\to\infty}{\longrightarrow} -1 \qquad \text{in } \mathbb{P}^*\text{- probability.}$$

$$\mathscr{R}_n^X \approx n e^{-(\log n)^{\alpha-1}(1+o(1))} \approx R_n e^{-(\log n)^{\alpha-1}(1+o(1))}.$$



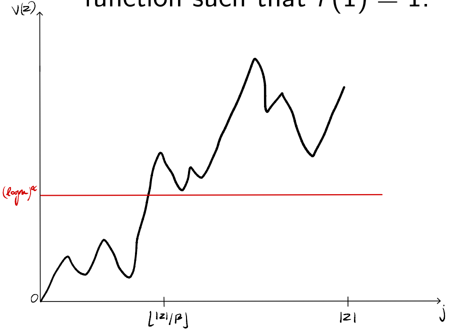
- Heavy range + high potential:

$$\mathscr{R}_n^{(b)} = \sum_{z\in\omega} \mathbb{1}_{\{\mathscr{L}_z^n \ge n^b,\, V(z_i)\ge(\log n)^\alpha \ \forall \lfloor \frac{|z|}{\beta}\rfloor \le i \le |z|\}} \quad \text{with } \beta \ge 1$$

$$\frac{\log \mathscr{R}_n^{(b)} - (1-b)\log n}{(\log n)^{\alpha-1}} \underset{n\to\infty}{\longrightarrow} -f(\beta) \qquad \text{in } \mathbb{P}^*\text{- probability.}$$

$$\mathscr{R}_n^{(b)} \approx \underset{n^{1-b}}{\cancel{n}}\, e^{-f(\beta)(\log n)^{\alpha-1}(1+o(1))} \approx R_n e^{-f(\beta)(\log n)^{\alpha-1}(1+o(1))} \times n^{-b}$$

with $f : [1,\infty) \longrightarrow (0,\infty)$ an increasing and continuous function such that $f(1) = 1$.



Sketch of proof

Recall

$$\mathscr{R}_n^{(b)} = \#\{z \in \omega; \mathscr{L}_z^n \geq n^b \text{ and } (V(z_1), \ldots, V(z)) \in E_n^{|z|}\}$$
$$= \sum_{z \in \omega} \mathbb{1}_{\{\mathscr{L}_z^n \geq n^b,\, (V(z_1), \ldots, V(z)) \in E_n^{|z|}\}}.$$

Goal: show that $\mathscr{R}_n^{(b)} \approx \mathbb{E}[\mathscr{R}_n^{(b)}] = n^{1-b} r_n^{1+o(1)}$ with high probability.

Upper bound: Markov inequality.

Lower bound: Bienaymé-Tchebychev type inequalities.

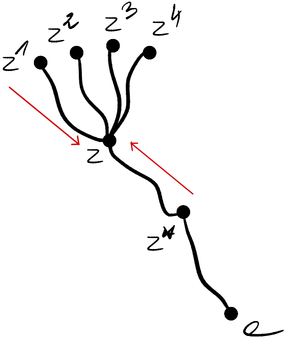
Edge local time and excursions above the root e

Introduce the edge local time $\mathscr{N}_z^m$ of z at m:

$$\mathscr{N}_z^m := \sum_{k=1}^{m} \mathbb{1}_{\{X_{k-1}=z^*, X_k=z\}}.$$

Let $T^0 = 0$ and for any $n \geq 1$, $T^n = \inf\{j > T^{n-1}; X_j = e\}$.

Remark: $\mathscr{L}_z^{T^n} = \mathscr{N}_z^{T^n} + \sum_{i=1}^{N_z} \mathscr{N}_{z^i}^{T^n}$



Introduce the edge local time N_z^m of z at m:

$$\mathscr{N}_z^m := \sum_{k=1}^{m} \mathbb{1}_{\{X_{k-1}=z^*, X_k=z\}}.$$

Let $T^0 = 0$ and for any $n \geq 1$, $T^n = \inf\{j > T^{n-1}; X_j = e\}$.

Remark: $\mathscr{L}_z^{T^n} = \mathscr{N}_z^{T^n} + \sum_{i=1}^{N_z} \mathscr{N}_{z^i}^{T^n}$.

Fact: strong bias towards the root e:

$$\frac{T^n}{n \log n} \underset{n\to\infty}{\longrightarrow} X \in (0, \infty) \quad \text{in } \mathbb{P}^*\text{- probability.}$$

$\Longrightarrow$ we can study $\mathscr{N}_z^{T^n}$ instead of $\mathscr{L}_z^n$.

Let

$$\mathscr{G}_n := \{z \in \omega; |z| \leq (\log n)^3 \text{ and } \overline{V}(z) \geq 3 \log n\}.$$

where $\overline{V}(z) = \max_{u \leq z} V(u)$.

Fact: Any vertex in $\mathscr{G}_n$ visited at least once during the first n excursions above the root e is actually visited during a single excursion.

Indeed, $E_z^n := \sum_{j=1}^n \mathbb{1}_{\{\exists k \in [T^{j-1}, T^j) : X_k = z\}} \sim \text{Bin}\big(n, \mathbb{P}^{\mathscr{E}}(T_x < T_e)\big)$ under $\mathbb{P}^{\mathscr{E}}$ and $\mathbb{P}^{\mathscr{E}}(T_x < T_e) \leq e^{-\overline{V}(z)}$ so we have

$$\mathbb{P}\big(\exists z \in \mathscr{G}_n : E_z^n \geq 2\big) \leq n^2 (\log n)^3 e^{-3 \log n} = \frac{(\log n)^3}{n}.$$

$\Longrightarrow \mathscr{R}_n(b) \gtrsim \sum_{j=1}^{n} \mathscr{R}_{j,n}(b)$ where

$$\mathscr{R}_{j,n}(b) := \sum_{z \in \mathscr{G}_n} \mathbb{1}_{\{\mathscr{N}_z^{T^j} - \mathscr{N}_z^{T^{j-1}} \geq n^b,\ (V(z_1),\dots,V(z)) \in E_n^{|z|}\}},$$

i.i.d with law $\mathscr{R}_{1,n}(b)$ under $\mathbb{P}^{\mathscr{E}}$.

$\mathbb{E}^{\mathscr{E}}[\sum_{j=1}^{n} \mathscr{R}_{j,n}(b)] = n\mathbb{E}^{\mathscr{E}}[\mathscr{R}_{1,n}(b)]$ and

$$\mathbb{E}^{\mathscr{E}}[\mathscr{R}_{1,n}(b)] = \sum_{z \in \mathscr{G}_n} \mathbb{P}^{\mathscr{E}}(\mathscr{N}_z^{T^1} \geq n^b) \mathbb{1}_{\{(V(z_1),\dots,V(z)) \in E_n^{|z|}\}}$$

$\mathbb{V}\mathrm{ar}^{\mathscr{E}}\left(\sum_{j=1}^{n} \mathscr{R}_{j,n}(b)\right) = n\mathbb{V}\mathrm{ar}^{\mathscr{E}}\left(\mathscr{R}_{1,n}(b)\right) \leq n\mathbb{E}^{\mathscr{E}}[\mathscr{R}_{1,n}(b)^2]$ and

$$\mathbb{E}^{\mathscr{E}}[\mathscr{R}_{1,n}(b)^2] = \sum_{z,u \in \mathscr{G}_n} \mathbb{P}^{\mathscr{E}}(\mathscr{N}_z^{T^1} \vee \mathscr{N}_u^{T^1} \geq n^b) \mathbb{1}_{\{(V(z_1),\dots,V(z)) \in E_n^{|z|}\}} \times \mathbb{1}_{\{(V(u_1),\dots,V(u)) \in E_n^{|u|}\}}.$$

•The law of $\mathscr{N}_z^{T^1}$:

$\mathbb{P}^{\mathscr{E}}(\mathscr{N}_z^{T^1} = 0) = \mathbb{P}^{\mathscr{E}}(T_z > T_e)$ and for all $k \geq 1$

$$\mathbb{P}^{\mathscr{E}}(\mathscr{N}_z^{T^1} = k) = \mathbb{P}^{\mathscr{E}}(T_z < T_e)\mathbb{P}^{\mathscr{E}}_{z^*}(T_z < T_e)^{k-1}\mathbb{P}^{\mathscr{E}}_{z^*}(T_z > T_e).$$

•The law of $\mathscr{N}_z^{T^1}$ conditionally given $(\mathscr{N}_v^{T^1})_{v \leq z}$:

$\mathscr{N}_z^{T^1} \sim \text{BinNeg}\left(\mathscr{N}_{z^*}^{T^1}, (e^{V(z^*)-V(z)})^{-1}\right)$ under $\mathbb{P}^{\mathscr{E}}$.