The Non-Planar Drainage Networks with Dependence and Their Scaling Limit

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September, 2021

- Consider the integer lattice $\mathbb{Z}^2$ and fix $p \in (0,1)$. Let $\{U_{\mathbf{v}} : \mathbf{v} \in \mathbb{Z}^2\}$ and $\{U_{\mathbf{v},\mathbf{w}} : \mathbf{v}, \mathbf{w} \in \mathbb{Z}^2, \mathbf{v}(2) < \mathbf{w}(2)\}$ be two independent collections of i.i.d. uniform $(0,1)$ random variables.

- A vertex $\mathbf{v} \in \mathbb{Z}^2$ is said to be *open* if $U_{\mathbf{v}} < p$, and it is *closed* otherwise. Let $\mathcal{V} := \{\mathbf{v} : \mathbf{v} \text{ is open}\}$.

Model 1 (Introduced by (Athreya et al., 2008))

$h(\mathbf{u}) =?$

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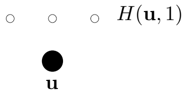

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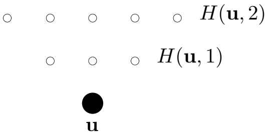

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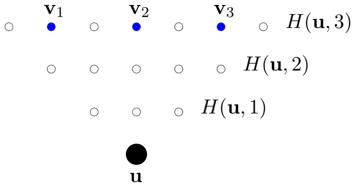

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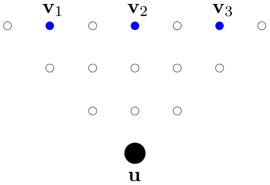

$$U_{\mathbf{u},\mathbf{v}_1} < \min\{U_{\mathbf{u},\mathbf{v}_2}, U_{\mathbf{u},\mathbf{v}_3}\}$$

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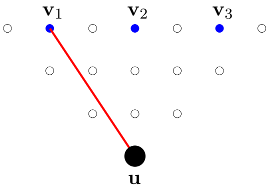

$$U_{\mathbf{u},\mathbf{v}_1} < \min\{U_{\mathbf{u},\mathbf{v}_2}, U_{\mathbf{u},\mathbf{v}_3}\} \implies h(\mathbf{u}) = \mathbf{v}_1$$

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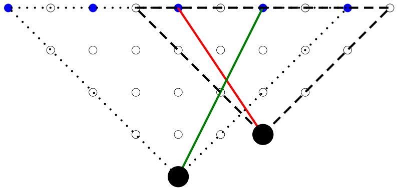

The graph is non-planar.

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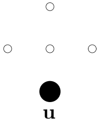

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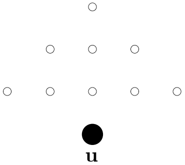

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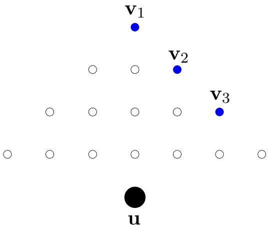

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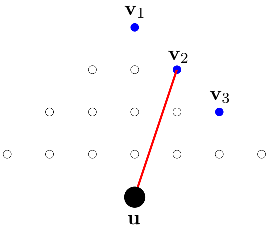
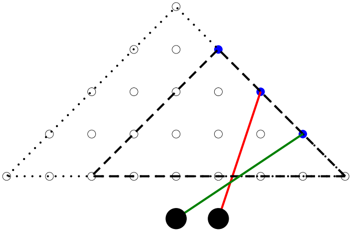


$$U_{\mathbf{u},\mathbf{v}_2} < \min\{U_{\mathbf{u},\mathbf{v}_1}, U_{\mathbf{u},\mathbf{v}_3}\} \implies h(\mathbf{u}) = \mathbf{v}_2$$

Model 2 (Originated from (Roy et al., 2016))



The graph is non-planar.

Now look at the graph as a collection of particles, each particle is doing a random walk. For this purpose, consider the horizontal axis as the space and the vertical axis as the time.

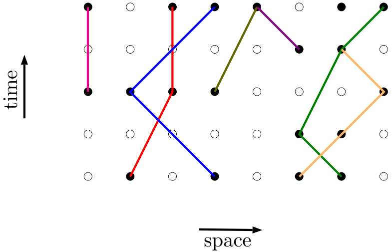


Figure: *A possible realization in Model 1.*

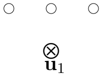
How can the motion of one or more particles be followed over time?

To answer this question, a process starting at a finite number of points is constructed. For some $k \geq 1$, fix vertices $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ with $\mathbf{u}_1(2) = \mathbf{u}_2(2) = \cdots = \mathbf{u}_k(2)$. In this presentation, the process started from just one point is, intuitively, described because its extension to $k > 1$ arbitrary points is straightforward.

The Construction of Model 1

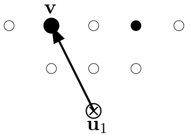
$$\mathbf{u}_{1,0} = \mathbf{u}_1$$
$$t_{1,0} = 1$$

The Construction of Model 1



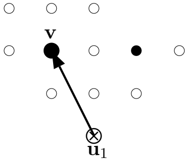
$$\mathbf{u}_{1,1} = \mathbf{u}_{1,0} = \mathbf{u}_1$$
$$t_{1,1} = t_{1,0} + 1 = 2$$

The Construction of Model 1



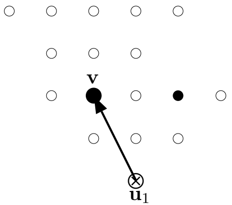
$$\mathbf{u}_{1,2} = \mathbf{v} = h_1(\mathbf{u}_1)$$
$$t_{1,2} = 1$$

The Construction of Model 1

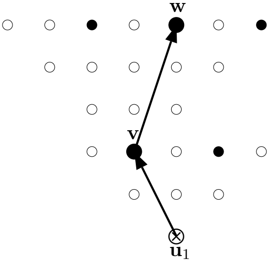


$$\mathbf{u}_{1,3} = \mathbf{u}_{1,2} = \mathbf{v}$$
$$t_{1,3} = t_{1,2} + 1 = 2$$

The Construction of Model 1



$$\mathbf{u}_{1,4} = \mathbf{u}_{1,3} = \mathbf{v}$$
$$t_{1,4} = t_{1,3} + 1 = 3$$

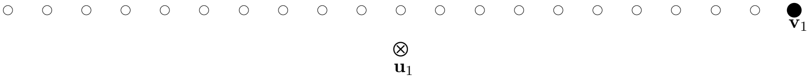


$$\mathbf{u}_{1,5} = \mathbf{w} = h_2(\mathbf{u}_1)$$
$$t_{1,5} = 1$$

The Construction of Model 2

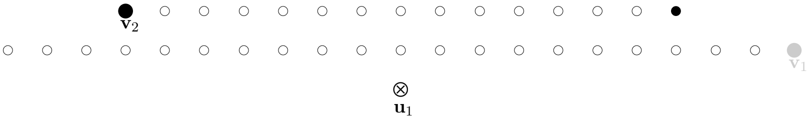
$$\otimes$$
$$\mathbf{u}_1$$

$$\mathbf{c}_{1,0} = \mathbf{u}_1$$
$$d_{1,0} = 0$$
$$\Delta_{1,0} = \emptyset$$
$$\pi_{1,0} = (\mathbf{u}_1)$$

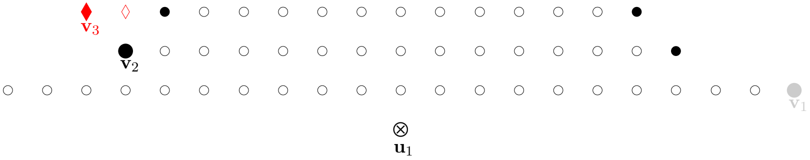


$$\mathbf{c}_{1,1} = \mathbf{v}_1$$
$$d_{1,1} = |\mathbf{u}_1(1) - \mathbf{v}_1(1)| = 10$$
$$\Delta_{1,1} = \{(\langle \mathbf{u}_1, \mathbf{v}_1 \rangle, 10)\}$$
$$\pi_{1,1} = (\mathbf{u}_1)$$

The Construction of Model 2

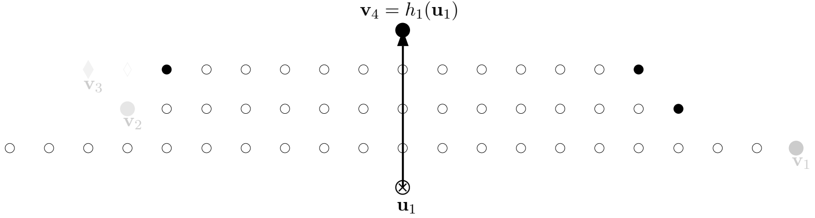


$$\mathbf{c}_{1,2} = \mathbf{v}_2$$
$$d_{1,2} = |\mathbf{u}_1(1) - \mathbf{v}_2(1)| = 7$$
$$\Delta_{1,2} = \{(\langle \mathbf{u}_1, \mathbf{v}_2 \rangle, 7)\}$$
$$\pi_{1,2} = (\mathbf{u}_1)$$



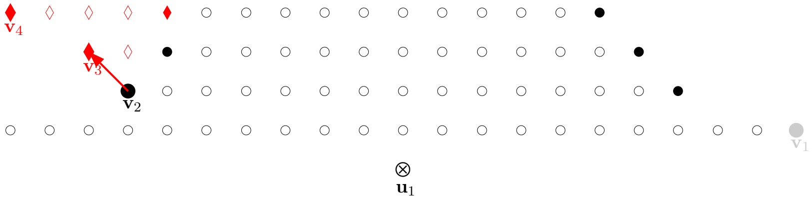
$$\mathbf{c}_{1,3} = \mathbf{v}_2$$
$$d_{1,3} = d_{1,2} - 1 = 6$$
$$\Delta_{1,3} = \{(\langle \mathbf{u}_1, \mathbf{v}_2 \rangle, 6), (\langle \mathbf{v}_2, \mathbf{v}_3 \rangle, 1)\}$$
$$\pi_{1,3} = (\mathbf{u}_1)$$

The Construction of Model 2



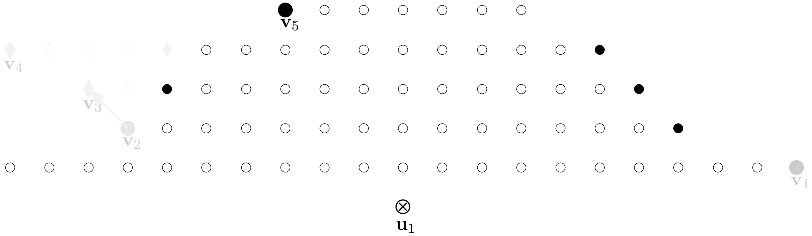
$$\mathbf{c}_{1,4} = \mathbf{v}_4$$
$$d_{1,4} = |\mathbf{u}_1(1) - \mathbf{v}_4(1)| = 0$$
$$\Delta_{1,4} = \emptyset$$
$$\pi_{1,4} = (\mathbf{u}_1, \mathbf{v}_4)$$

The Construction of Model 2

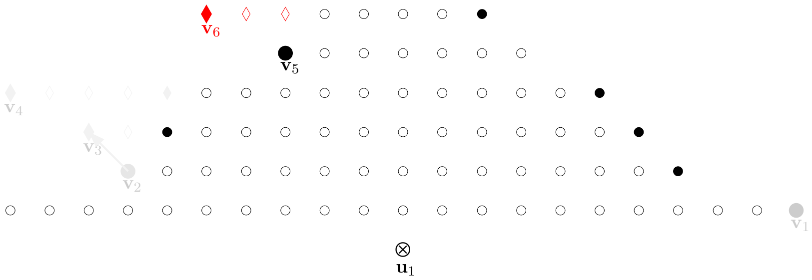


$$\mathbf{c}_{1,4} = \mathbf{v}_2$$
$$d_{1,4} = d_{1,3} - 1 = 5$$
$$\Delta_{1,4} = \{(\langle \mathbf{u}_1, \mathbf{v}_2 \rangle, 5), (\langle \mathbf{v}_2, \mathbf{v}_3 \rangle, 0), (\langle \mathbf{v}_3, \mathbf{v}_4 \rangle, 2)\}$$
$$\pi_{1,4} = (\mathbf{u}_1)$$

The Construction of Model 2

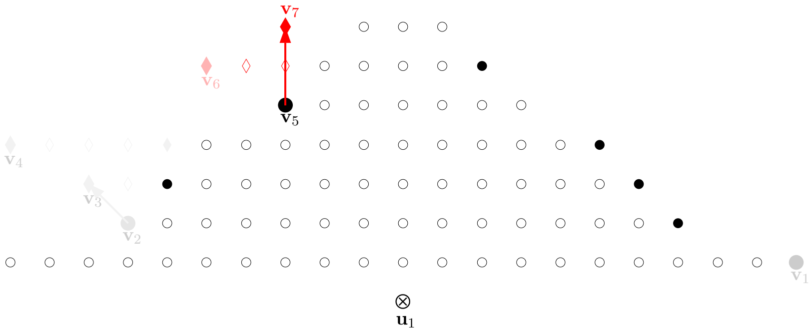


$$\mathbf{c}_{1,5} = \mathbf{v}_5$$
$$d_{1,5} = |\mathbf{u}_1(1) - \mathbf{v}_5(1)| = 3$$
$$\Delta_{1,5} = \{(\langle \mathbf{u}_1, \mathbf{v}_5 \rangle, 3)\}$$
$$\pi_{1,5} = (\mathbf{u}_1)$$



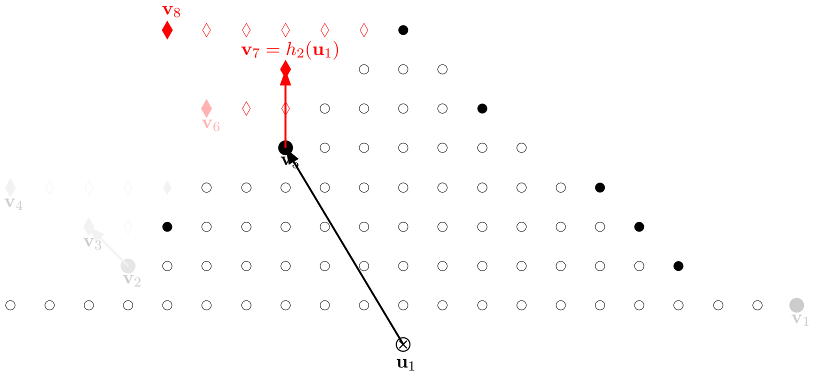
$$\mathbf{c}_{1,6} = \mathbf{v}_5$$
$$d_{1,6} = d_{1,5} - 1 = 2$$
$$\Delta_{1,6} = \{(\langle \mathbf{u}_1, \mathbf{v}_5 \rangle, 2), (\langle \mathbf{v}_5, \mathbf{v}_6 \rangle, 2)\}$$
$$\pi_{1,6} = (\mathbf{u}_1)$$

The Construction of Model 2



$$\mathbf{c}_{1,7} = \mathbf{v}_5$$
$$d_{1,7} = d_{1,6} - 1 = 1$$
$$\Delta_{1,7} = \{(\langle \mathbf{u}_1, \mathbf{v}_5 \rangle, 1), (\langle \mathbf{v}_5, \mathbf{v}_7 \rangle, 0)\}$$
$$\pi_{1,7} = (\mathbf{u}_1)$$

The Construction of Model 2



$$\mathbf{c}_{1,8} = \mathbf{v}_5$$
$$d_{1,8} = d_{1,7} - 1 = 0$$
$$\Delta_{1,8} = \{(\langle \mathbf{v}_7, \mathbf{v}_8 \rangle, 3)\}$$
$$\pi_{1,8} = (\mathbf{u}_1, \mathbf{v}_5, \mathbf{v}_7)$$

The Construction

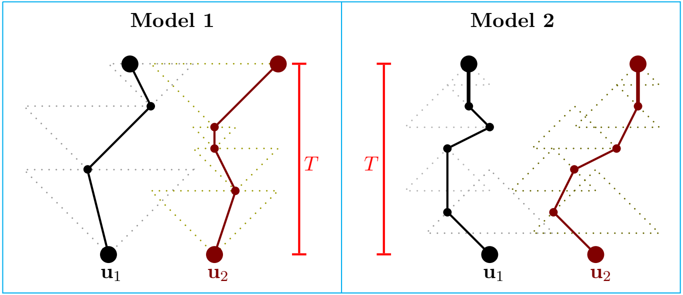
Taking

$$\mathfrak{X}_{i,m} := \begin{cases} (\mathbf{u}_{i,m}, t_{i,m}) & \text{for Model 1,} \\ (\mathbf{c}_{i,m}, d_{i,m}, \Delta_{i,m}, \pi_{i,m}) & \text{for Model 2,} \end{cases}$$

for $1 \leq i \leq k$ and $m \geq 0$, from the construction it is clear that, the process $\{(\mathfrak{X}_{1,m}, \ldots, \mathfrak{X}_{k,m}) : m \geq 0\}$ is a **Markov process** on the state space $(\mathbb{Z}^2 \times \mathbb{N})^k$ for Model 1 and $(\mathbb{Z}^2 \times (\mathbb{N} \cup \{0\}) \times \mathrm{P} \times \sqcap)^k$ for Model 2, where P is the set of all subsets of $\{(\langle \mathbf{u}, \mathbf{v} \rangle, x) : \mathbf{u}, \mathbf{v} \in \mathbb{Z}^2, \mathbf{u}(2) < \mathbf{v}(2), x \in \mathbb{N} \cup \{0\}\}$ and

$$\sqcap = \big\{(\mathbf{w}_1, \ldots, \mathbf{w}_n) \in \mathbb{Z}^2 \times \cdots \times \mathbb{Z}^2 : n \in \mathbb{N}, \mathbf{w}_1(2) < \cdots < \mathbf{w}_n(2)\big\}.$$

Regeneration Time



$$T = T(\mathbf{u}_1, \mathbf{u}_2) := \begin{cases} \inf\{m \geq 1 : t_{1,m} = t_{2,m} = 1\} & \text{for Model 1,} \\ \inf\{m \geq 1 : \Delta_{1,m} = \Delta_{2,m} = \emptyset\} & \text{for Model 2.} \end{cases}$$

Regeneration Time

The random variable T has exponentially decaying tail probabilities.

Proposition

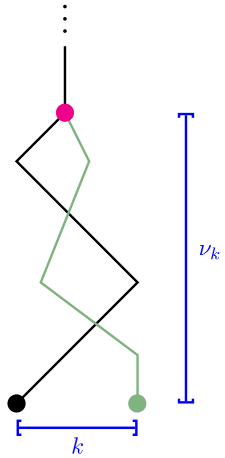
There exist positive constants C_1 and C_2 such that

$$\mathbb{P}\{T \geq n\} \leq C_1 \exp\{-C_2 n\},$$

for all real positive number n.

Coalescence Time

The graphs are trees almost surely, but we also show that:



$$\mathbb{P}\{\nu_k \geq n\} \leq \frac{C|k|}{\sqrt{n}}$$

For this purpose, we use the following theorem by Coupier et al. (2021):

Theorem

Let $\{W_n : n \geq 0\}$ be a discrete time adapted process with respect to a filtration $\{\mathcal{G}_n : n \geq 0\}$ taking values in $\{0, 1, \dots\}$. Suppose there exist positive constants M_0, C_3, C_4, C_5 and C_6 such that for all $n \geq 0$,

(a) *there exists an event $A_{n+1} \in \mathcal{G}_{n+1}$ such that on the event $\{W_n > M_0\}$, we have $\mathbb{P}\{A^c_{n+1} \mid \mathcal{G}_n\} \leq C_3/W_n^{2+\varepsilon}$ for some $\varepsilon \in (0, 1]$, and $\mathbb{E}[(W_{n+1} - W_n)\mathbf{1}(A_{n+1}) \mid \mathcal{G}_n] = 0$;*

(b) *on the event $\{W_n \in (0, M_0]\}$, $\mathbb{E}[W_{n+1} - W_n \mid \mathcal{G}_n] \leq C_4$;*

(c) *for any $m > 0$, there exists $p_m > 0$ such that on the event $\{W_n \in (0, m]\}$, $\mathbb{P}\{W_{n+1} = 0 \mid \mathcal{G}_n\} \geq p_m$;*

(d) *on the event $\{W_n > M_0\}$, we have $\mathbb{E}[(W_{n+1} - W_n)^2 \mid \mathcal{G}_n] \geq C_5$ and $\mathbb{E}[|W_{n+1} - W_n|^3 \mid \mathcal{G}_n] \leq C_6$.*

Then, if $\nu := \inf\{n \geq 1 : W_n = 0\}$, there is a positive constant C_7 such that for any $y \geq 1$ and positive real number m, we have

$$\mathbb{P}\{\nu \geq m \mid W_0 = y\} \leq \frac{C_7 y}{\sqrt{m}}.$$

Coalescence Time

For each $m \in \mathbb{Z}$, let $\mathcal{F}_m := \sigma\big(\{U_{\mathbf{z}}, U_{\mathbf{v},\mathbf{w}} : \mathbf{z}, \mathbf{v}, \mathbf{w} \in \mathbb{Z}^2, \mathbf{z}(2) \le m, \mathbf{v}(2) < \mathbf{w}(2) \le m\}\big)$. Consider the process starting from two arbitrary vertices $\mathbf{u}_1, \mathbf{u}_2 \in \mathbb{Z}^2$ with $\mathbf{u}_1(2) = \mathbf{u}_2(2)$ where, without loss of generality, $\mathbf{u}_1(2) = 0$. For $n \ge 0$, define $T_n := T_n(\mathbf{u}_1, \mathbf{u}_2)$ as the time of the n-th joint regeneration and

$$Z_n = Z_n(\mathbf{u}_1, \mathbf{u}_2) := |g_n(\mathbf{u}_1)(1) - g_n(\mathbf{u}_2)(1)|,$$

where $g_n(\mathbf{u}_1)$ and $g_n(\mathbf{u}_2)$ are the positions of the paths started at $\mathbf{u}_1$ and $\mathbf{u}_2$ at time T_n, respectively. We can easily find that $\{Z_n : n \ge 0\}$ is an $\{\mathcal{F}_{T_n} : n \ge 0\}$ adapted process. So taking

$$n_{\mathbf{u}_1,\mathbf{u}_2} := \inf\{n \ge 1 : Z_n = 0\},$$

which is the first joint regeneration step in which two paths starting from $\mathbf{u}_1$ and $\mathbf{u}_2$ coalesce, we apply the theorem of Coupier et al. (2021) to obtain

$$\mathbb{P}\big\{n_{\mathbf{u}_1,\mathbf{u}_2} \ge k \mid Z_0 = y\big\} \le C\frac{y}{\sqrt{k}}, \tag{1}$$

for all $y \ge 1$ and positive real number k and some positive constant C.

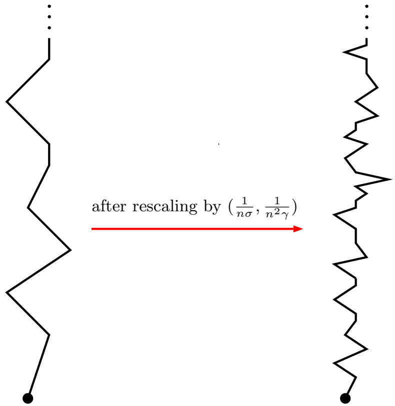
Consider each path as a collection of piecewise linear edges in such a way that a path starting from $\mathbf{u}$ is the map $\pi_{\mathbf{u}} : [\mathbf{u}(2), \infty) \to \mathbb{R}$ with $\pi_{\mathbf{u}}(h_k(\mathbf{u})(2)) = h_k(\mathbf{u})(1)$ and it is linear on $[h_k(\mathbf{u})(2), h_{k+1}(\mathbf{u})(2)]$ for each $k \geq 0$.

Consider a path $\pi \in \mathcal{X}$ with the starting time σ_π. For each $n \in \mathbb{N}$ and some normalization constants $\sigma, \gamma > 0$, the scaled path $\pi^{(n)}(\sigma, \gamma)$ is defined by

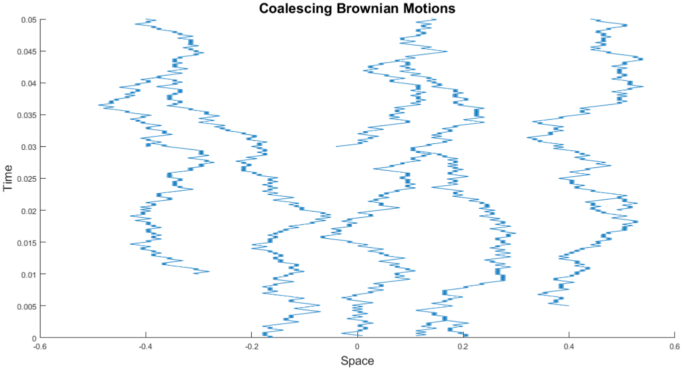
$$\pi^{(n)}(\sigma, \gamma) : [\sigma_\pi/(n^2\gamma), \infty) \to \mathbb{R}$$

such that $\pi^{(n)}(\sigma, \gamma)(t) = \pi(n^2\gamma t)/(n\sigma)$. We denote the collection of scaled paths by $\chi_n(\sigma, \gamma) := \{\pi_{\mathbf{u}}^{(n)}(\sigma, \gamma) : \mathbf{u} \in \mathcal{V}\}$.

Taking $\sigma = [\mathbb{V}\mathrm{ar}(\pi_{\mathbf{0}}(T_1(\mathbf{0}))]^{\frac{1}{2}}$ and $\gamma = \mathbb{E}[T_1(\mathbf{0})]$, it is proved that $\pi^{(n)}_{\mathbf{0}}(\sigma,\gamma) \Rightarrow B_{\mathbf{0}}$ as $n \to \infty$, where $B_{\mathbf{x}}$ is the Brownian motion (with unit diffusion constant) starting from $\mathbf{x} \in \mathbb{R}^2$. As a result, if $(\mathbf{u}_n(1)/(n\sigma), \mathbf{u}_n(2)/(n^2\gamma)) \xrightarrow{\mathbb{P}} \mathbf{u}$, then $\pi^{(n)}_{\mathbf{u}_n}(\sigma,\gamma) \Rightarrow B_{\mathbf{u}}$ as $n \to \infty$.



For a finite number of paths, the limit is **coalescing Brownian motions**: a system of Brownian motions so that every two paths move independently until they meet, and then, they coalesce and move independent from the remaining paths.



Main Result

Theorem

For each of the models, there exist $\sigma := \sigma(p)$ and $\gamma := \gamma(p)$ such that as $n \to \infty$, $\bar{\chi}_n(\sigma, \gamma)$ converges in distribution to the (standard) Brownian web $\mathcal{W}$.

Intuitively, the Brownian web is a collection of 1-dimensional coalescing Brownian motions starting from every point (x, t) in the space-time plane $\mathbb{R}^2$. It arises naturally as the diffusive scaling limit of a system of 1-dimensional coalescing random walks starting from every vertex of the space-time lattice $\mathbb{Z}^2_{\text{even}} := \{(i, j) \in \mathbb{Z}^2 : i + j \text{ is even}\}$.

First, **Arratia** (1979, 1981) considered the Brownian web to study the scaling limit of the voter models. Next, **Tóth** and **Werner** (1998) also considered the Brownian web for their study of the true self-repelling motion.

Later, **Fontes**, **Isopi**, **Newman** and **Ravishankar** (2002, 2004) introduced it as a random variable taking values in the space of compact sets of paths.

Brownian Web

Briefly, consider the following complete separable metric spaces:

- $(\mathbb{R}^2_c, \rho)$: a space of points which is the completion of $\mathbb{R}^2$ under the metric ρ;
- (Π, d): a space of paths with specified starting points;
- $(\mathcal{H}, d_{\mathcal{H}})$: the Hausdorff metric space of compact subsets of Π.

Let $\mathcal{B}_{\mathcal{H}}$ be the Borel σ-algebra generated by the metric $d_{\mathcal{H}}$. The Brownian web $\mathcal{W}$ is an $(\mathcal{H}, \mathcal{B}_{\mathcal{H}})$ valued random variable whose distribution is uniquely characterized by the following three properties:

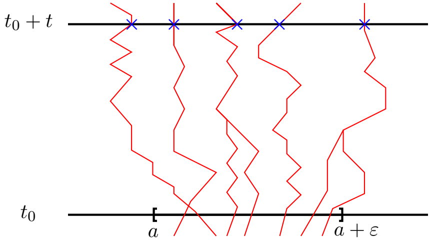
- for each deterministic point $\mathbf{x} \in \mathbb{R}^2$, there is a unique path $\tilde{\pi}_{\mathbf{x}} \in \mathcal{W}$ starting from $\mathbf{x}$ almost surely;
- for each $k \geq 1$ and any finite set of deterministic points $\mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathbb{R}^2$, the collection $\{\tilde{\pi}_{\mathbf{x}_1}, \ldots, \tilde{\pi}_{\mathbf{x}_k}\}$ is distributed as coalescing Brownian motions starting from $\mathbf{x}_1, \ldots, \mathbf{x}_k$;
- for any countable deterministic dense set $\mathcal{D} \subseteq \mathbb{R}^2$, $\mathcal{W}$ is the closure of $\{\tilde{\pi}_{\mathbf{x}} : \mathbf{x} \in \mathcal{D}\}$ in (Π, d) almost surely.

The method of proof follows Theorem 5.1, Proposition B.1 of **(Fontes et al., 2004)** and Theorem 1.4 of **(Newman et al., 2005)**, where four conditions (I_1), (B_1'), (T_1) and (E_1') need to be checked.

($\mathbf{I}_1$) For all $\mathbf{y} \in \mathbb{R}^2$, there exist $\zeta_n^{\mathbf{y}} \in \bar{\chi}_n(\sigma, \gamma)$ such that for any finite set of points $\mathbf{y}_1, \ldots, \mathbf{y}_k$ from a deterministic countable dense set $\mathcal{D}$ of $\mathbb{R}^2$, $(\zeta_n^{\mathbf{y}_1}, \ldots, \zeta_n^{\mathbf{y}_k}) \Rightarrow (W_{\mathbf{y}_1}, \ldots, W_{\mathbf{y}_k})$ as $n \to \infty$, where $(W_{\mathbf{y}_1}, \ldots, W_{\mathbf{y}_k})$ is the coalescing Brownian motions (with unit diffusion constant) starting from k points $\mathbf{y}_1, \ldots, \mathbf{y}_k$.

($\mathbf{B}_1'$) For all $\beta > 0$,

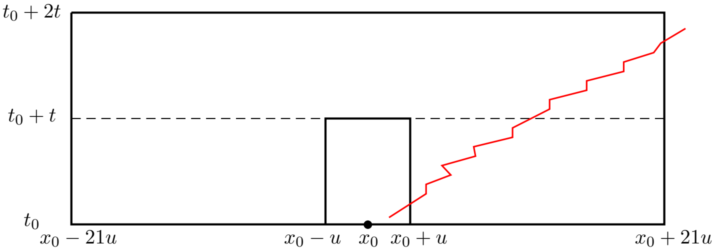
$$\limsup_{\varepsilon \to 0^+} \limsup_{n \to \infty} \sup_{t > \beta} \sup_{(a, t_0) \in \mathbb{R}^2} \mathbb{P}\{\eta_{\bar{\chi}_n(\sigma,\gamma)}(t_0, t; a, a+\varepsilon) \geq 2\} = 0.$$



In this realization, $\eta_{\bar{\chi}_n(\sigma,\gamma)}(t_0, t; a, a+\varepsilon)$ is 5.

(T$_1$) Let $\Lambda_{L,T} := [-L, L] \times [-T, T]$. Then, for every $u, L, T \in (0, \infty)$,

$$\limsup_{t \to 0^+} \frac{1}{t} \limsup_{n \to \infty} \sup_{(x_0, t_0) \in \Lambda_{L,T}} \mathbb{P}\{A_{\bar{\chi}_n(\sigma,\gamma)}(x_0, t_0; u, t)\} = 0.$$



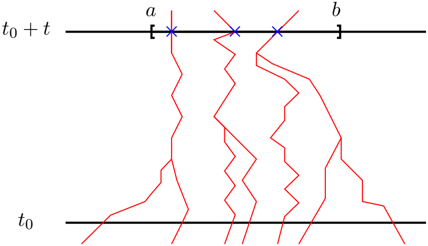
A realization of the event $A_{\bar{\chi}_n(\sigma,\gamma)}(x_0, t_0; u, t)$.

not-to-scale

The Method of Proof

(E$_1'$) For every $t_0 \in \mathbb{R}$, if Z^{t_0} is any subsequential limit of $\{\bar{\chi}_n^{t_0} : n \in \mathbb{N}\}$, where $\bar{\chi}_n^{t_0}$ is the subset of paths in $\bar{\chi}_n(\sigma, \gamma)$ which start before or at time t_0, then for all $t, a, b \in \mathbb{R}$ with $t > 0$ and $a < b$,

$$\mathbb{E}[\hat{\eta}_{Z^{t_0}}(t_0, t; a, b)] \leq \mathbb{E}[\hat{\eta}_{\mathcal{W}}(t_0, t; a, b)].$$



In this realization, $\hat{\eta}_\Gamma(t_0, t; a, b)$ is 3.

Similar Studies

Convergence to the Brownian web for a system with (possible) crossing paths:

- Newman, Ravishankar and Sun (2005)
- Coletti and Valle (2014)

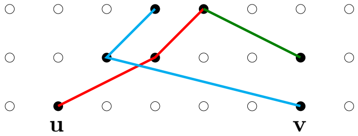
Convergence to the Brownian web for a system with non-Markovian nature:

- Roy, Saha and Sarkar (2016)
- Coupier, Saha, Sarkar and Tran (2021)

Next Result

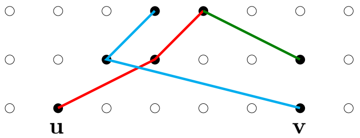
Our method of proof can also be applied to the study of the generalized Howard's model (Coletti and Valle, 2014) to relax the condition they need to show convergence to the Brownian web.

Let $\{\zeta_{\mathbf{v}} : \mathbf{v} \in \mathbb{Z}^2\}$ be another independent collection of i.i.d. random variables taking values in $\mathbb{N}$ according to a probability mass function q.



$\zeta_{\mathbf{u}}$ takes value 2 and $\zeta_{\mathbf{v}}$ takes value 3

Let $\{\zeta_{\mathbf{v}} : \mathbf{v} \in \mathbb{Z}^2\}$ be another independent collection of i.i.d. random variables taking values in $\mathbb{N}$ according to a probability mass function q.



$\zeta_{\mathbf{u}}$ takes value 2 and $\zeta_{\mathbf{v}}$ takes value 3

Coletti and Valle have shown that if q has a finite range (that means there exists a set $F \subseteq \mathbb{N}$ such that $|F| < \infty$ and $q(n) \neq 0$ if and only if $n \in F$), then $\bar{\chi}_n$ (the closure of the collection of all suitably rescaled paths) converges in distribution to the (standard) Brownian web $\mathcal{W}$ as $n \to \infty$. However, our method just needs that q **has a finite 6-th moment**.

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Thank You For Your Attention!