The time constant for Bernoulli percolation is Lipschitz continuous strictly above p_c

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Percolation

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- Graph $(\mathbb{Z}^d, \mathbb{E}^d)$, $d \geq 2$.
- $(B(e))_{e \in \mathbb{E}^d}$: i.i.d. family of Bernoulli random variable of parameter $p \in [0, 1]$.
- $B(e) = 1 \implies e$ is open.
- $B(e) = 0 \implies e$ is closed.

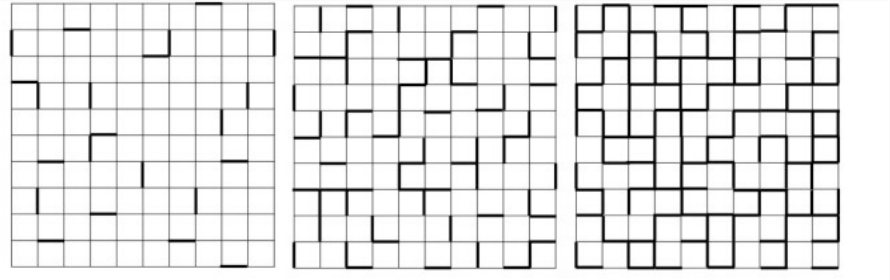

Figure 1: Simulation of percolation for parameters $p = 0.1$; 0.3 and 0.6

Percolation probability

- Random graph $\mathcal{G}_p = (\mathbb{Z}^d, \{e \in \mathbb{E}^d : B(e) = 1\})$.
- $\mathcal{C}_p(0)$: the connected component of 0 in $\mathcal{G}_p$.

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- $\theta(0) = 0$
- $\theta(1) = 1$
- $p \mapsto \theta(p)$ is nondecreasing

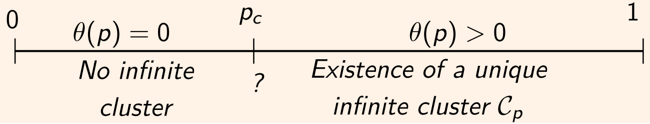
Phase transition

Definition (Critical parameter)

$$p_c = \sup\{\, p : \theta(p) = 0 \,\}$$

Phase transition at $p_c \in]0,1[$:

Theorem (Broadbendt-Hammersley 57-59,...)



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What is the asymptotic value of $T_G(0, nx)$?

First passage percolation : Definition of the time constant

Theorem (Hammersley-Welsh 65, Kingman 73-75)

Under some conditions on G, we have

$$\forall x \in \mathbb{Z}^d \qquad \lim_{n\to\infty} \frac{T_G(0,nx)}{n} = \mu_G(x) \textit{ almost surely and in } L^1 .$$

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Regularity of μ_G in G ?

Theorem (Cox 81,…, Garet-Marchand-Proccacia-Théret 17)

The map $G \mapsto \mu_G$ is continuous.

Time constant in the Bernoulli case

Graph distance

We are interested in the random metric induced by $\mathcal{G}_p$ when $p > p_c$. We define for x and y in $\mathbb{Z}^d$

$$\mathcal{D}_p(x, y) = \inf \{|\gamma| : \gamma \text{ path that joins } x \text{ and } y \text{ in } \mathcal{G}_p\}$$

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Set $G_p = p\delta_1 + (1-p)\delta_\infty$. We can couple $(t(e))_{e\in\mathbb{E}^d}$ with $(B(e))_{e\in\mathbb{E}^d}$ by setting $B(e) = \mathbb{1}_{t(e)=1}$ so that

$$\mathcal{D}_p = T_{G_p}\,.$$

First passage percolation : Definition of the time constant for the graph distance

Theorem (Cerf-Théret 14)

For $p > p_c$, for any $x \in \mathbb{Z}^d$, there exists $\mu_p(x) > 0$ such that

$$\lim_{n\to\infty} \frac{\mathcal{D}_p(\widetilde{0}, \widetilde{nx})}{n} = \mu_p(x) \text{ almost surely and in } L^1$$

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Theorem (D. 18)

Let $p_0 > p_c$*, there exists a positive constant* C *(depending on* p_0*) such that*

$$\forall p, q \in [p_0, 1] \qquad \sup_{\|x\|=1} |\mu_p(x) - \mu_q(x)| \leq C|q-p| \log|q-p| \,.$$

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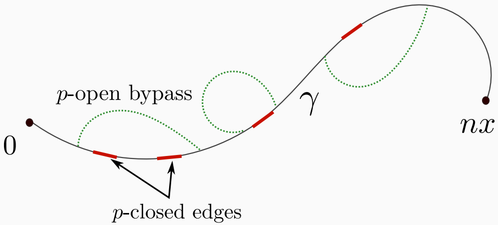

Figure 2: Build a p-open path upon a q-open path for $q > p > p_c$

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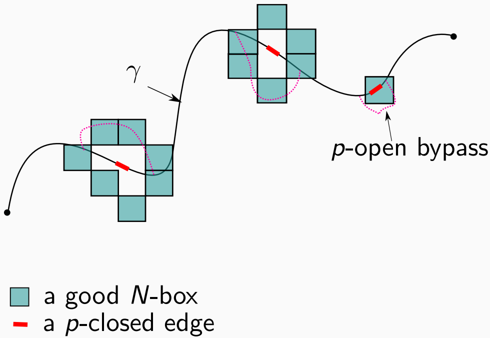
$$\mathcal{D}_p(0, nx) \leq |\gamma'| \leq |\gamma| + |\gamma' \setminus \gamma| \leq \mathcal{D}_q(0, nx) + |\gamma' \setminus \gamma|$$

If we prove that $|\gamma' \setminus \gamma| \leq C_0|q-p|n$ then

$$\mu_p \leq \mu_q + C_0|q-p|.$$

First approach: renormalization



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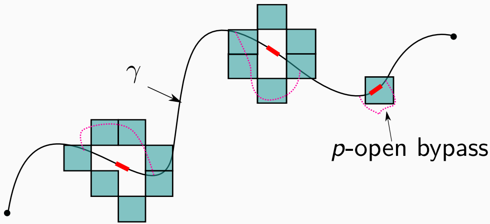

a good N-box
a p-closed edge

Divide the lattice into boxes of mesoscopic size N. A good box is a box that has good connectivity property. Being a good box is something very likely for N large.Two cases :

1. Bad edge in good box
2. Bad edge in bad box

A different approach

Let $q > p > p_c$. γ is the q-geodesic between 0 and nx. We don't reveal which edges need to be bypassed. For each $e \in \gamma$, we define $c(e)$ the cost to bypass e such that:

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We have

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To build $c(e)$ we need a multiscale renormalisation. We need to consider an infinite number of scales of box at the same time. A good box at scale $k+1$ is good if it does not contain too much bad boxes at scale k.

Thank you for your attention !