Time averages for kinetic Fokker-Planck equations

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September 8, 2021

Acknowledgements

- Article available at https://arxiv.org/pdf/2106.12801.pdf
- This research project is funded by the European Union's Horizon 2020 research and innovation program under the Marie Sklodowska-Curie grant agreement No 754362.
- Partial support has been obtained from the EFI ANR-17-CE40-0030 Project of the French National Research Agency.

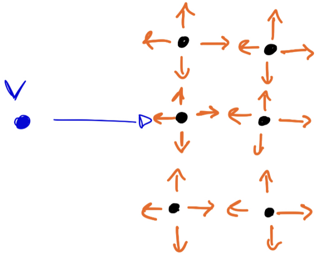
Notation

- t time in $[0, \infty)$,
- x space in $[0, L]^d$, periodic boundary conditions,
- v velocity in $\mathbb{R}^d$,
- $f = f(t, x, v)$, distribution function (probability density).
- $\langle v \rangle^2 = 1 + v^2$, Chinese bracket,
- $\gamma(v)$ Gaussian probability density, $d\gamma = \gamma \, dv$.

Foundation of the model

Evolution of a system of particles $(x_i, v_i)_{i=1,\cdots,N}$ moving and interacting with a random background force W_N :

$$\begin{cases} \dot{x}_i = v_i; \\ \dot{v}_i = -v_i + W_N(v_i). \end{cases}$$



Langevin dynamics

Limit stochastic process as $N \to \infty$:

$$\begin{cases} dX_t = V_t \, dt; \\ dV_t = -V_t \, dt + \sqrt{2} \, dW_t. \end{cases} \tag{LD}$$

The process W_t is a standard Brownian motion (in v).
Evolution (statistical) of positions and velocities of the *particles* (molecules, electrons, stars, agents, $\cdots$).
Description at a mesoscopic level.
Example: plasmas, particles crossing an irregular medium.

From Langevin to VFP

- Let L^* be the generator associated with Langevin dynamics:

$$\langle L^\star (X_0, V_0), f \rangle = \lim_{t \to 0} \int f(x, v) dX_t dV_t.$$

- Dual operator

$$Lf = -v \cdot \nabla_x f + \nabla_v \cdot (\nabla_v f + vf) = -Tf + Qf.$$

- Densities of solutions to (LD) are solutions to

$$\partial_t f = Lf. \tag{VFP}$$

- The operator L is the kinetic Fokker-Planck or Vlasov-Fokker-Planck operator = transport and collisions.

KOU equations and remarks

Let γ be the normalized Gaussian and consider $h = f/\gamma$. Then, h solves the Kinetic Ornstein-Uhlenbeck equation

$$\partial_t h + v \cdot \nabla_x h = \Delta_v h - v \cdot \nabla_v h = \Delta_\gamma h. \tag{KOU}$$

- linear and mass-preserving kinetic equation. Henceforth, null-mass solutions only.
- degenerate dynamics: diffusion acts on the v variable only.
- the Ornstein-Uhlenbeck operator satisfies

$$-\int (\Delta_\gamma h)\, h \, dx \, d\gamma = \int |\nabla_v h|^2 \, dx \, d\gamma,$$

- a priori we control weak norms in velocity only.

Questions

1. Steady states?
2. Existence, uniqueness, regularity?
3. Approach to equilibrium? Rates? Explicit/optimal estimates?

Answers to 1 and 2

- h steady (null mass) solution if and only if $h = 0$.
- For all $h_0 \in \mathrm{L}^2(dxd\gamma)$ there exists a unique classic solution to (KOU) starting at h_0. More general data allowed. Known at least since Kolmogorov 1934.
- Smoothing effect of the equation due to hypoellipticity, see Hörmander 1967.
- Hypoellipticity carries regularity from velocity to space directions, thanks to a twist in the phase space.

Fundamental energy estimate

Lyapunov functional for the evolution:

$$\frac{1}{2}\frac{d}{dt}\|h_t\|^2_{\mathrm{L}^2(dxd\gamma)} = \iint h_t \, \Delta_\gamma h_t \, dx \, d\gamma = -\iint |\nabla_v h_t|^2 \, dx \, d\gamma \leq 0.$$

Not sufficient to prove

$$h_t \to 0, \quad \text{as } t \to \infty$$

for all h_0 initial data.

How to prove it? How fast? Decay rates?

Conclusion in the space-homogeneous case

Assume independence from x in this slide only (overdamped Langevin dynamics).
The energy estimate is

$$\frac{1}{2}\frac{d}{dt}\|h_t\|^2_{\mathrm{L}^2(d\gamma)} = \int h_t\,\Delta_\gamma h_t\,d\gamma = -\int |\nabla_v h_t|^2\,d\gamma.$$

Via Gauss-Poincaré inequality (Nash 1965):

$$-\int |\nabla_v h_t|^2\,d\gamma \leq -\|h_t\|^2_{\mathrm{L}^2(d\gamma)}.$$

Therefore

$$\|h_t\|^2_{\mathrm{L}^2(d\gamma)} \leq \|h_0\|^2_{\mathrm{L}^2(d\gamma)}\,\mathrm{e}^{-2t},$$

sharp exponential decay estimate.

The problem of hypocoercivity

How to recover a decay result in the fully inhomogeneous case?

- coercivity is available along velocity directions, lacking along space directions,
- hypocoercivity is the name of a class of techniques aimed at recovering decay in the missing directions,
- term coined by T. Gallay, appeared first in Villani 2006,
- need to separate decay from regularity issues.

A very important toy model

One-particle kinetic model:

$$\frac{d}{dt}\begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}.$$

Intuition for key papers of Arnold-Erb 2014,
Dolbeault-Mouhot-Schmeiser 2015.
How to prove convergence to $(0, 0)$ without solving the ODE? :)

Strategy of hypocoercivity

1. twist of the reference norm (L^2 or H^1), creating an equivalent norm;
2. Gronwall estimate in the new norm;
3. exponential decay in the reference norm, via norm equivalence.
4. typical estimate:

$$\|h_t\| \leq C\, e^{-\lambda t}\, \|h_0\|,$$

5. $C > 1$ whenever the diffusion is degenerate. $C = 1$ in case of coercivity in all variables (as in the space homogeneous case).

H^1 hypocoercivity

- Villani 2006, Villani 2009, Dolbeault-Li 2018, Baudoin 2013-2020,
- cover a broad class of degenerate PDEs,
- explicit results in H^1-norm and Wasserstein distance,
- consequent rates of convergence in L^2-norm are suboptimal, more direct calculations are required,
- example of twisted norm:

$$\|h\|^2 = \|h\|^2_{\mathrm{H}^1(dxdv)} + \epsilon \nabla_x h \cdot \nabla_v h.$$

L^2 hypocoercivity

- DMS 2015, Bouin-Dolbeault-Mouhot-Mischler-Schmeiser 2019, Arnold-Achleitner-Carlen 2016-2018-..., Bernard-Fathi-Levitt-Stoltz 2020, ...
- direct L^2 calculations for a general class of PDEs,
- consistency with diffusion limits (DMS),
- rates of convergence are generally off by one order of magnitude, but very accurate/optimal in many interesting cases: AE 2014, AAC 2018, Arnold-Dolbeault-Schmeiser-Wöhrer 2021,
- twisted norm in DMS:

$$\|f\|^2 \approx \|f\|^2_{\mathrm{L}^2(dxdv)} + \int f \left(\epsilon_1 \frac{\Delta_x}{\epsilon_2 + \Delta_x}\right) f \, dxdv.$$

Recent approaches involving weak norms

- Armstrong-Mourrat 2019, Cao-Lu-Wang 2020.
- Weak velocity norms of solutions are considered, reference space
$$h \in \mathrm{L}^2(dtdx; H^1(d\gamma)),$$
with
$$(\partial_t + v \cdot \nabla_x)h \in \mathrm{L}^2(dtdx; H^{-1}(d\gamma)).$$
- Generalized Poincaré inequality (AM 2019)
$$\|h\|^2_{\mathrm{L}^2(dtdxd\gamma)} \leq C_1 \|\nabla_v h\|^2_{\mathrm{L}^2} + C_2 \|(\partial_t + v \cdot \nabla_x)h\|^2_{\mathrm{L}^2(\mathrm{H}^{-1}(d\gamma))}.$$
- hypocoercivity via a discrete Gronwall estimate.

Explicit generalized Poincaré inequalities

1. Introduce $\rho = \int h \, d\gamma$.
2. Lions' Lemma (J.-L. Lions '60 - Amrouche et alii 2014):

$$\|\rho\|^2_{\mathrm{L}^2} \leq C_L \|\nabla_{t,x}\rho\|^2_{\mathrm{H}^{-1}(dtdx)},$$

3. Averaging Lemma (AM 2019, B. 2021)

$$\|\nabla_{t,x}\rho\|^2_{\mathrm{H}^{-1}(dtdx)} \leq d_2 \|\nabla_v h\|^2_{\mathrm{L}^2(dtdxd\gamma)},$$

4. Explicit Poincaré inequality:

$$\|h\|^2_{\mathrm{L}^2(\mathrm{dtdxd}\gamma)} \leq C_P \|\nabla_v h\|^2_{\mathrm{L}^2(dtdxd\gamma)}.$$

Message: velocity derivatives control the whole norm of the solution, if integrated in time.

Explicit decay for time averages (B. 2021)

Let $\tau > 0$ small, then

$$\frac{d}{dt}\int_t^{t+\tau} \|h\|^2_{\mathrm{L}^2(dxd\gamma)}\,dt = -2\|\nabla_v h\|^2_{\mathrm{L}^2(dtdxd\gamma)},$$

and,

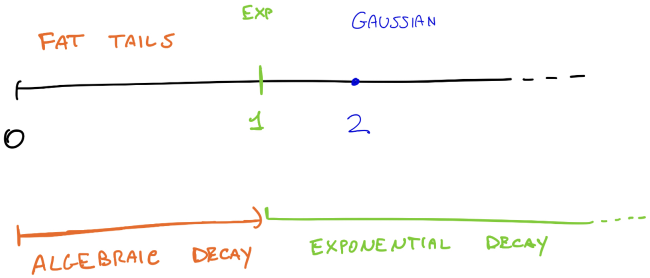
$$-2\|\nabla_v h\|^2_{\mathrm{L}^2(dtdxd\gamma)} \le -\frac{2}{C_P}\int_t^{t+\tau} \|h\|^2_{\mathrm{L}^2(dxd\gamma)}\,dt.$$

Therefore

$$\tau^{-1}\int_t^{t+\tau} \|h\|^2_{\mathrm{L}^2(dxd\gamma)}\,dt \le \|h_0\|^2_{\mathrm{L}^2}\, e^{-\frac{2}{C_P}\,t}.$$

Bonus: extension to a class of kinetic Fokker-Planck equations

- Langevin dynamics: Gaussian local equilibria (due to Brownian motion).
- more general local equilibria? Decay rates depend on their growth at infinity (Bouin-Dolbeault-Lafleche-Schmeiser 2020, B. 2021):



Thank you for your attention.