New fermion measures from map enumeration

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Work in progress with Guillaume Chapuy (CNRS & IRIF)
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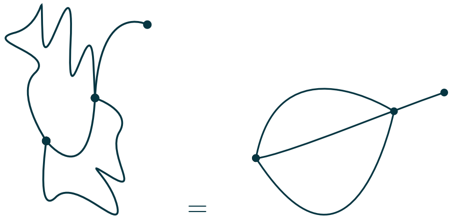
Random Networks and Interacting Particle Systems, 6 September 2021

Maps are graphs embedded in surfaces, defined as a set of vertices and edges and their oriented incidence relations.



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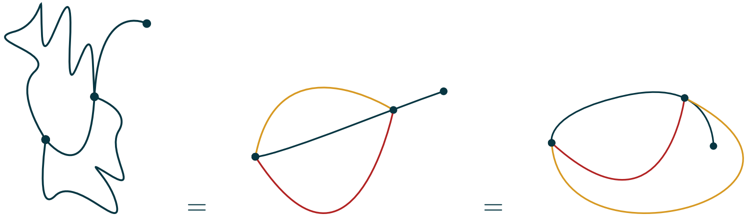

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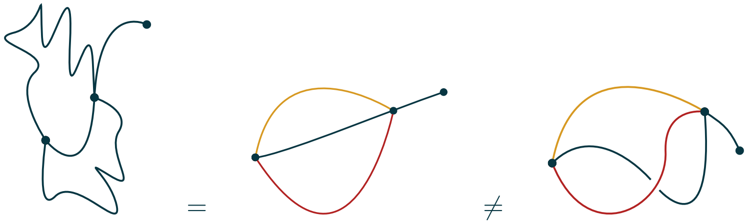

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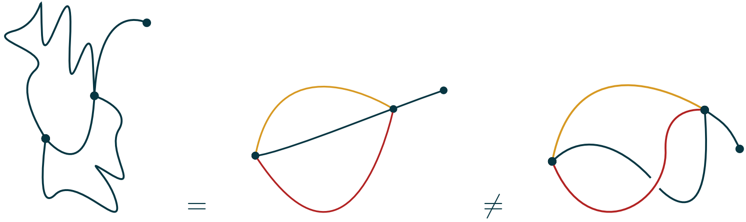
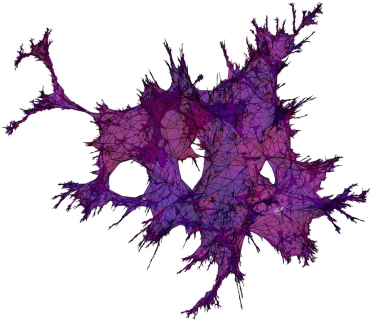


The genus g of a map is the minimal number of holes needed in the surface it's drawn on. It satisfies the Euler characteristic relation

$$\#\text{vertices} - \#\text{edges} + \#\text{faces} = 2 - 2g.$$

Maps have **universal** properties:

- e.g. universal asymptotic enumeration for maps, triangulations, ... with n vertices as $n \to \infty$.
- e.g. universal local limits for large random maps



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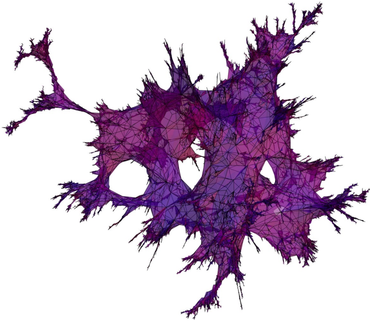

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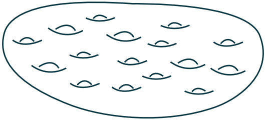

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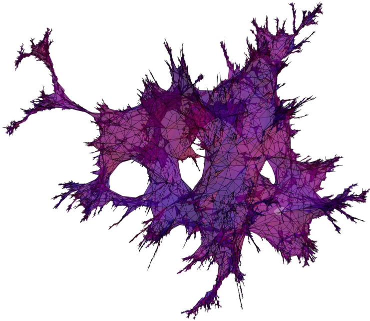

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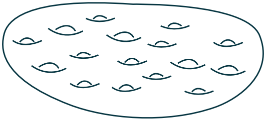
Theorem (Budzinski–Louf, 2019)

Let $\mathcal{T}_{n,g_n}$ be the triangulations with n vertices and genus $g_n = \lfloor \theta n \rfloor$. Then, as $n \to \infty$,

$$|\mathcal{T}_{n,g_n}| = n^{2g_n} \exp[f(\theta)n + o(n)].$$



We consider a family of maps constructed from constellations
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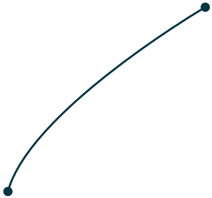

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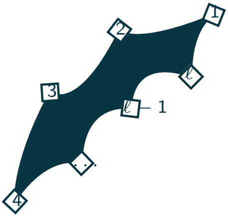

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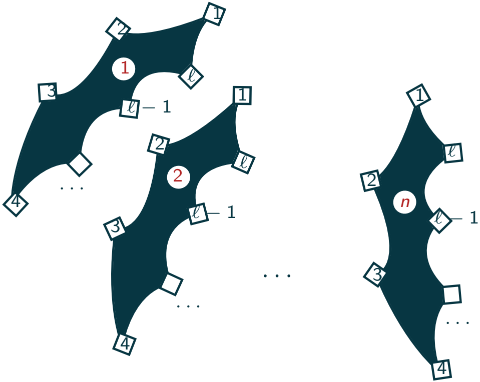

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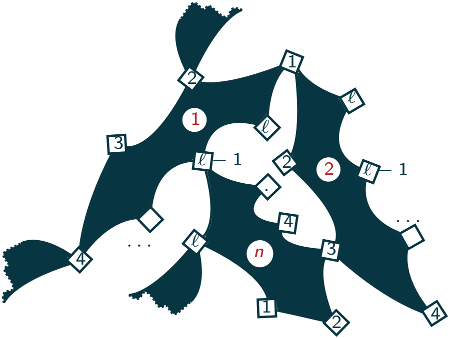

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$$\{\sigma_1, \sigma_2, \ldots, \sigma_\ell, \phi\} \subset S_n$$

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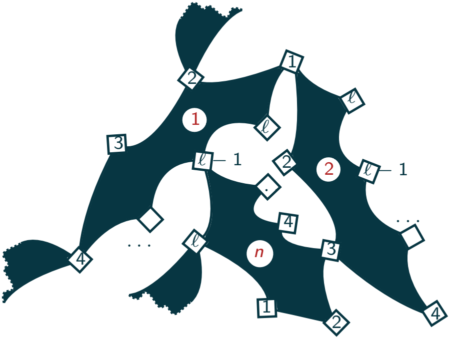

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- fix each $\sigma_i = \tau_i$ to be a transposition
- fix $\phi = \mathrm{id}$
- draw vertices on hyperedges and edges on hypervertices

These maps are in the same universality class as maps of all degree (Duchi, Poulalhon & Schaeffer, 2014).

Consider the enumeration of the resulting maps,

$$H_{n,\ell} = \frac{1}{n!} \#\{\tau_1, \tau_2, \ldots, \tau_\ell\} \subset S_n$$
$$\tau_1 \cdot \tau_2 \cdots \tau_\ell = \mathrm{id}\,.$$

By the ELSV formula, $H_{n,\ell}$ are the classical **Hurwitz numbers**; we call this family Hurwitz maps.

Frobenius' formula

The number of factorizations of the identity on S_n by ℓ transpositions can be expressed as a sum over irreducible representations V^λ of S_n,

$$H_{n,\ell} = \frac{1}{n!} \#\{\tau_1 \cdot \tau_2 \cdots \tau_\ell = \mathrm{id} \in S_n\} = \frac{1}{n!^2} \sum_{\lambda \vdash n} (\dim V^\lambda)^2 \prod_{i=1}^{\ell} \frac{\chi^\lambda(\tau_i)}{\dim V^\lambda}$$

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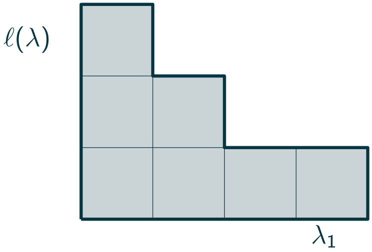
The irreducible representations and conjugacy classes of S_n are indexed by partitions $\lambda \vdash n$, $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \lambda_{\ell(\lambda)})$, $\lambda_1 + \lambda_2 + \ldots + \lambda_{\ell(\lambda)} = n$.



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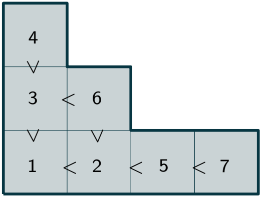
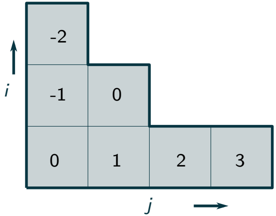
- d_λ is the number of **standard Young tableaux** of shape λ



- $c(\square_{i,j}) = j - i$ is the **contents**



$*$ When ℓ is odd, $H_{n,\ell} = 0$.

$$H_{n,\ell} = \frac{1}{n!}\#\{\tau_1 \cdot \tau_2 \cdots \tau_\ell = \mathrm{id} \in S_n\} = \frac{1}{n!^2}\sum_{\lambda \vdash n} d_\lambda^2 \Big(\sum_{\square \in \lambda} c(\square)\Big)^\ell$$

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$H_{n,\ell}$, counting unconnected Hurwitz maps, can be reframed as the partition function of random partitions of n under the Hurwitz measure

$$\mathbb{P}_{n,\ell}(\lambda) = \frac{1}{H_{n,\ell}}\frac{1}{n!^2} d_\lambda C_\lambda^\ell, \qquad C_\lambda = \sum_{\square \in \lambda} c(\square).$$

Goal: Study these random partitions in the "high genus" regime

$$\ell_n = 2\lfloor \theta n \rfloor.$$

Theorem (Okounkov, 2000)

The generating function

$$\tau(\beta, q) = \sum_{n,\ell} q^n \frac{\beta^\ell}{\ell!} e^{nu} H_{n,\ell}$$

satisfies the Toda equation

$$e^{-u} \frac{\partial^2}{\partial u^2} \log \tau(\beta, q) = \frac{\tau(u+\beta, q)\tau(u-\beta, q)}{\tau(\beta, q)^2}.$$

- This leads to recurrence relations for the connected Hurwitz numbers $h_{n,\ell}$ generated by $\log \tau$ (Dubrovin, Yang & Zagier, 2017).
- Okounkov's approach is what's particularly interesting:

> *"we remark that the generating function for double Hurwitz numbers is almost by definition a certain matrix element in the infinite wedge space. It is a well known result of the Kyoto school that such matrix elements are τ-functions of integrable hierarchies."*

The infinite wedge space is built up from the vacuum

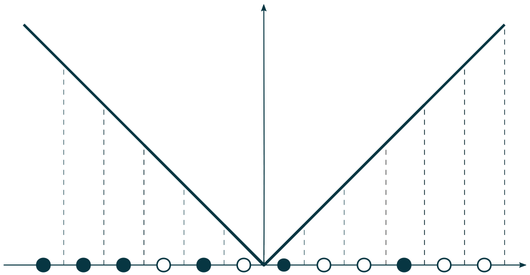
$$\cdots \wedge \underset{-\frac{7}{2}}{\bullet} \wedge \underset{-\frac{5}{2}}{\bullet} \wedge \underset{-\frac{3}{2}}{\bullet} \wedge \underset{-\frac{1}{2}}{\bullet} \wedge \underset{\frac{1}{2}}{\circ} \wedge \underset{\frac{3}{2}}{\circ} \wedge \underset{\frac{5}{2}}{\circ} \wedge \cdots \qquad = |\emptyset\rangle$$

with the following toolkit:

- Creation:
  $\psi_k(\cdots \wedge \underset{k}{\circ} \wedge \cdots) = (-1)^{\#\{\bullet_{j>k}\}} \cdots \wedge \underset{k}{\bullet} \wedge \cdots, \qquad \psi_k(\cdots \wedge \underset{k}{\bullet} \wedge \cdots) = 0$
- Annihilation:
  $\psi_k^*(\cdots \wedge \underset{k}{\bullet} \wedge \cdots) = (-1)^{\#\{\bullet_{j>k}\}} \cdots \wedge \underset{k}{\circ} \wedge \cdots, \qquad \psi_k^*(\cdots \wedge \underset{k}{\circ} \wedge \cdots) = 0$
- Indicator: $\psi_k \psi_k^* |\mathfrak{S}\rangle = \begin{cases} |\mathfrak{S}\rangle, & \underset{k}{\bullet} \text{ in } |\mathfrak{S}\rangle \\ 0, & \underset{k}{\circ} \text{ in } |\mathfrak{S}\rangle \end{cases}$
- Jumps: $a_{\pm 1} = \sum_{k \in \mathbb{Z} + \frac{1}{2}} \psi_{k \mp 1} \psi_k^*$
- Anticommutation:
  $\psi_k \psi_\ell + \psi_\ell \psi_k = \psi_k^* \psi_\ell^* + \psi_\ell^* \psi_k^* = 0, \qquad \psi_k \psi_\ell^* + \psi_\ell^* \psi_k = \delta_{k\ell}$

Up to charge, each fermion configuration maps to a partition, by

$$\mathfrak{S}(\lambda) = \{\lambda_i - i + \frac{1}{2}, i \in \mathbb{Z}_{>0}\}.$$



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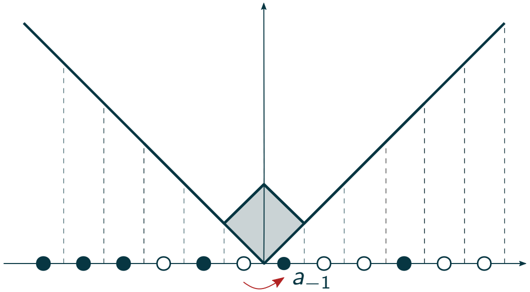

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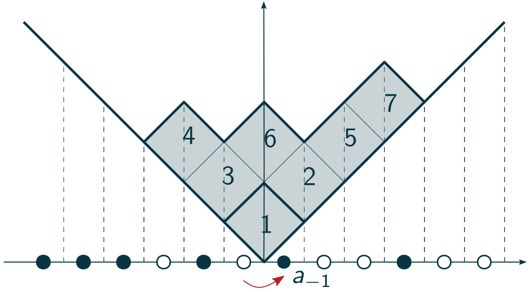

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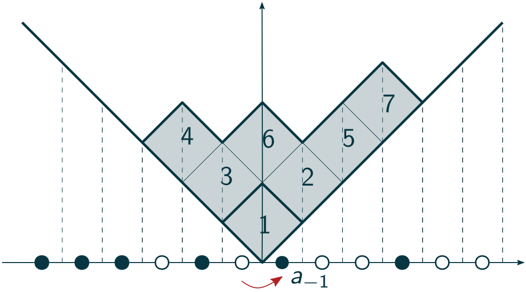

The vacuum expectation value

$$\langle \emptyset | \, e^{\theta a_{+1}} \left(\prod_{k \in \mathfrak{S}(\lambda)} \psi_k \psi_k^* \right) e^{\theta a_{-1}} \, | \emptyset \rangle = d_\lambda^2 \frac{\theta^{2|\lambda|}}{|\lambda|!^2}$$

counts the standard Young tableaux of shape λ.

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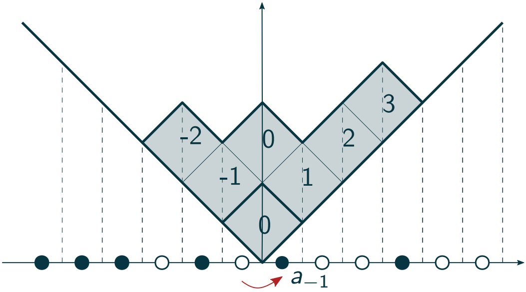


The sum of contents $C_\lambda = \sum_{\square \in \lambda} c(\square)$ is the eigenvalue of the operator

$$F_2 = \sum_{k \in \mathfrak{S}(\lambda)} \frac{k^2}{2} : \psi_k \psi_k^* :, \qquad : \psi_k \psi_k^* := \begin{cases} \psi_k \psi_k^*, & k > 0 \\ \psi_k^* \psi_k & k < 0 \end{cases}$$

The unconnected Hurwitz numbers $H_{n,\ell}$ are generated by

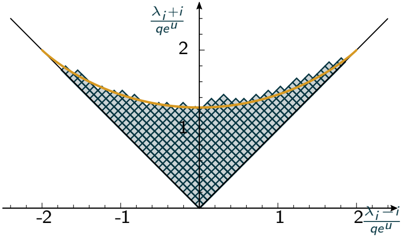
$$\tau(\beta, q) = \sum_{\lambda} \langle\emptyset|\, e^{\theta a_{+1}} \left(\prod_{k \in \mathfrak{S}(\lambda)} \psi_k \psi_k^* \right) e^{\beta F_2} e^{\theta a_{-1}} \,|\emptyset\rangle \Big|_{\theta = q e^{u/2}}$$

$$= \sum_{n,\ell} q^n \frac{\beta^\ell}{\ell!} e^{nu} \sum_{\lambda \vdash n} d_\lambda^2 C_\lambda^\ell.$$

- At $\beta = 0$, the Poissonized Plancherel measure

$$\mathbb{P}(\lambda) = \frac{1}{\tau(0,q)} d_\lambda^2 \frac{q^{|\lambda|} e^{u|\lambda|}}{|\lambda|!^2}$$

  defines a **determinantal point process**, in which every correlation function can be expressed as a determinant of a kernel (Borodin, Okounkov & Olshanski, 1999).



$*$ This is one way to find the rather trivial result $H_{n,0} = \frac{1}{n!}$.

Consider a shuffling n numbers by ℓ random exchanges

$$\mathbb{P}(\mathrm{id}) = \frac{1}{n}, \qquad \mathbb{P}(\tau_i) = \frac{1}{n^2} \qquad \forall\, \tau_i \in S_n.$$

$$(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, \dots, n-1, n)$$

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& \vdots \\
\sigma_\ell: \quad & (9, 17, 6, 28, n-1, 4, 31, 8, 25, 1, 3, n, 13, 21, \ldots, 5, 11)
\end{aligned}$$

The random partition σ_ℓ is distributed by the convolution $\mathbb{P}^{*\ell}(\sigma)$.

- **Question:** For what ℓ_n is σ_{ℓ_n} a uniform random permutation (Diaconis & Shahshahani, 1981) ?

Theorem (Diaconis–Shahshahani, 1981)

The measure on S_n from a random walk of length ℓ by transpositions with $\mathbb{P}(\tau) = \frac{1}{n^2}$ approaches the uniform measure U as $n \to \infty$ if $\ell > \frac{1}{2} n \log n$. In particular, the total variation distance is bounded as

$$\|\mathbb{P}^{*\ell} - U\| \leq b \exp\left[\frac{n \log n - 2\ell}{n}\right].$$

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$\mathbb{P}^{*\ell}$ is **Fourier transformed** to the representations of S_n, with

$$\rho(\mathbb{P}) = \sum_{\sigma \in S_n} \mathbb{P}(\sigma)\rho(\sigma), \quad \rho(\mathbb{P}^{*\ell}) = \rho(\mathbb{P})^\ell, \quad \mathbb{P}(\sigma) = \frac{1}{n!} \sum_{\lambda \vdash n} d_\lambda \operatorname{tr} \rho(\sigma)^* \mathbb{P}(\rho),$$

leading to an upper bound

$$\|\mathbb{P}^{*\ell} - U\|^2 \leq \sum_{\lambda \vdash n} d_\lambda^2 \left(\frac{1}{n} + \frac{n-1}{n} C_\lambda\right)^{2\ell}.$$

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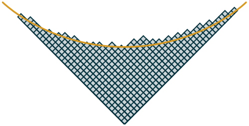
The cut-off in the total variation distance at $2\ell > n \log n$ is due to the contributions from

$$\boldsymbol{\lambda} = (n), \qquad \boldsymbol{\lambda} = (1, 1, \ldots, 1);$$

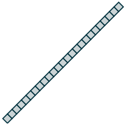
the Hurwitz random partitions concentrate to them as $n \to \infty$.

Limit shapes as $n \to \infty$ are known for Hurwitz random measures $\mathbb{P}_{n,\ell}$, with:

- $\ell = 0$ (Plancherel measure)



- $\ell_n > n \log n$ (random shuffling)



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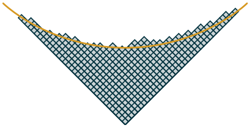

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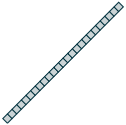
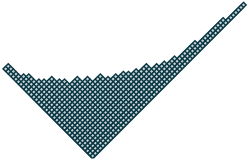


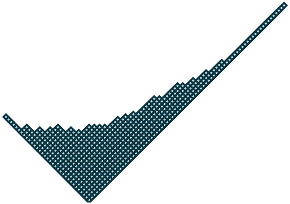
In the related model of random maps, these correspond to trivial cases:

- for $\ell = 0$, there are no edges to construct maps
- for $\ell_n > n \log n$, the maps saturate and edges are left unconnected.

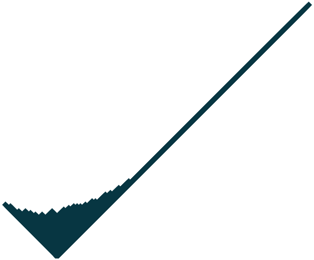
The intermediate regime corresponds to the map enumeration problem.



$\ell_n \sim n^{1/2}$



$\ell_n \sim n^{2/3}$



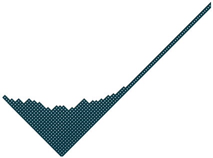
$\ell_n \sim n^{3/4}$

Lemma (Chapuy–Louf–W., 2021+)

Let λ be a random partition of n distributed by the Hurwitz measure $\mathbb{P}_{n,\ell_n}$ with $\ell_n = 2\lfloor \theta n \rfloor$ for $\theta \geq 1$. As $n \to \infty$, λ converges to the limit shape

$$\boldsymbol{\lambda} = \left(\frac{2\ell_n}{\log n}, \tilde{\lambda} \right)$$

where $\tilde{\lambda}$ is the limit shape under the Plancherel measure $\mathbb{P}_{n-\frac{2\ell_n}{\log n},0}$.



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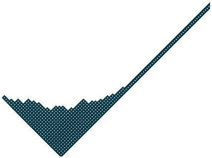


This is proven by

- varying λ_1 with $\lambda_2, \lambda_3, \ldots, \lambda_{\ell(\lambda)}$ fixed to maximize

$$d_\lambda^2 C_\lambda^{\ell_n}$$

- showing λ_1 is the only large part
- showing that with λ_1, $\tilde{\lambda} = \lambda \backslash \lambda_1$ should maximize

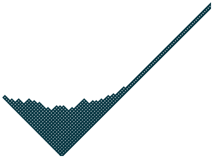
$$d_{\tilde{\lambda}}^2.$$

Theorem (Chapuy–Louf–W., 2021+)

Where $\ell_n = 2\lfloor \theta n \rfloor$, the classical Hurwitz numbers are approximated by

$$H_{n,\ell_n} = \exp\left[2\ell_n(\log \ell_n - \log\log n) - \ell_n(2 + 2\log 2) + o(n)\right]$$

as $n \to \infty$.



Theorem (Chapuy–Louf–W., 2021+)

Where $\ell_n = 2\lfloor \theta n \rfloor$, the unconnected classical Hurwitz numbers are approximated by

$$H_{n,\ell_n} = \exp\left[2\ell_n(\log \ell_n - \log\log n) - \ell_n(2 + 2\log 2) + o(n)\right]$$

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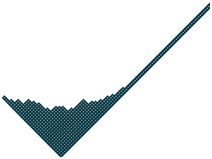
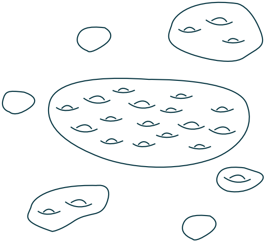


The partition function H_{n,ℓ_n} is bounded below by

$$H_{n,\ell_n} \geq d_{\boldsymbol{\lambda}}^2 C_{\boldsymbol{\lambda}}^{\ell_n} = e^{2\ell_n(\log \ell_n - \log\log n) - \ell_n(2+2\log 2) + o(n)}$$

and bounded above in turn by showing that

$$\sum_{\mu \neq \boldsymbol{\lambda} \vdash n} d_\mu^2 C_\mu^{\ell_n} \leq \frac{1}{n^{kn^{1/2}}} d_{\boldsymbol{\lambda}}^2 C_{\boldsymbol{\lambda}}^{\ell_n}.$$

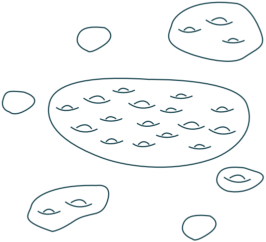


H_{n,ℓ_n} do not really count high genus Hurwitz maps; for our original problem we need to find the connected numbers $h_{n,\ell}$.

- Is there a high genus **giant component**? Is

  $$H_{n,\ell} = \sum_{m,k} \binom{n-1}{m-1} \binom{\ell}{k} h_{m,k} H_{n-m,\ell-k}$$

  dominated by $(m,k) = (c_1 n, c_2 \ell)$?



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- . . . no.

Rather, we can hope to extract approximate asymptotics for $h_{n,\ell}$ from $H_{n,\ell}$ at smaller ℓ_n.

Conclusions & Perspectives

Via the Hurwitz numbers, a particular high genus map enumeration problem can be reframed as a model of random integer partitions. This involves an integrable generalization of the Plancherel measure, which also appears in random walks on the symmetric group. Using this approach, we have found asymptotics of $H_{n,\ell_n \sim n}$ to $e^{o(n)}$.

This is work in progress, and we are still looking at

- $H_{n,\ell}$ in the full intermediate regime, $0 < \ell < n \log n$
- approximating the connected numbers $h_{n,\ell}$
- the corresponding fermion model.

Thank you for your attention!