Coalescing and branching simple exclusion and Fredrickson-Andersen models[1]

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Coalescing Random Walks with Neighbour Births

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CRWNB representation

Random walk jumping along each edge at rate 1.

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Coalescing independent random walks jumping along each edge at rate 1 **and giving birth to a particle at each neighbour independently at rate** β.

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- On $\mathbb{Z}^d$ for $\beta > 0$ – limit shape, cutoff [Bramson,Griffeath'80,81; Durrett,Griffeath'82]
- On $\mathbb{Z}$ for $\beta \to 0$ Brownian net [Sun,Swart'08]

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- (B) a particle fills the adjacent hole with rate $p/(2-p)$;
- (C) two particles coalesce at uniformly chosen of the two positions at rate $2(1-p)/(2-p)$.

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- CBSEP is the same as CRWNB with $\beta = p/(1-p)$ slowed down by a factor $(1-p)/(2-p)$.
- Nice dual model (in two distinct ways).
- Lots of embedded random walks (even more than those in the CRWNB representation).

Mixing times

Let $h_\omega^t(\cdot) = P_\omega^t(\cdot)/\mu(\cdot)$ be the density of the law of CBSEP started at ω w.r.t. the reversible measure μ.

Mixing times

$h_\omega^t(\cdot) = P_\omega^t(\cdot)/\mu(\cdot)$

Let $\|f\|_q = \left(\int f^q \,\mathrm{d}\mu\right)^{1/q} = (\mu(f^q))^{1/q}$ for $q \in [1, \infty]$.

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$h^t_\omega(\cdot) = P^t_\omega(\cdot)/\mu(\cdot)$
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$$\forall q \in [1, \infty], \quad T_q \leqslant O\left(\log\log\frac{1}{\mu_*}\right) T_{\mathrm{Sob}},$$

$\mu_* = \min_\omega \mu(\omega)$; T_{Sob} is 'the inverse rate of decay of entropy'

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- It's also $2|V|\mathcal{R}_{x,y}$, where $\mathcal{R}_{x,y}$ is the resistance between x, y.
- $T^{x,y}_{\mathrm{meet}}$ is the expected meeting time of x and y.
- In all examples we will encounter (and many others) we have

$$T_{\mathrm{meet}} := \frac{1}{|V|^2}\sum_{x,y} T^{x,y}_{\mathrm{meet}} \asymp \frac{1}{|V|^2}\sum_{x,y} T^{x,y}_{\mathrm{com}} \asymp \max_{x,y} T^{x,y}_{\mathrm{meet}} \asymp \max_{x,y} T^{x,y}_{\mathrm{com}} =: T_{\mathrm{com}}$$

and these are known up to a constant factor (or better).

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Theorem (Martinelli, Toninelli, H.'20)

Let $p_n = \Theta(1/n)$ and $G_n = (V_n, E_n)$ be a sequence of 'nice'[a] graphs with $|V_n| = n$. Then

$$\Omega(T_{\mathrm{meet}}) \leqslant T_{\mathrm{Sob}}^{\mathrm{CBSEP}} \leqslant O(T_{\mathrm{com}} \log n).$$

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Corollary

If G_n is the d-dimensional torus, then

$$\Omega(n^2) \leqslant T_{\mathrm{Sob}}^{\mathrm{CBSEP}} \leqslant O(n^2 \log n) \qquad d = 1$$
$$\Omega(n \log n) \leqslant T_{\mathrm{Sob}}^{\mathrm{CBSEP}} \leqslant O(n \log^2 n) \qquad d = 2$$
$$\Omega(n) \leqslant T_{\mathrm{Sob}}^{\mathrm{CBSEP}} \leqslant O(n \log n) \qquad d \geqslant 3$$

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– No (known) embedded random walks.
– Not well understood even for $p = 1/10$ on $\mathbb{Z}$.

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T_{Sob} is the smallest constant such that

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Corollary

$$T_{\mathrm{Sob}}^{\mathrm{FA1f}} \leqslant O(d_{\max}/p)\, T_{\mathrm{Sob}}^{\mathrm{CBSEP}}$$

Corollary

With $p=\Theta(1/n)$ on the torus of dimension d, for all $q\geqslant 1$

$$T_q^{\mathrm{FA}}\leqslant O(\log n)\,T_{\mathrm{Sob}}^{\mathrm{FA}}\leqslant O(n\log n)\,T_{\mathrm{Sob}}^{\mathrm{CBSEP}}\leqslant\begin{cases}O(n^3\log^2 n) & d=1\\ O(n^2\log^3 n) & d=2\\ O(n^2\log^2 n) & d\geqslant 3\end{cases}$$

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With $p = \Theta(1/n)$ *on the torus of dimension* d*, for all* $q \geqslant 1$

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- Simpler proof.
- Stronger mixing notion.
- General graphs and choices of p.

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Definition (j-neighbour bootstrap percolation)

Each vertex $v \in \mathbb{Z}^d$ such that there are at least j neighbouring particles becomes filled at rate 1.

Bootstrap percolation and FA2f

Theorem (Gravner,Holroyd'08+Morris,H.'19)

For $d = j = 2$ bootstrap percolation w.h.p. the origin becomes filled at time

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Theorem (Martinelli,Toninelli,H.'20+)

For $d = j = 2$ FA w.h.p. the origin becomes filled at time

$$\exp\left(\frac{\pi^2}{9p} + \frac{O(\log(1/p))^3}{\sqrt{p}}\right).$$

Thank you.

?

Theorem

There exists $c > 0$ s.t. for any $p_n \to 0$

$$T_{\mathrm{Sob}} \leqslant c \max\left(\frac{d_{\mathrm{avg}} d_{\mathrm{max}}^2}{d_{\mathrm{min}}^2} T_{\mathrm{mix}}^{\mathrm{rw}} \log(n), \left(\max_y \bar{\mathcal{R}}_y\right) n|\log(p_n)|\right),$$

where $T_{\mathrm{mix}}^{\mathrm{rw}}$ is the mixing time of the lazy simple random walk on G.

[Alon-Kozma'18+Lee-Yau'98]

Theorem (Balogh, Bollobás, Duminil-Copin, Morris'12+Uzzell'19)

For $d \geqslant j \geqslant 2$ bootstrap percolation there exists an explicit constant[a] *$\lambda(d,j) > 0$ such that w.h.p. the filling time τ of the origin satisfies*

$$\exp^{j-1}\left(\frac{\lambda(d,j)-o(1)}{p^{1/(d-j+1)}}\right) \leqslant \tau \leqslant \exp^{j-1}\left(\frac{\lambda(d,j)}{p^{1/(d-j+1)}} - \frac{\Omega(1)}{p^{1/(2(d-j+1))}}\right).$$

[a] This notation is not the standard one in bootstrap percolation.

Theorem (Cancrini, Martinelli, Roberto, Toninelli'08+H., Martinelli, Toninelli'20+)

For $d \geqslant j \geqslant 3$ FA w.h.p. the filling time satisfies the same inequalities. For $d > j = 2$ FA instead

$$\exp\left(\frac{d\cdot\lambda(d,2)-o(1)}{p^{1/(d-1)}}\right) \leqslant \tau \leqslant \exp\left(\frac{d\cdot\lambda(d,2)}{p^{1/(d-1)}} + \frac{O(\log^3 p)}{p^{1/(2(d-1))}}\right).$$