An interpretation of random network dynamics as generalized exclusion processes

Jens Fischer

Université Toulouse III - Paul Sabatier;
University of Potsdam

September 8, 2021

Random Network Dynamics - A reduced Echo Chamber model

A reduced echo chamber model

We consider discrete-time dynamics on discrete state spaces!

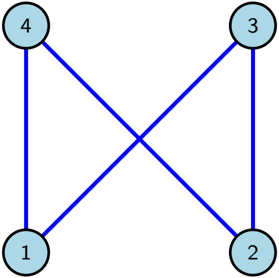

Figure: A reduced Echo Chamber model $S = (S_t)_{t \in \mathbb{N}}$.

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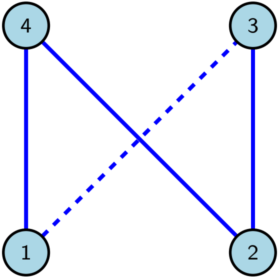

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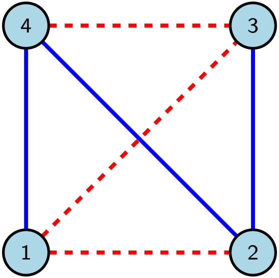

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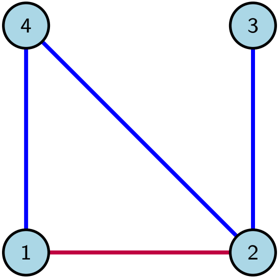
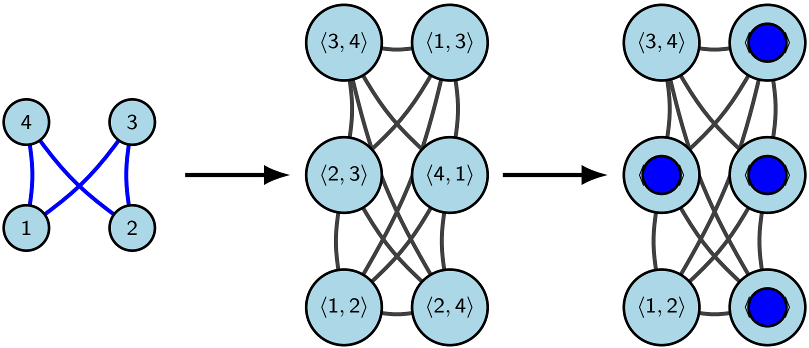


Figure: A reduced Echo Chamber model $S = (S_t)_{t \in \mathbb{N}}$.

Particle configuration representation on line graph



$n = 4$

$m = 4$

$\bar{n} = \frac{n(n-1)}{2} = 6$

$k = m = 4$

Figure: Translation of existing edges (blue) in G to occupied sites (blue) in the line graph L. Existing edges are interpreted as k particles occupying sites on L.

Exclusion processes on graphs

Definition

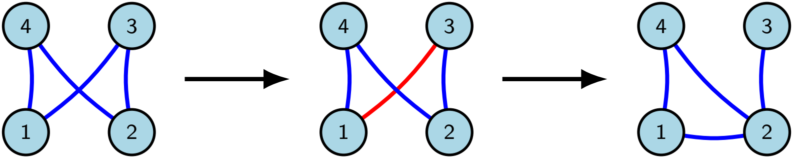
Let $L = (V, E)$ be a simple connected graph with $|V| = \bar{n} \in \mathbb{N}$ and $k \in \{1, \ldots, \bar{n} - 1\}$. Denote by P a stochastic matrix. **An exclusion process** of k particles in discrete time $\eta_k := (\eta_{k;t})_{t \in \mathbb{N}}$ on L is a Markov chain on the set of configurations

$$\mathcal{S}_k = \{\eta \in \{0,1\}^V \mid |\eta| = k\}$$

defined by the transition matrix $Q = (q_{\eta,\mu})_{\eta,\mu \in \{0,1\}^V}$ given for $\eta, \mu \in \{0,1\}^V$ by

$$q_{\eta,\mu} = \begin{cases} P(v,w) 1_{\substack{\eta(v)=1=\mu(w), \eta(w)=0=\mu(v), \\ \eta(u)=\mu(v) \forall u \notin \{v,w\}}}, & \eta \neq \mu \\ 1 - \sum_{\mu \neq \eta} q_{\eta,\mu}, & \eta = \mu. \end{cases}$$

Translation of dynamics to line graph



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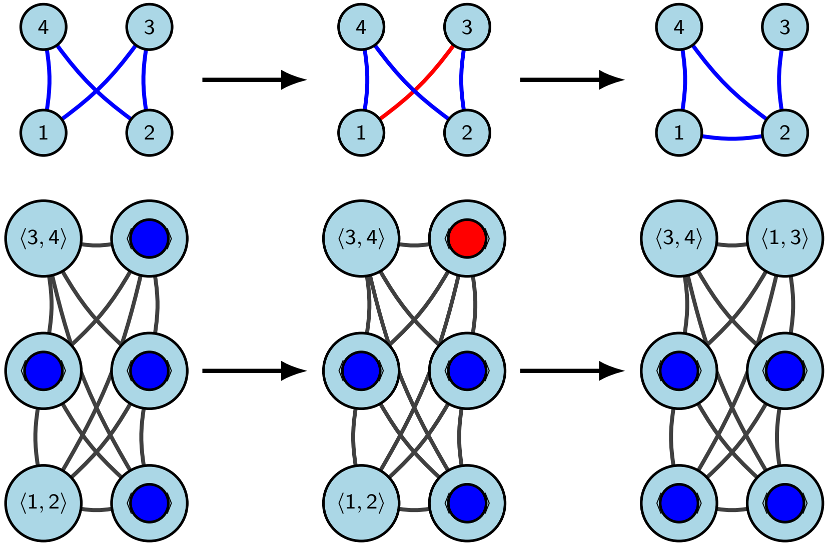

Generalized Exclusion processes on graphs

Definition

Let $L = (V, E)$ be a simple connected graph with $|V| = \bar{n} \in \mathbb{N}$ and $k \in \{1, \ldots, \bar{n} - 1\}$. Denote by $(P^\eta)_{\eta \in \{0,1\}^V,\, |\eta| = k}$ a family of stochastic matrices. **A generalized exclusion process** in discrete time $\eta_k := (\eta_{k;t})_{t \in \mathbb{N}}$ of k particles on L is a Markov chain on the set of configurations

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Canonical state space of η_k

k-particle graph (kPG)

Definition

Let $L = (V, E)$ be a simple graph and consider for $k \in \{1, \ldots, |V|\}$ the graph $\mathfrak{L}_k = (\mathfrak{V}_k, \mathfrak{E}_k)$ with $\mathfrak{V}_k = \{\mathfrak{v} \subseteq V || \mathfrak{v}| = k\}$ and $\langle \mathfrak{v}, \mathfrak{w} \rangle \in \mathfrak{E}_k$ if and only if

$$\mathfrak{v} \triangle \mathfrak{w} = \{v, w\},\ \langle v, w \rangle \in E.$$

We call $\mathfrak{L}_k$ the k-**particle graph (kPG)** associated to L.

Visualization of $\mathfrak{L}_k$

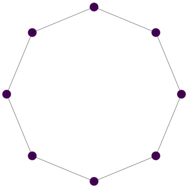

Figure: The cycle graph $L = (V, E)$ with $|V| = \bar{n} = 8$.

Visualization of $\mathfrak{L}_k$

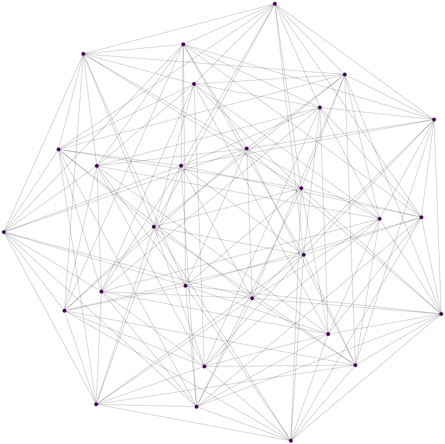

Figure: $\mathfrak{L}_k$ associated to cycle $L = (V, E)$ with $|V| = \bar{n} = 8$ and $k = 2$.

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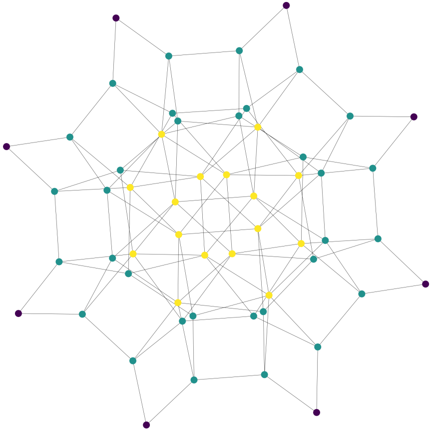

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Visualization of $\mathfrak{L}_k$

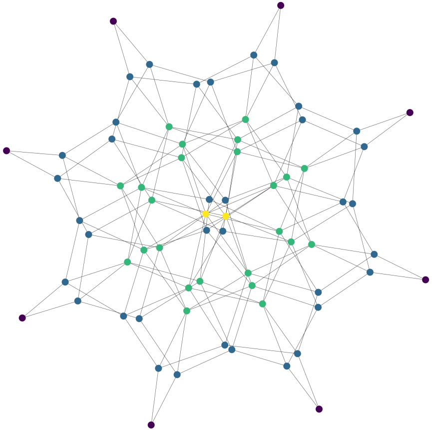

Figure: $\mathfrak{L}_k$ associated to cycle $L = (V, E)$ with $|V| = \bar{n} = 8$ and $k = 4$.

An associated Markov chain $\mathfrak{S}_k$

Lemma

Let $L = (V, E)$ be a connected simple graph $k \in \{1, \ldots, |V| - 1\}$. Then, there is a Markov chain $\mathfrak{S}_k$ on $\mathfrak{L}_k$ such that if L is a line-graph of some graph G the chain $\mathfrak{S}_k$ is equal in law to the reduced Echo Chamber model process.

The importance of vertex induced sub-graphs

Vertex induced sub-graphs

Definition

Let $L = (V, E)$ be any simple graph and $\mathfrak{v} \subseteq V$. Then the graph $L_{\mathfrak{v}} = (\mathfrak{v}, E_{\mathfrak{v}})$ with $\langle v, w \rangle \in E_{\mathfrak{v}}$ if and only if $v, w \in \mathfrak{v}$ and $\langle v, w \rangle \in E$ is called the vertex induced subgraph of L on $\mathfrak{v}$.
We denote by $\deg^{L_{\mathfrak{v}}}(v)$ the degree of $v \in \mathfrak{v}$ in $L_{\mathfrak{v}}$.

Echo chamber model as a Markov chain on $\mathfrak{L}_k$

Theorem (Cattiaux, F.)

Let L be a connected simple graph, $\bar{n} = |V|$ and $k \in \{1, \ldots, \bar{n}-1\}$. The transition matrix $P_k^{\triangle}$ of $\mathfrak{S}_k$ on $\mathfrak{L}_k$ satisfies

$$p_{k;\mathfrak{v},\mathfrak{w}}^{\triangle} = \begin{cases} \dfrac{1}{k}\dfrac{1}{\deg(v) - \deg^{L_\mathfrak{v}}(v) + 1}, & \mathfrak{v}\triangle\mathfrak{w} = \{v, w\}; \langle v, w\rangle \in E, \\ \displaystyle\sum_{v\in\mathfrak{v}} \frac{1}{k}\frac{1}{\deg(v) - \deg^{L_\mathfrak{v}}(v) + 1}, & \mathfrak{v} = \mathfrak{w}, \\ 0, & \textit{otherwise.} \end{cases}$$

Lumpability and stationary distribution

Theorem (Cattiaux, F., Roelly)

Let $L = (V, E)$ be a simple connected graph with $\bar{n} = |V|$ and $k \in \{1, \ldots, \bar{n} - 1\}$. Define for $\mathfrak{v}, \mathfrak{w} \in \mathfrak{V}_k$ the equivalence relation $\mathfrak{v} \sim \mathfrak{w}$ if and only if $L_{\mathfrak{v},\mathfrak{v}^c} \overset{\sim}{=} L_{\mathfrak{w},\mathfrak{w}^c}$. Write $[\mathfrak{v}_i] := \{\mathfrak{u} \in \mathfrak{V}_k | \mathfrak{u} \sim \mathfrak{v}_i\}$ the equivalence class of $\mathfrak{v}_i$ and denote by l the number of distinct equivalence classes. Then, the Markov chain $\mathfrak{S}_k$ is strongly lumpable with respect to the partition $\{[\mathfrak{v}_1], \ldots, [\mathfrak{v}_l]\}$.

Lumpability and stationary distribution

Theorem (Cattiaux, F., Roelly)

Let $L = (V, E)$ be a $\bar{d}$-regular graph with $\bar{n} = |V|$ and $k \in \{1, \dots, \bar{n} - 1\}$. Denote by π_k the stationary distribution of $\mathfrak{S}_k$. Then, for all equivalence classes $[\mathfrak{v}]$ under the equivalence relation $\sim$ we have that all $\mathfrak{v}, \mathfrak{w} \in [\mathfrak{v}]$ satisfy the identity $\pi_k(\mathfrak{v}) = \pi_k(\mathfrak{w})$.

Convergence speed to equilibrium on $\bar{d}$-regular graphs

L as a $\bar{d}$-regular graph

If L is a line graph of some underlying graph G where G has n vertices, then L is a strongly regular graph with

$$L = \mathrm{srg}\left(\frac{n(n-1)}{2}, 2(n-2), n-2, 4\right).$$

We reduce, therefore, for this section our considerations to $\bar{d}$-regular graphs.

Convergence speed to equilibrium

Theorem

Let L be a connected $\bar{d}$-regular graph on $\bar{n}$ vertices and $k \in \{1, \ldots, \bar{n}-1\}$ with $\bar{d}+k+1 \leq \bar{n}$. Denote for $\mathfrak{v}, \mathfrak{w} \in \mathfrak{V}_k$ and $l \in \mathbb{N}$ by $\omega_l^{\mathfrak{L}_k}(\mathfrak{v}, \mathfrak{w})$ the number of walks of length l from $\mathfrak{v}$ to $\mathfrak{w}$ along the edges in $\mathfrak{L}_k$. Then there is a constant $C(L,k)$ such that for $\kappa := \operatorname{diam}(\mathfrak{L}_k)$ and $\varepsilon > 0$ the transition matrix $P_k^{\triangle}$ satisfies

$$\sup_{\mathfrak{v} \in \mathfrak{V}_k} \sum_{\mathfrak{w} \in \mathfrak{V}_k} |p_{k;\mathfrak{v},\mathfrak{w}}^{\triangle;(n)} - \pi(\mathfrak{w})| \leq 2(1-\varepsilon)^{\lfloor \frac{n}{\kappa} \rfloor}, \quad n \geq 1. \tag{1}$$

Convergence speed to equilibrium

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We know C explicitly but it is too large for a slide.

Properties of $\mathfrak{S}_k$

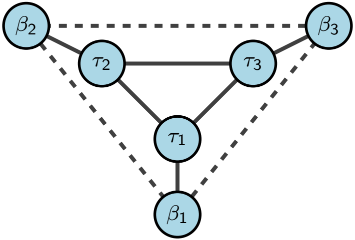


Figure: The tri-star $\mathcal{T}$.

Note: What follows can be easily generalized for general cycles instead of the triangle $\{\tau_1, \tau_2, \tau_3\}$.

Reversibility of $\mathfrak{S}_k$

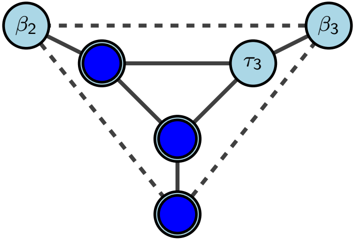
Theorem (F., Cattiaux, 2020, forthcoming)

Let L be any simple connected $\bar{d}$-regular graph on $\bar{n}$ vertices with $\bar{d} \in \{3, \ldots, \bar{n} - 3\}$. Assume that L contains a tri-star $\mathcal{T}$. Then, the Markov chain $\mathfrak{S}_k$ is reversible on $\mathfrak{L}_k$ if and only if $k \in \{1, 2, \bar{n} - 2, \bar{n} - 1\}$.

Reversibility of $\mathfrak{S}_k$

Idea of the proof:

Construct a counterexample using 3 particles and Kolmogorov's criterion.



□

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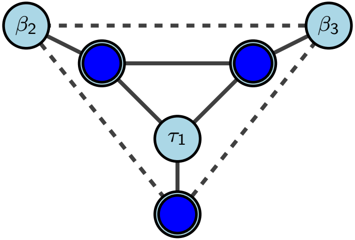

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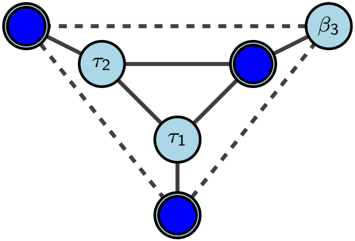

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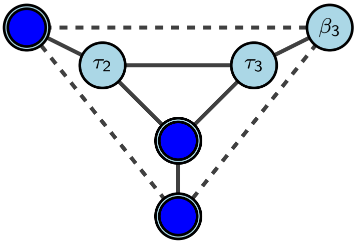

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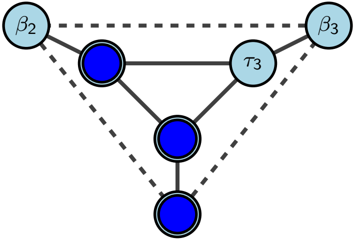

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Idea: Moving out of a crowded neighborhood is more probable then moving back into this specific neighborhood.

□

Reversibility of $\mathfrak{S}_k$

Idea of the proof:

Construct a counterexample using 3 particles and Kolmogorov's criterion.
Idea: Moving out of a crowded neighborhood is more probable then moving back into this specific neighborhood.
This is not the case for the classical exclusion process! □

Reversibility on complete graphs

Corollary (Cattiaux, F.)

Let L be a strongly regular graph with parameters $(\bar{n}, \bar{d}, \alpha, \beta)$ with $\alpha \geq 1$. Then, the process $\mathfrak{S}_k$ is reversible if and only if $\bar{d} \in \{2, \bar{n}-2, \bar{n}-1\}$ or $k \in \{1, 2, \bar{n}-2, \bar{n}-1\}$.

Reversibility in social conflict model

Corollary (Cattiaux, F.)

Consider the process $S = (S_t)_t$ on a population of n individuals with k relationships. Then the associated process $\mathfrak{S}_k$ is reversible if and only if $n = 3$ or $k \in \left\{1, 2, \frac{n(n-1)}{2} - 2, \frac{n(n-1)}{2} - 1\right\}$.

The probability go from one set of relationships back to the same set depends highly on the order in which single relationships are dissolved and recreated.

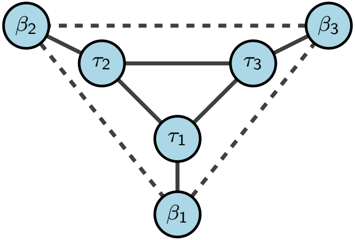

Figure: The tri-star $\mathcal{T}$ **does never exist in bipartite graphs!**

Conclusions

Outlook

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- Analyze complete Echo Chamber model,

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- Generalized exclusion processes in random absorbing env.,

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- Analyze complete Echo Chamber model,
- Generalized exclusion processes in random absorbing env.,
- Establish links between the geometry of a graph and exclusion processes.