The jump of the clique chromatic number for random graphs

Joint work with Dieter Mitsche and Lutz Warnke

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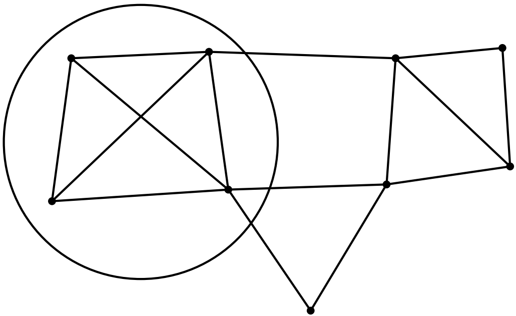
The setting

- The Erdős-Rényi random graph $G(n, p)$.
- Proper (vertex-)coloring - no two adjacent vertices have the same color $\Rightarrow$ chromatic number, denoted $\chi(\cdot)$.
- Proper clique coloring - no maximal clique (with more than one vertex) contains vertices in only one color $\Rightarrow$ clique chromatic number, denoted $\chi_c(\cdot)$.

Some examples



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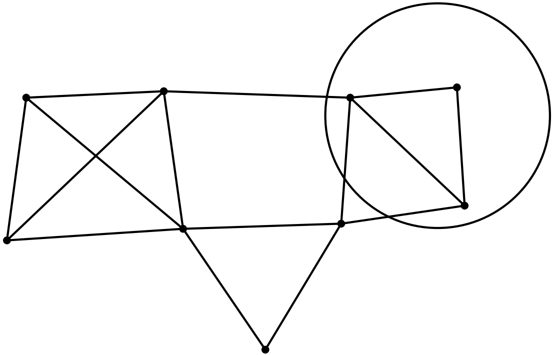

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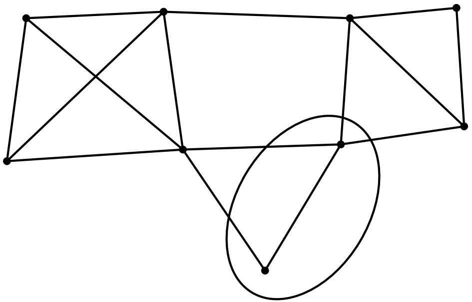

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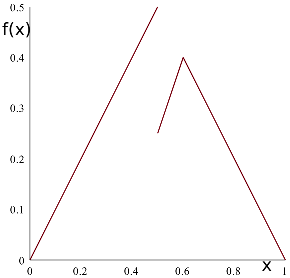
In 2016, McDiarmid, Mitsche and Pralat establish that, for every $x \in (0, 1/2) \cup (1/2, 1)$ and $np = n^{x+o(1)}$, whp $\chi_c(G(n,p)) = n^{f(x)+o(1)}$.



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Probability that an edge does not participate in a triangle (i.e. is not covered) : $(1 - p^2)^{n-2} \approx \exp(-np^2)$.

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Upper bound on χ_c : $\chi_c(G) \leq \chi(G)$.

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But there are $\binom{n}{s}$ vertex sets of size s $\Rightarrow$ look for s satisfying $\binom{n}{s}\exp(-s^2p\exp(-np^2)/2) \approx 1$ (later called $s_{\max}$).

The main result

Theorem (L., Mitsche, Warnke, 2021)

For every $p \in [n^{-1/2}, (\log n)^{1/2} n^{-1/2}/2]$, *with high probability*

$$\chi_c(G(n,p)) = \Theta\left(\frac{np\exp(-np^2)}{\log(np)}\right).$$

Highlights of the proof - the upper bound

The upper bound relies on the following result :

Theorem (Joret, Micek, Reed, Smid, 2020)

For every $\varepsilon > 0$ *there is* $\Delta_\varepsilon > 0$ *such that every graph G with maximum degree* $\Delta \geq \Delta_\varepsilon$ *has clique chromatic number at most* $(1+\varepsilon)\Delta/\log \Delta$*.*

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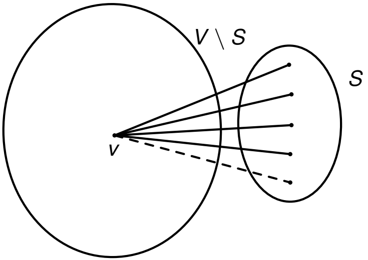

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Typical bounded difference inequality : "Functions of a family of many i.i.d.r.v. that admit large differences with very small probability are well concentrated."

An open problem

The correct whp order of $\chi_c(G(n,p))$ is still not known for all values of p. Our work reduced the gaps to $O(\log n)$ for all values of p.

In particular, what is the correct order for $\chi_c(G(n,p))$ for $(\log n)^{3/5} n^{-2/5} \ll p \ll (\log n)^{-1}$?

Thank you ! Any questions ?