Small deviation estimates and small ball probabilities for geodesics in last passage percolation

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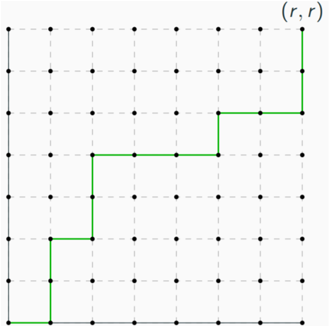
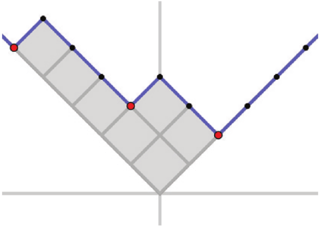
Exponential last Passage Percolation on $\mathbb{Z}^2$

- Have i.i.d. random variables $w_{i,j} \sim \mathrm{Exp}(1)$ on the vertices. The weight of a path is the sum of the values of the traversed vertices in $\mathbb{Z}^2$.
- $T(u, v)$ is the maximum weight of up-right paths going from u to v.
- For convenience, $T(n) = T((0,0),(n,n))$.
- Satisfies the recursion

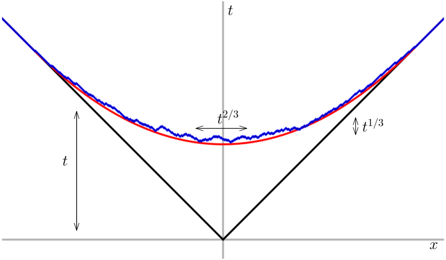
$$T(u, v) = \max\{T(u, v - e_1), T(u, v - e_2)\} + w_v.$$





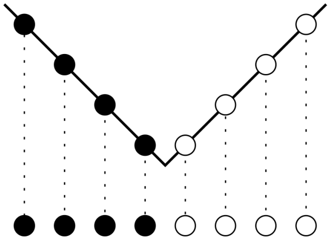
Kardar-Parisi-Zhang (KPZ)

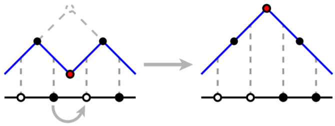
- General class of models of random growth.
- 3 : 2 : 1 scaling for time : space : fluctuations.



Connection to the TASEP

- Totally Asymmetric Exclusion Process.
- Start with a configuration of particles and holes on $\mathbb{Z} + \frac{1}{2}$.
- Vertices have i.i.d. $\mathrm{Exp}(1)$ clocks which signal the respective particle to attempt a jump to its right.
- A jump is successful if there is a hole to the right of a particle.
- If a particle moves from $i + \frac{1}{2}$ to $(i+1) + \frac{1}{2}$, then flip the wedge on the line $\{x = i + 1\}$.
- Exponential LPP corresponds to the TASEP started from the step initial condition.





Exponential LPP: Properties

- Limit shape: $\frac{\mathbb{E}T(\mathbf{0},\alpha(m,n))}{\alpha} \to (\sqrt{m}+\sqrt{n})^2$ as $\alpha \to \infty$.
- $\frac{T(n)-4n}{n^{1/3}}$ converges in distribution to a multiple of the GUE Tracy-Widom distribution, which has negative mean.
- [Ledoux, Rider '10], [Basu, Ganguly, Hegde, Krishnapur '19]: For all $y < \delta n^{2/3}$ and for all large n,

$$C_1 e^{-c_1 y^{3/2}} \leq \mathbb{P}(T(n) - 4n > yn^{1/3}) \leq C_2 e^{-c_2 y^{3/2}},$$

$$C_3 e^{-c_3 y^{3}} \leq \mathbb{P}(T(n) - 4n < -yn^{1/3}) \leq C_4 e^{-c_4 y^{3}}.$$

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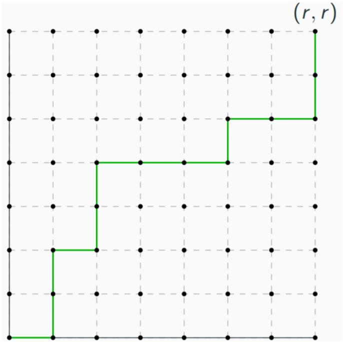
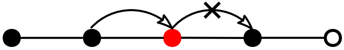
- Optimal in the exponent.

Geodesics in Exponential LPP

- The almost surely unique path attaining $T(u, v)$ is called the geodesic.
- Semi-infinite geodesics are related to the trajectory of a second class particle in the TASEP starting from the stationary initial condition.





Exponential LPP: Transversal Fluctuations

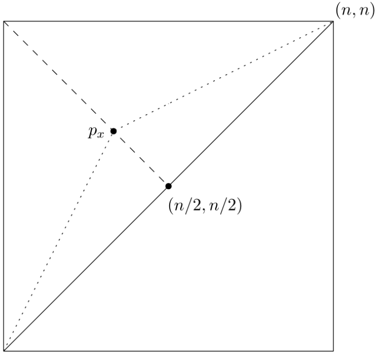
- Let $\mathcal{A}_\alpha$ be the event that the geodesic for $T(n)$ stays in a strip of width α about the line $\{x = y\}$.
- [Johansson '99] $\mathbb{P}(\mathcal{A}_{n^{2/3+\epsilon}}) \to 1$ and $\mathbb{P}(\mathcal{A}_{n^{2/3-\epsilon}}) \to 0$ as $n \to \infty$.
- [Basu, Sidoravicius, Sly '16], [Basu, Ganguly, Zhang '19], $\mathbb{P}((\mathcal{A}_{rn^{2/3}})^c) \leq C_1 e^{-c_1 r^3}$.
- [Hammond, Sarkar '18] $\mathbb{P}((\mathcal{A}_{rn^{2/3}})^c) \geq C_2 e^{-c_2 r^3}$.
- [Balász, Cator Seppäläinen '06], [Busani, Ferrari '20] Similar estimates for $\mathbb{P}((\mathcal{A}_{rn^{2/3}})^c)$.

Exponential LPP: Transversal Fluctuations

- How far does the geodesic for $T(n)$ venture from the line $\{x = y\}$?.
- By the limit shape result, for $p_x = (n/2 - x, n/2 + x)$,

$$\mathbb{E}T((0,0), p_x) + \mathbb{E}T(p_x, (n,n)) \sim 4n - C\frac{x^2}{n}.$$

- For typical transversal fluctutations, heuristically $\frac{x^2}{n} \sim n^{1/3}$ and thus $x \sim n^{2/3}$.

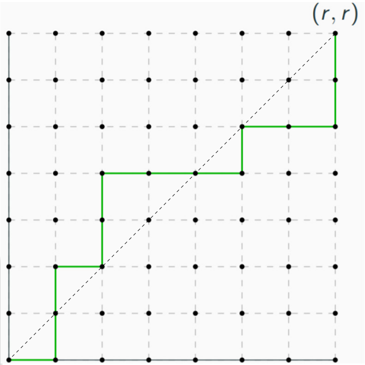


Small ball probabilities for the geodesic

- Estimates for $\mathbb{P}(\mathcal{A}_{\delta n^{2/3}})$ for small δ?
- The geodesic takes values in the same state spaces as the SRW bridge.
- For Brownian bridge: $\log \mathbb{P}\left(\sup_{t\in[0,1]} |B_t| \le \delta\right) \sim -\frac{\pi^2}{8}\delta^{-2}$.

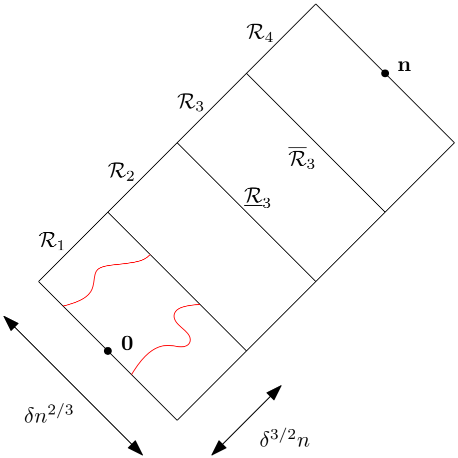
Theorem ([Basu, B. '20])

$$C_1 e^{-c_1\delta^{-3/2}} \le \mathbb{P}\left(\mathcal{A}_{\delta n^{2/3}}\right) \le C_2 e^{-c_2\delta^{-3/2}}$$



Why $e^{-\delta^{-3/2}}$? Sketch for the upper bound

- Let $Y_i = \sup_{u \in \underline{\mathcal{R}}_i, v \in \overline{\mathcal{R}}_i} T(u, v)$
- $Y_i \sim 4\delta^{3/2} n + Z_i \delta^{1/2} n^{1/3}$ with $\mathbb{E} Z_i < -2c$.
- Let $S = \sum Z_i / \delta^{-3/2}$.
- $\sum Y_i \sim 4n + S\delta^{-1} n^{1/3}$.



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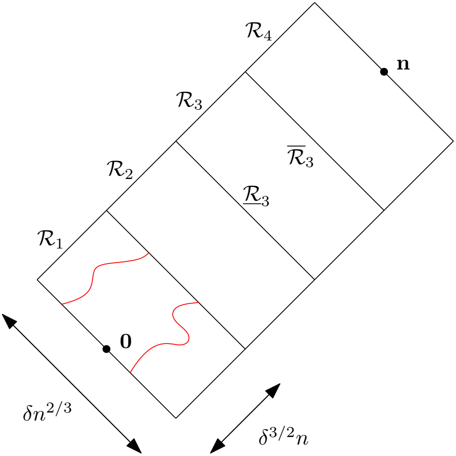
- Let $S = \sum Z_i / \delta^{-3/2}$.
- $\sum Y_i \sim 4n + S\delta^{-1} n^{1/3}$.
- $\mathbb{P}\left(T(n) < 4n - c\delta^{-1} n^{1/3}\right) \sim e^{-c_1 \delta^{-3}}$.
- $\mathbb{P}(S \geq -c) \sim e^{-c_2 \delta^{-3/2}}$.



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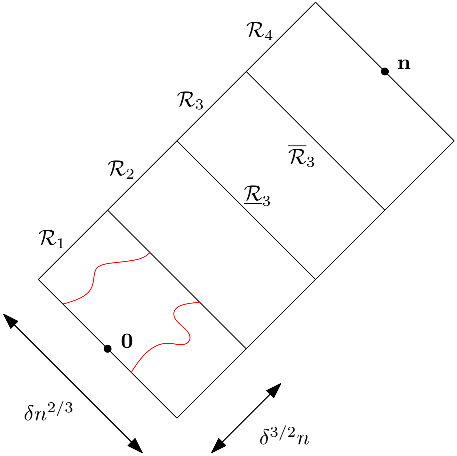
- $\sum Y_i \sim 4n + S\delta^{-1} n^{1/3}$.
- $\mathbb{P}\left(T(n) < 4n - c\delta^{-1} n^{1/3}\right) \sim e^{-c_1 \delta^{-3}}$.
- $\mathbb{P}(S \geq -c) \sim e^{-c_2 \delta^{-3/2}}$.
- The proof of the lower bound involves forcing the geodesic to stay inside by making the environment in the central $\delta n^{2/3}$ strip favorable and putting unfavorable barriers straddling the strip.

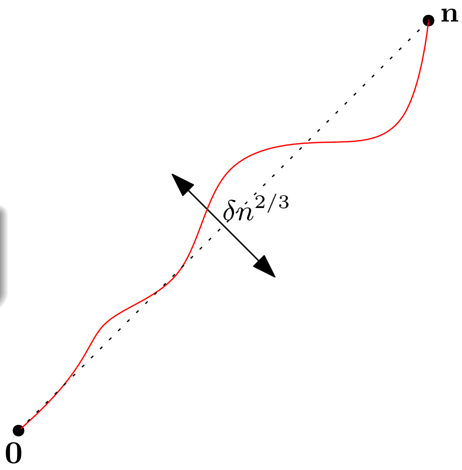


One point estimates

- Let $\mathcal{E}_n$ denote the event that the geodesic passes through a transversal segment of length $\delta n^{2/3}$ about the point $(n, n)/2$.

Theorem ([Basu, B. '20])

$$C_1\delta \leq \mathbb{P}(\mathcal{E}_n) \leq C_2\delta$$



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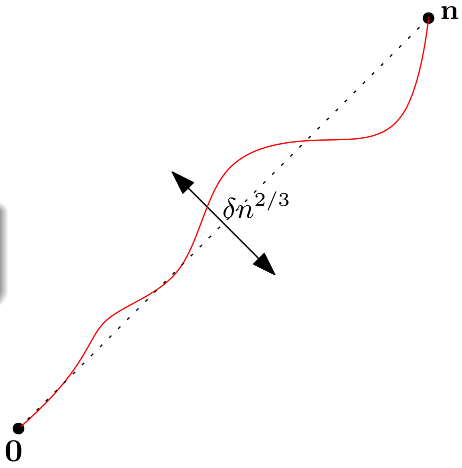
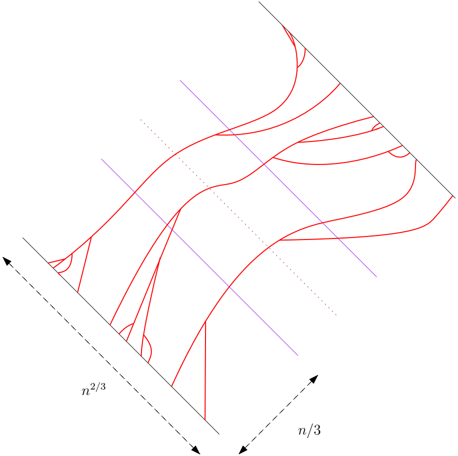
- In fact, can take $\delta = n^{-2/3}$ and obtain that the probability that the geodesic passes through a particular point $\sim n^{-2/3}$.



Coalescence of geodesics

- [Basu, Hoffman, Sly '18] Consider a $n \times n^{2/3}$ rectangle and look at the geodesics between the two short sides. Denoting the number of distinct geodesic traces in the middle third by N_n, we have $\mathbb{P}(N_n \geq \ell) \leq Ce^{-c\ell^{\epsilon}}$.



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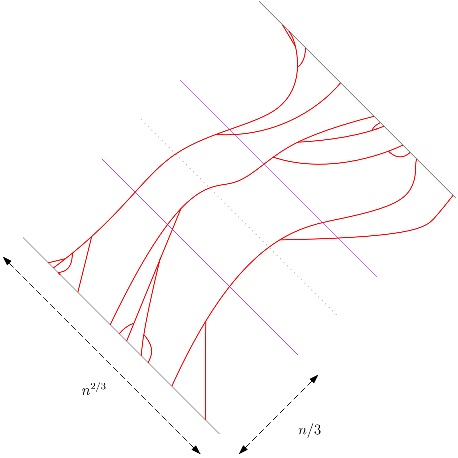
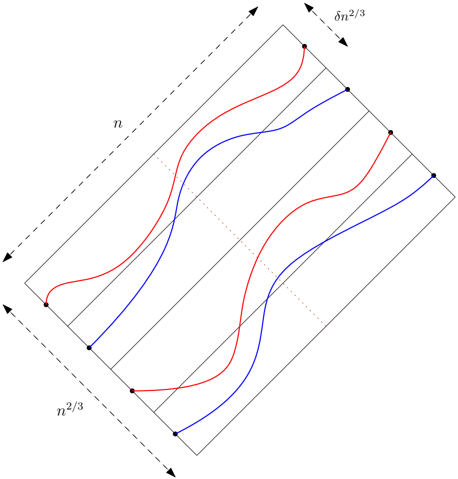
- Denoting the number of intersections of the geodesics with the mid-line by L_n, we have $\mathbb{E}L_n \leq C$.



One point estimate– upper bound proof sketch

- Divide a $n \times n^{2/3}$ rectangle into δ^{-1} many $n \times \delta n^{2/3}$ rectangles and consider the geodesics between the midpoints of their short sides.
- With M_n denoting the number of small rectangles satisfying the one-point condition, we have $M_n \leq L_n$.
- By translational invariance and coalescence, $\mathbb{E}M_n = \delta^{-1}\mathbb{P}(\mathcal{E}_n) \leq \mathbb{E}L_n \leq C$.



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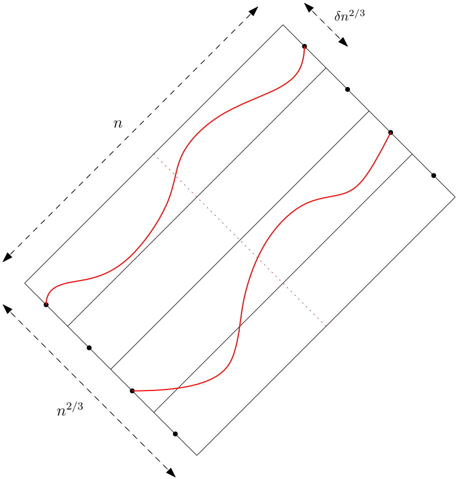

Passing through the scaling limit

- Analogous to the convergence of SRW bridge to Brownian bridge, geodesics in exponential LPP converge to geodesics in the directed landscape constructed in [Dauvergne, Ortmann, Virág '18].
- [Dauvergne, Sarkar, Virág '20] Geodesics in the directed landscape have finite $3/2$ variation.
- The small ball and one point estimates pass to the limit and give corresponding results for the directed landscape geodesics.

Questions?