On the asymptotics for the minimal distance between extreme vertices in a generalised Barak — Erdős graph

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Presentation Outline

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Simple Erdős – Rényi Graph

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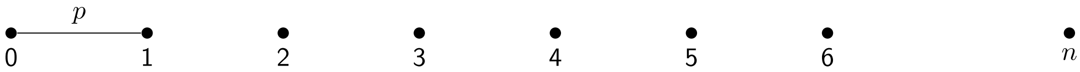

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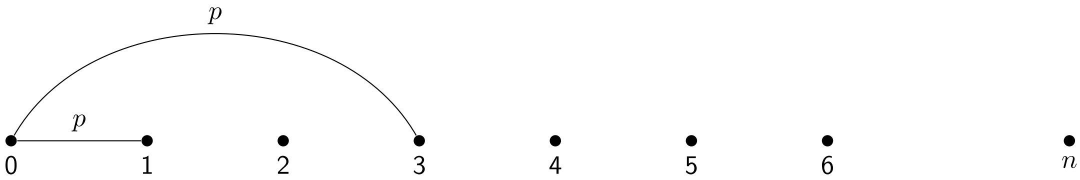

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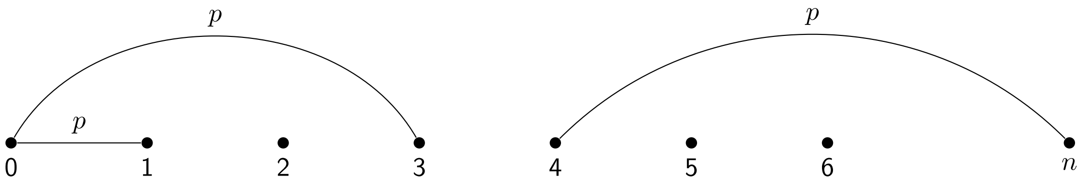

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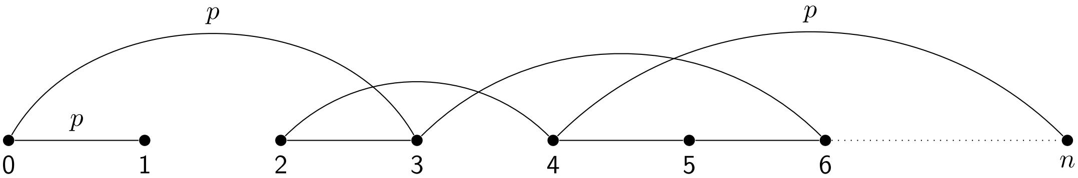

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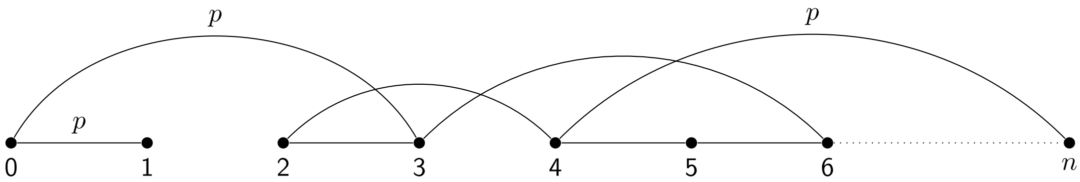

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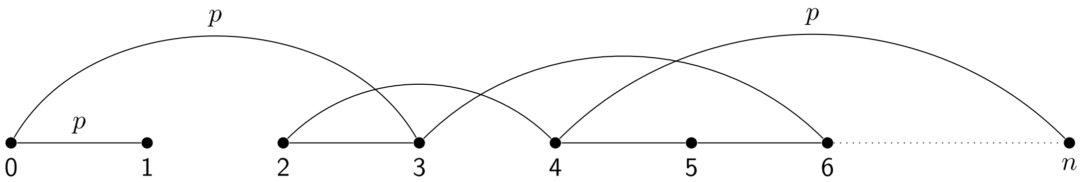

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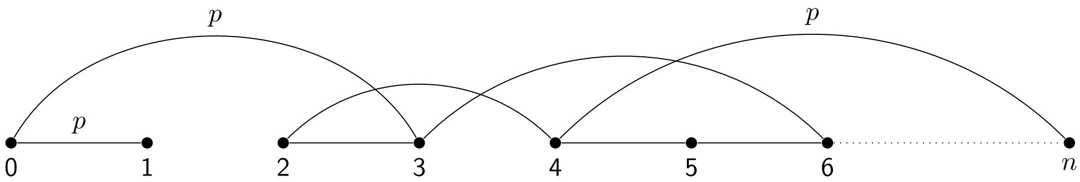

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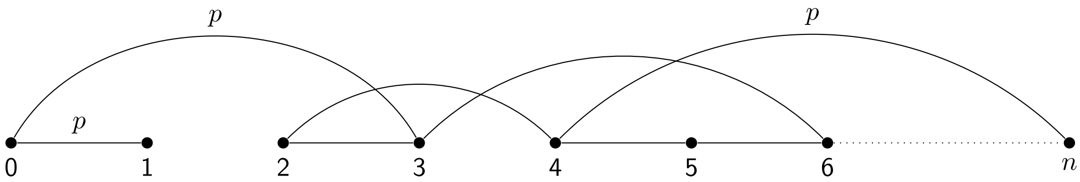

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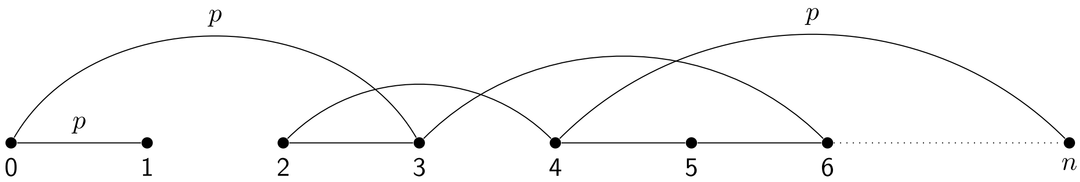

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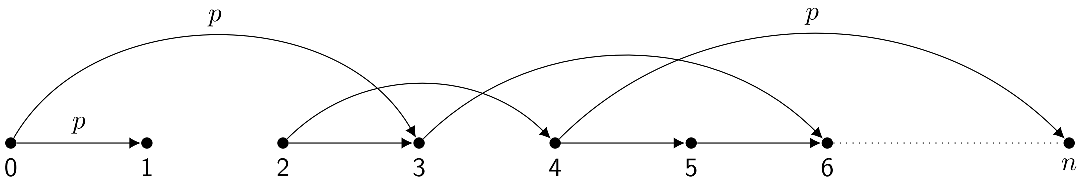

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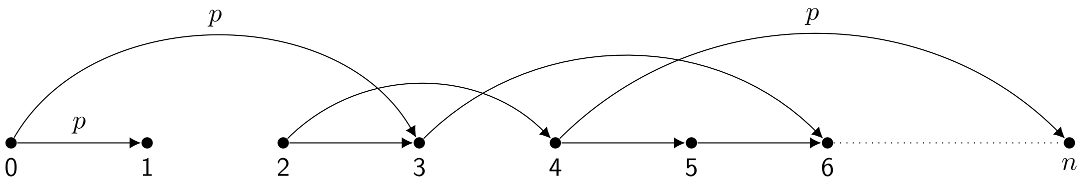

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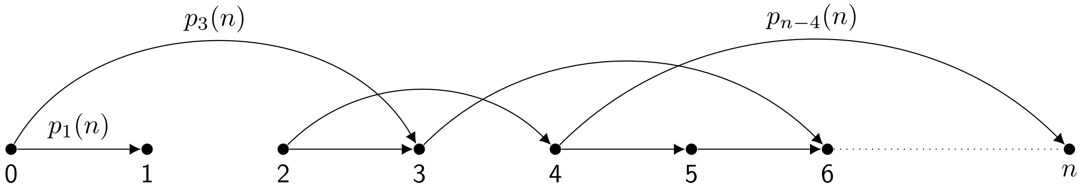

$$\mathcal{G}_n = (\mathcal{V}_n, \mathcal{E}_n), \text{ where } \mathbb{P}\left((i,j) \in \mathcal{E}_n\right) = p_{j-i}(n) \in [0,1] \text{ for } i < j$$

Key assumptions:

- Oriented edges from smaller vertices to bigger are presented *independently* of each other.
- $p_m(n)$ is the function of n and m.

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- T. (2018) studied the behaviour of the minimal distance between extreme vertices.

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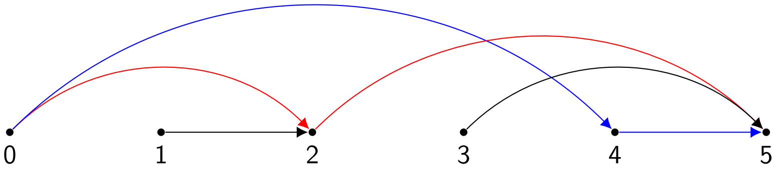

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Example: $p_m(n) = \theta n^{-\alpha} m^{-\beta}$, $n, m \geq 1$, where $\alpha, \beta \geq 0$.

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Theorem (B. Mallein, 2021)

- Let $\gamma \in (1 - \frac{1}{k-1}, 1 - \frac{1}{k})$ for some $k \geq 1$. Then

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- Let $\gamma = 1 - \frac{1}{k}$. Then

$$l_n \xrightarrow{d} l_\infty,$$

as $n \to \infty$, where $\mathbb{P}(l_\infty = k) = 1 - \mathbb{P}(l_\infty = k+1) = 1 - \exp\{-\frac{\theta^k}{(k-1)!}\}$

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- Let $\gamma \geq 1$ (> 1). Then
$$l_n \xrightarrow{p} \infty \text{ (a.s.)}$$
as $n \to \infty$.

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From the previous results, coupling consideration and the fact that for any $n, m \geq 1$,

$$p_n(n) \leq p_n(m) \leq p_m(m)$$

follows the following fact

Results for GBE

Theorem (T., 2021)

Let $\mathcal{G}_n$ be a GBE

- Assume that $\gamma \in (1 - \frac{1}{k-1}, 1 - \frac{1}{k})$ for some $k \geq 1$. Then

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- Assume that $\gamma = 1 - \frac{1}{k}$ for some $k \geq 1$. Then

$$\mathbb{P}\left(l_n \in \{k, k+1\}\right) \to 1$$

as $n \to \infty$.

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- **Inhomogeneous Case:** if $\gamma \in (1,2)$, then $\mathbb{P}(l_n = \infty) \to ?$ as $n \to \infty$.

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- **GBE:** if $\gamma = 1 - \frac{1}{k}$, then Is there a weak limit for l_n and what is its distribution?

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Questions?