A Probabilistic Broadcast Mechanism on Random Geometric Graphs

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Random Networks and Interacting Particle Systems

9th September, 2021

Motivation

$n = 7$ packets

X_1

X_2

X_3

$X_1 + X_2$

$X_2 + X_3$

$X_3 + X_1$

$X_1 + X_2 + X_3$

Source has n coded packets

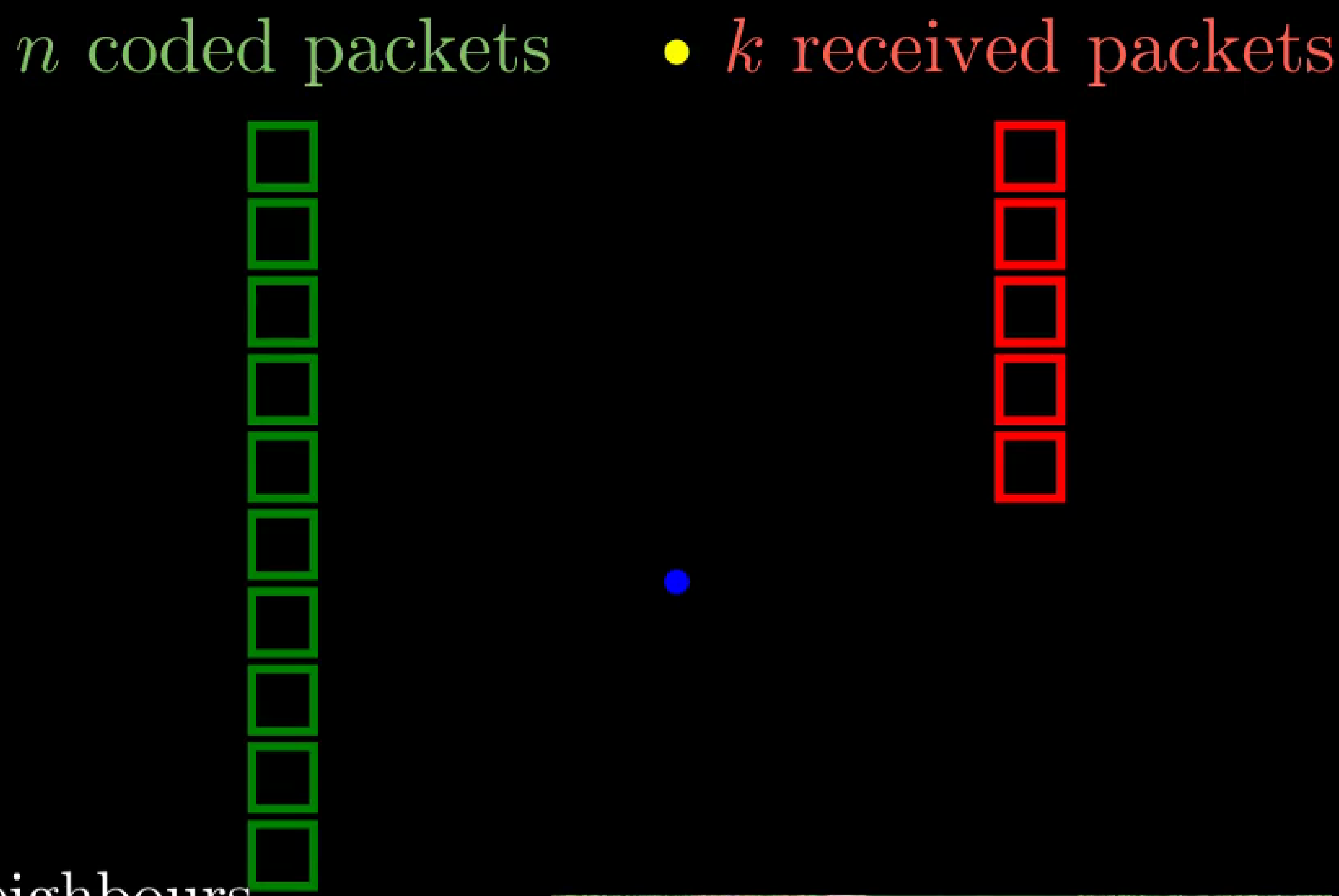
Broadcast information in the network

with minimal number of transmissions

Probabilistic Forwarding with Coding

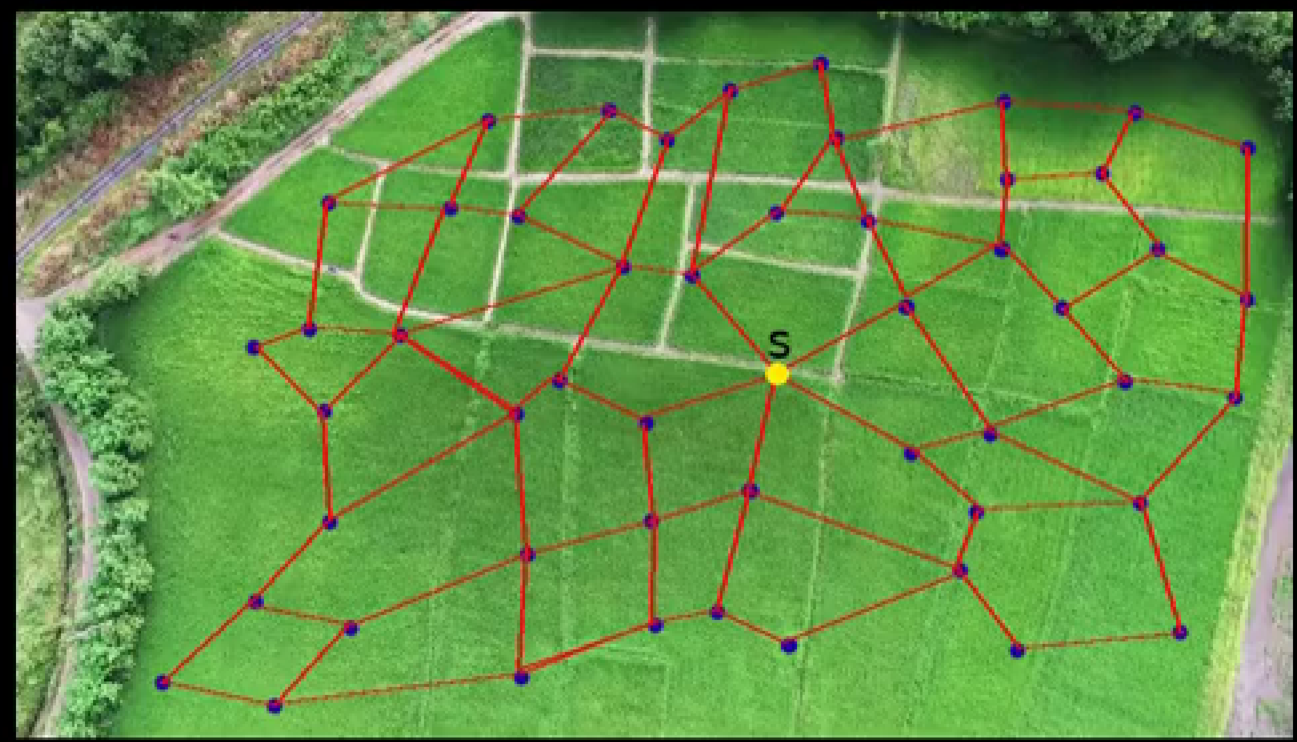
Coding scheme

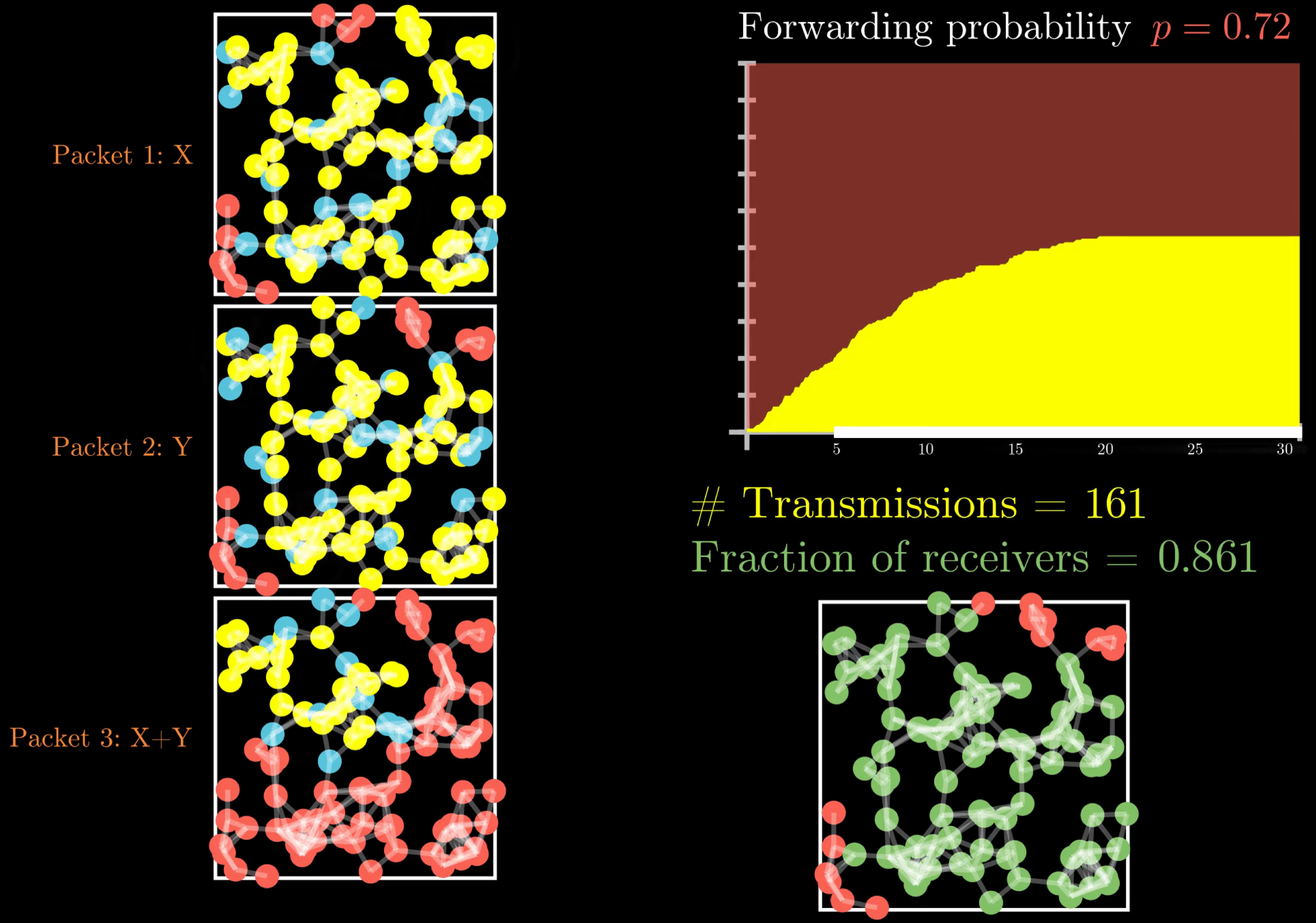
- Source has n coded packets.
- Code is such that reception of any k out of the n coded packets by any node, suffices to recover the information from the source.



Probabilistic forwarding of coded packets

- Source transmits all n coded packets to its one-hop neighbours.
- Other nodes transmit each packet w.p. p, do nothing w.p. $1 - p$.
- Each packet is forwarded independently of other packets and other nodes.





Packet 1: X
Packet 2: Y
Packet 3: X+Y
Forwarding probability p = 0.72
5
10
15
20
25
30
Transmissions = 161
Fraction of receivers = 0.861

Formal Problem Statement

Given

- a connected graph with N nodes
- number of coded packets, n
- number of packets to receive for decoding, k
- δ close to 0
- retransmission probability p

Define

$$\mathcal{R}_{k,n} = \{ \text{ nodes that receive at least k out of n coded packets } \}$$

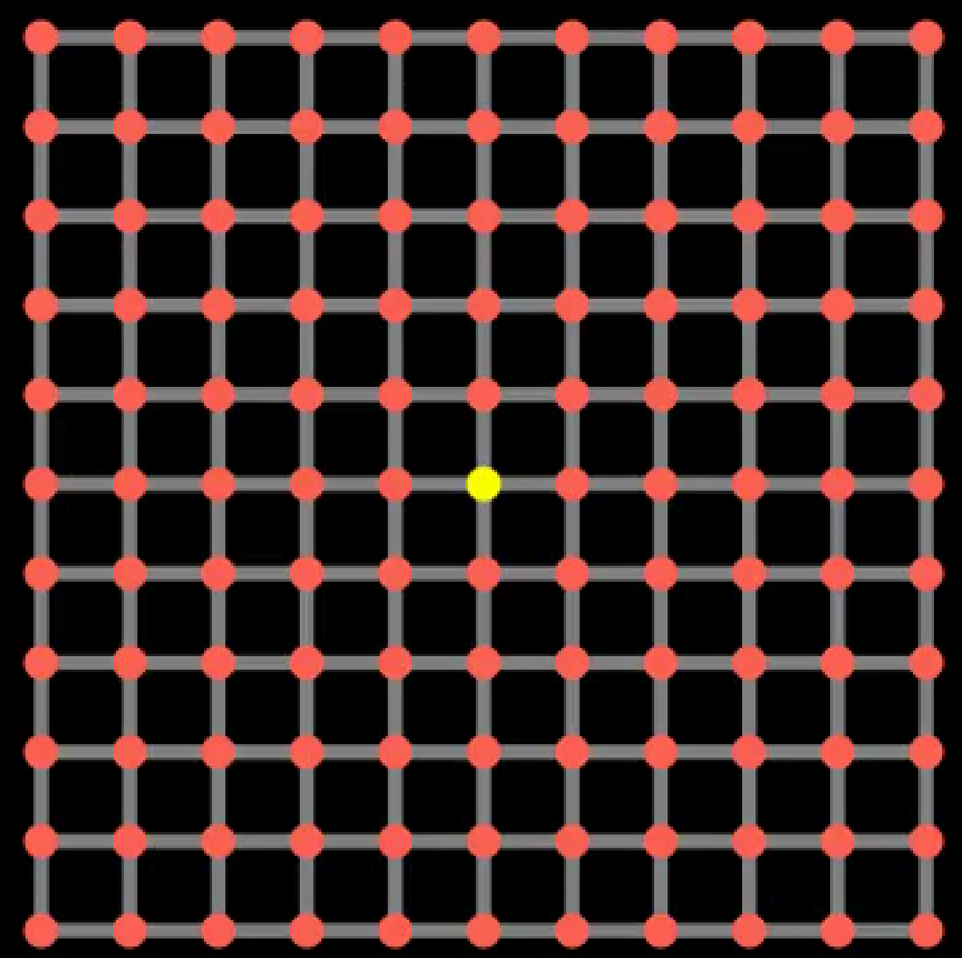
$|\mathcal{R}_{k,n}| = R_{k,n}$: number of successful receivers

Want to find

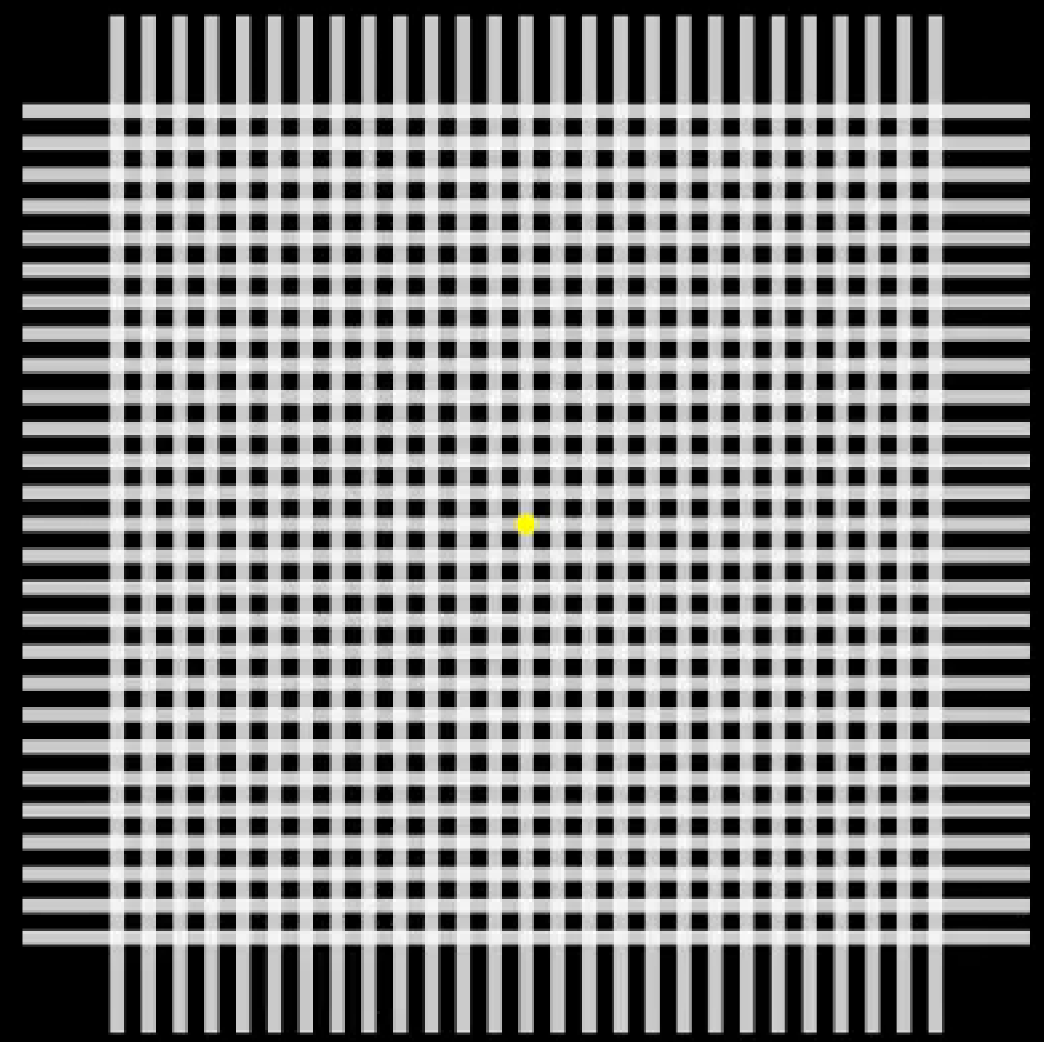
- $p_{k,n,\delta} =$ minimum p such that $\mathbb{E}_p\left[\frac{R_{k,n}}{N}\right] \geq 1 - \delta.$ (near broadcast)
- $\tau_{k,n,\delta} = \mathbb{E}_{p_{k,n,\delta}}$ [total # transmissions over all N nodes]

On Grids

Probabilistic forwarding on the $m \times m$ grid Γ_m

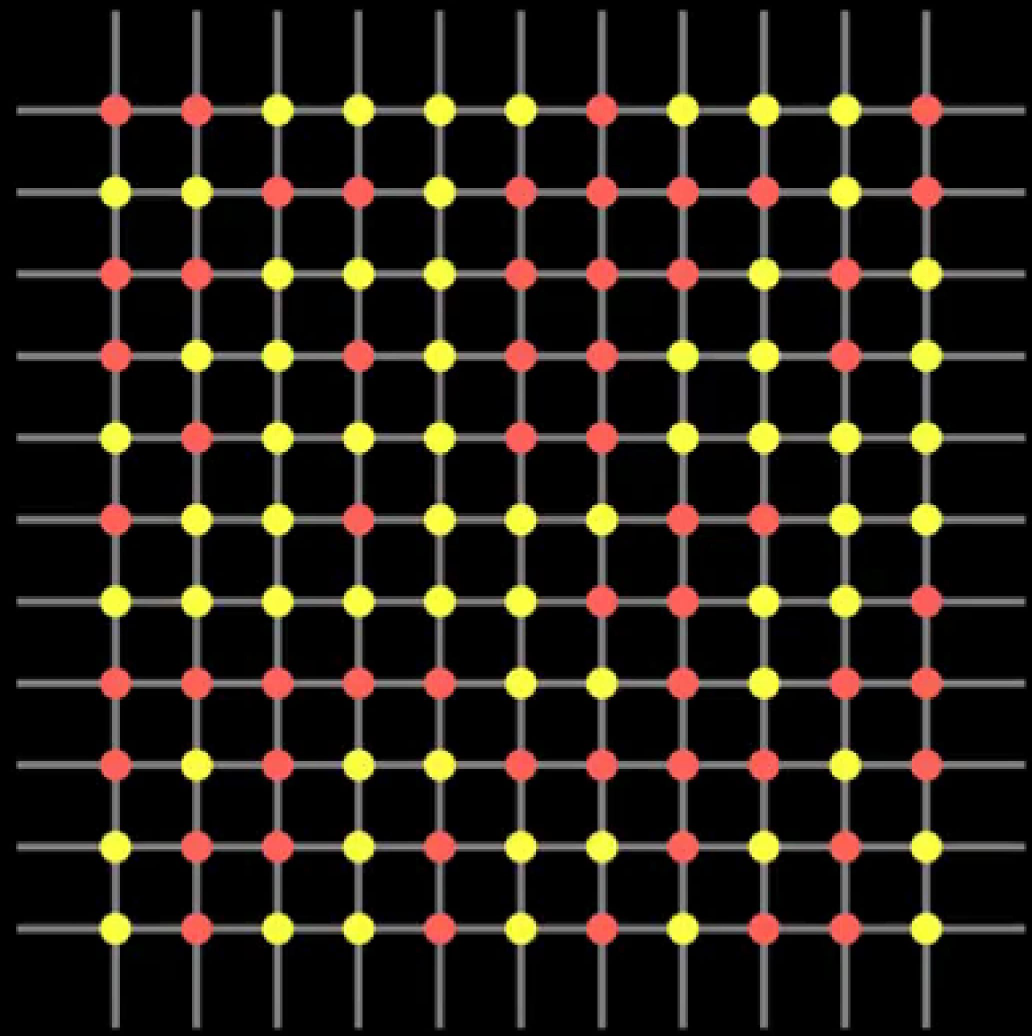


Probabilistic forwarding on the $\mathbb{Z}^2$ lattice



We will use the site percolation process on $\mathbb{Z}^2$ to obtain estimates of $p_{k,n,\delta}$ and $\tau_{k,n,\delta}$

Site percolation on $\mathbb{Z}^2$- Transmitters



- Associate each vertex (site) u of $\mathbb{Z}^2$ with a Ber(p) r.v. X_u. The vertex is open if $X_u = 1$; else closed.

- For two open sites u and v, v is said to be in the component of u $(v \in C_u)$, if there is a path of open sites from u to v.

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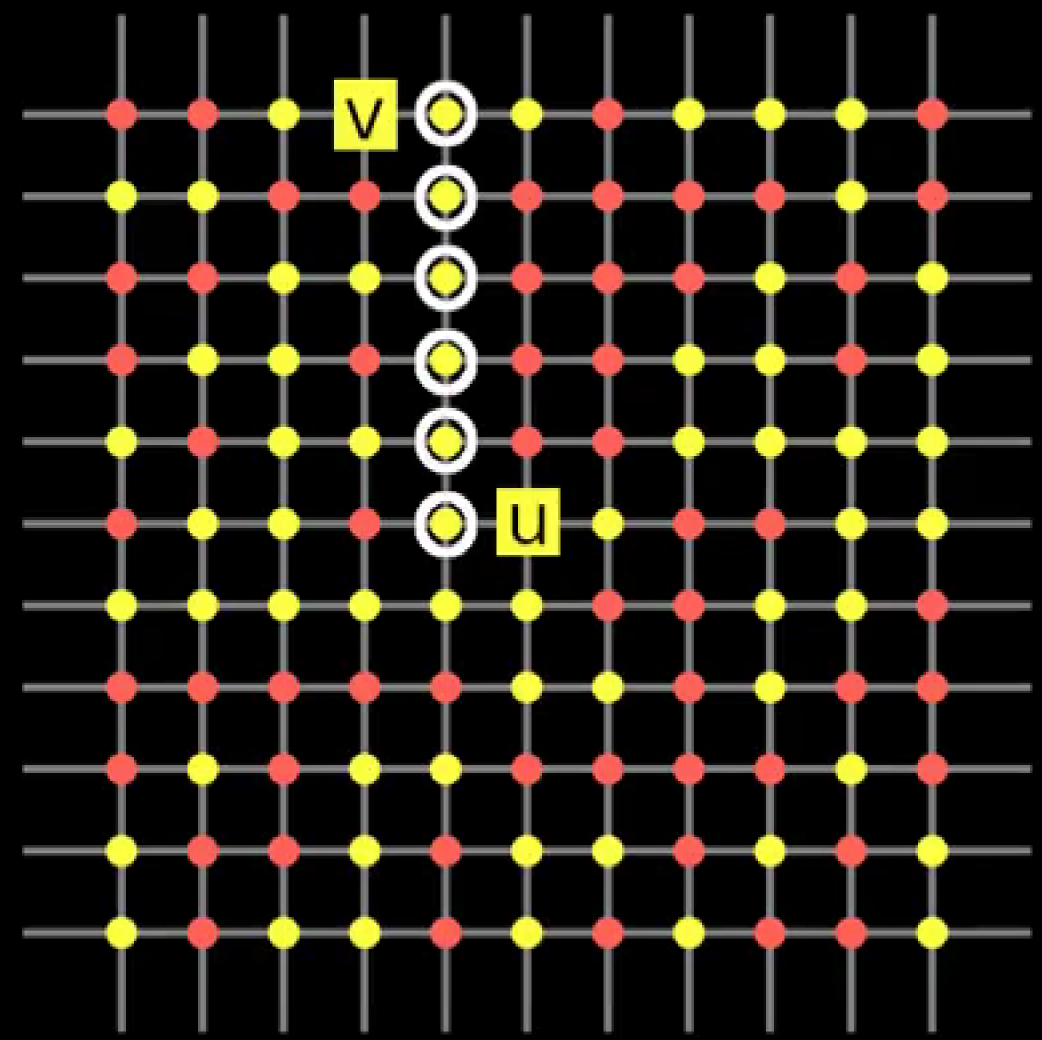
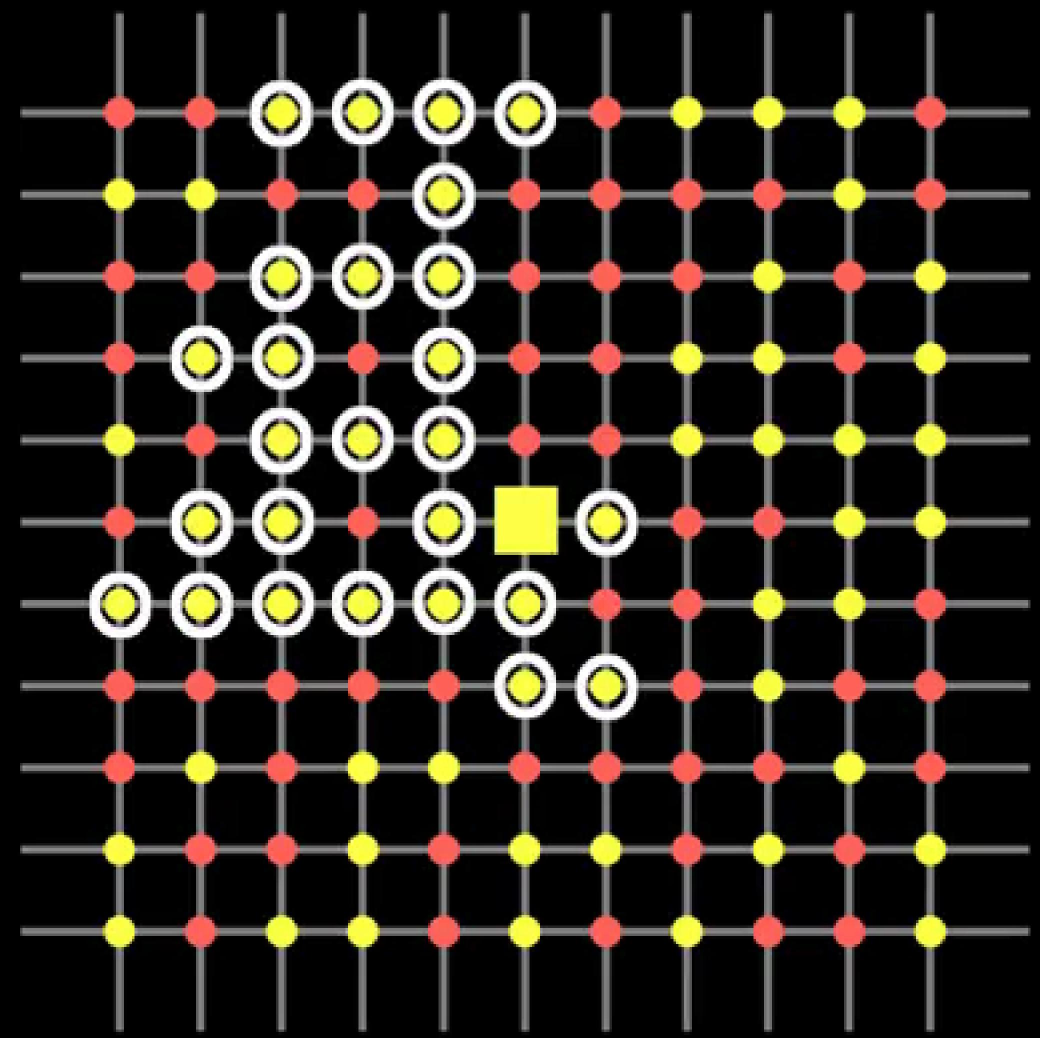


- For two open sites u and v, v is said to be in the component of u ($v \in C_u$), if there is a path of open sites from u to v.

- Probabilistic forwarding of a single packet over $\mathbb{Z}^2$ is modelled by site percolation on $\mathbb{Z}^2$ conditioned on the origin $\mathbf{0}$ being open.



- Nodes transmitting the j**th packet** (for fixed $j \in [n]$) may be viewed as open sites in the component of the origin. Call this cluster of nodes as $C_{\mathbf{0},j}$.

- The total number of transmissions is simply $\sum_{j=1}^{n} |C_{\mathbf{0},j}|$.

Site percolation on $\mathbb{Z}^2$- Receivers

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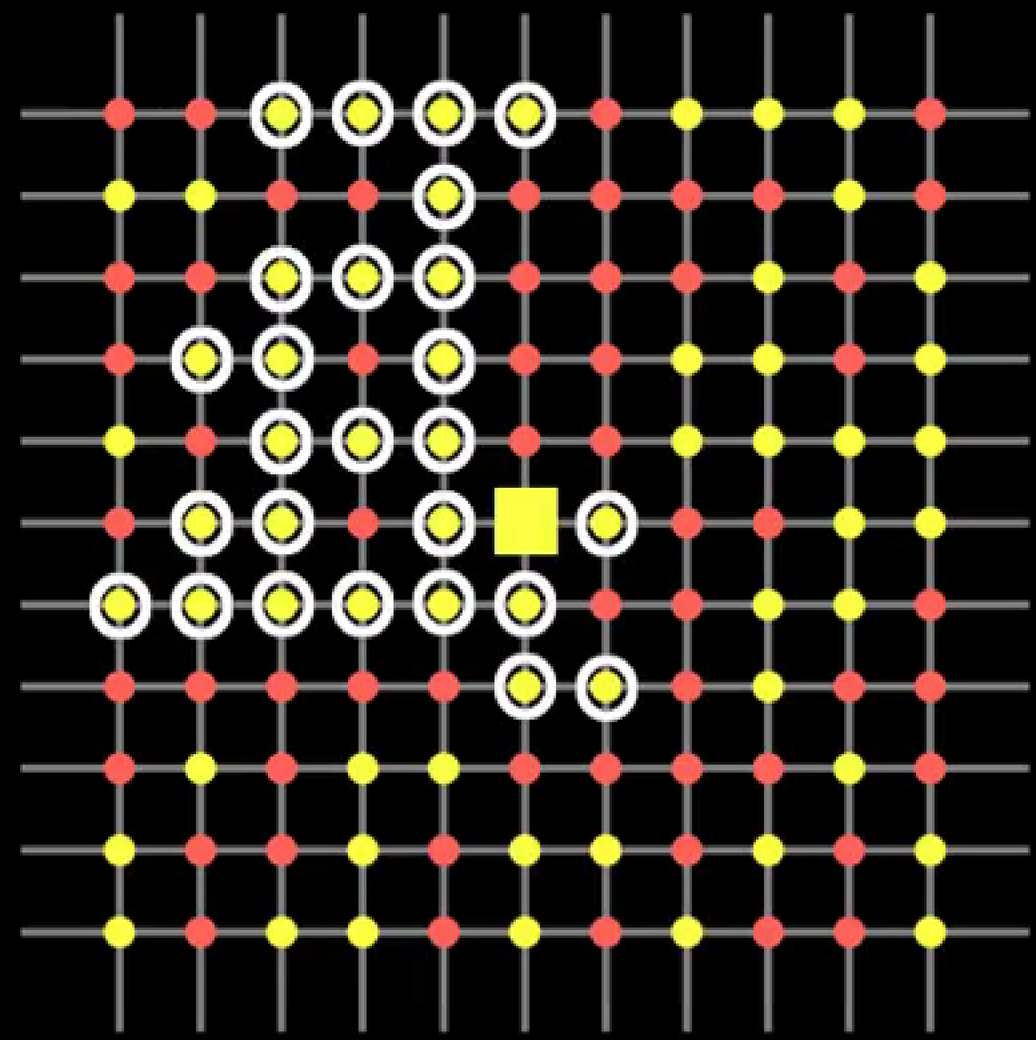
- The total number of transmissions is simply $\sum_{j=1}^{n} |C_{\mathbf{0},j}|$.

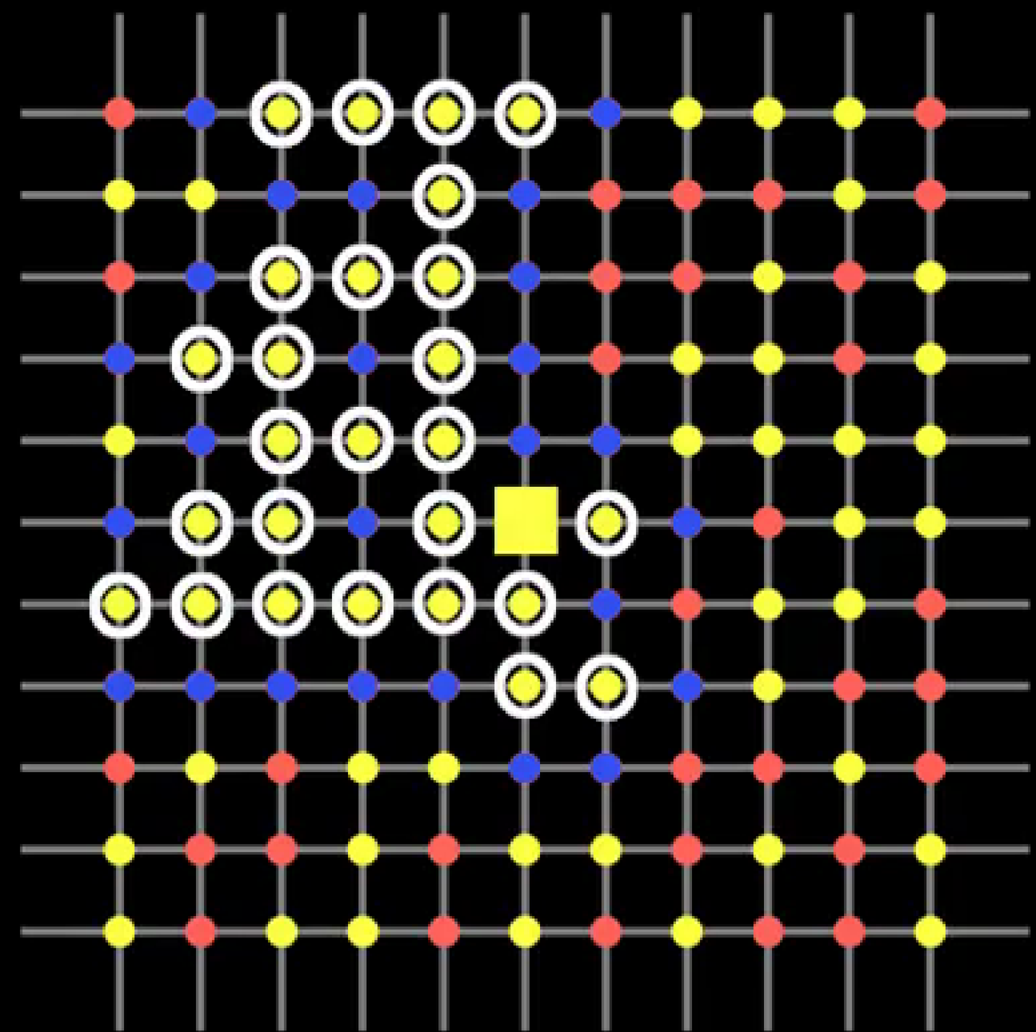
- The boundary, $\partial C_{\mathbf{0},j}$ is the set of all closed sites which are adjacent to a site in $C_{\mathbf{0},j}$.

- The set $C_{\mathbf{0},j}^{\text{ext}} := C_{\mathbf{0},j} \cup \partial C_{\mathbf{0},j}$ is called the **extended cluster** of the origin.

Transmitters $\Leftrightarrow$ open cluster of the origin

Receivers $\Leftrightarrow$ extended cluster of the origin





Site percolation

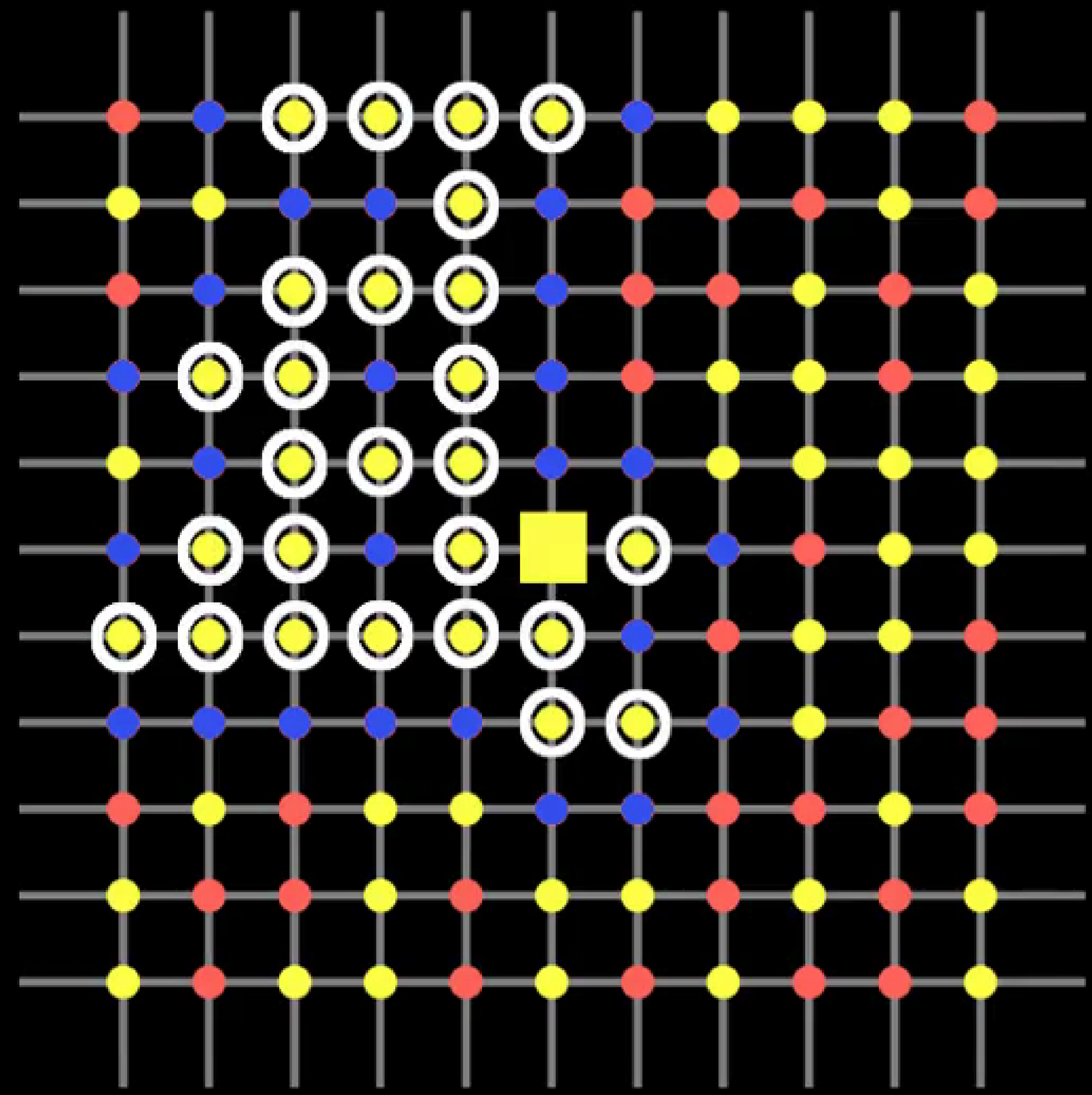
For site percolation on $\mathbb{Z}^2$, there exists $p_c \in (0,1)$ s.t. for $p > p_c$,

- There exists a unique infinite open cluster (IOC), C, almost surely.

$p_c \approx 0.59$ for site percolation

- Hence, there also exists a unique infinite extended cluster (IEC), C^{ext} a.s.

- $\theta(p) :=$ percolation probability, i.e., $\mathbb{P}(\mathbf{0} \in C)$
- $\theta^{\text{ext}}(p) :=$ extended probability, i.e., $\mathbb{P}(\mathbf{0} \in C^{\text{ext}})$



Lemma: $\theta^{\text{ext}}(p) = \dfrac{\theta(p)}{p}$

Proof: $\{\mathbf{0} \in C\} = \{\mathbf{0} \in C^{\text{ext}} \text{ and } \mathbf{0} \text{ is open}\}$

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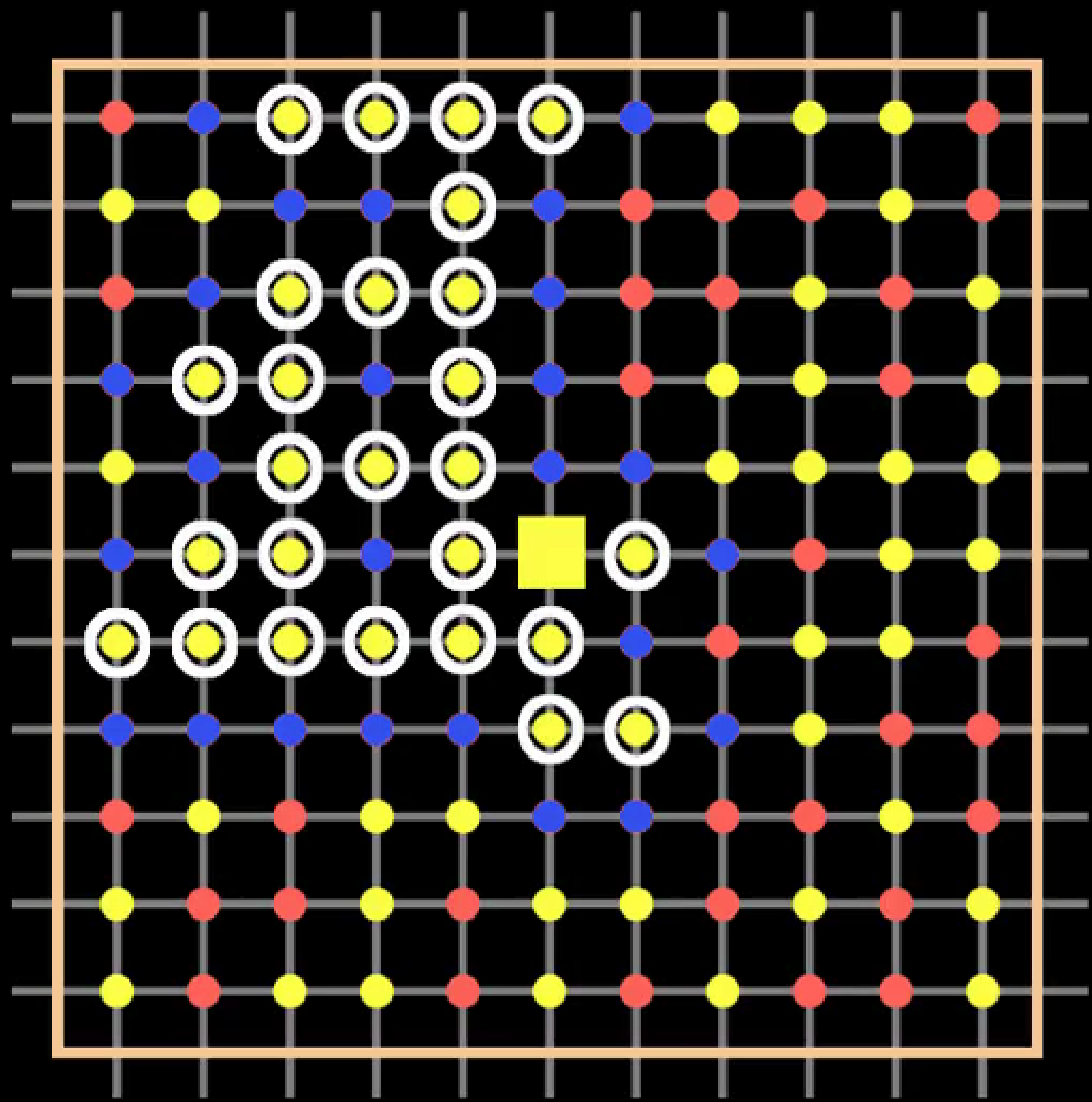

Ergodic theorems

- $\lim_{m\to\infty} \frac{|C \cap \Gamma_m|}{m^2} = \theta(p)$ a.s. and in L^1

- $\lim_{m\to\infty} \frac{|C^{\text{ext}} \cap \Gamma_m|}{m^2} = \theta^{\text{ext}}(p)$ a.s. and in L^1

Lemma: $\theta^{\text{ext}}(p) = \frac{\theta(p)}{p}$

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Site Percolation and Probabilistic Forwarding

- Prob. forwarding of a single packet over $\mathbb{Z}^2$ is modelled using site percolation on $\mathbb{Z}^2$ conditioned on the origin $\mathbf{0}$ being open.

- n packets $\leftrightarrow$ n independent site percolation with $\mathbf{0}$ open in all.

- $\mathcal{R}_{k,n}(\Gamma_m) := \{$sites in Γ_m that receive at least k out of n pkts$\}$

- We are interested in finding

$$p_{k,n,\delta} = \min \left\{ p \;\middle|\; \mathbb{E}_p \left[\frac{1}{m^2} \left| \mathcal{R}_{k,n}(\Gamma_m) \right| \right] \geq 1 - \delta \right\}$$

Theorem

For $p > p_c$, we have

$$\lim_{m \to \infty} \mathbb{E} \left[\frac{1}{m^2} \left| \mathcal{R}_{k,n}(\Gamma_m) \right| \right] = \mathbb{P}(Y \geq k),$$

where $Y \sim \text{Bin}\left(n, (\theta^{\text{ext}}(p))^2\right)$

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Intuition

For $k = n = 1$, receivers $\Leftrightarrow C_{\mathbf{0}}^{\text{ext}}$

$$(\theta^{\text{ext}}(p))^2 = \theta^{\text{ext}}(p) \times \theta^{\text{ext}}(p)$$

$$\lim_{m\to\infty} \mathbb{E}\left[\frac{|C^{\text{ext}} \cap \Gamma_m|}{m^2}\right] \qquad \mathbb{P}(\mathbf{0} \in C^{\text{ext}})$$

For multiple packets,

$$\mathbb{P}(Y \geq k) = \sum_{t=k}^{n} \sum_{\substack{T\subseteq[n] \\ |T|=t}} \theta_{k,t}^{\text{ext}}(p) \left(\theta^{\text{ext}}(p)\right)^t \left(1 - \theta^{\text{ext}}(p)\right)^{n-t}$$

$\mathbb{P}(\mathbf{0} \in$ IECs indexed by T only)

$$\mathbb{P}(\mathbf{0} \in C_{k,t}^{\text{ext}}) = \sum_{j=k}^{t} \binom{t}{j} \left(\theta^{\text{ext}}(p)\right)^j \left(1 - \theta^{\text{ext}}(p)\right)^{t-j}$$

$C_{k,t}^{\text{ext}} :=$ {sites in at least k out of the t IECs }

Site Percolation and Probabilistic Forwarding

$$p_{k,n,\delta} = \min \left\{ p \,\middle|\, \mathbb{E}_p\left[\frac{1}{m^2}\left|\mathcal{R}_{k,n}(\Gamma_m)\right|\right] \geq 1-\delta \right\}$$

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where $Y \sim \mathrm{Bin}\left(n, (\theta^{\mathrm{ext}}(p))^2\right)$

$$\tau_{k,n,\delta} \approx nm^2\theta(p_{k,n,\delta})\theta^{\mathrm{ext}}(p_{k,n,\delta})$$

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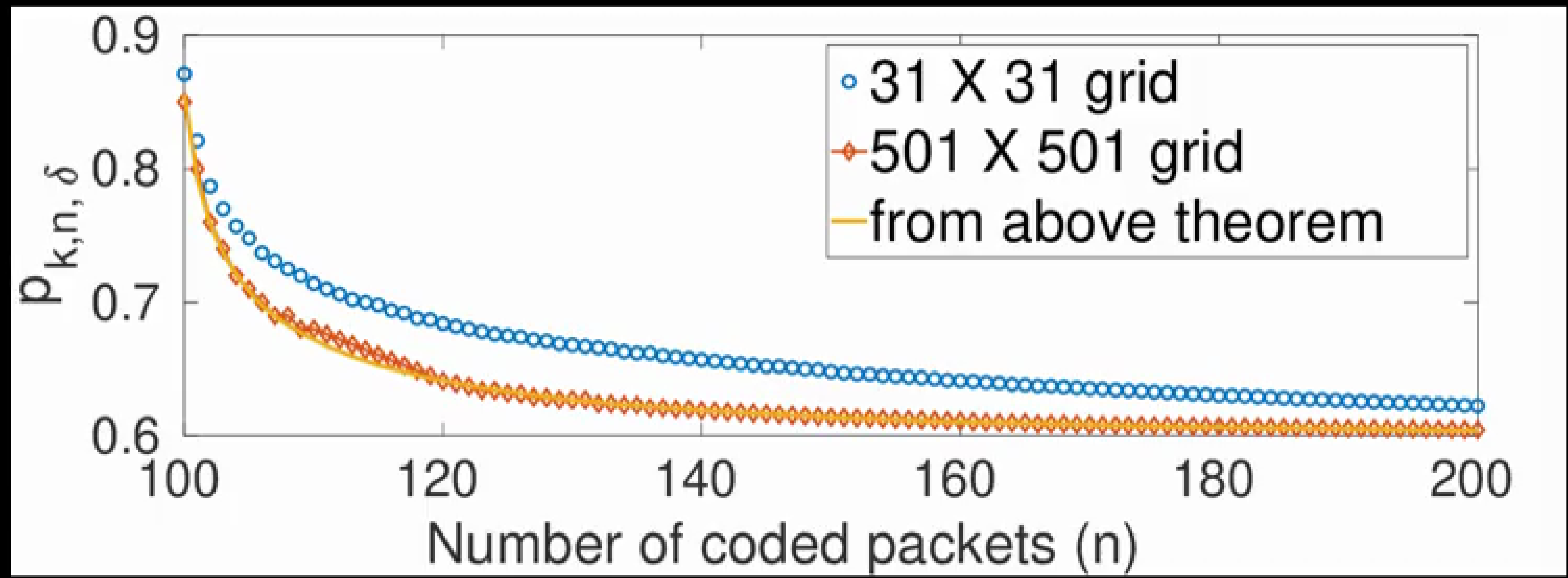
Comparison with simulations

$$p_{k,n,\delta} \approx \min \left\{ p \;\middle|\; \mathbb{P}(Y \geq k) \geq 1 - \delta \right\}$$

where $Y \sim \text{Bin}\left(n, (\theta^{\text{ext}}(p))^2\right)$



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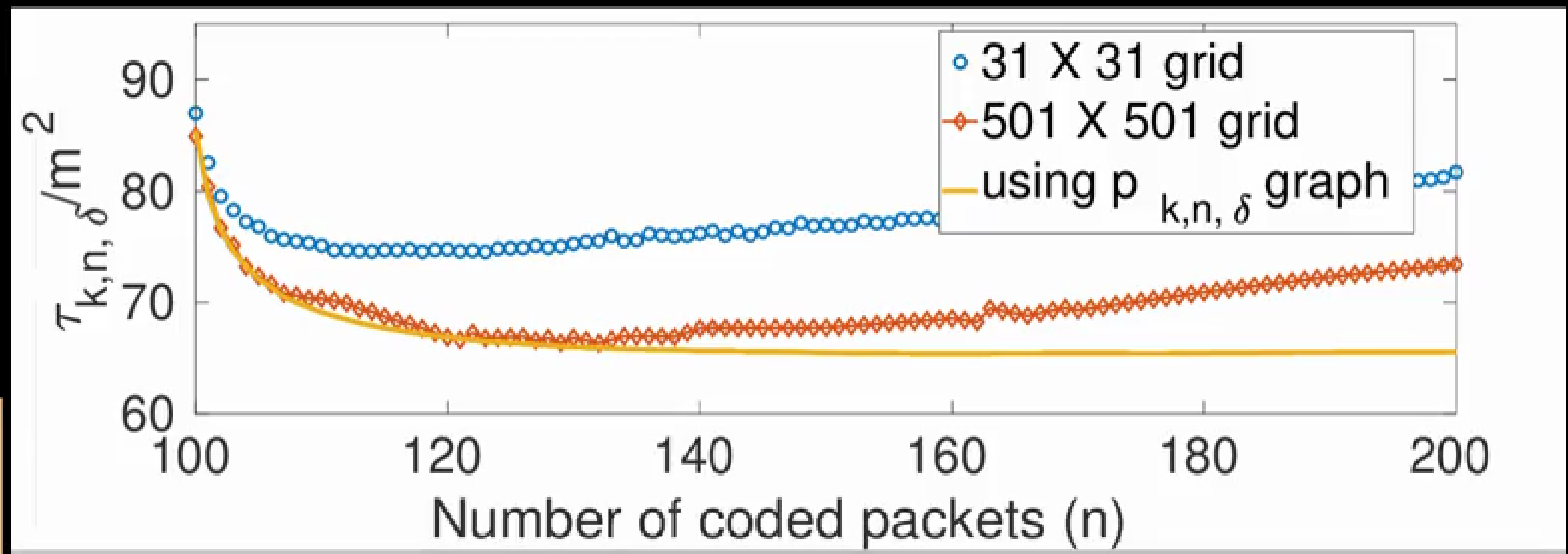


Conclusion: Introducing coded packets with probabilistic forwarding on the grid reduces the expected number of transmissions while ensuring a near-broadcast.

Random Geometric Graphs

What is an RGG?

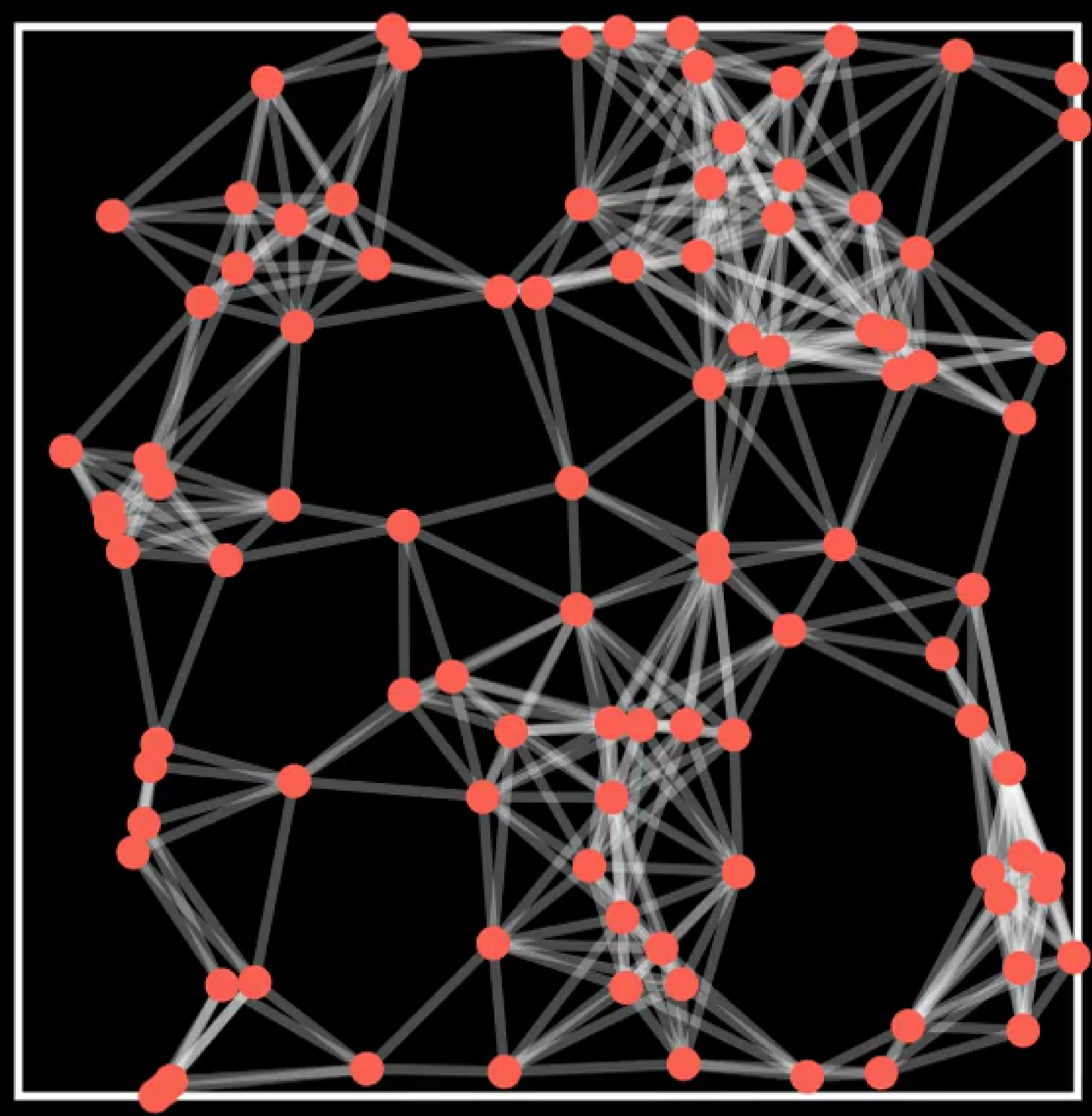
Intensity: λ

Generating $G_m \sim \mathrm{RGG}(\lambda)$ on $\Gamma_m = \left[\frac{-m}{2}, \frac{m}{2}\right]^2$

- Sample the number of points, $N \sim \mathrm{Poi}(\lambda m^2)$.

- Choose points $X_1, X_2, \cdots, X_N$ uniformly and independently from Γ_m. These form the points of a Poisson point process, Φ , and constitute the vertex set of the RGG.

- Place an edge between any two vertices which are within unit distance of each other.



Formulation

Where is the source?

- Include source at the origin.
- $\Phi^{\mathbf{0}} = \Phi \cup \{\mathbf{0}\}$; Resulting graph $G_m^{\mathbf{0}}$
- Palm probability $\mathbb{P}^{\mathbf{0}}(\cdot) = \mathbb{P}(\Phi^{\mathbf{0}} \in \cdot)$

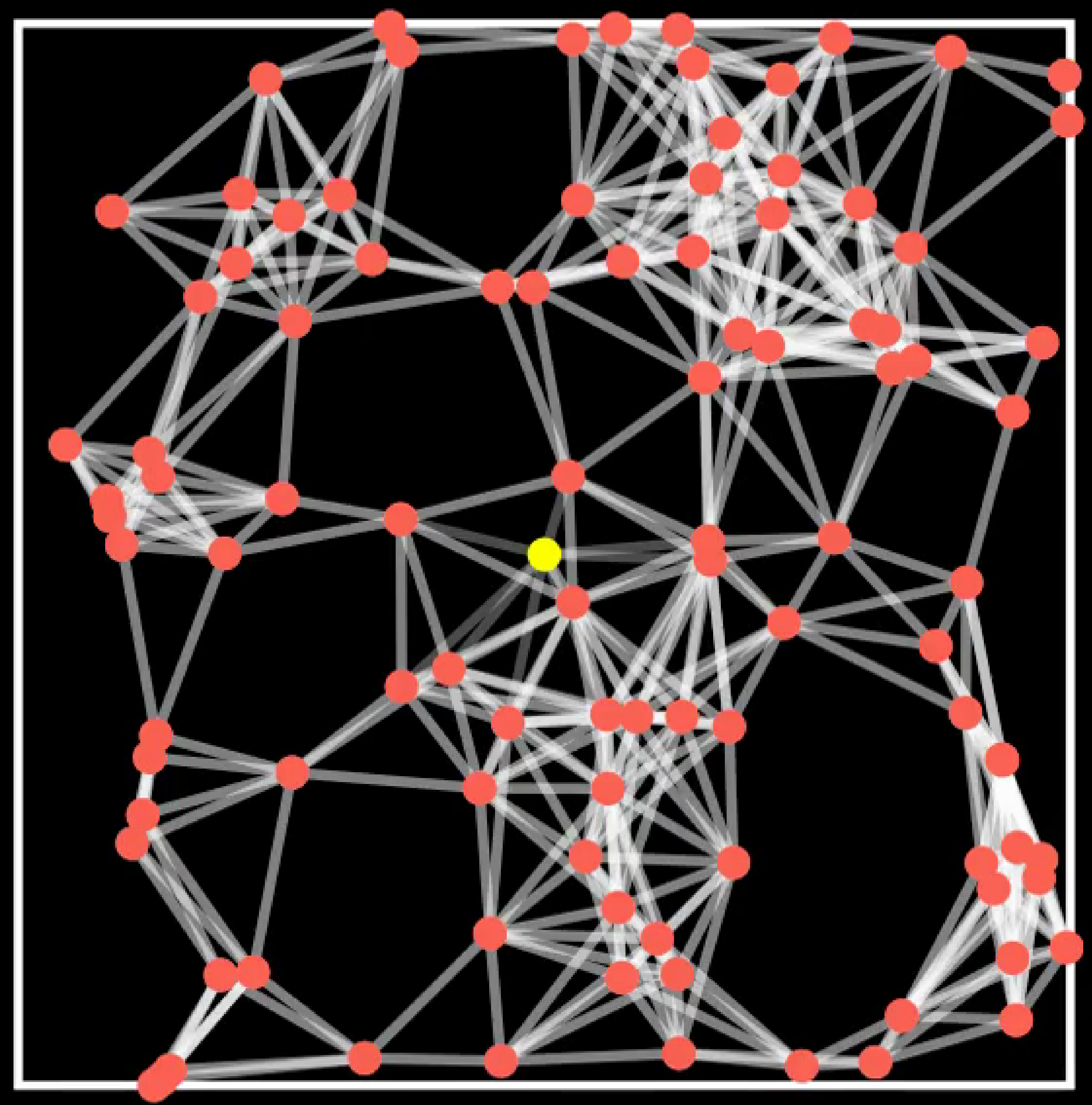
Is it always connected?

- Component of the origin, $C_{\mathbf{0}} \equiv C_{\mathbf{0}}(G_m^{\mathbf{0}})$.

$R_{k,n}(G_m^{\mathbf{0}})$ - Successful receivers within $C_{\mathbf{0}}$

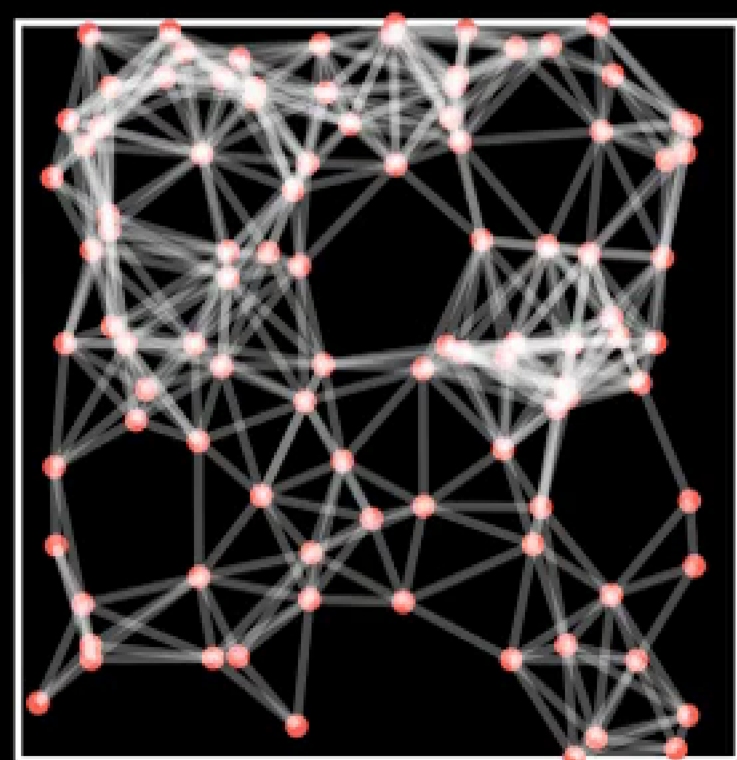
$$p_{k,n,\delta} = \min \left\{ p \,\middle|\, \mathbb{E}\left[\frac{R_{k,n}(G_m^{\mathbf{0}})}{|C_{\mathbf{0}}(G_m^{\mathbf{0}})|}\right] \geq 1 - \delta \right\}$$

$$\tau_{k,n,\delta} = \mathbb{E}\left[\text{total } \# \text{ transmissions}\right]$$

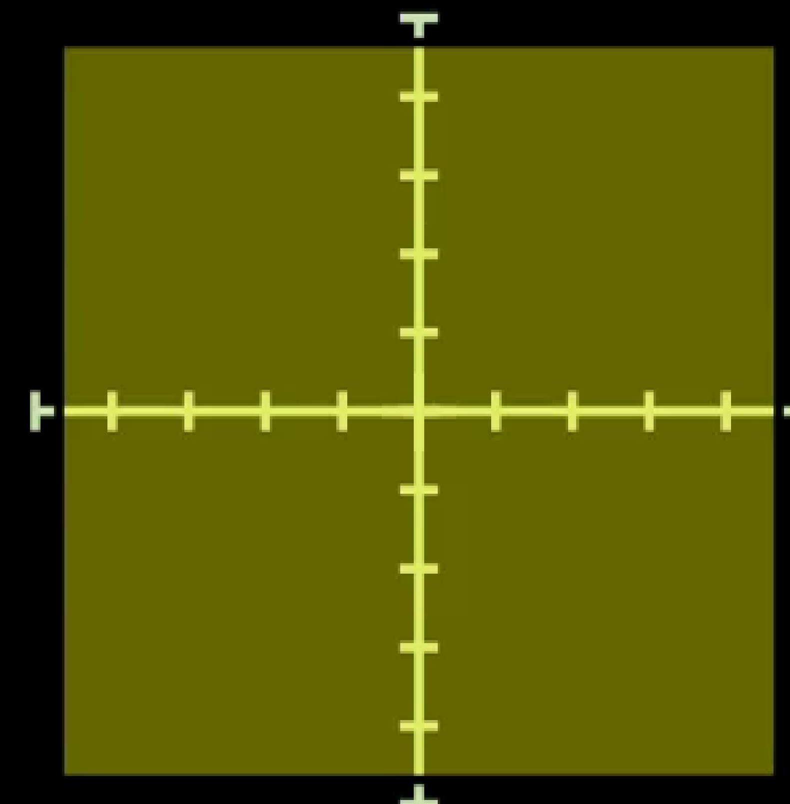


Idea for Analysis

Probabilistic forwarding on RGG within Γ_m, i.e., $G_m^{\mathbf{0}}$

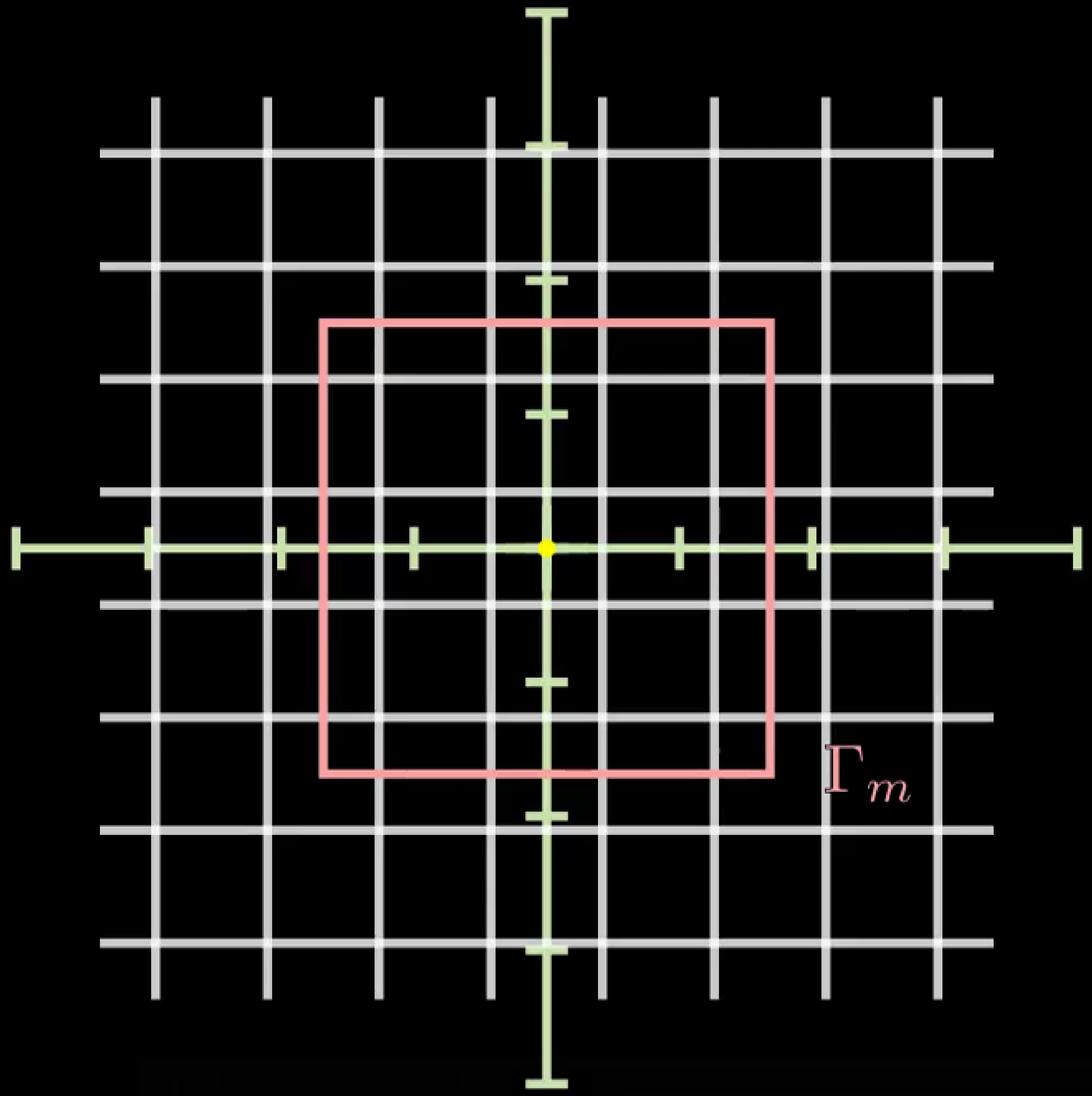


Probabilistic forwarding on RGG over $\mathbb{R}^2$, $\mathcal{G}^{\mathbf{0}}$



We will use ideas from continuum percolation and ergodic theory to obtain estimates of $p_{k,n,\delta}$ and $\tau_{k,n,\delta}$

RGG on the $\mathbf{R}^2$ plane



- Create a tiling of the $\mathbf{R}^2$ plane.
- Generate independent Poisson point process of intensity λ on each tile.
- Add a point at the origin.
- Connect nodes within unit distance to obtain $\mathcal{G}^{\mathbf{0}}$.

Continuum percolation

- There exists a critical intensity, λ_c s.t. for $\lambda > \lambda_c$ there exists a unique infinite cluster, C .
- Percolation probability:

$$\theta(\lambda) = \mathbb{P}^{\mathbf{0}}(\mathbf{0} \in C)$$

- Ergodic theorem: For $\lambda > \lambda_c$,

$$\frac{|C \cap \Gamma_m|}{\lambda m^2} \xrightarrow{m \to \infty} \theta(\lambda) \qquad \mathbb{P}\text{-a.s..}$$

Palm probabilities

- Ergodic theorems: $\mathbb{P}$ a.s. results; $\mathbb{P}$ - distribution of Φ
- We need w.r.t. $\mathbb{P}^{\mathbf{0}}$; distribution of $\Phi^{\mathbf{0}} = \Phi \cup \{\mathbf{0}\}$

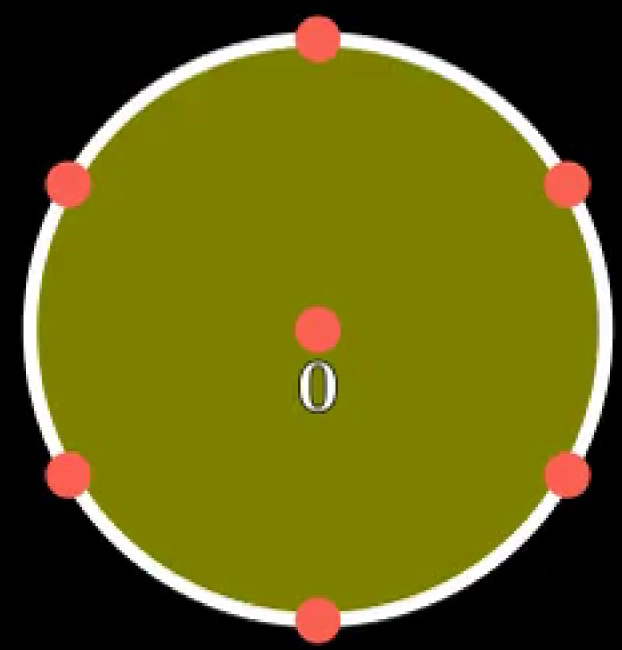
An example: Let $\lambda > \lambda_c$ and $\mathcal{G} \sim \mathrm{RGG}(\lambda)$

$C(\Phi)$: infinite cluster in $\mathcal{G}$, $C(\Phi^{\mathbf{0}})$: infinite cluster in $\mathcal{G}^{\mathbf{0}}$

$$\frac{|C(\Phi^{\mathbf{0}}) \cap \Gamma_m|}{\lambda m^2} \geq \frac{|C(\Phi) \cap \Gamma_m|}{\lambda m^2}$$

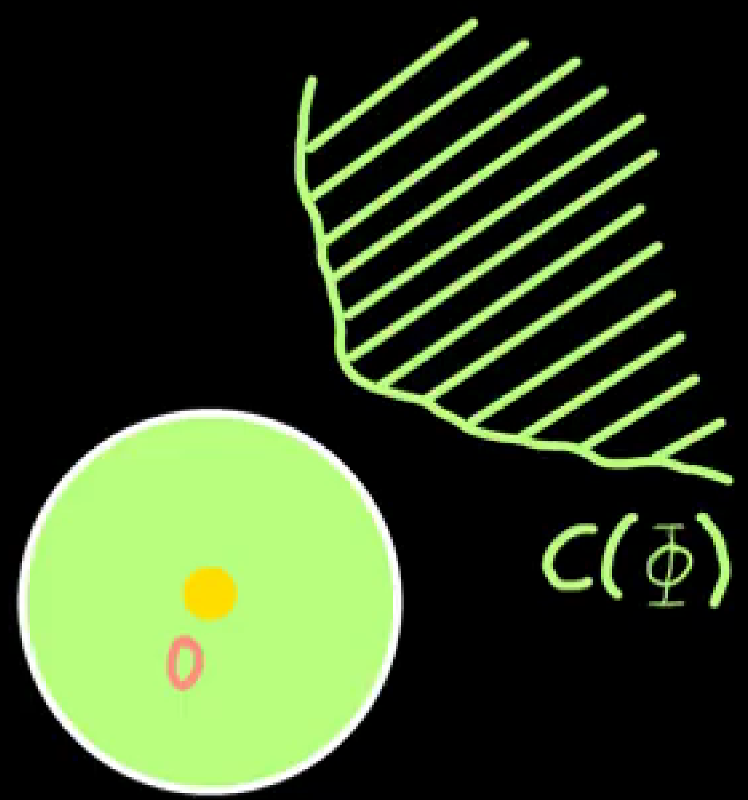
We will show,

$$\lim_{m\to\infty} \mathbf{E}^{\mathbf{0}}\left[\frac{|C \cap \Gamma_m|}{\lambda m^2}\right] = \lim_{m\to\infty} \mathbf{E}\left[\frac{|C \cap \Gamma_m|}{\lambda m^2}\right]$$



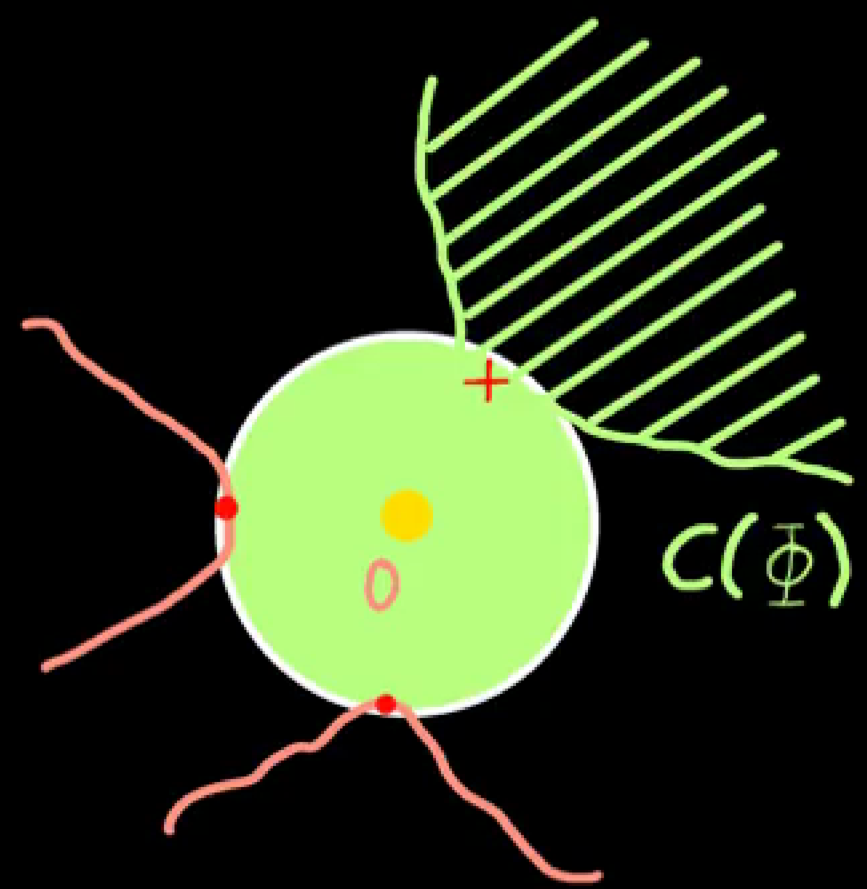
$K \leq 6$ a.s.

Case 1:



$$\frac{|C(\Phi^{\mathbf{0}}) \cap \Gamma_m|}{\lambda m^2} = \frac{|C(\Phi) \cap \Gamma_m|}{\lambda m^2}$$

Case 2:



$$\frac{|C(\Phi^{\mathbf{0}}) \cap \Gamma_m|}{\lambda m^2} = \frac{|C(\Phi) \cap \Gamma_m|}{\lambda m^2} + \sum_{i=1}^{K} \frac{|C_i \cap \Gamma_m|}{\lambda m^2}$$

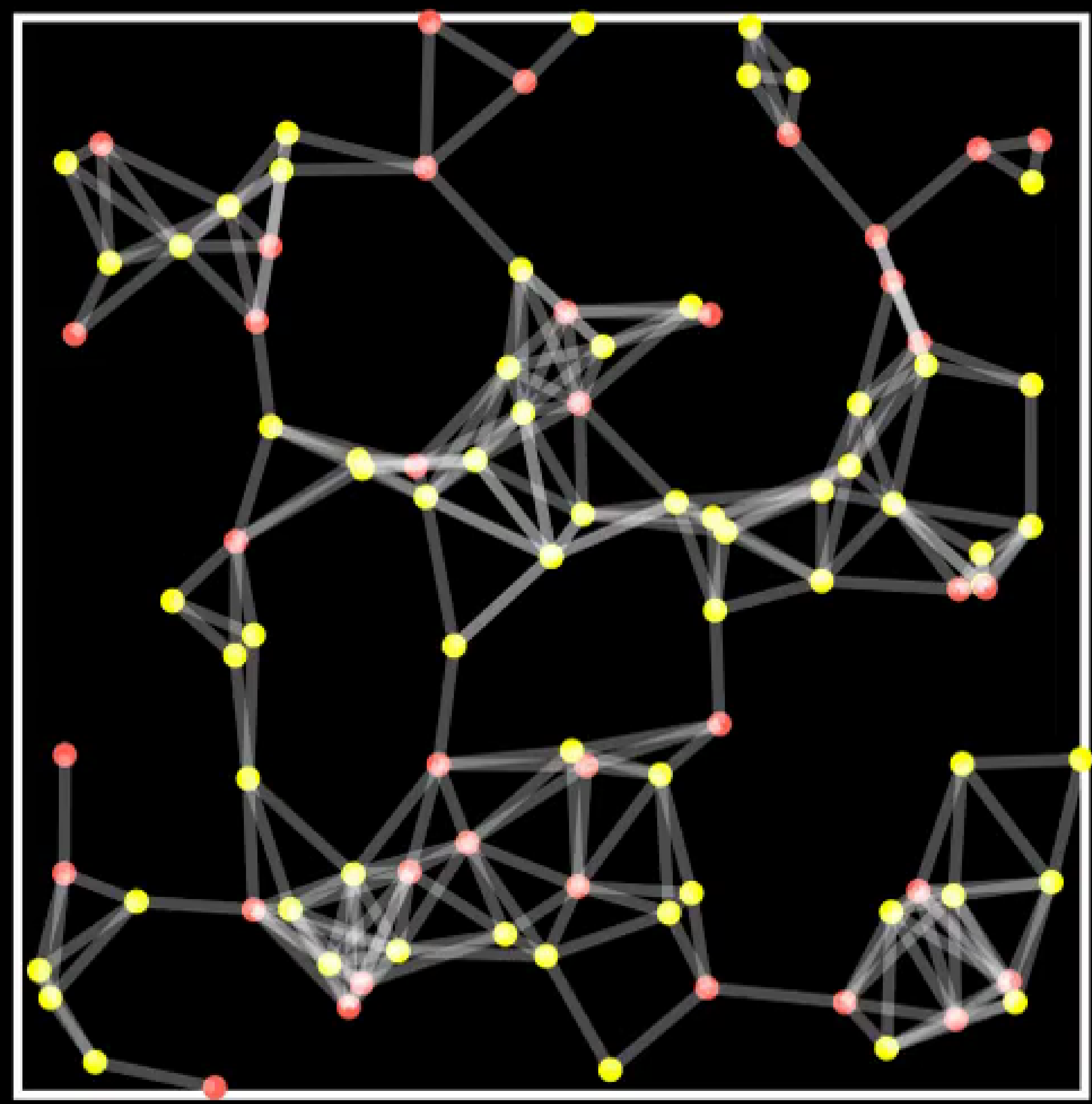
Prob. Forwarding and Marked Point Process

Marked point processes

- Associate each point, X_u, of Φ with a mark $Z(X_u) \in \mathbb{K}$ - space of marks
- $\mathbb{P}(Z \in \cdot|\Phi) \overset{\text{iid}}{\sim} \Pi(\cdot)$
- $\Pi(\cdot)$ - Mark distribution

Single packet probabilistic forwarding

- $\mathbb{K} = \{0, 1\}$, Π - $\mathrm{Ber}(p)$
- Transmitters $\Leftrightarrow C_{\mathbf{0}}^{+}$
- Receivers $\Leftrightarrow \{$nodes in Φ^{-} adjacent to $C_{\mathbf{0}}^{+}\} \cup C_{\mathbf{0}}^{+}$



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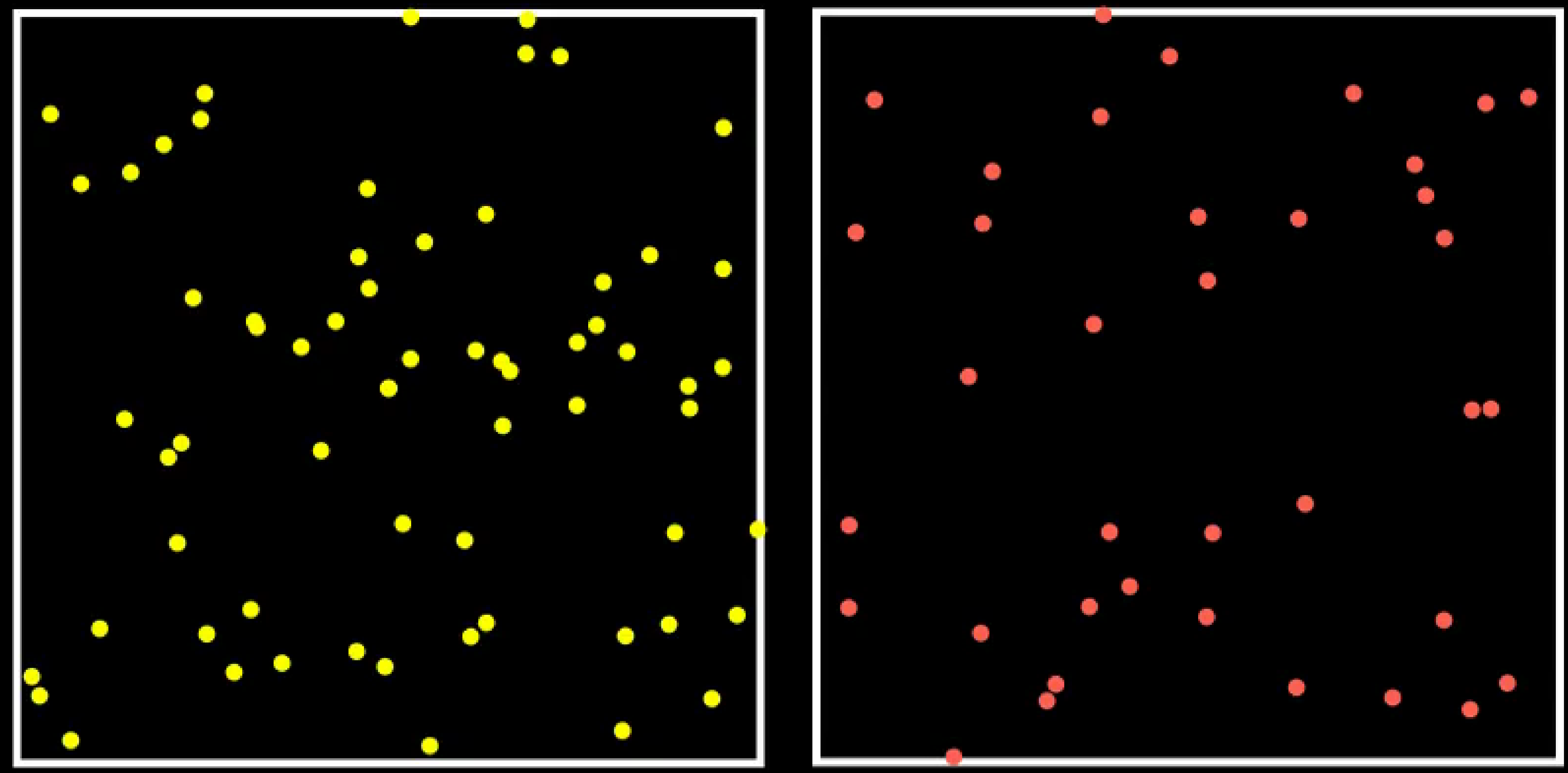

Mark: $Z = 1$ — PPP: Φ^{+} — Int.: λp

Mark: $Z = 0$ — PPP: Φ^{-} — Int.: $\lambda(1-p)$

Prob. Forwarding and Marked Point Process

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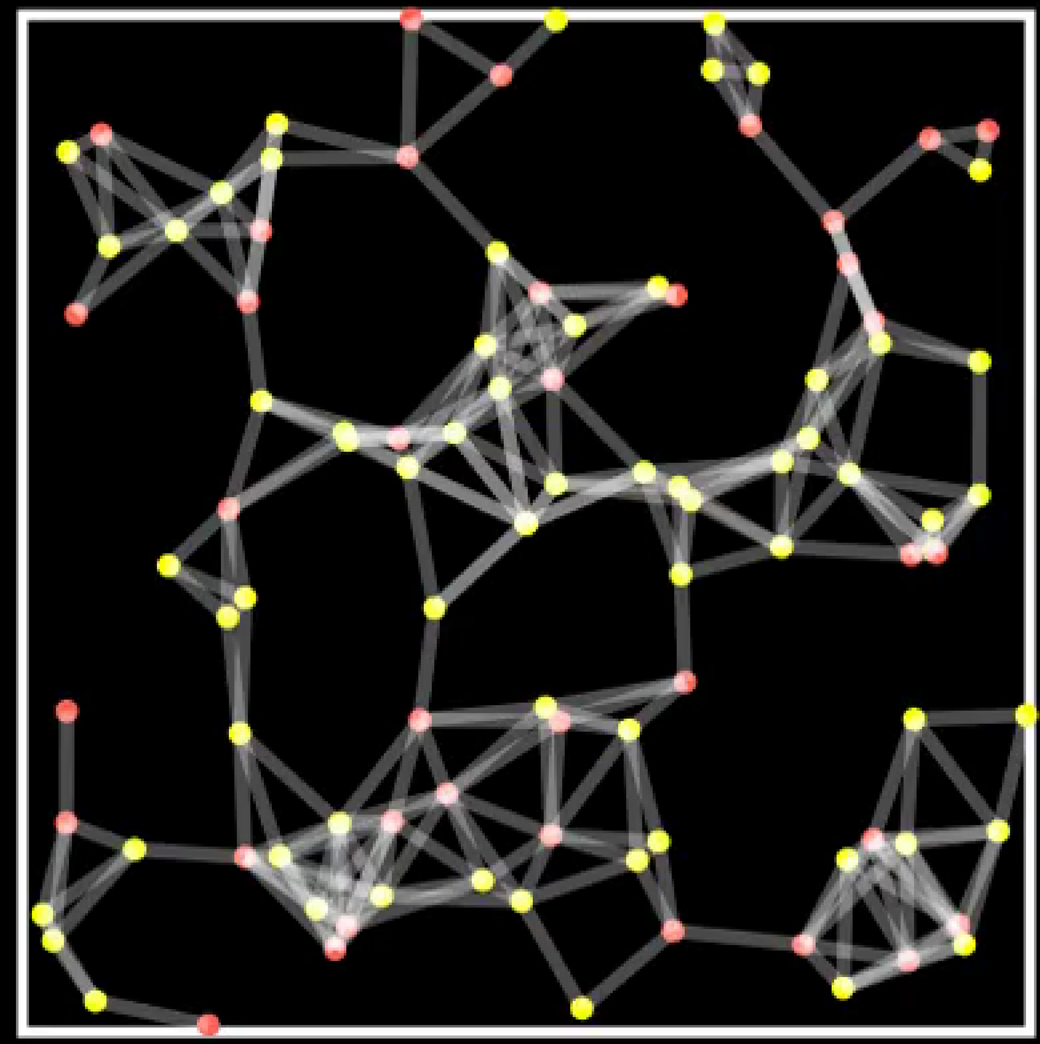

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Mark: $Z = 1$	Mark: $Z = 0$
PPP: Φ^{+}	PPP: Φ^{-}
Int.: λp	Int.: $\lambda(1-p)$

Infinite cluster in Φ^{+}: C^{+}

Infinite extended cluster: C^{ext}

Probabilistic forwarding of n pkts

- $\mathbb{K} = \{0, 1\}^{n}$
- Marks $\mathbf{Z} = (Z_1, Z_2, \cdots, Z_n)$, where $Z_i(\cdot) \overset{\text{ind}}{\sim} \text{Ber}(p)\ \forall i$

Main results

$$p_{k,n,\delta} = \min \left\{ p \;\middle|\; \mathbb{E}\left[\frac{R_{k,n}(G_m^{\mathbf{0}})}{|C_{\mathbf{0}}(G_m^{\mathbf{0}})|}\right] \geq 1 - \delta \right\}$$

Define

$C_{k,n}^{\text{ext}} = \{$nodes present in at least k out of n IECs$\}$

Theorem*: For $\lambda p > \lambda_c$,

$$\lim_{m\to\infty} \mathbb{E}\left[\frac{R_{k,n}(G_m^{\mathbf{0}})}{|C_{\mathbf{0}}(G_m^{\mathbf{0}})|}\right] = \frac{1}{\theta(\lambda)} \sum_{t=k}^{n} \sum_{\substack{T\subseteq[n] \\ |T|=t}} \theta_{k,t}^{\text{ext}}\, \mathbb{P}^{\mathbf{0}}(\mathbf{0} \in \text{ IECs indexed by } T \text{ only}).$$

where $\theta_{k,n}^{\text{ext}} \equiv \theta_{k,n}^{\text{ext}}(\lambda, p) = \mathbb{P}^{\mathbf{0}}(\mathbf{0} \in C_{k,n}^{\text{ext}})$

For $\lambda p_{k,n,\delta} > \lambda_c$, $\tau_{k,n,\delta} \approx n m^2 \lambda p_{k,n,\delta} \;(\theta(\lambda p_{k,n,\delta}))^2$

$$= n\left(\lambda p_{k,n,\delta} m^2\right) \theta(\lambda p_{k,n,\delta}) \times \theta(\lambda p_{k,n,\delta})$$

Thank you !!