Branching Brownian motion conditioned on small maximum

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Abstract

We consider a standard binary branching Brownian motion on the real line. It is known that the maximal position M_t among all particles alive at time t, shifted by $m_t = \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t$ converges in law to a randomly shifted Gumbel variable. Derrida and Shi [DS17a] conjectured the precise asymptotic behaviour of the corresponding lower deviation probability $\mathbb{P}(M_t \leq \sqrt{2}\alpha t)$ for $\alpha < 1$. We verify their conjecture, and describe the law of the branching Brownian motion conditioned on having a small maximum.

1 Introduction

We consider a one-dimensional standard binary branching Brownian motion. It is a continuous-time particle system on the real line which is constructed as follows. It starts with one individual located at the origin at time 0 that moves according to a standard Brownian motion. After an independent exponential time of parameter 1, the initial particle dies and gives birth to 2 children that start at the position their parent occupied at its death. These 2 children then move according to independent Brownian motions and give birth independently to their own children at rate 1. The particle system keeps evolving in this fashion for all time.

For all $t \geq 0$, we denote by $N(t)$ the collection of the individuals alive at time t. For any $u \in N(t)$ and $s \leq t$, let $X_u(s)$ denote the position at time s of the individual u or its ancestor alive at that time. The maximum of the branching Brownian motion at time t is defined as $M_t := \max\{X_u(t) : u \in N(t)\}$.

The asymptotic behaviour of M_t as $t \to \infty$ has been subjected to intense study, partly due to its link to the F-KPP reaction-diffusion equation, defined as

$$\partial_t u = \frac{1}{2}\Delta u - u(1-u). \tag{1.1}$$

Precisely, McKean [McK75] showed that the function $(t, x) \mapsto u(x, t) = \mathbb{P}(M_t \leq x)$ is the unique solution of (1.1) with initial condition $u(x, 0) = \mathbb{1}_{\{x>0\}}$.

Using this representation, Bramson [Bra83] proved that for all $z \in \mathbb{R}$,

$$\lim_{t\to\infty} \mathbb{P}(M_t \leq m_t + z) = \lim_{t\to\infty} u(m_t + z, t) = w(z), \tag{1.2}$$

where $m_t = \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t$ and w is the slowest travelling wave solution of the F-KPP equation, which satisfies

$$\frac{1}{2}w'' + \sqrt{2}w' - w(1-w) = 0.$$

Observe that $(t, x) \mapsto w(x - \sqrt{2}t)$ is a solution to (1.1). Lalley and Sellke [LS87] later provided the following representation for w as

$$w(z) := \mathbb{E}[e^{-C_0 e^{-\sqrt{2}z} D_\infty}], \tag{1.3}$$

where $C_0 > 0$ is a constant and D_∞ is an a.s. positive random variable, constructed as the almost sure limit of the so-called derivative martingale, defined for all $t \geq 0$ by

$$D_t := \sum_{u \in N(t)} (\sqrt{2}t - X_u(t)) e^{\sqrt{2}X_u(t) - 2t}.$$

The upper large deviations of the branching Brownian motion (i.e. estimating the asymptotic decay of $\mathbb{P}(M_t \geq \sqrt{2}\alpha t)$ for $\alpha \geq 1$) were first investigated by Chauvin and Rouault [CR88, CR90], who obtained tight estimates of $\mathbb{P}(M_t \geq \sqrt{2}\alpha t)$ for $\alpha \geq 1$. It is now known (see e.g. [Bov17, Lemma 9.7]) that

$$\mathbb{P}(M_t \geq \sqrt{2}\alpha t) \sim \begin{cases} \frac{\Upsilon(\alpha)}{\sqrt{4\pi}\alpha} t^{-1/2} e^{-(\alpha^2-1)t}, & \text{if } \alpha > 1, \\ \frac{3C_0}{2\sqrt{2}} t^{-3/2} \log t, & \text{if } \alpha = 1, \end{cases} \quad \text{as } t \to \infty. \tag{1.4}$$

Here C_0 is the same constant as in (1.3), and Υ is a non-decreasing bounded function on $(1, \infty)$ that can be rewritten as the probability for a Brownian motion with drift $\alpha - 1$ to stay above a random barrier [BBCM18, Theorem 1.2]. Similar tight estimates were recently obtained for the upper large deviations of branching random walks [BM19, GH18].

The aim of this article is to obtain precise lower deviations estimates for the maximum of the branching Brownian motion, i.e. the asymptotic behaviour of the probability $\mathbb{P}(M_t \leq \sqrt{2}\alpha t)$ for all $\alpha < 1$. The same question recently arose in the context of branching random walks [GH18, CH20]. Derrida and Shi [DS17b] obtained the following estimates on the exponential decay

$$\mathbb{P}(M_t \leq \sqrt{2}\alpha t) = \begin{cases} e^{-2(\sqrt{2}-1)(1-\alpha)t + o(t)}, & \text{for } 1 - \sqrt{2} < \alpha < 1, \\ e^{-(1+\alpha^2)t + o(t)}, & \text{for } \alpha \leq 1 - \sqrt{2}, \end{cases} \quad \text{as } t \to \infty.$$

In particular, some transition occurs at $\alpha_c = -\gamma := 1 - \sqrt{2} \approx -0.414$, where the large deviations rate function exhibits a second order phase transition (see Figure 1). Derrida and Shi [DS17a] also conjectured the existence of a positive constant $C^{(1)}$ such that for all $\alpha < 1$,

$$\mathbb{P}(M_t \leq \sqrt{2}\alpha t) \sim \begin{cases} C^{(1)} \left(\frac{\alpha - \alpha_c}{\sqrt{2}}\right)^{\frac{3\gamma}{2}} t^{\frac{3\gamma}{2}} e^{-2\gamma(1-\alpha)t}, & \text{if } \alpha > \alpha_c, \\ \frac{\Phi(\alpha)}{\sqrt{4\pi}} t^{-\frac{1}{2}} e^{-(1+\alpha^2)t}, & \text{if } \alpha < \alpha_c, \end{cases} \tag{1.5}$$

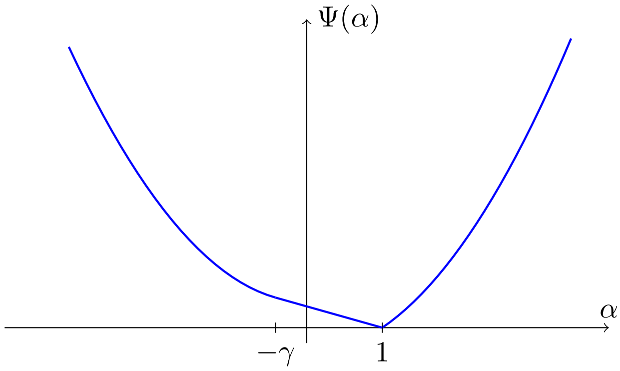


Figure 1: Rate function for the maximal displacement of the branching Brownian motion, defined as $\Psi(\alpha) = \lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}(M_t \approx \sqrt{2}\alpha t)$ (c.f. [DS17a]). Note the second order phase transition occurring at position $-\gamma = 1 - \sqrt{2}$.

where

$$\Phi(\alpha) =: -\frac{1}{\alpha} + \sqrt{2}\int_0^\infty \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y e^{(1-\alpha^2)s+\sqrt{2}\alpha y} u(y,s)^2 \in (0,\infty). \tag{1.6}$$

We shall prove in Section 4 that $\Phi(\alpha)$ is finite.

In this work, we prove the conjecture (1.5) of Derrida and Shi. Additionally, we obtain the precise asymptotic decay of $\mathbb{P}(M_t \leq \sqrt{2}\alpha t)$ in the critical case $\alpha = 1 - \sqrt{2}$ as well. We also describe the law of the branching Brownian motion conditioned on the large deviation event $\{M_t \leq \sqrt{2}\alpha t\}$ for all $\alpha < 1$, exhibiting the typical behaviour of a branching Brownian motion realizing this large deviation.

This behaviour is governed by the value of the first branching time τ, defined as the time at which the initial ancestor of the process dies, i.e.

$$\tau := \inf\{t \geq 0 : \#N(t) \geq 2\}.$$

On the event $\{M_t \leq \sqrt{2}\alpha t\}$, τ will typically be of order t and, up to some normalization, will converge in distribution as $t \to \infty$. Moreover, conditionally on τ and $\{M_t \leq \sqrt{2}\,\alpha t\}$, the position $X_\emptyset(\tau)$ at which the initial particle branches into children tight around its median $x_\tau \sim -c\tau$ for some $c > 0$.

We also describe, under the probability $\mathbb{P}(\cdot | M_t \leq \sqrt{2}\alpha t)$, the asymptotic behaviour of the point measure

$$\mathcal{E}_t(\alpha) := \sum_{u\in N(t)} \delta_{X_u(t)-\sqrt{2}\alpha t}, \quad t \geq 0,$$

which is the extremal process at time t of the conditioned branching Brownian motion, i.e. the position of particles that are within distance $O(1)$ from the maximal position.

The extremal process of the branching Brownian motion (without conditioning) has been previously studied by Arguin, Bovier and Kistler [ABK13] and Aïdékon, Berestycki, Brunet and Shi [ABBS13]. Writing $\mathcal{E}_t := \sum_{u\in N(t)} \delta_{X_u(t)-m_t}$ the extremal process of the branching Brownian motion, they showed that as $t \to \infty$,

$$(\mathcal{E}_t, M_t - m_t) \Longrightarrow (\mathcal{E}, \max_{x \in \mathcal{E}} x),$$

where $\mathcal{E}$ is a randomly shifted decorated Poisson point process with exponential intensity, and $\Longrightarrow$ denotes convergence in distribution for the topology of vague convergence of random measures. In the rest of the article, we always consider the convergence of random measure with the topology of vague convergence, i.e. that $\mathcal{E}_t \to \mathcal{E}$ if $\langle \mathcal{E}_t, \phi \rangle \to \langle \mathcal{E}, \phi \rangle$ for all continuous compactly supported function ϕ.

Precisely, the limiting extremal point process $\mathcal{E}$ point process can be constructed as

$$\mathcal{E} := \sum_{x \in \mathcal{P}} \sum_{y \in \mathcal{D}_x} \delta_{x+y},$$

where C_0, D_∞ are the quantities defined in (1.3), and conditioned on D_∞, $\mathcal{P}$ is a Poisson point process with intensity $C_0 \sqrt{2} D_\infty e^{-\sqrt{2}x} \mathrm{d}x$ and conditioned on $\mathcal{P}$, $(\mathcal{D}_x, x \in \mathcal{P})$ are i.i.d. decorated point processes in $(-\infty, 0]$ with an atom at 0, which we refer to as the decoration of the branching Brownian motion. In particular, observe that

$$\max \mathcal{E} = \max \mathcal{P} \overset{(d)}{=} \frac{1}{\sqrt{2}} \left(G - \log(C_0 \sqrt{2} D_\infty) \right),$$

where $\max \mathcal{A}$ is the largest position occupied by an atom of the point process $\mathcal{A}$, and G is a random variable independent of D_∞ with standard Gumbel distribution. Therefore the distribution function of $\max \mathcal{E}$ is exactly $w(\cdot)$, in view of (1.3).

Remark 1.1. In this article, we choose to focus on branching Brownian motions with binary branching, to keep the proofs as simple as possible. Up to minor changes, one can assume the number of children made by an individual at death to be i.i.d. integer-valued random variables of law ν. As long as $\nu(0) = 0$ (i.e. the process never gets extinct) and $\sum_{k=1}^{\infty} k (\log k)^2 \nu(k) < \infty$ (an integrability condition guaranteeing the non-degeneracy of the limit D_∞), we expect similar results to hold.

Before stating our main result, we quickly recall the heuristics given in [DS17b] to explain the asymptotic decay of $\mathbb{P}(M_t \leq \sqrt{2}\alpha t)$ in (1.5) in the next section.

1.1 Heuristics behind the conjecture (1.5)

Recall that τ is the first branching time of the process and $X_\emptyset(\tau)$ the position of the particle at that first branching time. As particles behave independently after they branched, the probability of observing an unusually low maximum decays sharply after each branching event. Therefore, to maximize the possibility that $M_t \leq \sqrt{2}\alpha t$, a good strategy is to make the first branching time as large as possible. Recalling that $\mathbb{P}(\tau > s) = e^{-s}$ and that we expect exponential decay in t, it is reasonable to conjecture that $\tau \approx \lambda_\alpha t$ conditioned on $\{M_t \leq \sqrt{2}\alpha t\}$, for some $\lambda_\alpha \in [0, 1]$. Additionally, after that branching time, particle should behave as regular branching Brownian motions with length $t - \tau$, therefore the maximal position at time t should be around level $X_\emptyset(\tau)+\sqrt{2}(t-\tau)$, which has to be lower than $\sqrt{2}\alpha t$. This yields the condition $X_\emptyset(\tau) \leq \sqrt{2}\alpha t + \sqrt{2}(\tau - t)$.

Then, with B a standard Brownian motion, observe that

$$\begin{aligned}\mathbb{P}(\tau \approx \lambda t, X_\emptyset(\tau) \leq \sqrt{2}\alpha t + \sqrt{2}(\tau - t)) &\approx e^{-\lambda t}\mathbb{P}(B_{\lambda t} \leq \sqrt{2}\alpha t + \sqrt{2}(\lambda - 1)t) \\ &\approx \exp\left(-t\left(\lambda + \frac{(\alpha + (\lambda - 1))^2}{\lambda}\right)\right).\end{aligned}$$

Thus, to maximize this probability, one has to choose the parameter $\lambda_\alpha \in [0,1]$ that minimizes the quantity

$$\lambda + \frac{(\alpha - (1-\lambda))^2}{\lambda}.$$

Note that if $\alpha > -\gamma = 1 - \sqrt{2}$, this minimum is attained for $\lambda_\alpha = \frac{(1-\alpha)}{\sqrt{2}} \in [0,1]$, whereas if $\alpha \leq \gamma$, this minimum is attained at $\lambda_\alpha = 1$.

As a result, we expect three different behaviours for the branching Brownian motion conditioned on having a maximum smaller than $\sqrt{2}\alpha t$, depending on whether α is larger than, smaller than, or equal to $-\gamma$. In the first case, the branching time should happen at some intermediate time in the process, and the branching Brownian motion after this first branching time should behave as a regular process, conditioned on an event of positive probability. If $\alpha < -\gamma$, then one expects the process not to branch until the very end of the process, which allows an explicit description of the extremal process in that case. In the intermediate case $\alpha = -\gamma$, the branching time should be such that $t - \tau$ is large, but negligible with respect to t. In this setting, the behaviour of the process after that time should not be too different from the case $\alpha > -\gamma$.

These heuristics describing the lower large deviations for the maximal displacement closely match the one used for lower deviations of similar branching processes. For example, in [BGMS14], the law of a Galton-Watson process conditioned on the limiting martingale being small is described as a Galton-Watson process with minimal branching until a given generation, which then behaves as typical process after that generation. Similarly, in [CH20], a branching random walk with an unusually small minimum is described as a process in which particles produced as few children as possible during the first few branches in as few children as possible, which all drift to a low position, from which they start independent unconditioned branching random walks.

1.2 Main results

We now state our main results, which completely validate the above heuristics of [DS17b]. With careful analysis of the first branching time and position, we are able to give an equivalent for the lower large deviations of the maximal displacement. We are also able to describe exactly the joint convergence in law of $(\tau, X_\emptyset(\tau), M_t, \mathcal{E}_t(\alpha))$ conditioned on $\{M_t \leq \sqrt{2}\alpha t\}$.

We begin with the case $\alpha > -\gamma$.

Theorem 1.2. *Assume that $\alpha \in (-\gamma, 1)$. Then, as $t \to \infty$, we have*

$$\mathbb{P}(M_t \le \sqrt{2}\alpha t) \sim C^{(1)}(v_\alpha t)^{\frac{3\gamma}{2}} e^{-2\gamma(1-\alpha)t}, \tag{1.7}$$

where $C^{(1)} := \frac{1}{2}\int_{\mathbb{R}} e^{-\sqrt{2}\gamma z} w(z)^2 \mathrm{d}z \in (0,\infty)$ and $v_\alpha := \frac{\gamma+\alpha}{\sqrt{2}} \in (0,1)$. Furthermore, conditioned on $\{M_t \le \sqrt{2}\alpha t\}$,

$$\left(\frac{\tau - \frac{(1-\alpha)}{\sqrt{2}} t}{\sqrt{t \frac{(1-\alpha)}{4\sqrt{2}}}}, X_\emptyset(\tau) - (\sqrt{2}\alpha t - m_{t-\tau}), M_t - \sqrt{2}\alpha t \right) \Longrightarrow (\xi, -\chi, -E), \tag{1.8}$$

where ξ and (χ, E) are independent, with ξ a standard Gaussian random variable and E an exponential random variable with parameter $\sqrt{2}\gamma$. The joint distribution of (χ, E) is given by

$$\mathbb{P}(\chi \le x, E \ge y) = \frac{1}{2C^{(1)}} e^{-\sqrt{2}\gamma y} \int_{-\infty}^{x-y} e^{-\sqrt{2}\gamma z} w(z)^2 \mathrm{d}z, \quad x \in \mathbb{R},\ y \in \mathbb{R}_+.$$

Moreover, we have, jointly with the convergence in (1.8),

$$\mathcal{E}_t(\alpha) \Longrightarrow \mathcal{E}^- := \sum_{x \in \mathcal{E}_1 \cup \mathcal{E}_2} \delta_{x-\chi}, \tag{1.9}$$

where given χ, $\mathcal{E}_1$ and $\mathcal{E}_2$ are i.i.d. point processes distributed as $\mathcal{E}$ conditioned on $\{\max \mathcal{E} \le \chi\}$.

Remark 1.3. The finiteness of the constant $C^{(1)}$ defined above can be checked using that $w(z) \sim Ce^{\sqrt{2}\gamma z}$ as $z \to -\infty$ (see e.g. [ABK11]).

We now consider the case $\alpha < -\gamma$. In this setting, the total number of particles in the process at time t remains tight, allowing the following description of the process, conditioned on the large deviations event.

Theorem 1.4. *Assume that $\alpha \in (-\infty, -\gamma)$. Then, as $t \to \infty$, we have*

$$\mathbb{P}(M_t \le \sqrt{2}\alpha t) \sim \frac{\Phi(\alpha)}{\sqrt{4\pi}} t^{-\frac{1}{2}} e^{-(1+\alpha^2)t}. \tag{1.10}$$

Moreover, conditioned on $\{M_t \le \sqrt{2}\alpha t\}$,

$$\left(t - t \wedge \tau, \sqrt{2}\alpha t - X_\emptyset(t \wedge \tau), M_t - \sqrt{2}\alpha t\right) \Longrightarrow (\xi_\alpha, -\chi_\alpha, -E_\alpha), \tag{1.11}$$

where ξ_α is distributed as

$$\frac{1}{-\alpha\Phi(\alpha)} \delta_0(\mathrm{d}s) + \frac{1}{\Phi(\alpha)} \int_{\mathbb{R}} e^{\sqrt{2}\alpha z + (1-\alpha^2)s} u(z,s)^2 \mathrm{d}z \mathrm{d}s,$$

E_α is distributed as an exponential random variable with parameter $-\sqrt{2}\alpha$, and the joint distribution of $(\xi_\alpha, \chi_\alpha, E_\alpha)$ is given by

$$\begin{aligned}\mathbb{P}(\xi_\alpha \le x_1, \chi_\alpha \le x_2, E_\alpha \ge x_3) \\ = \frac{1}{\Phi(\alpha)}\Big(\mathbb{1}_{\{x_3 < x_2\}} \int_{x_3}^{x_2} \sqrt{2} e^{\sqrt{2}\alpha z} \mathrm{d}z \\ + \sqrt{2}\int_0^{x_1} \mathrm{d}s \int_{-\infty}^{x_2 - x_3} e^{\sqrt{2}\alpha(x_3+z)+(1-\alpha^2)s} u(z,s)^2 \mathrm{d}z\Big),\end{aligned}$$

for any $x_1, x_3 \in \mathbb{R}_+$ and $x_2 \in \mathbb{R}$. Further, we have jointly

$$\mathcal{E}_t(\alpha) \Longrightarrow \mathcal{E}_\infty(\alpha) := \delta_{-\chi_\alpha} \mathbb{1}_{\{\xi_\alpha = 0\}} + \mathbb{1}_{\{\xi_\alpha > 0\}} \sum_{x \in \mathcal{B}_1 \cup \mathcal{B}_2} \delta_{x - \chi_\alpha}, \tag{1.12}$$

where given $(\xi_\alpha, \chi_\alpha)$, $\mathcal{B}_1$ and $\mathcal{B}_2$ are i.i.d. copies of $\sum_{u \in N(\xi_\alpha)} \delta_{X_u(\xi_\alpha)}$ conditioned on $\{M_{\xi_\alpha} \le \chi_\alpha\}$.

Remark 1.5. Recall that $\Phi(\alpha)$ is defined in (1.6). Observe that the law of $t - t \wedge \tau$ has a Dirac mass at 0, corresponding to the probability that no branching occurs in the time interval $[0, t]$.

Theorems 1.2 and 1.4 verify the conjecture of Derrida and Shi, and as expected in the heuristics, the behaviour of the conditioned process is very different depending on the sign of $\alpha + \gamma$. We end with a description in the boundary case $\alpha = -\gamma$, which except for the asymmetric fluctuations on the time τ is similar to the case $\alpha > -\gamma$.

Theorem 1.6. *If $\alpha = -\gamma$, as $t \to \infty$, we have*

$$\mathbb{P}(M_t \le \sqrt{2}\alpha t) \sim C^{(2)} t^{3\gamma/4} e^{-(1+\gamma^2)t}, \tag{1.13}$$

where

$$C^{(2)} := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}_+} u^{3\gamma/2} e^{-2u^2} \mathrm{d}u \int_{\mathbb{R}} e^{-\sqrt{2}\gamma z} w(z)^2 \mathrm{d}z = \frac{C^{(1)} \Gamma(\frac{3\sqrt{2}-1}{4})}{\sqrt{2\pi} 2^{\frac{3\sqrt{2}-1}{4}}}.$$

Moreover, conditioned on $\{M_t \le \sqrt{2}\alpha t\}$,

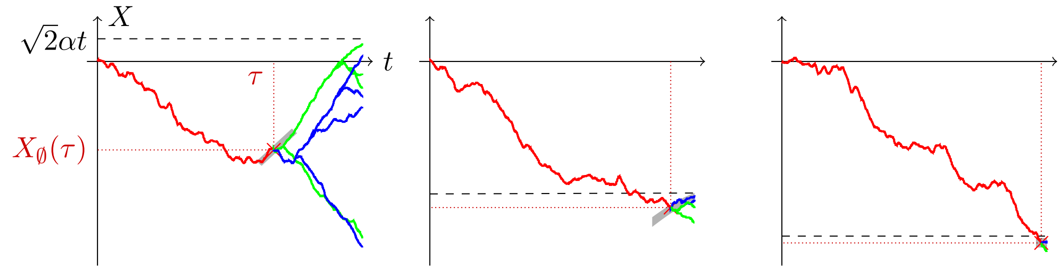
$$\left(\frac{t-\tau}{\sqrt{t}}, X_\emptyset(\tau) - (\sqrt{2}\alpha t - m_{t-\tau}), M_t - \sqrt{2}\alpha t\right) \Longrightarrow (\xi_\alpha, -\chi, -E), \tag{1.14}$$

where ξ_α and (χ, E) are independent, (χ, E) have same law as in Theorem 1.2 and ξ_α is a positive random variable with density $2^{-3(\sqrt{2}+1)/4}\Gamma((3\sqrt{2}-1)/4) u^{3\gamma/2} e^{-2u^2} \mathrm{d}u$. Further, we have jointly

$$\mathcal{E}_t(\alpha) \Longrightarrow \mathcal{E}^-, \tag{1.15}$$

where $\mathcal{E}^-$ is the same as in Theorem 1.2.

We draw, in Figure 2, schemes of the expected behaviour of the branching Brownian motion conditioned to stay below $\sqrt{2}\alpha t$ if $\alpha > -\gamma$ (Theorem 1.2), $\alpha = -\gamma$ (Theorem 1.6), or $\alpha < -\gamma$ (Theorem 1.4).



(a) Theorem 1.2. (b) Theorem 1.6. (c) Theorem 1.4

Figure 2: Scheme of the first branching time in different conditioning scenarios. The initial particle is drawn in red, its two offspring giving birth to the green and blue subtrees respectively. The typical branching zone is drawn as a grey area. Its width is of order $t^{1/2}$ and its height of order 1 in cases (a) and (b).

Remark 1.7. In fact, we could look closer around the phase transition $\alpha_c = -\gamma$ and obtain the following results by a straightforward adaptation of the reasoning used in Section 5. We leave the proof to interested readers. Let $a : \mathbb{R}_+ \to \mathbb{R}$ with $a_t = o(t)$.

1. If $a_t = o(\sqrt{t})$, then

$$\mathbb{P}(M_t \leq -\sqrt{2}\gamma t + a_t) \sim C^{(2)} t^{3\gamma/4} e^{-2\sqrt{2}\gamma t + \sqrt{2}\gamma a_t}. \tag{1.16}$$

2. If $a_t = a\sqrt{t}$ with $a \in \mathbb{R}$, there exists a positive function $a \mapsto C(a)$ such that

$$\mathbb{P}(M_t \leq -\sqrt{2}\gamma t + a_t) \sim C(a) t^{3\gamma/4} e^{-2\sqrt{2}\gamma t + \sqrt{2}\gamma a_t}. \tag{1.17}$$

3. If $\lim_{t\to\infty} \frac{a_t}{\sqrt{t}} = \infty$ and $a_t = o(t)$, then there exist $C^{(3)}, C^{(4)} > 0$ such that

$$\mathbb{P}(M_t \leq -\sqrt{2}\gamma t + a_t) \sim C^{(3)} a_t^{3\gamma/2} e^{-2\sqrt{2}\gamma t + \sqrt{2}\gamma a_t}, \tag{1.18}$$

$$\mathbb{P}(M_t \leq -\sqrt{2}\gamma t - a_t) \sim C^{(4)} (t/a_t)^{3\gamma/2+1} t^{-1/2} e^{-2\sqrt{2}t - \sqrt{2}\gamma a_t - \frac{a_t^2}{4t}}. \tag{1.19}$$

Finally, we consider the lower moderate deviations for the maximum, i.e. the asymptotic behaviour of the probability of the event $\{M_t \leq m_t - a_t\}$ where $\lim_{t\to\infty} a_t = \infty$ and $a_t = o(t)$. As expected from the heuristic, in that case the first branching time happens at a time of order a_t, and the process after that first branching time is a branching Brownian motion conditioned on an event of positive probability. More precisely, the following result holds.

Theorem 1.8. *If $a_t = o(t)$ and $\lim_{t\to\infty} a_t = \infty$, then as $t \to \infty$,*

$$\mathbb{P}(M_t \leq m_t - a_t) \sim C^{(1)} e^{-\sqrt{2}\gamma a_t}. \tag{1.20}$$

Moreover, conditioned on $\{M_t \le m_t - a_t\}$*,*

$$\left(\frac{\tau - \frac{1}{2}a_t}{\sqrt{a_t/8}}, X_\emptyset(\tau) - (\sqrt{2}\tau - a_t), M_t - (m_t - a_t)\right) \Longrightarrow (\xi, -\chi, -E), \tag{1.21}$$

and jointly,

$$\sum_{u \in N(t)} \delta_{X_u(t) - (m_t - a_t)} \Longrightarrow \mathcal{E}^-, \tag{1.22}$$

where $(\xi, \chi, E, \mathcal{E}^-)$ *is the same as in Theorem 1.2.*

Note that in this theorem, (1.20) is already known in the literature (see [ABK13]); our contribution consists in the joint convergence in distribution described in (1.21–1.22).

The main idea behind the proof of all these results is the decomposition of the branching Brownian motion at its first branching point. More precisely, the cumulative distribution function of the maximal displacement, defined as $u(z, s) = \mathbb{P}(M_s \le z)$, for all $s \ge 0$ and $z \in \mathbb{R}$, satisfies

$$u(z, t) = e^{-t}\mathbb{P}(B_t \le z) + \int_0^t \mathrm{d}s \int_{\mathbb{R}} \mathbb{P}(B_s \in \mathrm{d}y) e^{-s} u(z - y, t - s)^2, \tag{1.23}$$

where $(B_t)_{t \ge 0}$ is a standard Brownian motion. This formula allows us to bootstrap close to optimal bounds on $u(\sqrt{2}t - a_t, t)$ from a priori bounds, using Laplace's method (see e.g. [DZ98, Chapter 4]). This allows us to obtain equivalents for different regimes as $t, a_t \to \infty$.

Observe that (1.23) is a simple consequence of the Markov property applied at the first branching time of the branching Brownian motion. Indeed, at time t, the original ancestor did not split with probability e^{-t}, in which case its position is distributed as a Gaussian random variable with variance t. Otherwise, the ancestor died at time s with probability $e^{-s}\mathrm{d}s$, in which case the maximum of the branching Brownian motion at time t has the same law as the maximum of two independent branching Brownian motions at time $t - s$, shifted by the position of the ancestor at time s, which is distributed as B_s.

We use (1.23) to show that with high probability, conditioned on $\{M_t \le \sqrt{2}t - a_t\}$, the first branching time has to happen at some specific time and position with high probability. Depending on the growth rate of a_t, this first branching time can be either $o(t)$ (Theorem 1.8), of order $\lambda t + O(t^{1/2})$ for some $\lambda \in [0, 1]$ (Theorem 1.2), or of order $t - o(t)$ (Theorems 1.4 and 1.6).

The rest of the paper is organized as follows. In Section 2, we state some well-known results on branching Brownian motion and show some rough bounds of $u(z, t)$. In Section 3, we treat the case where $\alpha \in (-\gamma, 1)$ and prove Theorem 1.2 and Theorem 1.8 in that context. Section 4 is devoted to proving Theorem 1.4. In Section 5, we consider the critical case and prove Theorem 1.6. The proofs of some technical lemmas are postponed to Appendix A.

In this paper, we write $f(t) \sim g(t)$ as $t \to \infty$ to denote $\lim_{t \to \infty} \frac{f(t)}{g(t)} = 1$. As usual, $f(t) = o_t(g(t))$ means $\lim_{t \to \infty} \frac{f(t)}{g(t)} = 0$. The quantities $(C_i)_{i \in \mathbb{N}}$ and $(c_i)_{i \in \mathbb{N}}$ represent positive constants, and c, C are non-specified positive constants, that might change from line to line, taken respectively small enough and large enough.

2 Preliminary results and well-known facts

In this section, we recall previously known results on the maximum and extremal process of branching Brownian motions. We source most of the results stated here from the book of Bovier [Bov17] for convenience, and refer the reader to it for the origins of these results. Using these results, we obtain first order estimates on $u(z,t) = \mathbb{P}(M_t \le z)$.

Let $\mathcal{C}_c^+(\mathbb{R})$ be the set of non-negative continuous functions on $\mathbb{R}$ with compact support. We begin by recalling that by [Bov17, Proposition 2.22], for any family $(\mathcal{D}_t)_{t\in[0,\infty]}$ of point processes, the joint convergence in law of $(\mathcal{D}_t, \max \mathcal{D}_t)$ to $(\mathcal{D}_\infty, \max \mathcal{D}_\infty)$ is equivalent to

$$\forall \phi \in \mathcal{C}_c^+(\mathbb{R}),\ \forall z \in \mathbb{R},\ \lim_{t\to\infty} \mathbb{E}\left[e^{-\int \phi \mathrm{d}\mathcal{D}_t}; \max \mathcal{D}_t \le z\right] = \mathbb{E}\left[e^{-\int \phi \mathrm{d}\mathcal{D}_\infty}; \max \mathcal{D}_\infty \le z\right], \tag{2.1}$$

writing $\mathbb{E}[X; A]$ for $\mathbb{E}[X \mathbb{1}_A]$, with X a random variable and A a measurable event.

For all $\phi \in \mathcal{C}_c^+(\mathbb{R})$, we denote by

$$u_\phi : (z,t) \in \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{E}\left[e^{-\sum_{u\in N(t)} \phi(X_u(t)-z)}; M_t \le z\right] = \mathbb{E}\left[\prod_{u\in N(t)} f_\phi(z - X_u(t))\right], \tag{2.2}$$

where $f_\phi : y \mapsto e^{-\phi(-y)}\mathbb{1}_{\{y\ge 0\}}$. Recall that u_ϕ is the unique solution of the F-KPP partial differential equation (1.1) with initial condition f_ϕ, i.e.

$$\begin{cases} \partial_t u = \frac{1}{2}\Delta u - u(1-u), \\ u_\phi(z,0) = f_\phi(z), \text{ for all } z \in \mathbb{R}. \end{cases} \tag{2.3}$$

We remark that the cumulative distribution function of M_t is given by $u(z,t) = u_0(z,t)$.

By (2.1), the joint convergence in law of the centred extremal process and maximal displacement of the branching Brownian motion can be rewritten as the following pointwise convergence

$$\forall \phi \in \mathcal{C}_c^+(\mathbb{R}), \forall z \in \mathbb{R}, \lim_{t\to\infty} u_\phi(m_t + z, t) = w_\phi(z), \tag{2.4}$$

where $w_\phi(z) := \mathbb{E}\left[e^{-\int \phi(\cdot - z)\mathrm{d}\mathcal{E}}; \max \mathcal{E} \le z\right]$.

Moreover, convergence (2.4) in fact holds uniformly on compact sets, by [Bov17, Lemma 5.5 and Theorem 5.9]. Let $K > 0$. As $\min_{z\in[-K,K]} w_\phi(z) > 0$, this uniform convergence result implies that

$$\lim_{t\to\infty} \sup_{|z|\le K} \frac{|u_\phi(m_t + z, t) - w_\phi(z)|}{w_\phi(z)} = 0. \tag{2.5}$$

Applying the above result to the function $\phi \equiv 0$ gives that uniformly on $z \in [-K, K]$, $u(m_t + z, t) = w(z)(1 + o(1))$ as $t \to \infty$.

Let $(B_t, t \geq 0)$ be a standard Brownian motion. We recall the following classical asymptotic on the tail of the standard Gaussian variable (see e.g. [Bov17, Lemma 1.1]). For any $z > 0$,

$$\frac{1}{z\sqrt{2\pi}} e^{-z^2/2}(1 - 2z^{-2}) \leq \mathbb{P}(B_1 > z) \leq \frac{1}{z\sqrt{2\pi}} e^{-z^2/2}. \tag{2.6}$$

It follows that for any $z > 0$ and $t > 0$,

$$\int_z^{\infty} \frac{e^{-\frac{y^2}{2t}}}{\sqrt{2\pi t}} \mathrm{d}z = \int_{-\infty}^{-z} \frac{e^{-\frac{y^2}{2t}}}{\sqrt{2\pi t}} \mathrm{d}z = \mathbb{P}(B_t > z) \leq \frac{\sqrt{t}}{z\sqrt{2\pi}} e^{-\frac{z^2}{2t}}. \tag{2.7}$$

Observe that (2.5) gives tight bounds on $u(z,t)$ for z in a neighbourhood of m_t. We use the above equation (2.7) to give cruder bounds on $u(z,t)$ outside of this neighbourhood. At time t, the system contains $\#N(t) \geq 1$ individuals, the positions of which are distributed with the same law as B_t. Therefore, for any $t \geq 0$ and $z \in \mathbb{R}$,

$$u(z,t) = \mathbb{P}(M_t \leq z) \leq \mathbb{P}(B_t \leq z),$$

which using (2.7), yields for $z < 0$:

$$u(z,t) \leq \frac{\sqrt{t}}{-z\sqrt{2\pi}} e^{-\frac{z^2}{2t}}. \tag{2.8}$$

This straightforward upper bound, combined with the lower deviation results of Derrida and Shi [DS17b] gives us the following lemma.

Lemma 2.1. *For any $\beta \geq 1$ and $\varepsilon > 0$, there exists $t_{\varepsilon,\beta} > 1$ such that for any $t \geq t_{\varepsilon,\beta}$,*

$$u(\sqrt{2}at, t) \leq \begin{cases} 1, & \text{if } a \geq 1; \\ e^{-2\gamma(1-a)t + \varepsilon t}, & \text{if } -\gamma \leq a < 1; \\ e^{-(1+a^2)t + \varepsilon t}, & \text{if } -\beta \leq a < -\gamma; \\ e^{-a^2 t}, & \text{if } a < -\beta. \end{cases} \tag{2.9}$$

Proof. Let $\beta \geq 1$. We begin by noting that $u(z,t) \leq 1$ for any $z \in \mathbb{R}$ and $t \geq 0$. Additionally, by (2.8), for any $a < -1$, we have

$$u(\sqrt{2}at, t) \leq \frac{1}{-2a\sqrt{\pi t}} e^{-a^2 t} \leq e^{-a^2 t}, \tag{2.10}$$

for all $t \geq 1$. To complete the proof, it is therefore enough to bound $u(\sqrt{2}at, t)$ for $a \in [-\beta, 1)$.

We first reformulate Derrida and Shi's result [DS17b, Theorem 1] as follows:

$$\lim_{t\to\infty} \frac{1}{t} \log u(\sqrt{2}at, t) = \psi(a) := \begin{cases} 0, & \text{if } a \geq 1; \\ -2\gamma(1-a), & \text{if } -\gamma \leq a < 1; \\ -(1+a^2), & \text{if } a < -\gamma. \end{cases} \tag{2.11}$$

Note that being a cumulative distribution function for any $t \geq 0$, the function $z \mapsto u(z,t)$ is non-decreasing. Thus, both $\frac{\log u(\sqrt{2}at,t)}{t}$ and $\psi(a)$ are non-decreasing in $a \in \mathbb{R}$, and moreover ψ is continuous. By Dini's theorem, the convergence in (2.11) holds uniformly on any compact sets in $\mathbb{R}$, hence in particular on $[-\beta, 1]$. As a result, for all $\varepsilon > 0$,there exists $t_{\varepsilon,\beta} > 1$ such that for all $t \geq t_{\varepsilon,\beta}$, we have

$$\sup_{a\in[-\beta,1]} \left| \frac{1}{t} \log u(\sqrt{2}at, t) - \psi(a) \right| \leq \varepsilon. \tag{2.12}$$

We then deduce (2.9) from (2.12) and (2.10). □

Next, we recall [CH20, Theorem 1.7], that gives a tight estimate on the moderate lower deviations of the maximal displacement: for any sequence (a_t) such that $\lim_{t\to\infty} a_t = \infty$ and $a_t = o(t)$,

$$\mathbb{P}(M_t \leq m_t - a_t) = e^{-\sqrt{2}(\gamma+o_t(1))a_t}.$$

To complete this section, we strengthen the above estimate into the following non-asymptotic upper bound for $u(m_t - z, t)$.

Lemma 2.2. *For any $\delta \in (0,1)$, there exist $K_\delta \geq 1$ and $T_\delta \geq 1$, such that for any $t \geq T_\delta$ and any $z \geq K_\delta$,*

$$u(m_t - z, t) = \mathbb{P}(M_t \leq m_t - z) \leq c_\delta e^{-\sqrt{2}\gamma(1-\delta)z}, \tag{2.13}$$

with $c_\delta > 1$ a constant depending on δ.

The idea of the proof of this result is mainly borrowed from the proof of Theorem 3.2 (Case 2) in [GH18]. We apply the Markov property at some intermediate time, and observe that either there is an anomalously small number of particles alive at that time, or all of the particles alive at that time must satisfy Lemma 2.1. The detailed proof is postponed to Appendix A.1.

3 The case $-\gamma < \alpha < 1$: proof of Theorems 1.2 and 1.8

In this section, we treat the case when $1 - \sqrt{2} < \alpha < 1$ and prove Theorem 1.2. The proof of Theorem 1.8, which can be though of as $\alpha = 1 - o_t(1)$ can be obtained in a very similar fashion. We thus feel free to omit it.

Recall that if $-\gamma < \alpha < 1$, we expect the existence of $\lambda_\alpha \in (0,1)$ such that with high probability the first branching time of the branching Brownian motion conditioned on the event $\{M_t \leq \sqrt{2}\alpha t\}$ is close to $\lambda_\alpha t$. In this situation, we have $\lambda_\alpha = \frac{1-\alpha}{\sqrt{2}}$.

Let $\phi \in \mathcal{C}_c^+(\mathbb{R})$. Applying the Markov property at the first branching time τ, we obtain that u_ϕ satisfies (1.23) as well:

$$\begin{aligned} u_\phi(z,t) &= e^{-t}\,\mathbb{E}\left(e^{-\phi(B_t - z)}; B_t \leq z\right) + \int_0^t e^{-s}\mathrm{d}s \int_{\mathbb{R}} \mathbb{P}(B_s \in \mathrm{d}y) u_\phi(z-y, t-s)^2 &(3.1)\\ &=: U_1^\phi(z,t) + U_2^\phi(z,t). &(3.2)\end{aligned}$$

Note that $U_1^\phi(z,t)$ is the contribution to $u_\phi(z,t) = \mathbb{E}\left(e^{-\sum_{u\in N(t)}\phi(X_u(t)-z)}; M_t \leq z\right)$, which comes from the event $\{\tau > t\}$ on which no branching occurred. As we write u for u_0, we denote by U_1 and U_2 the quantities defined above with $\phi \equiv 0$.

Using standard computations for the Brownian motion, we first give an uniform estimate of $U_1^\phi(\sqrt{2}\alpha t, t)$ for $\alpha \in [0,1)$, as well as an exact asymptotic for $\alpha < 0$.

Lemma 3.1. *Let $\phi \in \mathcal{C}_c^+(\mathbb{R})$. For $\alpha \geq 0$, we have*

$$\frac{e^{-\|\phi\|_\infty}}{2}e^{-t} \leq U_1^\phi(\sqrt{2}\alpha t, t) \leq e^{-t}. \tag{3.3}$$

Moreover, for $\alpha < 0$, we have

$$\lim_{t\to\infty} \sqrt{t}e^{(1+\alpha^2)t}U_1^\phi(\sqrt{2}\alpha t, t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^0 e^{-\phi(y)-\sqrt{2}\alpha y}\mathrm{d}y. \tag{3.4}$$

In particular, for $\alpha < 0$ we have $U_1(\sqrt{2}\alpha t, t) \sim \frac{e^{-(1+\alpha^2)t}}{\sqrt{4\pi t}|\alpha|}$ as $t \to \infty$.

Proof. We have $U_1^\phi(\sqrt{2}\alpha t, t) = e^{-t}\,\mathbb{E}(e^{-\phi(B_t-\sqrt{2}\alpha t)}; B_t \leq \sqrt{2}\alpha t)$. For all $\alpha \geq 0$, as ϕ is non-negative, we have $1 \geq \mathbb{E}(e^{-\phi(B_t-\sqrt{2}\alpha t)}; B_t \leq \sqrt{2}\alpha t) \geq e^{-\|\phi\|_\infty}\mathbb{P}(B_t \leq 0)$, which is enough to prove (3.3).

Additionally, for $\alpha < 0$, by Girsanov transform, we then have

$$\begin{aligned} U_1^\phi(\sqrt{2}\alpha t, t) &= e^{-(1+\alpha^2)t}\,\mathbb{E}\left(e^{-\phi(B_t)-\sqrt{2}\alpha B_t}; B_t \leq 0\right) \\ &= \frac{e^{-(1+\alpha^2)t}}{\sqrt{2\pi t}}\int_{-\infty}^0 e^{-y^2/2t-\phi(y)-\sqrt{2}\alpha y}\mathrm{d}y. \end{aligned}$$

ùthe dominated convergence theorem yields

$$\lim_{t\to\infty}\int_{-\infty}^0 e^{-y^2/2t-\phi(y)-\sqrt{2}\alpha y}\mathrm{d}y = \int_{-\infty}^0 e^{-\phi(y)-\sqrt{2}\alpha y}\mathrm{d}y,$$

which completes the proof. □

Remark 3.2. For all $\alpha \in (-\gamma, 1)$, we have $u(\sqrt{2}\alpha t, t) = e^{-2\gamma(1-\alpha)t(1+o_t(1))}$ by [DS17b]. Moreover, observe that

$$2\gamma(1-\alpha) < \begin{cases} 1, & \text{if } 0 \leq \alpha < 1; \\ 1+\alpha^2, & \text{if } -\gamma < \alpha < 0. \end{cases}$$

As a result, Lemma 3.1 shows that $U_1(\sqrt{2}\alpha t, t) = o_t(1)u(\sqrt{2}\alpha t, t)$ for all $\alpha \in (-\gamma, 1)$, which by (3.2) implies that $u(\sqrt{2}\alpha t, t) \sim U_2(\sqrt{2}\alpha t, t)$ as $t \to \infty$.

We thus turn to study $U_2^\phi(\sqrt{2}\alpha t, t)$, which is by definition

$$U_2^\phi(\sqrt{2}\alpha t, t) = \int_0^t \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}z \frac{e^{-(t-s)-\frac{(m_s+z-\sqrt{2}\alpha t)^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}} u_\phi(m_s+z,s)^2.$$

Recalling that $v_\alpha = \frac{\gamma+\alpha}{\sqrt{2}} = 1 - \frac{1-\alpha}{\sqrt{2}} \in (0,1)$, our next lemma consists in the observation that most of the mass on this double integral is carried by $\{(s,z) : |s - v_\alpha t| \leq A\sqrt{t}, |z - m_s| \leq K\}$, with A, K large enough constants. This is consistent with Theorem 1.2, and can be thought of as a proof of the tightness of the family of variables

$$\left\{(t^{-1/2}(\tau - (1-v_\alpha)t), X_\emptyset(\tau) - (\sqrt{2}\alpha t - m_{t-\tau}), M_t - \sqrt{2}\alpha t, \mathcal{E}_t), t \geq 0\right\}.$$

For any Borel sets $I \subset [0,t]$ and $B \subset \mathbb{R}$, let

$$U_2^\phi(\sqrt{2}\alpha t, t, I, B) := \int_I \mathrm{d}s \int_B \mathrm{d}z \frac{e^{-(t-s)-\frac{(m_s+z-\sqrt{2}\alpha t)^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}} u_\phi(m_s+z,s)^2.$$

Lemma 3.3. *Let $\alpha \in (-\gamma, 1)$, we set $I_{t,A} = \left[v_\alpha t - A\sqrt{t}, v_\alpha t + A\sqrt{t}\right] \cap [0,t]$ for all $A, t > 0$. For all $\phi \in \mathcal{C}_c^+(\mathbb{R})$, we have*

$$\limsup_{K\to\infty} \limsup_{t\to\infty} \frac{e^{2\gamma(1-\alpha)t}}{t^{3\gamma/2}} \left[U_2^\phi(\sqrt{2}\alpha t, t) - U_2^\phi(\sqrt{2}\alpha t, t, I_{t,A}, [-K,K])\right] = o_A(1). \tag{3.5}$$

The proof of Lemma 3.3 is postponed to Appendix A.2. A consequence of this result is that the asymptotic behaviour of $U_2^\phi(\sqrt{2}\alpha t, t)$ as $t \to \infty$ is captured by the following lemma, that is used to complete the proof of Theorem 1.2.

Lemma 3.4. *Let $\alpha \in (-\gamma, 1)$, we set $I_{t,a,b} = \left[v_\alpha t + a\sqrt{t}, v_\alpha t + b\sqrt{t}\right] \cap [0,t]$ for all $a < b \in \mathbb{R}$. Then for all $a < b$ and $a' < b'$, we have*

$$\lim_{t\to\infty} \frac{e^{2\gamma(1-\alpha)t}}{(v_\alpha t)^{3\gamma/2}} U_2(\sqrt{2}\alpha t, t, I_{t,a,b}, [a',b']) = \int_a^b \frac{e^{-\frac{2\sqrt{2}r^2}{1-\alpha}}}{\sqrt{2\pi\frac{1-\alpha}{\sqrt{2}}}} \mathrm{d}r \int_{a'}^{b'} e^{-\sqrt{2}\gamma} w_\phi(z)^2 \mathrm{d}z. \tag{3.6}$$

Proof. Recall that we can write

$$U_2^\phi(\sqrt{2}\alpha t, t, I_{t,a,b}, [a',b']) = \int_{v_\alpha t + a\sqrt{t}}^{v_\alpha t + b\sqrt{t}} \mathrm{d}s \frac{e^{-(t-s)}}{\sqrt{2\pi(t-s)}} \int_{a'}^{b'} e^{-\frac{(\sqrt{2}\alpha t - m_s - z)^2}{2(t-s)}} u_\phi(m_s+z,s)^2 \mathrm{d}z.$$

By the uniform convergence (2.5), we observe that uniformly in $s \in I_{t,a,b}$,

$$\int_{a'}^{b'} e^{-\frac{(\sqrt{2}\alpha t - m_s - z)^2}{2(t-s)}} u_\phi(m_s+z,s)^2 \mathrm{d}z \sim \int_{a'}^{b'} e^{-\frac{(\sqrt{2}\alpha t - m_s - z)^2}{2(t-s)}} w_\phi(z)^2 \mathrm{d}z,$$

as $t\to\infty$. Then, with the change of variable $s=ut$, we have

$$U_2(\sqrt{2}\alpha t, t, I_{t,a,b}, [a',b'])$$
$$\sim \int_{v_\alpha+\frac{a}{\sqrt{t}}}^{v_\alpha+\frac{b}{\sqrt{t}}} \mathrm{d}u \frac{te^{-tg_\alpha(u)}}{\sqrt{2\pi t(1-u)}} \int_{a'}^{b'} e^{\frac{u-\alpha}{1-u}(\frac{3}{2}\log(ut)-\sqrt{2}z)-\frac{\left(\frac{3}{2\sqrt{2}}\log(ut)-z\right)^2}{2t(1-u)}} w_\phi(z)^2\mathrm{d}z,$$

as $t\to\infty$, by setting

$$g_\alpha(u) : u\in(0,1)\mapsto (1-u)+\frac{(\alpha-u)^2}{1-u}. \tag{3.7}$$

Note that uniformly in $z\in[a',b']$ and in $u\in[v_\alpha+\frac{a}{\sqrt{t}}, v_\alpha+\frac{b}{\sqrt{t}}]$, as $t\to\infty$ we have

$$\frac{\left(\frac{3}{2\sqrt{t}}\log(ut)-z\right)^2}{2t(1-u)} = o_t(1)$$
$$\text{and}\quad \frac{u-a'}{1-u}\left(\frac{3}{2}\log(ut)-\sqrt{2}z\right) = \frac{3\gamma}{2}\log(v_\alpha t)-\sqrt{2}\gamma z+o_t(1).$$

It then follows that as $t\to\infty$,

$$U_2(\sqrt{2}\alpha t, t, I_{t,a,b}, [a',b']) \sim \frac{(v_\alpha t)^{3\gamma/2}}{\sqrt{2\pi(1-v_\alpha)}} \int_{v_\alpha+\frac{a}{\sqrt{t}}}^{v_\alpha+\frac{b}{\sqrt{t}}} e^{-tg_\alpha(u)}\sqrt{t}\mathrm{d}u \int_{a'}^{b'} e^{-\sqrt{2}\gamma z} w_\phi(z)^2\mathrm{d}z. \tag{3.8}$$

We estimate that quantity by doing an asymptotic expansion of g_α around v_α.

By change of variable $r=\sqrt{t}(u-v_\alpha)$, we have

$$\int_{v_\alpha+\frac{a}{\sqrt{t}}}^{v_\alpha+\frac{b}{\sqrt{t}}} e^{-tg_\alpha(u)}\sqrt{t}\mathrm{d}u = \int_a^b e^{-tg_\alpha(v_\alpha+\frac{r}{\sqrt{t}})}\mathrm{d}r.$$

We note that g_α is smooth and strictly convex, and attains its minimum of $2\gamma(1-\alpha)$ at $u=v_\alpha$. By Taylor's expansion at v_α, we have as $|h|\downarrow 0$,

$$g_\alpha(v_\alpha+h)-g_\alpha(v_\alpha) = g'(v_\alpha)h+\frac{1}{2}g''(v_\alpha)h^2+o(h^2) = \frac{2\sqrt{2}}{1-\alpha}h^2+o(h^2). \tag{3.9}$$

Hence,

$$\int_{v_\alpha+\frac{a}{\sqrt{t}}}^{v_\alpha+\frac{b}{\sqrt{t}}} e^{-tg_\alpha(u)}\sqrt{t}\mathrm{d}u = e^{-2\gamma(1-\alpha)t}\int_a^b e^{-\frac{2\sqrt{2}}{1-\alpha}r^2+o_t(1)}\mathrm{d}r \sim e^{-2\gamma(1-\alpha)t}\int_a^b e^{-\frac{2\sqrt{2}}{1-\alpha}r^2}\mathrm{d}r,$$

as $t\to\infty$ by dominated convergence. In view of (3.8), this is enough to complete the proof of (3.6). $\square$

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. For all $\phi \in \mathcal{C}_c^+(\mathbb{R})$, $x_1, x_2 \in \mathbb{R}$ and $x_3 \geq 0$, we set

$$\begin{aligned} &F_t(\phi; x_1, x_2, x_3) \\ &\quad := \mathbb{E}\left(e^{-\int \phi \mathrm{d}\mathcal{E}_t(\alpha)}; \tfrac{\tau-(1-v_\alpha)t}{\sqrt{t}} \leq x_1, X_\emptyset(\tau) \geq \sqrt{2}\alpha t - m_{t-\tau} - x_2, M_t \leq \sqrt{2}\alpha t - x_3 \right), \end{aligned}$$

and we shall study the asymptotic behaviour of this quantity as $t \to \infty$. Applying the Markov property at time τ, we have

$$F_t(\phi; x_1, x_2, x_3) = U_2^{\tau_{x_3}\phi}\left(\sqrt{2}\alpha t, t, \left[v_\alpha t - x_1\sqrt{t}, t\right], (-\infty, x_2]\right),$$

with $\tau_{x_3}\phi : y \mapsto \phi(y - x_3)$. Therefore, using Lemma 3.3 gives

$$\begin{aligned} &\lim_{t\to\infty} e^{2\gamma(1-\alpha)t}(v_\alpha t)^{-3\gamma/2} F_t(\phi; x_1, x_2, x_3) \\ &\qquad = \int_{-x_1}^{A} \frac{e^{-\frac{2\sqrt{2}r^2}{1-\alpha}}}{\sqrt{2\pi\frac{1-\alpha}{\sqrt{2}}}}\mathrm{d}r \int_{-K}^{x_2} e^{-\sqrt{2}\gamma z} w_\phi(z - x_3)^2 \mathrm{d}z + o_A(1) + o_K(1), \end{aligned}$$

with the $o_A(1)$ term being uniform in K, using that by definition, $w_{\tau_x\phi} = w_\phi(\cdot - x)$. Hence, letting $K \to \infty$ then $A \to \infty$, we conclude that

$$\lim_{t\to\infty} \frac{e^{2\gamma(1-\alpha)t}}{(v_\alpha t)^{3\gamma/2}} F_t(\phi; x_1, x_2, x_3) = \int_{-x_1}^{\infty} \frac{e^{-\frac{2\sqrt{2}r^2}{1-\alpha}}}{\sqrt{2\pi\frac{1-\alpha}{\sqrt{2}}}}\mathrm{d}r \int_{-\infty}^{x_2} e^{-\sqrt{2}\gamma z} w_\phi(z - x_3)^2 \mathrm{d}z. \tag{3.10}$$

Using this result, we can now complete the proof of Theorem 1.2.

We begin by proving (1.7). By (1.23), we have

$$\mathbb{P}(M_t \leq \sqrt{2}\alpha t) = U_1(\sqrt{2}\alpha t, t) + U_2(\sqrt{2}\alpha t, t) = U_2(\sqrt{2}\alpha t, t) + o(t^{3\gamma/2} e^{-(1+\alpha^2)t}),$$

using Lemma 3.1. Applying then Lemma 3.3, for all $A, K > 0$ we have

$$\begin{aligned} &\lim_{t\to\infty} (v_\alpha t)^{-3\gamma/2} e^{(1+\alpha^2)t} \mathbb{P}(M_t \leq \sqrt{2}\alpha t) \\ &\qquad = \lim_{t\to\infty} (v_\alpha t)^{-3\gamma/2} e^{(1+\alpha^2)t} F_t(0; A, K, 0) + o_A(1) + o_K(1). \end{aligned}$$

Hence, letting $K \to \infty$ then $A \to \infty$, by the monotone convergence theorem, (3.10) yields

$$\lim_{t\to\infty} (v_\alpha t)^{-3\gamma/2} e^{(1+\alpha^2)t} \mathbb{P}(M_t \leq \sqrt{2}\alpha t) = \int_{\mathbb{R}} \frac{e^{-\frac{2\sqrt{2}r^2}{1-\alpha}}}{\sqrt{2\pi\frac{1-\alpha}{\sqrt{2}}}}\mathrm{d}r \int_{\mathbb{R}} e^{-\sqrt{2}\gamma z} w(z)^2 \mathrm{d}z = C^{(1)}.$$

We now turn to the proof of (1.8). By (3.10), for all $x_1, x_2 \in \mathbb{R}$ and $x_3 \in \mathbb{R}_+$, we have

$$\begin{aligned} &\mathbb{P}\left(\tau \leq \frac{1-\alpha}{\sqrt{2}} t + x_1\sqrt{t}, X_\emptyset(\tau) \geq \sqrt{2}\alpha t - m_{t-\tau} - x_2, M_t \leq \sqrt{2}\alpha t - x_3 \right) \\ &\quad = F_t(0; x_1, x_2, x_3) \sim (v_\alpha t)^{3\gamma/2} e^{-(1+\alpha^2)t} \int_{-x_1}^{\infty} \frac{e^{-\frac{2\sqrt{2}r^2}{1-\alpha}}}{\sqrt{2\pi\frac{1-\alpha}{\sqrt{2}}}}\mathrm{d}r \int_{-\infty}^{x_2} e^{-\sqrt{2}\gamma z} w(z - x_3)^2 \mathrm{d}z, \end{aligned}$$

which we can rewrite as

$$\lim_{t\to\infty} \frac{F_t(0;x_1,x_2,x_3)}{\mathbb{P}(M_t \le \sqrt{2}\alpha t)} = \frac{1}{2C^{(1)}} \int_{-\infty}^{x_1} \mathrm{d}r \frac{e^{-\frac{2\sqrt{2}r^2}{1-\alpha}}}{\sqrt{2\pi\frac{1-\alpha}{4\sqrt{2}}}} \times e^{-\sqrt{2}\gamma x_3} \int_{-\infty}^{x_2-x_3} e^{-\sqrt{2}\gamma z} w(z)^2 \mathrm{d}z,$$

which completes the proof of (1.8).

We finally turn to the proof of (1.9), i.e. the joint convergence in distribution of the extremal process seen from $\sqrt{2}\alpha t$, conditioned on $\{M_t \le \sqrt{2}\alpha t\}$. By a straightforward adaptation of [Bov17, Proposition 2.2], to obtain this weak convergence, it is enough to obtain for all $\phi \in \mathcal{C}_c^+(\mathbb{R})$ and $x_1, x_2 \in \mathbb{R}$, $x_3 \ge 0$ the convergence

$$\lim_{t\to\infty} \mathbb{E}\left[e^{-\int \phi \mathrm{d}\mathcal{E}_t(\alpha)}; \tau \le (1-v_\alpha)t + x_1\sqrt{t}, X_\emptyset(\tau) \ge \sqrt{2}\alpha t - m_{t-\tau} - x_2 \middle| M_t \le \sqrt{2}\alpha t\right]$$
$$= \mathbb{E}\left[e^{-\int \phi \mathrm{d}\mathcal{E}^-}; \chi \le x_2\right] \mathbb{P}\left(\xi \le x_1\sqrt{\tfrac{4\sqrt{2}}{1-\alpha}}\right).$$

By (3.10) and (1.9), we have immediately that

$$\lim_{t\to\infty} \frac{F(\phi,x_1,x_2,0)}{\mathbb{P}(M_t \le \sqrt{2}\alpha t)} = \frac{1}{2C^{(1)}} \int_{-\infty}^{x_1} \mathrm{d}r \frac{e^{-\frac{2\sqrt{2}r^2}{1-\alpha}}}{\sqrt{2\pi\frac{1-\alpha}{4\sqrt{2}}}} \times \int_{-\infty}^{x_2} e^{-\sqrt{2}\gamma z} w_\phi(z)^2 \mathrm{d}z.$$

Observe that according to the definition of $\mathcal{E}^-$, writing $\mathcal{E}$ for the limiting extremal process of the unconditioned branching Brownian motion, we have

$$\begin{aligned}\mathbb{E}\left[e^{-\int \phi \mathrm{d}\mathcal{E}^-}; \chi \le x_2\right] &= \int_{-\infty}^{x_2} \mathbb{E}\left[e^{-\sum_{x\in\mathcal{E}} \phi(x-z)} \middle| \max \mathcal{E} \le z\right]^2 \mathbb{P}(\chi \in \mathrm{d}z)\\ &= \frac{1}{2C^{(1)}} \int_{-\infty}^{x_2} e^{-\sqrt{2}\gamma z} w_\phi(z)^2 \mathrm{d}z,\end{aligned}$$

which is therefore enough to end the proof. $\square$

Theorem 1.8 is obtained following a similar line of proof as Theorem 1.2. The principal difference is that the Laplace method in the proof of Lemma 3.4 has to be applied with a maximum obtained on the boundary of the interval of definition. All other estimates follow with straightforward modifications, by replacing $1-\alpha$ by $a_t/\sqrt{2}t$.

4 The case $\alpha < -\gamma$: proof of Theorem 1.4

We now treat the case of $\alpha < -\gamma$. We use in this section that, conditioned on the event $\{M_t \le \sqrt{2}\alpha t\}$, with high probability no branching occurs before time $t - O(1)$. We use this observation to prove Theorem 1.4, using the same decomposition of $u_\phi(t,x)$ as $U_1^\phi + U_2^\phi$ as in the previous section. Contrarily to the previous section, U_1^ϕ and U_2^ϕ are of the same order of magnitude.

Note that the asymptotic behaviour of $U_1^\phi(\sqrt{2}\alpha t, t)$ is given by Lemma 3.1. To study the asymptotic behaviour of $U_2^\phi(\sqrt{2}\alpha t, t)$, we begin by showing that $t-\tau$ is tight conditioned on $\{M_t \le \sqrt{2}\alpha t\}$.

Lemma 4.1. *Assume that $\alpha < -\gamma$, then for all $\phi \in \mathcal{C}_c^+(\mathbb{R})$ we have*

$$\lim_{A\to\infty}\lim_{t\to\infty}\sqrt{t}e^{(1+\alpha^2)t}U_2^{\phi}(\sqrt{2}\alpha t, t, [A,t], \mathbb{R}) = 0. \tag{4.1}$$

The proof of Lemma 4.1 is postponed to Appendix A.3. The next lemma completes the description of the asymptotic of U_2^{ϕ}.

Lemma 4.2. *If $\alpha < -\gamma$, then for any $x > 0$ and $-\infty \le c < d \le \infty$ we have*

$$\lim_{t\to\infty}\sqrt{t}e^{(1+\alpha^2)t}U_2^{\phi}(\sqrt{2}\alpha t, t, [0,x], (c,d)) = \int_0^x\int_c^d e^{\sqrt{2}\alpha y}u_\phi(y,s)^2 e^{(1-\alpha^2)s}\mathrm{d}y\mathrm{d}s. \tag{4.2}$$

Moreover, we have

$$\int_0^\infty\int_{\mathbb{R}} e^{(1-\alpha^2)s+\sqrt{2}\alpha y}u(y,s)^2\mathrm{d}y\mathrm{d}s < \infty. \tag{4.3}$$

Proof. Observe that we can rewrite

$$\begin{aligned}
U_2^{\phi}(\sqrt{2}\alpha t, t, [0,x], [c,d]) &= \int_0^x \mathrm{d}s\int_c^d \mathrm{d}y\frac{e^{-(t-s)-\frac{(\sqrt{2}\alpha t - y)^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}}u_\phi(y,s)^2\\
&= \frac{e^{-(1+\alpha^2)t}}{\sqrt{2\pi t}}\int_0^x\mathrm{d}s\int_c^d\mathrm{d}y\sqrt{\frac{t}{t-s}}e^{(1-\alpha^2)s}e^{\sqrt{2}\alpha y-\frac{(\sqrt{2}\alpha s-y)^2}{2(t-s)}}u_\phi(y,s)^2\\
&\sim\frac{e^{-(1+\alpha^2)t}}{\sqrt{2\pi t}}\int_0^x\mathrm{d}s\int_c^d\mathrm{d}y e^{(1-\alpha^2)s}e^{\sqrt{2}\alpha y-\frac{(\sqrt{2}\alpha s-y)^2}{2(t-s)}}u_\phi(y,s)^2,
\end{aligned}$$

as $t\to\infty$. Then, by the monotone convergence theorem, as $t\to\infty$ we have

$$\int_0^x\int_c^d e^{\sqrt{2}\alpha y-\frac{(\sqrt{2}\alpha s-y)^2}{2(t-s)}}u_\phi(y,s)^2e^{(1-\alpha^2)s}\mathrm{d}y\mathrm{d}s \to \int_0^x\int_c^d e^{\sqrt{2}\alpha y}u_\phi(y,s)^2e^{(1-\alpha^2)s}\mathrm{d}y\mathrm{d}s,$$

which completes the proof of (4.2).

The rest of the proof is devoted to show that $\int_0^\infty\int_{\mathbb{R}} e^{(1-\alpha^2)s+\sqrt{2}\alpha y}u(y,s)^2\mathrm{d}s\mathrm{d}y < \infty$. As a first step, we bound for any $s \ge 0$ the quantity $I_s := \int_{\mathbb{R}} e^{\sqrt{2}\alpha y}u(y,s)^2\mathrm{d}y$. First, by (2.8), for all $y \le 0$ we have $0 \le u(y,s) \le \frac{s^{1/2}}{|y|}e^{-y^2/2s}\wedge 1$, therefore

$$I_s \le \int_{-\infty}^1 e^{\sqrt{2}\alpha y}\mathrm{d}y + \int_1^\infty e^{\sqrt{2}\alpha y}e^{-y^2/s}\mathrm{d}y \le \frac{e^{\sqrt{2}\alpha}}{-\sqrt{2}\alpha} + s\sqrt{\pi s}e^{\alpha^2 s},$$

by (2.6). As a result, for all $A > 0$, we have

$$\int_0^A\int_{\mathbb{R}} e^{(1-\alpha^2)s+\sqrt{2}\alpha y}u(y,s)^2\mathrm{d}s\mathrm{d}y = \int_0^A e^{(1-\alpha^2)s}I_s\mathrm{d}s < \infty. \tag{4.4}$$

To complete the proof of (4.3), it is enough to bound $\int_A^\infty\int_{\mathbb{R}} e^{(1-\alpha^2)s+\sqrt{2}\alpha y}u(y,s)^2\mathrm{d}y\mathrm{d}s$ for $A \ge 1$ large enough. Recall that $u(y,s) = \mathbb{P}(M_s \le y)$ is close to 1 for $y \gg \sqrt{2}s$ and to 0 for $y \ll \sqrt{2}s$. Observe that

$$\int_A^\infty \int_{\sqrt{2}s}^\infty e^{(1-\alpha^2)s+\sqrt{2}\alpha y} u(y,s)^2 \mathrm{d}y\mathrm{d}s \leq \int_A^\infty \int_{\sqrt{2}s}^\infty e^{(1-\alpha^2)s+\sqrt{2}\alpha y} \mathrm{d}y\mathrm{d}s$$
$$= \frac{1}{-\sqrt{2}\alpha} \int_A^\infty e^{(1-\alpha^2)s+2\alpha s} \mathrm{d}s < \infty, \tag{4.5}$$

using that for all $\alpha < -\gamma$, $1 - \alpha^2 + 2\alpha < 0$. Therefore, we only need to bound

$$\int_A^\infty \int_{-\infty}^{\sqrt{2}s} e^{(1-\alpha^2)s+\sqrt{2}\alpha y} u(y,s)^2 \mathrm{d}y\mathrm{d}s = \int_A^\infty \int_{-\infty}^{1} e^{(1-\alpha^2)s+2x\alpha s} u(\sqrt{2}xs,s)^2 \sqrt{2}s \mathrm{d}x\mathrm{d}s,$$

by change of variable $y = \sqrt{2}sx$. We now apply Lemma 2.1 to bound $u(\sqrt{2}xs,s)^2$ for s large enough, depending on the region to which x belongs.

Let $\varepsilon > 0$, that will be taken small enough later on, and $\beta > 1-\alpha/2 > 1$. We assume that $A > t_{\varepsilon,\beta}$, and we bound the above integral using (2.9). First, for x in the interval $[-\gamma, 1]$, we have

$$\begin{aligned}
&\int_A^\infty \int_{-\gamma}^{1} e^{(1-\alpha^2)s+2\alpha x s} u(\sqrt{2}xs,s)^2 \sqrt{2}s \mathrm{d}x\mathrm{d}s \\
\leq &\int_A^\infty \int_{-\gamma}^{1} e^{(1-\alpha^2)s+2\alpha x s} e^{-4\gamma(1-x)s+2\varepsilon s} \sqrt{2}s \mathrm{d}x\mathrm{d}s \\
\leq &\int_A^\infty \sqrt{2}s e^{(1-\alpha^2-4\gamma+2\varepsilon)s} \int_{-\gamma}^{1} e^{(2\alpha+4\gamma)xs} \mathrm{d}x\mathrm{d}s \\
\leq &\begin{cases}
\frac{1}{\sqrt{2}(\alpha+2\gamma)} \int_A^\infty e^{(1-\alpha^2+2\alpha)s+2\varepsilon s} \mathrm{d}s, & \text{if } -2\gamma < \alpha < -\gamma; \\
\sqrt{2}(1+\gamma) \int_A^\infty s e^{(1-\alpha^2+2\alpha)s+2\varepsilon s} \mathrm{d}s, & \text{if } \alpha = -2\gamma; \\
\frac{1}{-\sqrt{2}(\alpha+2\gamma)} \int_A^\infty e^{(1-\alpha^2-4\gamma-2\alpha\gamma-4\gamma^2)s+2\varepsilon s} \mathrm{d}s, & \text{if } \alpha < -2\gamma.
\end{cases}
\end{aligned}$$

As $1 - \alpha^2 - 4\gamma - 2\alpha\gamma - 4\gamma^2 < 0$ for all $\alpha < -2\gamma$, we conclude that for all $\varepsilon > 0$ small enough,

$$\int_A^\infty \int_{-\gamma}^{1} e^{(1-\alpha^2)s+2\alpha x s} u(\sqrt{2}xs,s)^2 \sqrt{2}s \mathrm{d}x\mathrm{d}s < \infty. \tag{4.6}$$

We then consider the case $x \in [-\beta, -\gamma]$. In fact

$$\begin{aligned}
&\int_A^\infty \int_{-\beta}^{-\gamma} e^{(1-\alpha^2)s+2\alpha x s} u(\sqrt{2}xs,s)^2 \sqrt{2}s \mathrm{d}x\mathrm{d}s \\
\leq &\int_A^\infty \int_{-\beta}^{-\gamma} e^{(1-\alpha^2)s+2\alpha x s} e^{-2(1+x^2)s+2\varepsilon s} \sqrt{2}s \mathrm{d}x\mathrm{d}s \\
\leq &\int_A^\infty \sqrt{2}s e^{-(1+\alpha^2/2-2\varepsilon)s} \int_{-\beta}^{-\gamma} e^{-2(x-\alpha/2)^2 s} \mathrm{d}x\mathrm{d}s \\
\leq &\int_A^\infty \sqrt{\frac{\pi s}{2}} e^{-(1+\alpha^2/2-2\varepsilon)s} \mathrm{d}s < \infty,
\end{aligned} \tag{4.7}$$

for all $\varepsilon < 1/2$. Similarly, for $x < -\beta$:

$$\begin{aligned}\int_A^\infty \int_{-\infty}^{-\beta} e^{(1-\alpha^2)s+2\alpha xs} u(\sqrt{2}xs,s)^2 \sqrt{2}s\mathrm{d}x\mathrm{d}s &\le \int_A^\infty \sqrt{2}se^{(1-\alpha^2/2)s} \int_{-\infty}^{-\beta} e^{-2(x-\alpha/2)^2 s}\mathrm{d}x\mathrm{d}s \\ &\le \int_A^\infty \sqrt{2}se^{(1-\alpha^2/2)s} \int_{-\infty}^{-\beta-\alpha/2} e^{-2y^2 s}\mathrm{d}y\mathrm{d}s.\end{aligned}$$

Using that $-\beta - \alpha/2 < -1$, we have for all $s > 0$:

$$\int_{-\infty}^{-\beta-\alpha/2} e^{-2y^2 s}\mathrm{d}y \le \int_1^\infty e^{-2y^2 s}\mathrm{d}y \le \frac{1}{4s}e^{-2s},$$

by (2.7), yielding

$$\int_A^\infty \int_{-\beta}^{-\gamma} e^{(1-\alpha^2)s+2\alpha xs} u(\sqrt{2}xs,s)^2 \sqrt{2}s\mathrm{d}x\mathrm{d}s \le \frac{1}{2\sqrt{2}} \int_A^\infty e^{(-1-\alpha^2/2)s} < \infty. \tag{4.8}$$

Consequently, using (4.5–4.8), for any $A > 0$ large enough

$$\int_A^\infty \int_{-\infty}^{1} e^{(1-\alpha^2)s+2x\alpha s} u^2(\sqrt{2}xs,s)\sqrt{2}s\mathrm{d}x\mathrm{d}s < \infty,$$

which, with (4.4), completes the proof of (4.3). □

We now complete the proof of Theorem 1.4 by proving the joint convergence in law of the first branching time and position, and the shifted extremal process, conditioned on $\{M_t \le \sqrt{2}\alpha t\}$.

Proof of Theorem 1.4. We begin by observing that by Lemma 3.1, we have

$$\begin{aligned}&\mathbb{E}\left(e^{-\int \phi \mathrm{d}\mathcal{E}_t(\alpha)}; \tau \ge t, M_t \le \sqrt{2}\alpha t - z\right) \\ &\qquad = \mathbb{E}\left(e^{-\phi(B_t - \sqrt{2}\alpha t)}; B_t \le \sqrt{2}\alpha t - z\right) \sim \frac{e^{-(1+\alpha^2)t}}{\sqrt{2\pi t}} \int_{-\infty}^{-z} e^{-\phi(y)-\sqrt{2}\alpha y}\mathrm{d}y, \end{aligned}\tag{4.9}$$

as $t \to \infty$. Similarly to the proof of Theorem 1.2, the key to the proof of this theorem is the determination of the asymptotic behaviour of

$$\begin{aligned}&F_t(\phi; x_1, x_2, x_3) \\ &\qquad := \mathbb{E}\left(e^{-\int \phi \mathrm{d}\mathcal{E}_t(\alpha)}; \tau \in [t - x_1, t], X_\emptyset(\tau \wedge t) \le \sqrt{2}\alpha t - x_2, M_t \le \sqrt{2}\alpha t - x_3\right),\end{aligned}$$

as $t \to \infty$.

Using the branching property at time τ, we observe that

$$F_t(\phi; x_1, x_2, x_3) = U_2^{\tau_{x_3}\phi}(\sqrt{2}\alpha t, t, [0, x_1], [x_2, \infty)),$$

and therefore, by Lemma 4.2 we have

$$\lim_{t\to\infty} t^{1/2}e^{(1+\alpha^2)t}F_t(\phi;x_1,x_2,x_3) = \int_0^{x_1} e^{(1-\alpha^2)s}\int_{x_2}^{\infty} e^{\sqrt{2}\alpha y}u_\phi(y-x_3,s)^2\mathrm{d}y\mathrm{d}s. \tag{4.10}$$

We now use this formula to prove Theorem 1.4.

We begin with the proof of (1.10). Observe that by (3.2), we have

$$\mathbb{P}(M_t \leq \sqrt{2}\alpha t) = U_1(\sqrt{2}\alpha t, t) + U_2(\sqrt{2}\alpha t, t).$$

Using (4.10) with $x_2 = -\infty$ and $x_1 = A$ together with Lemma 4.1, we have letting $t \to \infty$ then $A \to \infty$:

$$\lim_{t\to\infty} t^{1/2}e^{(1+\alpha^2)t}U_2(\sqrt{2}\alpha t, t) = \int_0^{\infty} e^{(1-\alpha^2)s}\int_{\mathbb{R}} e^{\sqrt{2}\alpha y}u(y-x_3,s)^2\mathrm{d}y\mathrm{d}s,$$

which, together with (4.9), implies $\mathbb{P}(M_t \leq \sqrt{2}\alpha t) \sim \frac{\Phi(\alpha)}{\sqrt{4\pi t}}e^{-(1+\alpha^2)t}$ as $t \to \infty$.

Then, to prove (1.11), it is enough to observe that

$$\lim_{t\to\infty} \frac{F_t(0;x_1,x_2,x_3)}{\mathbb{P}(M_t \leq \sqrt{2}\alpha t)} = \frac{\sqrt{4\pi}}{\Phi(\alpha)}\int_0^{x_1} e^{(1-\alpha^2)s}e^{\sqrt{2}\alpha x_3}\int_{-\infty}^{x_2-x_3} e^{\sqrt{2}\alpha y}u_\phi(y,s)^2\mathrm{d}y\mathrm{d}s,$$

by (4.10). This proves that $(t - t\wedge\tau, \sqrt{2}\alpha t - X_\emptyset(t\wedge\tau), \sqrt{2}\alpha t - M_t)$ jointly converge in distribution as $t \to \infty$.

We now prove the convergence of the extremal process $\mathcal{E}_t(\alpha)$. For any $\phi \in \mathcal{C}_c^+(\mathbb{R})$, using again the decomposition at first branching time of the branching Brownian motion, we have

$$\mathbb{E}\left[e^{-\int \phi \mathrm{d}\mathcal{E}_t(\alpha)}; M_t \leq \sqrt{2}\alpha t\right] = U_1^\phi(\sqrt{2}\alpha t, t) + U_2^\phi(\sqrt{2}\alpha t, t),$$

which, by (4.10) and (1.10), yields

$$\lim_{t\to\infty} \mathbb{E}\left[e^{-\int \phi \mathrm{d}\mathcal{E}_t(\alpha)}\middle| M_t \leq \sqrt{2}\alpha t\right] = \frac{\sqrt{2}}{\Phi(\alpha)}\left(\int_{-\infty}^0 e^{-\phi(z)-\sqrt{2}\alpha z}\mathrm{d}z + \int_0^\infty \mathrm{d}s\int_{\mathbb{R}} e^{\sqrt{2}\alpha z+(1-\alpha^2)s}u_\phi(z,s)^2\mathrm{d}z\right).$$

As $u(z,s) = \mathbb{E}\left(e^{-\sum_{u\in N(t)}\phi(X_u(s)-z)}; M_s \leq z\right)$, we observe that we can rewrite this limit as

$$\int_{\mathbb{R}_+\times\mathbb{R}} \mathbb{E}\left[\exp\left(-\sum_{u\in N(s)}\phi(X_u(s)-z)\right)\middle| M_s \leq z\right]^2 \mathbb{P}(\xi_\alpha \in \mathrm{d}s, \chi_\alpha \in \mathrm{d}z) = \mathbb{E}[e^{-\int \phi \mathrm{d}\mathcal{E}_\infty(\alpha)}],$$

proving that $\mathcal{E}_t(\alpha)$ converges weakly to $\mathcal{E}_\infty(\alpha)$ in $\mathbb{P}(\cdot|M_t \leq \sqrt{2}\alpha t)$-distribution. In the same spirit as in the proof of Theorem 1.2, one could obtain the joint convergence in distribution of $\mathcal{E}_t(\alpha)$ with $(t - t\wedge\tau, X_\emptyset(t\wedge\tau))$, thus completing the proof of (1.12). □

5 The critical case $\alpha = -\gamma$: Proof of Theorem 1.6

We consider in this section the case $\alpha = -\gamma$ and prove Theorem 1.6. According to this theorem, the first branching time should occur around time $t - O(t^{1/2})$ with high probability. We use again (1.23) to compute the asymptotic behaviour of $\mathbb{P}(M_t \leq -\sqrt{2}\gamma t)$, and decompose the integral (3.1) onto sub-intervals of interest to prove the joint convergence in distribution of the first branching time and position and the extremal process of the branching Brownian motion.

Recall that, by (3.2), we have $u(-\sqrt{2}\gamma t, t) = U_1(-\sqrt{2}\gamma t, t) + U_2(-\sqrt{2}\gamma t, t)$, and by Lemma 3.1, we have, as $t \to \infty$,

$$U_1(-\sqrt{2}\gamma t, t) \sim \frac{1}{\sqrt{4\pi t}} e^{-(1+\gamma^2)t} \ll t^{3\gamma/4} e^{-(1+\gamma^2)t}. \tag{5.1}$$

Therefore, to complete the proof of (1.13), it is enough to show that

$$U_2(-\sqrt{2}\gamma t, t) \sim C^{(2)} t^{3\gamma/4} e^{-(1+\gamma^2)t}, \text{ as } t \to \infty. \tag{5.2}$$

Moreover, for any $0 < a < b < t$ and $a' < b'$, we set

$$U_2(-\sqrt{2}\gamma t, t, [a,b], [a',b']) := \int_a^b e^{-(t-s)} ds \int_{a'}^{b'} \frac{e^{-\frac{(z+m_s+\sqrt{2}\gamma t - a_t)^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}} u(m_s+z,s)^2 \mathrm{d}z.$$

Equation (5.2) follows from the next two lemmas.

Lemma 5.1. *For all $A, K > 0$, we have*

$$\limsup_{t\to\infty} \frac{e^{(1+\gamma^2)t}}{t^{3\gamma/4}} \left| U_2(-\sqrt{2}\gamma t, t) - U_2(-\sqrt{2}\gamma t, t, [\sqrt{t}/A, A\sqrt{t}], [-K,K]) \right| = o_A(1) + o_K(1).$$

This lemma allows to localise the first branching time and position, conditioned on $\{M_t \leq -\sqrt{2}\gamma t\}$. We note that it is similar to the proof of Lemma 3.3, and postpone its proof to Appendix A.4. The next lemma gives a more detailed estimate of the time at which this branching event occurs.

Lemma 5.2. *For any $0 < a < b$ and $c < d$ fixed, for all $\phi \in \mathcal{C}_c^+(\mathbb{R})$, we have*

$$\lim_{t\to\infty} \frac{e^{(1+\gamma^2)t}}{t^{3\gamma/4}} U_2^{\phi}(-\sqrt{2}\gamma t, t, [a\sqrt{t}, b\sqrt{t}], [a',b']) = \frac{1}{\sqrt{2\pi}} \int_a^b r^{3\gamma/2} e^{-2r^2} \mathrm{d}r \int_{a'}^{b'} e^{-\sqrt{2}\gamma z} w_\phi(z)^2 \mathrm{d}z.$$

Proof. This proof is similar to the proofs of Lemmas 3.4 and 4.2. By (2.5), we have

$$\begin{aligned} U_2^{\phi}(-\sqrt{2}\gamma t, t, [a\sqrt{t}, b\sqrt{t}], [a',b']) &= \int_{a\sqrt{t}}^{b\sqrt{t}} e^{-(t-s)} \mathrm{d}s \int_{a'}^{b'} \frac{e^{-\frac{(z+m_s+\sqrt{2}\gamma t)^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}} u_\phi(m_s+z,s)^2 \mathrm{d}z \\ &\sim \int_{a\sqrt{t}}^{b\sqrt{t}} e^{-(t-s)} \mathrm{d}s \int_{a'}^{b'} \frac{e^{-\frac{(z+m_s+\sqrt{2}\gamma t)^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}} w_\phi(z)^2 \mathrm{d}z, \end{aligned}$$

as $t \to \infty$. Note that $1+\gamma^2 = 2\sqrt{2}\gamma$. Hence by simple calculations we obtain

$$\begin{aligned}
&U_2^{\phi}(-\sqrt{2}\gamma t, t, [a\sqrt{t}, b\sqrt{t}], [a', b']) \\
&\qquad\qquad \sim e^{-(1+\gamma^2)t} \int_{a\sqrt{t}}^{b\sqrt{t}} s^{3\gamma/2} e^{-\frac{2s^2}{t}} \frac{\mathrm{d}s}{\sqrt{2\pi t}} \int_{a'}^{b'} e^{-\sqrt{2}\gamma z} w_{\phi}^2(z)\mathrm{d}z, \quad \text{as } t \to \infty.
\end{aligned}$$

With a change of variable $s = r\sqrt{t}$, the proof is now complete. $\square$

Proof of (1.13). Recall that it is enough to prove (5.2). For all $t > 0$ and $A, K > 0$, we have

$$\begin{aligned}
U_2(-\sqrt{2}\gamma t, t) = \Big(& U_2(-\sqrt{2}\gamma t, t) - U_2(-\sqrt{2}\gamma t, t, [\sqrt{t}/A, A\sqrt{t}], [-K, K])\Big) \\
&+ U_2(-\sqrt{2}\gamma t, t, [\sqrt{t}/A, A\sqrt{t}], [-K, K]).
\end{aligned}$$

Therefore, using Lemmas 5.1 and 5.2, we obtain

$$\begin{aligned}
&\lim_{t\to\infty} t^{-3\gamma/4} e^{(1+\gamma^2)t} U_2(-\sqrt{2}\gamma t, t) \\
&\qquad\qquad = o_A(1) + o_K(1) + \frac{1}{\sqrt{2\pi}} \int_{1/A}^{A} r^{3\gamma/2} e^{-2r^2} \mathrm{d}r \int_{-K}^{K} e^{-\sqrt{2}\gamma z} w^2(z)\mathrm{d}z,
\end{aligned}$$

which converges to $C^{(2)}$ as $A, K \to \infty$. $\square$

We now complete the proof of Theorem 1.6 by proving (1.14) and (1.15).

Proof of (1.14) *and* (1.15). For any $x_1, x_3 \in \mathbb{R}_+$ and $x_2 \in \mathbb{R}$, applying the Markov property at time τ, and using (5.1), we have

$$\begin{aligned}
&\mathbb{P}\left(\tau \geq t - x_1\sqrt{t}, X_{\emptyset}(\tau) \geq (-\sqrt{2}\gamma t + m_{t-\tau}) - x_2, M_t \leq -\sqrt{2}\gamma t - x_3\right) \\
&= \int_{t-x_1\sqrt{t}}^{t} e^{-r}\mathrm{d}r \int_{(-\sqrt{2}\gamma t - m_{t-r}) - x_2}^{\infty} \mathbb{P}(B_r \in \mathrm{d}y) u^2(-\sqrt{2}\gamma t - x_3 - y, t - r) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + o_t(1) t^{3\gamma/4} e^{-(1+\gamma^2)t} \\
&= \int_0^{x_1\sqrt{t}} \mathrm{d}s \int_{-\infty}^{x_2 - x_3} \frac{e^{-\frac{(z+m_s+\sqrt{2}\gamma t + x_3)^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}} u^2(m_s + z, s)\mathrm{d}z + o_t(1) t^{3\gamma/4} e^{-(1+\gamma^2)t},
\end{aligned}$$

using the change of variables $s = t - r$ and $z = -\sqrt{2}\gamma t - x_3 - y - m_s$. We now apply Lemma 5.1 to obtain

$$\begin{aligned}
&\mathbb{P}\left(\tau \geq t - x_1\sqrt{t}, X_{\emptyset}(\tau) \geq (-\sqrt{2}\gamma t + m_{t-\tau}) - x_2, M_t \leq -\sqrt{2}\gamma t - x_3\right) \\
&= (o_{A,t}(1) + o_{K,t}(1) + o_t(1)) t^{3\gamma/4} e^{-(1+\gamma^2)t} \\
&\qquad + \int_{\sqrt{t}/A}^{x_1\sqrt{t}} \mathrm{d}s \int_{-K}^{x_2 - x_3} \frac{e^{-\frac{(z+m_s+\sqrt{2}\gamma t + x_3)^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}} u^2(m_s + z, s)\mathrm{d}z.
\end{aligned}$$

Then Lemma 5.2 gives

$$\mathbb{P}\left(\tau \geq t - x_1\sqrt{t}, X_\emptyset(\tau) \geq (-\sqrt{2}\gamma t + m_{t-\tau}) - x_2, M_t \leq -\sqrt{2}\gamma t - x_3\right)$$
$$\sim \frac{1}{\sqrt{2\pi}} t^{3\gamma/4} e^{-(1+\gamma^2)t} \int_0^{x_1} r^{3\gamma/2} e^{-2r^2} \mathrm{d}r \int_{-\infty}^{x_2 - x_3} e^{-\sqrt{2}\gamma(z+x_3)} w^2(z)\mathrm{d}z \text{ as } t \to \infty,$$

which, together with (1.13) proves (1.14).

We now turn to the proof of (1.15). For any $\phi \in \mathcal{C}_c^+(\mathbb{R})$, using again the Markov property at time τ, we have

$$\begin{aligned}
&\mathbb{E}[e^{-\int \phi \mathrm{d}\mathcal{E}_t(-\gamma)}; M_t \leq -\sqrt{2}\gamma t] \\
=&e^{-t}\,\mathbb{E}[e^{-\phi(B_t+\sqrt{2}\gamma t)}; B_t \leq -\sqrt{2}\gamma t] \\
&+ \int_0^t e^{-r}\mathrm{d}r \int_{\mathbb{R}} \mathbb{P}(B_r \in \mathrm{d}y)\,\mathbb{E}\left[e^{-\sum_{u\in N(t-r)} \phi(X_u(t-r)+y+\sqrt{2}\gamma t)}; M_{t-r} \leq -\sqrt{2}\gamma t - y\right]^2.
\end{aligned}$$

On the one hand, by (3.4),

$$e^{-t}\,\mathbb{E}[e^{-\phi(B_t+\sqrt{2}\gamma t)}; B_t \leq -\sqrt{2}\gamma t] \leq e^{-t}\mathbb{P}(B_t \leq -\sqrt{2}\gamma t) = o_t(1) t^{3\gamma/4} e^{-(1+\alpha^2)t}.$$

On the other hand, using again Lemmas 5.1 and 5.2, we obtain

$$\begin{aligned}
&\int_0^t e^{-r}\mathrm{d}r \int_{\mathbb{R}} \mathbb{P}(B_r \in \mathrm{d}y)\,\mathbb{E}[e^{-\sum_{u\in N(t-r)} \phi(X_u(t-r)+y+\sqrt{2}\gamma t)}; M_{t-r} \leq -\sqrt{2}\gamma t - y]^2 \\
\sim& t^{3\gamma/4} e^{-(1+\gamma^2)t} \int_0^\infty r^{3\gamma/2} e^{-2r^2} \frac{dr}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\sqrt{2}\gamma z} w_\phi^2(z)\mathrm{d}z,
\end{aligned}$$

as $t \to \infty$. It thus follows, using (1.13), that

$$\lim_{t\to\infty} \mathbb{E}\left[e^{-\int \phi d\mathcal{E}_t(-\gamma)} | M_t \leq -\sqrt{2}\gamma t\right] = \frac{1}{2C^{(1)}} \int_{\mathbb{R}} e^{-\sqrt{2}\gamma z} w_\phi^2(z)\mathrm{d}z = \mathbb{E}\left[e^{-\int \phi d\mathcal{E}^-}\right],$$

which, by [Bov17, Proposition 2.2] is enough to conclude (1.15). □

A Proof of Lemmas

We prove in this section some of the more technical lemmas, that are needed to complete the proofs.

A.1 Proof of Lemma 2.2

Recall that Lemma 2.2 consists in the following non-asymptotic estimate : for all $\delta > 0$, $\mathbb{P}(M_t \leq m_t - z) \leq c_\delta e^{-\sqrt{2}\gamma(1-\delta)z}$ for all $t, z \geq 1$.

Proof of Lemma 2.2. We begin by bounding $\mathbb{P}(M_t \leq m_t - z)$ for $1 < z \leq t$. Denote by $n(t) := \# N(t)$ the total number of particles alive at time t. As every individual gives birth at exponential rate to two children, the process $(n(t), t \geq 0)$ is a standard Yule process. Hence $\mathbb{P}(n(t) = k) = e^{-t}(1-e^{-t})^{k-1}$ for any $k \in \mathbb{N}$. Let $\eta \in (0, \delta)$ small enough, such that $J := \lfloor \frac{\sqrt{2}\gamma}{\eta} \rfloor \geq 1$. Observe that

$$\begin{aligned}\mathbb{P}(M_t \leq m_t - z) \leq & \mathbb{P}(n(Jz\eta) \leq z^3) + \mathbb{P}(n(Jz\eta) > z^3; M_t \leq m_t - z) \\ \leq & z^3 e^{-Jz\eta} + \sum_{k=1}^{J} \mathbb{P}(n((k-1)z\eta) \leq z^3 < n(kz\eta); M_t \leq m_t - z). \end{aligned} \tag{A.1}$$

Let $(W_t)_{t \geq 0}$ be a standard Brownian motion, independent of the branching Brownian motion. Using [GH18, Lemma 5.1], for any $0 < s < t$ and $x \in \mathbb{R}$, we have

$$\mathbb{P}(M_t \leq x) \leq \mathbb{P}\left(\max_{u \in N(s)} (W_s + M^u_{t-s}) \leq x \right),$$

where $M^u_y := \max_{v \in N(y+s), u \preccurlyeq v} X_v(y+s) - X_u(s)$, and $u \preccurlyeq v$ means that v is a descendant of u. For any $1 \leq k \leq J$, one has

$$\begin{aligned} & \mathbb{P}(n((k-1)z\eta) \leq z^3 < n(kz\eta); M_t \leq m_t - z) \\ \leq & \mathbb{P}(n((k-1)z\eta) \leq z^3 < n(kz\eta); \max_{u \in N(kz\eta)} (W_{kz\eta} + M^u_{t-kz\eta}) \leq m_t - z) \\ \leq & \mathbb{P}(n((k-1)z\eta) \leq z^3 < n(kz\eta); W_{kz\eta} \leq m_t - m_{t-kz\eta} - z) + \mathbb{P}(M_{t-kz\eta} \leq m_{t-kz\eta})^{z^3}. \end{aligned} \tag{A.2}$$

On the one hand, for $1 \leq k \leq J$, by (2.7),

$$\begin{aligned} & \mathbb{P}(n((k-1)z\eta) \leq z^3 < n(kz\eta); W_{kz\eta} \leq m_t - m_{t-kz\eta} - z) \\ \leq & \mathbb{P}(n((k-1)z\eta) \leq z^3) \mathbb{P}(W_{kz\eta} \leq -(1 - \sqrt{2}k\eta)z) \\ \leq & z^3 e^{-(k-1)z\eta} \frac{\sqrt{kz\eta}}{(1-\sqrt{2}k\eta)z} e^{-\frac{(1-\sqrt{2}k\eta)^2}{2k\eta} z}. \end{aligned}$$

As $(1 - \sqrt{2}k\eta)\sqrt{z} \geq (1 - 2\gamma)\sqrt{z} > \sqrt{J\eta}$ for $z > 100$, we deduce that

$$\begin{aligned} & \mathbb{P}(n((k-1)z\eta) \leq z^3 < n(kz\eta); W_{kz\eta} \leq m_t - m_{t-kz\eta} - z) \\ & \qquad \leq z^3 e^{\eta z} \exp\left\{ - \left[k\eta + \frac{(1-\sqrt{2}k\eta)^2}{2k\eta} \right] z \right\}. \end{aligned} \tag{A.3}$$

On the other hand, as $t - J\eta z \geq (1 - \sqrt{2}\gamma)t \to \infty$ as $t \to \infty$, using that $z \leq t$ and the convergence (2.4), there exist $t_0 \geq 1$ and $c_0 > 0$ such that for all $t \geq t_0$ and $1 \leq z \leq t$, one has $\mathbb{P}(M_{t-kz\eta} \leq m_{t-kz\eta}) \leq e^{-c_0} < 1$. Then,

$$\mathbb{P}(M_{t-kz\eta} \leq m_{t-kz\eta})^{z^3} \leq e^{-c_0 z^3}. \tag{A.4}$$

As a result, using (A.2), (A.3) and (A.4), for $t > t_0$ and $100 < z \le t$, (A.1) becomes that

$$
\begin{aligned}
&\mathbb{P}(M_t \le m_t - z)\\
\le& z^3 e^{\eta z} e^{-\sqrt{2}\gamma z} + J \sup_{1\le k\le J}\left(z^3 e^{\eta z}\exp\left(-\left[k\eta + \frac{(1-\sqrt{2}k\eta)^2}{2k\eta}\right]z\right) + e^{-c_0 z^3}\right)\\
\le& z^3 e^{\eta z} e^{-\sqrt{2}\gamma z} + \frac{\sqrt{2}\gamma}{\eta}\sup_{0<s<\sqrt{2}\gamma}\left(z^3 e^{\eta z}\exp\left(-\left[s + \frac{(1-\sqrt{2}s)^2}{2s}\right]z\right) + e^{-c_0 z^3}\right)\\
=& z^3 e^{\eta z} e^{-\sqrt{2}\gamma z} + \frac{\sqrt{2}\gamma}{\eta}\left(z^3 e^{\eta z} e^{-\sqrt{2}\gamma z} + e^{-c_0 z^3}\right).
\end{aligned}
$$

For $\delta \in (0,1)$ small enough, we could take $\eta = \sqrt{2}\gamma\delta/2$, $t \ge t_0$ and $z \in [K_\delta, t]$ such that

$$
\mathbb{P}(M_t \le m_t - z) \le c_\delta e^{-\sqrt{2}\gamma(1-\delta)z}.
$$

Up to enlarging the constant c_δ, this equation will hold for all $1 \le z \le t$.

We now bound $\mathbb{P}(M_t \le m_t - z)$ with $z \ge t$. We apply (2.9) and obtain that for $z \ge t \ge t_{\varepsilon,\beta}$,

$$
\begin{aligned}
\mathbb{P}(M_t \le m_t - z) &\le u\left(\sqrt{2}\left(1-\tfrac{z}{\sqrt{2}t}\right)t, t\right)\\
&\le \begin{cases} e^{-\sqrt{2}\gamma z + \varepsilon t}, & \text{if } t \le z < 2t;\\ e^{-(1+(1-\frac{z}{\sqrt{2}t})^2)t + \varepsilon t}, & \text{if } 2t \le z \le \sqrt{2}(1+\beta)t;\\ e^{-(1-\frac{z}{\sqrt{2}t})^2 t}, & \text{if } z \ge \sqrt{2}(1+\beta)t. \end{cases}
\end{aligned}
$$

Note that $1 + a^2 \ge 2\gamma(1-a)$ for $a < 1-\sqrt{2}$. So, $(1+(1-\frac{z}{\sqrt{2}t})^2)t \ge \sqrt{2}\gamma z$ if $z \ge 2t$. We also have $\left(1-\frac{z}{\sqrt{2}t}\right)^2 t \ge \sqrt{2}\gamma z$ if $z \ge \sqrt{2}(1+\sqrt{2})t$. By taking $\beta = \sqrt{2}$ and $\varepsilon = \sqrt{2}\gamma\delta$, we thus get that for $t \ge t_{\varepsilon,\beta}$ and $z \ge t$,

$$
\mathbb{P}(M_t \le m_t - z) \le e^{-\sqrt{2}\gamma z + \varepsilon t} \le e^{-\sqrt{2}\gamma(1-\delta)z}.
$$

We hence conclude that for any $\delta \in (0,1)$, there exist $T_\delta = t_{\varepsilon,\beta} \vee t_0$ and $K_\delta \ge 1$ such that for any $t \ge t_\delta$ and $z \ge K_\delta$, (2.13) holds. Thus, up to enlarging again constant c_δ, the proof is now complete. □

A.2 Proof of Lemma 3.3

We assume here that $\alpha \in (-\gamma, 1)$. The aim of this section is to prove that for all $\phi \in \mathcal{C}_c^+(\mathbb{R})$, setting $I_{t,A} = [v_\alpha t - At^{1/2}, v_\alpha t + At^{1/2}]$, we have

$$
\lim_{A,K\to\infty}\limsup_{t\to\infty}\frac{e^{2\gamma(1-\alpha)t}}{t^{3\gamma/2}}\int_{(I_{t,A}\times[-K,K])^c} \mathrm{d}s\mathrm{d}z \frac{e^{-(t-s)-\frac{(m_s+z-\sqrt{2}\alpha t)^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}} u_\phi(m_s+z,s)^2 = 0. \quad (A.5)
$$

As ϕ is non-negative, we observe that

$$u_\phi(z,t) = \mathbb{E}\left(e^{-\sum_{u\in N(t)}\phi(X_u(t)-z)}; M_t \leq z\right) \leq \mathbb{P}(M_t \leq z) = u(z,t).$$

It is enough to prove that (A.5) holds for $\phi \equiv 0$.

Therefore, the objective of the section can be restated as follows: conditioned on $\{M_t \leq \sqrt{2}\alpha t\}$, we show that the first branching time τ is with high probability located around $(1-v_\alpha)t + O(\sqrt{t})$, and the position at which that particle branches satisfies $\sqrt{2}\alpha t - m_{t-\tau} + O(1)$ with high probability.

The idea of the proof is the following: we use (1.23) to rewrite u as the sum of U_1 and U_2. By Lemma 3.1, U_1 add a negligible contribution to u, so that

$$u(\sqrt{2}\alpha t, t) \approx \int_{[0,t]\times\mathbb{R}} \mathrm{d}s\mathrm{d}z \frac{e^{-(t-s)-\frac{(m_s+z-\sqrt{2}\alpha t)^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}} u(m_s+z,s)^2.$$

Moreover, by Lemma 3.4, a large contribution to u is carried by the regions of the form $I_{A,t}\times[-K,K]$, with $A>0$ and K large enough. We now use a priori domination estimates for u (e.g. Lemma 2.1) and methods similar to the proof of Laplace's method.

We decompose the proof of Lemma 3.3 into three parts, by considering the contribution of various domains of $[0,t]\times\mathbb{R}$.

A.2.1 Linear bounds on the first splitting time

As a first step towards the proof of Lemma 3.3, we show that for all $\varepsilon > 0$,

$$\mathbb{P}\left(|\tau - (1-v_\alpha)t| > \varepsilon t, M_t \leq \sqrt{2}\alpha t\right) \ll u(\sqrt{2}\alpha t, t).$$

Lemma A.1. *Let $\alpha \in (-\gamma, 1)$. For all $\varepsilon > 0$ small enough, we have*

$$\limsup_{t\to\infty} \frac{1}{t}\log U_2(\sqrt{2}\alpha t, t, [0,(v_\alpha-\varepsilon)t]) < -2\gamma(1-\alpha), \tag{A.6}$$

$$\limsup_{t\to\infty} \frac{1}{t}\log U_2(\sqrt{2}\alpha t, t, [(v_\alpha+\varepsilon)t, t]) < -2\gamma(1-\alpha). \tag{A.7}$$

To prove this result, we begin by bounding the probability that a split occurs at the very end of the process.

Lemma A.2. *Let $\alpha \in (-\gamma, 1)$. There exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$,*

$$\limsup_{t\to\infty} \frac{1}{t}\log U_2(\sqrt{2}\alpha t, t, [0,\varepsilon t]) < -2\gamma(1-\alpha). \tag{A.8}$$

Proof. Equation (A.8) can be rewritten as

$$\mathbb{P}(0 \leq t-\tau \leq \varepsilon t, M_t \leq \sqrt{2}\alpha t) \ll t^{3\gamma/2}e^{-2\gamma(1-\alpha)t}.$$

First note that $\mathbb{P}(0 \leq t-\tau \leq \varepsilon t, M_t \leq \sqrt{2}\alpha t) \leq \mathbb{P}(\tau \geq (1-\varepsilon)t) = e^{-(1-\varepsilon)t}$. Thus (A.8) holds for all α such that $2\gamma(1-\alpha) < 1$, i.e. $\alpha > -\gamma/2$, for all $\varepsilon > 0$ small enough.

We now assume that $\alpha \leq -\gamma/2 < 0$, and decompose the above probability as

$$\begin{aligned}\mathbb{P}(0 \leq t-\tau \leq \varepsilon t, M_t \leq \sqrt{2}\alpha t) \leq &\\ &\mathbb{P}(0 \leq t-\tau \leq \varepsilon t, X_\emptyset(\tau) \leq \sqrt{2}(\alpha+2\varepsilon)t, M_t \leq \sqrt{2}\alpha t)\\ &\quad + \mathbb{P}(0 \leq t-\tau \leq \varepsilon t, X_\emptyset(\tau) \geq \sqrt{2}(\alpha+2\varepsilon)t, M_t \leq \sqrt{2}\alpha t), \quad \text{(A.9)}\end{aligned}$$

and bound these two quantities separately.

First note that

$$\begin{aligned}\mathbb{P}(0 \leq t-\tau \leq \varepsilon t, X_\emptyset(\tau) \leq \sqrt{2}(\alpha+2\varepsilon)t, M_t \leq \sqrt{2}\alpha t)&\\ \leq \mathbb{P}(0 \leq t-\tau \leq \varepsilon t, X_\emptyset(\tau) \leq \sqrt{2}(\alpha+2\varepsilon)t)&\\ \leq \int_{(1-\varepsilon)t}^t e^{-s}\mathbb{P}(B_s \leq \sqrt{2}(\alpha+2\varepsilon)t)\mathrm{d}s \leq Ct^{-1/2}e^{-(1-\varepsilon)t}e^{-\frac{(\alpha+2\varepsilon)^2}{2(1-\varepsilon)}t}&,\end{aligned}$$

using (2.7). As $1+\frac{\alpha^2}{2} > 2\gamma(1-\alpha)$ for all $\alpha \in (-\gamma, -\gamma/2]$, we deduce that for all $\varepsilon > 0$ small enough, there exists $\delta > 0$ such that

$$\mathbb{P}(0 \leq t-\tau \leq \varepsilon t, X_\emptyset(\tau) \leq \sqrt{2}(\alpha+2\varepsilon)t) \leq Ce^{-2\gamma(1-\alpha)t-\delta t}. \tag{A.10}$$

We now turn to bounding the second probability in (A.9)

Using the Markov property at time τ, we bound it as

$$\begin{aligned}\mathbb{P}(0 \leq t-\tau \leq \varepsilon t, X_\emptyset(\tau) \geq \sqrt{2}(\alpha+2\varepsilon)t, M_t \leq \sqrt{2}\alpha t)&\\ \leq \int_{(1-\varepsilon)t}^t e^{-s}\,\mathbb{E}\left(u(t-s, \sqrt{2}\alpha t - B_s)^2 \mathbb{1}_{\left\{B_s \geq \sqrt{2}(\alpha+2\varepsilon)t\right\}}\right)\mathrm{d}s&.\end{aligned}$$

By Lemma 2.1, for all $s < \varepsilon t$ and $y \leq -2\varepsilon t$, we have $u(s,y) \leq e^{-y^2/2s}$, yielding

$$\begin{aligned}\mathbb{P}(0 \leq t-\tau \leq \varepsilon t, X_\emptyset(\tau) \geq \sqrt{2}(\alpha+2\varepsilon)t, M_t \leq \sqrt{2}\alpha t)&\\ \leq e^{-(1-\varepsilon)t}\int_0^{\varepsilon t}\int_{\sqrt{2}(\alpha+2\varepsilon)t}^\infty e^{-\frac{(\sqrt{2}\alpha t-y)^2}{s}}e^{-\frac{y^2}{2(t-s)}}\mathrm{d}y\mathrm{d}s&.\end{aligned}$$

Using that

$$-\frac{y^2}{2(t-s)} - \frac{(y-\sqrt{2}\alpha t)^2}{s} = -\frac{2\alpha^2t^2}{2t-s} - \frac{(y-\sqrt{2}\alpha\frac{2(t-s)}{2t-s})^2}{\frac{2(t-s)s}{2t-s}},$$

we obtain

$$\begin{aligned}\mathbb{P}(0 \leq t-\tau \leq \varepsilon t, X_\emptyset(\tau) \geq \sqrt{2}(\alpha+2\varepsilon)t, M_t \leq \sqrt{2}\alpha t)&\\ \leq e^{-(1+\alpha^2-\varepsilon)t}\int_0^{\varepsilon t}\int_{\sqrt{2}(\alpha+2\varepsilon-\frac{2(t-s)}{2t-s})t}^\infty e^{-\frac{z^2}{\frac{2(t-s)s}{2t-s}}}\mathrm{d}z\mathrm{d}s \leq \frac{\sqrt{2\pi}\varepsilon^{3/2}}{\sqrt{1-\varepsilon/2}}t^{3/2}e^{-(1+\alpha^2-\varepsilon)t}&.\end{aligned}$$

Therefore, as $1+\alpha^2 > 2\gamma(1-\alpha)$ for all $\alpha \in (-\gamma, 1)$, we conclude that for all $\varepsilon > 0$ small enough, there exists $\delta > 0$ such that

$$\mathbb{P}(0 \leq t-\tau \leq \varepsilon t, X_\emptyset(\tau) \geq \sqrt{2}(\alpha+2\varepsilon)t, M_t \leq \sqrt{2}\alpha t) \leq Ce^{-2\gamma(1-\alpha)t-\delta t}. \tag{A.11}$$

In view of (A.9), equations (A.10) and (A.11) show that there exists ε_0 so that for all $0 < \varepsilon < \varepsilon_0$, (A.8) holds. $\square$

We now bound the probability that the first splitting time in the branching Brownian motion occurs after time at a distance at least εt from the expected time $(1 - v_\alpha)t$.

Lemma A.3. *Let $\alpha \in (-\gamma, 1)$. There exists $\varepsilon_1 > 0$ such that for all $\varepsilon \in (0, \varepsilon_1)$,*

$$\limsup_{t\to\infty} \frac{1}{t} \log U_2(\sqrt{2}\alpha t, t, [\varepsilon t, (v_\alpha - \varepsilon)t]) < -2\gamma(1-\alpha), \tag{A.12}$$

$$\limsup_{t\to\infty} \frac{1}{t} \log U_2(\sqrt{2}\alpha t, t, [(v_\alpha + \varepsilon)t, t]) < -2\gamma(1-\alpha). \tag{A.13}$$

Proof. Let $a < b$ such that $[a, b] \subset (0, v_\alpha) \cup (v_\alpha, 1]$. By definition of U_2, we have

$$\begin{aligned} U_2(\sqrt{2}\alpha t, t, [at, bt]) &\leq \int_{at}^{bt} \int_{\mathbb{R}} \frac{\mathrm{d}z}{\sqrt{2\pi t}} e^{-(t-s)-\frac{(z-\sqrt{2}\alpha t)^2}{2(t-s)}} u(z, s)^2 \mathrm{d}z\mathrm{d}s \\ &\leq \int_a^b \int_{\mathbb{R}} e^{-t(1-r)-t\frac{(\sqrt{2}hr-\sqrt{2}\alpha)^2}{2(1-r)}} u(\sqrt{2}htr, tr)^2 \sqrt{2}t^{3/2} r\mathrm{d}r \frac{\mathrm{d}h}{\sqrt{2\pi}}, \end{aligned}$$

by change of variables $r = s/t$ and $h = z/\sqrt{2}s$. We then use Lemma 2.1 to bound $u(\sqrt{2}htr, r)$ uniformly in (h, r) for t large enough. For all $\delta > 0$ and $\beta \geq 1$, for all t large enough we have

$$U_2(\sqrt{2}\alpha t, t, [at, bt]) \leq \frac{t^{3/2}}{\sqrt{\pi}} \int_a^b \int_{\mathbb{R}} e^{-t\left(1-r+\frac{(hr-\alpha)^2}{1-r}+2\overline{\Psi}_\beta(h)-\delta\right)} \mathrm{d}h\mathrm{d}r, \tag{A.14}$$

where we set

$$\overline{\Psi}_\beta(a) := \begin{cases} 0, & \text{if } a \geq 1; \\ \sqrt{2}\gamma(1-a), & \text{if } -\gamma \leq a < 1; \\ (1+a^2), & \text{if } -\beta \leq a < -\gamma; \\ a^2, & \text{if } a < -\beta. \end{cases} \tag{A.15}$$

To complete this proof, it is therefore enough to prove that the right-hand side of (A.14) decays exponentially fast, at a rate larger than $2\gamma(1-\alpha)$. To do so, we decompose the integral over $\mathbb{R}$ into thee subsets : $(-\infty, -\beta)$, $[-\beta, 1]$ and $(1, \infty)$.

We first observe that on the interval $[1, \infty)$, by change of variable $v = hr - \alpha$, we have

$$\begin{aligned} \int_a^b \int_1^\infty e^{-t\left(1-r+\frac{(hr-\alpha)^2}{1-r}-\delta\right)} \mathrm{d}h\mathrm{d}r &\leq b \int_a^b \int_{r-\alpha}^\infty e^{-t(1-r+\frac{v^2}{1-r}-\delta)} \mathrm{d}v\mathrm{d}r \\ &\leq Ct^{-1/2} \left(\int_a^\alpha e^{-t(1-r)} \mathrm{d}r + \int_\alpha^b e^{-t(1-r+\frac{(r-\alpha)^2}{1-r})} \right) \mathrm{d}r, \end{aligned}$$

using (2.7) to bound the integrals over v. Hence, one straightforwardly obtains that

$$\limsup_{t\to\infty} \frac{1}{t} \log \int_a^b \int_1^\infty e^{-t\left(1-r+\frac{(hr-\alpha)^2}{1-r}-\delta\right)} \mathrm{d}h\mathrm{d}r \\ \leq \delta - \min(1-\alpha, g_\alpha(b)) < -2\gamma(1-\alpha), \quad \text{(A.16)}$$

for $\delta > 0$ small enough, where g_α is the function defined in (3.7), which attains its maximum at v_α with value $2\gamma(1-\alpha)$.

Similarly, as $[a,b] \times [-\beta, 1]$ is compact, we also have

$$\limsup_{t\to\infty} \frac{1}{t} \log \int_a^b \int_{-\gamma}^1 e^{-t\left(1-r+\frac{(hr-\alpha)^2}{1-r}+2\overline{\Psi}_\beta(h)-\delta\right)} \mathrm{d}h\mathrm{d}r$$
$$\leq \delta - \inf_{\substack{r\in[a,b]\\ h\in[-\beta,1]}} 1-r+\frac{(hr-\alpha)^2}{1-r}+2\overline{\Psi}_\beta(h) \leq \delta - \inf_{\substack{r\in[a,b]\\ h\in[-\beta,1]}} 1-r+\frac{(hr-\alpha)^2}{1-r}+2\sqrt{2}\gamma(1-h).$$

The function $(h,r) \in (-\infty, 1] \times [\varepsilon, 1] \mapsto 1-r+\frac{(hr-\alpha)^2}{1-r}+2\sqrt{2}\gamma(1-h)$ attaining its unique minimum at $(v_\alpha, 1)$, we conclude again that, choosing $\delta > 0$ small enough, we have

$$\limsup_{t\to\infty} \frac{1}{t} \log \int_a^b \int_{-\gamma}^1 e^{-t\left(1-r+\frac{(hr-\alpha)^2}{1-r}+2\overline{\Psi}_\beta(h)-\delta\right)} \mathrm{d}h\mathrm{d}r < -2\gamma(1-\alpha). \quad \text{(A.17)}$$

Finally, choosing $\beta > 0$ large enough so that the function $h \mapsto \frac{(hr-\alpha)^2}{1-r} + 2h^2$ is strictly decreasing on $(-\infty, -\beta]$, we have

$$\int_a^b \int_{-\infty}^{-\beta} e^{-t\left(1-r+\frac{(hr-\alpha)^2}{1-r}+2h^2-\delta\right)} \mathrm{d}h\mathrm{d}r \leq \int_a^b e^{-t\left(1-r+\frac{(-\beta r-\alpha)^2}{1-r}-\delta\right)} \mathrm{d}r \int_{-\infty}^{-\beta} e^{-2h^2 t} \mathrm{d}h$$
$$\leq Ct^{-1/2} e^{-2\beta^2 t} \int_a^b e^{-t\left(1-r+\frac{(-\beta r-\alpha)^2}{1-r}-\delta\right)} \mathrm{d}r \leq Ct^{-1/2} e^{-t(2\beta^2-\delta)},$$

which, if we choose β large enough, will be smaller than $e^{-t(2\gamma(1-\alpha)+\eta)}$ for some $\eta > 0$, for all t large enough. Using this estimate in combination with (A.16) and (A.17) allows us, by (A.14), to show that

$$\limsup_{t\to\infty} \frac{1}{t} \log U_2(\sqrt{2}\alpha t, t, [at, bt]) < -2\gamma(1-\alpha),$$

which completes the proof of (A.12) and (A.13). □

The proof of Lemma A.1 is then a combination of Lemmas A.2 and A.3.

A.2.2 Tightness of the normalized first splitting time

We now precise the estimates on $\mathbb{P}\left(|\tau - v_\alpha t| > A\sqrt{t}, M_t \leq \sqrt{2}\alpha t\right)$, bounding this quantity as $t \to \infty$ then $A \to \infty$.

Lemma A.4. *Given $\alpha \in (-\gamma, 1)$, we have*

$$\lim_{A\to\infty} \limsup_{t\to\infty} e^{2\gamma(1-\alpha)t} t^{-3\gamma/2} U_2(\sqrt{2}\alpha t, t, [0, v_\alpha t - A\sqrt{t}]) = 0; \tag{A.18}$$

$$\lim_{A\to\infty} \limsup_{t\to\infty} e^{2\gamma(1-\alpha)t} t^{-3\gamma/2} U_2(\sqrt{2}\alpha t, t, [v_\alpha t + A\sqrt{t}, t]) = 0. \tag{A.19}$$

As a first step, we show that with high probability, $|\tau - v_\alpha t| = o(t^{1/2}\log t)$ conditioned on the maximal displacement being small.

Lemma A.5. *Let $\alpha \in (-\gamma, 1)$. There exists $\varepsilon_\alpha > 0$ such that for all $\varepsilon \in (0, \varepsilon_\alpha)$, for all $\varrho > 0$ we have*

$$\limsup_{t\to\infty} t^\varrho e^{2\gamma(1-\alpha)t} U_2(\sqrt{2}\alpha t, t, [(v_\alpha - \varepsilon)t, v_\alpha t - \sqrt{t}\log t]) = 0; \tag{A.20}$$

$$\limsup_{t\to\infty} t^\varrho e^{2\gamma(1-\alpha)t} U_2(\sqrt{2}\alpha t, t, [v_\alpha t + \sqrt{t}\log t, (v_\alpha + \varepsilon)t]) = 0. \tag{A.21}$$

Proof. The two formulas being proved in a very similar way, we only prove the first one. Note that without loss of generality, one can choose $\varepsilon > 0$ small enough that $v_\alpha - 2\varepsilon > \min(\alpha, 0)$. By definition of U_2, we have

$$\begin{aligned}
&U_2(\sqrt{2}\alpha t, t, [(v_\alpha - \varepsilon)t, v_\alpha t - \sqrt{t}\log t]) \\
=& \int_{(v_\alpha-\varepsilon)t}^{v_\alpha t - \sqrt{t}\log t} \mathrm{d}s \int_{\mathbb{R}} \frac{\mathrm{d}z}{\sqrt{2\pi t}} e^{-(t-s) - \frac{(\sqrt{2}s + z - \sqrt{2}\alpha t)^2}{2(t-s)}} u(\sqrt{2}s + z, s)^2 \\
\leq& t^{1/2} \int_{v_\alpha - \varepsilon}^{v_\alpha - \frac{\log t}{\sqrt{t}}} \mathrm{d}u \int_{\mathbb{R}} \mathrm{d}z e^{-t(1-u) + \frac{(z + \sqrt{2}t(u-\alpha))^2}{2t(1-u)}} u(\sqrt{2}ut + z, ut)^2 \\
\leq& t^{1/2} \int_{v_\alpha - \varepsilon}^{v_\alpha - \frac{\log t}{\sqrt{t}}} \mathrm{d}u e^{-t g_\alpha(u)} \int_{\mathbb{R}} \mathrm{d}z e^{-\frac{z(2\sqrt{2}t(u-\alpha)+z)}{2t(1-u)}} u(\sqrt{2}ut + z, ut)^2,
\end{aligned}$$

with g_α the function defined in (3.7). Using (3.9), there exists $c > 0$ such that for all $r \in [v_\alpha - \varepsilon, v_\alpha]$, $g_\alpha(r) \leq g_\alpha(v_\alpha) - c(r - v_\alpha)^2$. Thus

$$\begin{aligned}
&e^{2\gamma(1-\alpha)t} U_2(\sqrt{2}\alpha t, t, [(v_\alpha - \varepsilon)t, v_\alpha t - \sqrt{t}\log t]) \\
&\quad \leq t \int_{-\varepsilon}^{-\frac{\log t}{t^{1/2}}} \mathrm{d}v e^{-ctv^2} \int_{\mathbb{R}} \mathrm{d}z e^{-\frac{z(2\sqrt{2}t(v_\alpha + v - \alpha) + z)}{2t(1 - v_\alpha - v)}} u(\sqrt{2}(v_\alpha + v)t + z, (v_\alpha + v)t)^2.
\end{aligned}$$

We now use Lemma 2.2, i.e. that for all $\delta > 0$ there exists $c_\delta > 0$ such that for all $t \geq 1$ and $z \in \mathbb{R}$, we have $u(m_t - z, t) \leq c_\delta e^{-\sqrt{2}\gamma(1-\delta)z_+}$. Therefore, up to a change of variables, for all $v \in [-\varepsilon, 0]$, writing $a_t(v) = \frac{3}{2\sqrt{2}} \log((v_\alpha + v)t)$, we have

$$
\begin{aligned}
&\int_{\mathbb{R}} \mathrm{d}z e^{-\frac{z(2\sqrt{2}t(v_\alpha+v-\alpha)+z)}{2t(1-v_\alpha-v)}} u(\sqrt{2}(v_\alpha+v)t+z,(v_\alpha+v)t)^2 \\
\le &\int_{\mathbb{R}} \mathrm{d}y e^{\frac{(y+a_t(v))(2\sqrt{2}t(v_\alpha+v-\alpha)-(y+a_t(v)))}{2t(1-v_\alpha-v)}} u(m_{(v_\alpha+v)t}-y,(v_\alpha+v)t)^2 \\
\le &c_\delta \int_{\mathbb{R}} \mathrm{d}y e^{\frac{(y+a_t(v))(2\sqrt{2}t(v_\alpha+v-\alpha)-(y+a_t(v)))}{2t(1-v_\alpha-v)}} e^{-2\sqrt{2}\gamma(1-\delta)y_+}.
\end{aligned}
$$

As a result, we get

$$
\begin{aligned}
&e^{2\gamma(1-\alpha)t} U_2(\sqrt{2}\alpha t, t, [(v_\alpha-\varepsilon)t, v_\alpha t - \sqrt{t}\log t]) \\
&\quad \le Ct^{1/2} \int_{-\varepsilon}^{-\frac{\log t}{t^{1/2}}} \mathrm{d}v e^{-ctv^2} \int_{\mathbb{R}} \mathrm{d}y e^{\frac{(y+a_t(v))(2\sqrt{2}t(v_\alpha+v-\alpha)-(y+a_t(v)))}{2t(1-v_\alpha-v)}} e^{-2\sqrt{2}\gamma(1-\delta)y_+}. \quad \text{(A.22)}
\end{aligned}
$$

We now bound this quantity in two different ways for $y \ge 0$ and $y \le 0$.

We first observe that for all $v \in [-\varepsilon, 0]$, using that $v_\alpha > \alpha + 2\varepsilon$,

$$
\begin{aligned}
&\int_{-\infty}^{0} \mathrm{d}y e^{\frac{(y+a_t(v))(2\sqrt{2}t(v_\alpha+v-\alpha)-(y+a_t(v)))}{2t(1-v_\alpha-v)}} \le \int_{-\infty}^{0} \mathrm{d}y e^{\frac{(y+a_t(v))(2\sqrt{2}t(v_\alpha+v-\alpha))}{2t(1-v_\alpha-v)}} \\
&\qquad \le \frac{1-v_\alpha-v}{\sqrt{2}(v_\alpha+v-\alpha)} ((v+v_\alpha)t)^{\frac{3}{2}\frac{v_\alpha+v-\alpha}{1-v_\alpha-v}} \le \frac{1-v_\alpha+\varepsilon}{2\sqrt{2}\varepsilon} (v_\alpha t)^{\frac{3}{2}\frac{v_\alpha-\alpha}{(1-v_\alpha)}}. \quad \text{(A.23)}
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\int_{0}^{\infty} \mathrm{d}y e^{\frac{(y+a_t(v))(2\sqrt{2}t(v_\alpha+v-\alpha)-(y+a_t(v)))}{2t(1-v_\alpha-v)}} e^{-2\sqrt{2}\gamma(1-\delta)y} \\
&\le e^{2\sqrt{2}\gamma(1-\delta)a_t(v)} \int_{a_t(v)}^{\infty} \mathrm{d}x e^{x\left(\sqrt{2}\frac{(v_\alpha-\alpha)}{(1-v_\alpha)}-2\sqrt{2}\gamma(1-\delta)\right)} \le \frac{1}{\sqrt{2}\gamma(1-2\delta)} (v_\alpha t)^{3\gamma(1-2\delta)/2}, \quad \text{(A.24)}
\end{aligned}
$$

for all $\delta > 0$ small enough, using that $v_\alpha - \alpha = \frac{\gamma}{\sqrt{2}}(1-\alpha) = \gamma(1-v_\alpha)$.

Hence, plugging (A.23) and (A.24) into (A.22), we deduce that there exist $C > 0$ and $\varrho > 0$ so that for all $t \ge 1$ large enough,

$$
\begin{aligned}
&e^{2\gamma(1-\alpha)t} U_2(\sqrt{2}\alpha t, t, [(v_\alpha-\varepsilon)t, v_\alpha t - \sqrt{t}\log t]) \\
&\qquad\qquad \le Ct^{\varrho} \int_{-\varepsilon}^{-\frac{\log t}{t^{1/2}}} \mathrm{d}v e^{-ctv^2} \le Ct^{\varrho} e^{-c(\log t)^2},
\end{aligned}
$$

which concludes the proof of (A.20). □

Proof of Lemma A.4. By Lemmas A.1 and A.5, to prove Lemma A.4, it is enough to bound for all t large enough, the quantities $U_2(\sqrt{2}\alpha t, t, [v_\alpha t - \sqrt{t}\log t, v_\alpha t - A\sqrt{t}])$ and $U_2(\sqrt{2}\alpha t, t, [v_\alpha t + A\sqrt{t} v_\alpha t - \sqrt{t}\log t])$ by $M(A)e^{-2\gamma(1-\alpha)t} t^{-3\gamma/2}$, with $A \mapsto M(A)$ a positive function converging to 0 as $A \to \infty$.

The proofs of (A.18) and (A.19) being very similar and symmetric, we only prove the second one. We write

$$
\begin{aligned}
&U_2(\sqrt{2}\alpha t, t, [v_\alpha t + A\sqrt{t}, v_\alpha t + \sqrt{t}\log t]) \\
\leq& t^{-1/2}\int_{v_\alpha t + A\sqrt{t}}^{v_\alpha t + \sqrt{t}\log t} \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}z e^{-(t-s)+\frac{(z-m_s)^2}{2(t-s)}} u(m_s+z,s)^2 \\
\leq& t^{1/2}e^{-2\gamma(1-\alpha)t}\int_{At^{-1/2}}^{t^{-1/2}\log t} \mathrm{d}v e^{-ctv^2}\int_{\mathbb{R}} \mathrm{d}y e^{\frac{(y+a_t(v))(2\sqrt{2}t(v_\alpha+v-\alpha)-(y+a_t(v)))}{2t(1-v_\alpha-v)}} e^{-2\sqrt{2}\gamma(1-\delta)y_+},
\end{aligned}
$$

with the same computations as the ones used to obtain (A.22), using Lemma 2.2.

Using that $|v| \leq t^{-1/2}\log t$, hence that $a_t(v) = \frac{3}{2\sqrt{2}}\log((v_\alpha+v)t) = a_t(0) + o_t(1)$, we obtain, for all t large enough:

$$
\begin{aligned}
&U_2(\sqrt{2}\alpha t, t, [v_\alpha t + A\sqrt{t}, v_\alpha t + \sqrt{t}\log t]) \\
\leq& 2c_\delta t^{1/2}e^{-2\gamma(1-\alpha)t}\int_{At^{-1/2}}^{t^{-1/2}\log t}\mathrm{d}v e^{-ctv^2}\int_{\mathbb{R}}\mathrm{d}y e^{\frac{(y+a_t)(2\sqrt{2}t(v_\alpha+v-\alpha)-(y+a_t)}{2t(1-v_\alpha-v)}} e^{-2\sqrt{2}\gamma(1-\delta)y_+}, \qquad \text{(A.25)}
\end{aligned}
$$

where $a_t = a_t(0) = \frac{3}{2}\log(v_\alpha t)$.

We then compute for all $|v| < t^{-1/2}\log t$,

$$
\begin{aligned}
\int_{-\infty}^0 \mathrm{d}y e^{\frac{(y+a_t)(2\sqrt{2}t(v_\alpha+v-\alpha)-(y+a_t))}{2t(1-v_\alpha-v)}} &\leq \int_{-\infty}^0 \mathrm{d}y e^{\sqrt{2}\frac{(y+a_t)(v_\alpha+v-\alpha)}{(1-v_\alpha-v)}} \\
&\leq \exp\left(\sqrt{2}a_t\frac{v_\alpha+v-\alpha}{1-v_\alpha-v}\right) \leq \exp\left(\sqrt{2}a_t\left(\frac{v_\alpha-\alpha}{1-v_\alpha}+Cv\right)\right),
\end{aligned}
$$

for all t large enough, using Taylor's expansion. Hence, with $(v_\alpha-\alpha)/(1-v_\alpha) = \gamma$, there exists $C > 0$ such that for all t large enough,

$$
\int_{-\infty}^0 \mathrm{d}y e^{\frac{(y+a_t)(2t(v_\alpha+v-\alpha)-(y+a_t))}{t(1-v_\alpha-v)}} \leq C(v_\alpha t)^{3\gamma/2}. \qquad \text{(A.26)}
$$

Similarly, we have

$$
\begin{aligned}
&\int_0^\infty \mathrm{d}y e^{\frac{(y+a_t)(2\sqrt{2}t(v_\alpha+v-\alpha)-(y+a_t))}{2t(1-v_\alpha-v)}} e^{-2\sqrt{2}\gamma(1-\delta)y} \\
\leq& e^{2\sqrt{2}\gamma(1-\delta)a_t}\int_{a_t}^\infty \mathrm{d}x e^{\frac{x(2\sqrt{2}t(v_\alpha+v-\alpha)-x)}{2t(1-v_\alpha-v)}-2\sqrt{2}\gamma(1-\delta)x} \\
\leq& (v_\alpha t)^{3\gamma(1-\delta)}\int_{a_t}^\infty \mathrm{d}x e^{\sqrt{2}x\left(\frac{v_\alpha+v-\alpha}{1-v_\alpha-v}-2\gamma(1-\delta)\right)} \leq C(v_\alpha t)^{3\gamma(1-\delta)}(v_\alpha t)^{\frac{3}{2}\frac{v_\alpha-v-\alpha}{1-v_\alpha}-3\gamma(1-\delta)}.
\end{aligned}
$$

Hence, using that $\frac{v_\alpha-v-\alpha}{1-v_\alpha} = \gamma + O(t^{-1/2}\log t)$, we obtain that for all t large enough

$$
\int_0^\infty \mathrm{d}y e^{\frac{(y+a_t)(2t(v_\alpha+v-\alpha)-(y+a_t))}{t(1-v_\alpha-v)}} \leq C(v_\alpha t)^{3\gamma/2}. \qquad \text{(A.27)}
$$

As a result, with (A.26) and (A.27), (A.25) becomes

$$
U_2(\sqrt{2}\alpha t, t, [v_\alpha t + A\sqrt{t}, v_\alpha t + \sqrt{t}\log t]) \leq Ct^{3\gamma/2}e^{-2\gamma(1-\alpha)t}\int_A^\infty e^{-cw^2}\mathrm{d}w.
$$

By dominated convergence, the proof of (A.19) is now complete. $\square$

A.2.3 Tightness of the centred splitting position

To complete the proof of Lemma 3.3, we prove that the position at which the first splitting occurs $X_\emptyset(\tau)$ is tightly concentrated around the position $\sqrt{2}\alpha t - m_{t-\tau}$, on the event $|\tau - v_\alpha t| \leq A\sqrt{t}$.

Lemma A.6. *For $1-\sqrt{2} < \alpha < 1$ and for any fixed $A > 0$,*

$$\lim_{K\to\infty}\lim_{t\to\infty} \frac{e^{2\gamma(1-\alpha)t}}{t^{3\gamma/2}} U_2(\sqrt{2}\alpha t, t, [v_\alpha t - A\sqrt{t}, v_\alpha + A\sqrt{t}], [-K,K]^c) = 0. \tag{A.28}$$

Proof. Let $K > 0$ and $A > 0$. We observe that with similar computations as in the proof of Lemma A.4, setting $a_t = \frac{3}{2\sqrt{2}}\log(v_\alpha t)$, we have

$$\begin{aligned}
&e^{2\gamma(1-\alpha)t} U_2(\sqrt{2}\alpha t, t, [v_\alpha t - A\sqrt{t}, v_\alpha t + A\sqrt{t}], [-K,K]^c) \\
\leq & Ct^{1/2} \int_{-At^{-1/2}}^{At^{-1/2}} \mathrm{d}v e^{-ctv^2} \int_{[-K,K]^c} \mathrm{d}y e^{\frac{(y+a_t)(2\sqrt{2}t(v_\alpha+v-\alpha)-(y+a_t))}{2t(1-v_\alpha-v)}} u(m_{(v_\alpha+v)t} - y, (v_\alpha+v)t) \\
\leq & Ct^{1/2} \int_{-At^{-1/2}}^{At^{-1/2}} \mathrm{d}v e^{-ctv^2} \int_{[-K,K]^c} \mathrm{d}y e^{\frac{(y+a_t)(2\sqrt{2}t(v_\alpha+v-\alpha)-(y+a_t))}{2t(1-v_\alpha-v)}} e^{-2\sqrt{2}\gamma(1-\delta)y_+},
\end{aligned}$$

where we used again Lemma 2.2.

We then observe, with similar computations as in the proof of Lemma A.4 again that

$$\begin{aligned}
&\int_{-\infty}^{-K} \mathrm{d}y e^{\frac{(y+a_t)(2\sqrt{2}t(v_\alpha+v-\alpha)-(y+a_t))}{2t(1-v_\alpha-v)}} \leq t^{3\gamma/2} e^{-(\gamma-\delta)K}, \\
&\int_{K}^{\infty} \mathrm{d}y e^{\frac{(y+a_t)(2\sqrt{2}t(v_\alpha+v-\alpha)-(y+a_t))}{2t(1-v_\alpha-v)}} e^{-2\sqrt{2}\gamma(1-\delta)y} \leq t^{3\gamma/2} e^{-(\gamma-\delta)K},
\end{aligned}$$

using that for all t large enough, $\left|\frac{v_\alpha - v - \alpha}{1-v_\alpha} - \gamma\right| \leq \delta$. Therefore, letting $t \to \infty$ then $K \to \infty$, we obtain, for all $A > 0$, that (A.28) holds. □

Lemma 3.3 is then a consequence of Lemmas A.1, A.4 and A.6.

A.3 Proof of Lemma 4.1

Similarly to the previous section, it is enough to prove Lemma 4.1 for $\phi \equiv 0$ by a straightforward domination argument. The proof is obtained in a similar, but slightly simple fashion.

Proof. Let $\alpha < -\gamma$ here. Note that by change of variable $y = \sqrt{2}as$ and (2.9), for any $\varepsilon > 0$ and $A \geq t_{\varepsilon,\beta}$, with $\beta = K\alpha$,

$$\begin{aligned}
U_2(\sqrt{2}\alpha t, t, [A,t]) =& \int_A^t \mathrm{d}s \int_{\mathbb{R}} \frac{e^{-(t-s)-\frac{(\sqrt{2}\alpha t - \sqrt{2}as)^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}} u^2(\sqrt{2}as, s)\sqrt{2}s\mathrm{d}a \\
\leq & \Sigma_1(A,t) + \Sigma_2(A,t) + \Sigma_3(A,t) + \Sigma_4(A,t),
\end{aligned}$$

where

$$\Sigma_1(A,t) := \int_A^t \mathrm{d}s \int_1^\infty \frac{e^{-(t-s)-\frac{(\sqrt{2}\alpha t-\sqrt{2}as)^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}}\sqrt{2}s\mathrm{d}a,$$

$$\Sigma_2(A,t) := \int_A^t \mathrm{d}s \int_{-\gamma}^1 \frac{e^{-(t-s)-\frac{(\sqrt{2}\alpha t-\sqrt{2}as)^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}}e^{-4\gamma(1-a)s+2\varepsilon s}\sqrt{2}s\mathrm{d}a,$$

$$\Sigma_3(A,t) := \int_A^t \mathrm{d}s \int_{K\alpha}^{-\gamma} \frac{e^{-(t-s)-\frac{(\sqrt{2}\alpha t-\sqrt{2}as)^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}}e^{-2(1+a^2)s+2\varepsilon s}\sqrt{2}s\mathrm{d}a,$$

$$\Sigma_4(A,t) := \int_A^t \mathrm{d}s \int_{-\infty}^{K\alpha} \frac{e^{-(t-s)-\frac{(\sqrt{2}\alpha t-\sqrt{2}as)^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}}e^{-2a^2 s}\sqrt{2}s\mathrm{d}a.$$

Recall g_α from (3.7). By change of variables $z = \sqrt{2}as - \sqrt{2}\alpha t$ and $s = ut$ and by (2.7), one gets that

$$\begin{aligned}\Sigma_1(A,t) &= \int_{A/t}^1 te^{-t(1-u)}\mathrm{d}u \int_{\sqrt{2}ut-\sqrt{2}\alpha t}^\infty \frac{e^{-\frac{z^2}{2t(1-u)}}}{\sqrt{2\pi(1-u)t}}\mathrm{d}z \\ &\le \int_{A/t}^1 te^{-t(1-u)}\frac{\sqrt{t(1-u)}}{\sqrt{2}(u-\alpha)t}e^{-\frac{(u-\alpha)^2}{1-u}t}\mathrm{d}u \le \frac{\sqrt{t}}{|\alpha|}\int_{A/t}^1 e^{-tg_\alpha(u)}\mathrm{d}u.\end{aligned}$$

Clearly, $g_\alpha(h) = g_\alpha(0) + g_\alpha'(0)h + o(h)$ as $|h| \to 0$. Note that $g_\alpha(0) = 1 + \alpha^2$ and $g_\alpha'(0) = (\alpha-1)^2 - 2 > 0$ for $\alpha < 1 - \sqrt{2}$. Note that, for any $u \in [\frac{A}{\sqrt{t}}, 1]$,

$$g_\alpha(u) \ge g_\alpha\left(\frac{A}{\sqrt{t}}\right) = g_\alpha(0) + (g_\alpha'(0) + o_t(1))\frac{A}{\sqrt{t}},$$

which implies that, for t sufficiently large,

$$\frac{\sqrt{t}}{|\alpha|}\int_{\frac{A}{\sqrt{t}}}^1 e^{-tg_\alpha(u)}\mathrm{d}u \le \frac{e^{-(1+\alpha^2)t}}{\sqrt{t}|\alpha|}te^{-\frac{Ag_\alpha'(0)}{2}\sqrt{t}} = o_t(1)\frac{e^{-(1+\alpha^2)t}}{\sqrt{t}}.$$

On the other hand, since $g_\alpha(h) = g_\alpha(0) + g_\alpha'(0)h + o(h)$ as $|h| \to 0$, then

$$\frac{\sqrt{t}}{|\alpha|}\int_{\frac{A}{t}}^{\frac{A}{\sqrt{t}}} e^{-tg_\alpha(u)}\mathrm{d}u = \frac{e^{-(1+\alpha^2)t}}{\sqrt{t}|\alpha|}\int_{\frac{A}{t}}^{\frac{A}{\sqrt{t}}} te^{-t(g_\alpha'(0)+o_t(1))u}\mathrm{d}u = o_{t,A}(1)\frac{e^{-(1+\alpha^2)t}}{\sqrt{t}|\alpha|}.$$

Thus $\Sigma_1(A,t) \le o_{t,A}(1)\frac{e^{-(1+\alpha^2)t}}{\sqrt{t}|\alpha|}$. Next, we shall handle $\Sigma_2(A,t)$. If $\alpha < -2\gamma$, then

$\gamma s - (\alpha+2\gamma)t > 0$. So, by change of variable $z = as - \alpha t + 2\gamma(s-t)$ and (2.7),

$$\begin{aligned}
\Sigma_2(A,t) &= \int_A^t e^{-(t-s)-4\gamma s+4\gamma\alpha t+4\gamma^2(t-s)+2\varepsilon s}\mathrm{d}s \int_{\gamma s-(\alpha+2\gamma)t}^{(1+2\gamma)s-(\alpha+2\gamma)t} \frac{e^{-\frac{z^2}{t-s}}}{\sqrt{2\pi(t-s)}}\sqrt{2}\mathrm{d}z \\
&\le \int_A^t e^{-(t-s)-4\gamma s+4\gamma\alpha t+4\gamma^2(t-s)+2\varepsilon s} \frac{\sqrt{t-s}}{\sqrt{2}\left(\gamma s-(\alpha+2\gamma)t\right)} e^{-\frac{(\gamma s-(\alpha+2\gamma)t)^2}{t-s}}\mathrm{d}s \\
&= \int_{A/t}^1 \frac{\sqrt{t(1-u)}}{\sqrt{2}\left(\gamma u-(\alpha+2\gamma)\right)} e^{-t[\frac{(\alpha+\gamma)^2}{1-u}-2\gamma(1+\gamma)(1-u)+2-2\alpha\gamma-2\varepsilon u]}\mathrm{d}u \\
&\le \frac{\sqrt{t}}{|\alpha|-2\gamma}\int_{A/t}^1 e^{-tg_{\alpha,\varepsilon}(u)}\mathrm{d}u,
\end{aligned}$$

where $g_{\alpha,\varepsilon}(u) = \frac{(\alpha+\gamma)^2}{1-u} - (1+\gamma^2)(1-u) + 2 - 2\alpha\gamma - 2\varepsilon u$. Observe that for $\varepsilon \in (0,1/2)$ and $u \in (0,1)$,

$$g'_{\alpha,\varepsilon}(u) = \frac{(\alpha+\gamma)^2}{(1-u)^2} + (1+\gamma^2) - 2\varepsilon \ge L_\varepsilon := (\alpha+\gamma)^2 + (1+\gamma^2) - 2\varepsilon,$$

and that $g_{\alpha,\varepsilon}(0) = \alpha^2+1$. Then, for any $h \in (0,1)$,

$$\min_{u\in[h,1]} g_{\alpha,\varepsilon}(u) \ge g_{\alpha,\varepsilon}(h) \ge \alpha^2+1+L_\varepsilon h.$$

This implies that if $\alpha < -2\gamma$, then

$$\Sigma_2(A,t) \le \frac{\sqrt{t}}{|\alpha|-2\gamma}\int_{A/t}^1 e^{-t(\alpha^2+1+L_\varepsilon u)}\mathrm{d}u = \frac{e^{-(1+\alpha^2)t}}{\sqrt{t}(|\alpha|-2\gamma)}\int_{A/t}^1 e^{-L_\varepsilon ut}t\mathrm{d}u,$$

which is $o_A(1)\frac{e^{-(1+\alpha^2)t}}{\sqrt{t}}$. If $-2\gamma \le \alpha < -\gamma$, then

$$\begin{aligned}
\Sigma_2(A,t) &\le \int_A^t \left(\int_{-\gamma}^1 \frac{e^{-(t-s)-\frac{(\sqrt{2}\alpha t-\sqrt{2}as)^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}} e^{-2\gamma(1-a)s+2\varepsilon s}\sqrt{2}sda\right)\mathrm{d}s \\
&= \int_A^t \left(\int_{-(\alpha+\gamma)t}^{(1+\gamma)s-(\alpha+\gamma)t} \frac{e^{-\frac{z^2}{t-s}}}{\sqrt{2\pi(t-s)}}\sqrt{2}\mathrm{d}z\right) e^{\gamma^2(t-s)+2\gamma\alpha t-(t-s)-2\gamma s+2\varepsilon s}\mathrm{d}s \\
&\le \int_A^t \frac{\sqrt{t-s}}{-(\alpha+\gamma)t} e^{-\frac{(\alpha+\gamma)^2t^2}{t-s}+\gamma^2(t-s)+2\gamma\alpha t-(t-s)-2\gamma s+2\varepsilon s}\mathrm{d}s \\
&= \int_{A/t}^1 \frac{\sqrt{t(1-u)}}{|\alpha|-\gamma} e^{-th(u)}\mathrm{d}u,
\end{aligned}$$

where in the first equality, we change variable $z = sa - \alpha t + \gamma(s-t)$, the second inequality holds by (2.7) and $h(u) = \frac{(\alpha+\gamma)^2}{1-u} - 2\varepsilon u + (1+\alpha^2) - (\alpha+\gamma)^2$. Note that for any $\varepsilon \in \left(0, \frac{(\alpha+\gamma)^2}{2}\right)$ and $u \in (0,1)$,

$$h'(u) = \frac{(\alpha+\gamma)^2}{(1-u)^2} - 2\delta \ge \widetilde{L}_\varepsilon := (\alpha+\gamma)^2 - 2\varepsilon > 0,$$

with $h(0) = \alpha^2 + 1$. It hence follows that if $-2\gamma \le \alpha < -\gamma$, then

$$\Sigma_2(A,t) \le \int_{A/t}^{1} \frac{\sqrt{t(1-u)}}{|\alpha| - \gamma} e^{-t(\alpha^2+1+\widetilde{L}_\varepsilon u)} \mathrm{d}u = o_A(1) \frac{e^{-(1+\alpha^2)t}}{\sqrt{t}}.$$

For $\Sigma_3(A,t)$, one sees that

$$\begin{aligned}\Sigma_3(A,t) &= \int_A^t e^{-t-s+2\varepsilon s - \frac{2\alpha^2 t^2}{2t-s}} \mathrm{d}s \int_{K\alpha}^{-\gamma} e^{-\frac{s(2t-s)}{t-s}(a - \frac{\alpha t}{2t-s})^2} \frac{s}{\sqrt{\pi(t-s)}} da \\ &\le \int_A^t \frac{\sqrt{s}}{\sqrt{2t-s}} e^{-t-s+2\varepsilon s - \frac{2\alpha^2 t^2}{2t-s}} \mathrm{d}s \le \frac{e^{-(1+\alpha^2)t}}{\sqrt{t}} \int_A^t \sqrt{s} e^{-(1-2\varepsilon)s} \mathrm{d}s,\end{aligned}$$

which is $o_A(1)\frac{e^{-(1+\alpha^2)t}}{\sqrt{t}}$ as long as $\varepsilon \in (0, 1/2)$. On the other hand,

$$\begin{aligned}\Sigma_4(A,t) &= \int_A^t \mathrm{d}s \int_{-\infty}^{K\alpha} \frac{e^{-(t-s)-\frac{(\sqrt{2}\alpha t - \sqrt{2} a s)^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}} e^{-2a^2 s} \sqrt{2} s da \\ &= \int_A^t e^{-t+s-\frac{2\alpha^2 t^2}{2t-s}} \mathrm{d}s \int_{-\infty}^{K\alpha} e^{-\frac{s(2t-s)}{t-s}(a - \frac{\alpha t}{2t-s})^2} \frac{s}{\sqrt{\pi(t-s)}} da \\ &= \int_A^t e^{-t+s-\frac{2\alpha^2 t^2}{2t-s}} \sqrt{\frac{s}{2t-s}} \mathrm{d}s \int_{-\infty}^{K\alpha - \frac{\alpha t}{2t-s}} e^{-\frac{s(2t-s)}{t-s} z^2} \frac{\mathrm{d}z}{\sqrt{\pi \frac{t-s}{s(2t-s)}}}.\end{aligned}$$

Choose $K > 1$ such that $(K-1)|\alpha| > 1$ and $K\alpha - \frac{\alpha t}{2t-s} < -1$. Then by (2.7),

$$\begin{aligned}\Sigma_4(A,t) &\le \int_A^t e^{-t+s-\frac{2\alpha^2 t^2}{2t-s}} \sqrt{\frac{s}{2t-s}} \frac{\sqrt{t-s}}{\sqrt{s(2t-s)}} e^{-\frac{s(2t-s)}{t-s}} \mathrm{d}s \\ &= \int_A^t \frac{\sqrt{t-s}}{2t-s} e^{-t-\frac{2\alpha^2 t^2}{2t-s}} e^{-s-\frac{s^2}{t-s}} \mathrm{d}s \le \frac{e^{-(1+\alpha^2)t}}{\sqrt{t}} \int_A^t e^{-s} \mathrm{d}s,\end{aligned}$$

as $\frac{1}{2t} \le \frac{1}{2t-s} \le \frac{1}{t}$ and $\sqrt{t-s} \le \sqrt{t}$. Therefore, $\Sigma_4(A,t) = o_A(1)\frac{e^{-(1+\alpha^2)t}}{\sqrt{t}}$. □

A.4 Proof of Lemma 5.1

Using again a domination argument, it is enough to prove Lemma 5.1 for $\phi \equiv 0$. We decompose it into the two following lemmas, that we prove one by one.

Lemma A.7.

$$\lim_{A\to\infty} \lim_{t\to\infty} \frac{e^{(1+\gamma^2)t}}{t^{3\gamma/4}} U_2(-\sqrt{2}\gamma t, t, [A\sqrt{t}, t]) = 0; \tag{A.29}$$

$$\lim_{A\to\infty} \lim_{t\to\infty} \frac{e^{(1+\gamma^2)t}}{t^{3\gamma/4}} U_2(-\sqrt{2}\gamma t, t, [0, \sqrt{t}/A]) = 0. \tag{A.30}$$

Lemma A.8. *For any $A > 0$ fixed,*

$$\lim_{K\to\infty}\lim_{t\to\infty}\frac{e^{(1+\gamma^2)t}}{t^{3\gamma/4}}U_2(-\sqrt{2}\gamma t,t,[\frac{1}{A}\sqrt{t},A\sqrt{t}],(-\infty,-K])=0 \tag{A.31}$$

and
$$\lim_{K\to\infty}\lim_{t\to\infty}\frac{e^{(1+\gamma^2)t}}{t^{3\gamma/4}}U_2(-\sqrt{2}\gamma t,t,[\sqrt{t}/A,A\sqrt{t}],[K,\infty))=0. \tag{A.32}$$

Proof of Lemma A.7. **Proof of (A.29):** Observe that

$$\begin{aligned}&U_2(-\sqrt{2}\gamma t,t,[A\sqrt{t},t])\\&\quad=U_2(-\sqrt{2}\gamma t,t,[A\sqrt{t},t],[-K,\infty)])+U_2(-\sqrt{2}\gamma t,t,[A\sqrt{t},t],(-\infty,-K))\\&\quad=:U_{(A.33)a}+U_{(A.33)b}.\end{aligned}\tag{A.33}$$

As $u(m_s+z,s)\le 1$, one sees that

$$\begin{aligned}U_{(A.33)a}&\le\int_{A\sqrt{t}}^{t}\mathrm{d}s\int_{-K}^{\infty}\mathrm{d}z\frac{e^{-(t-s)-\frac{(z+m_s+\sqrt{2}\gamma t)^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}}\\&=\int_{A\sqrt{t}}^{t}e^{-(t-s)}\mathrm{d}s\int_{-K+m_s+\sqrt{2}\gamma t}^{\infty}\frac{e^{-\frac{z^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}},\end{aligned}$$

which by (2.7) is bounded by

$$\begin{aligned}&\int_{A\sqrt{t}}^{t}e^{-(t-s)}\frac{\sqrt{t-s}}{-K+m_s+\sqrt{2}\gamma t}e^{-\frac{(K+m_s+\sqrt{2}\gamma t)^2}{2(t-s)}}\mathrm{d}s\\\le&c_4\frac{e^{-(1+\gamma^2)t}}{\sqrt{t}}\int_{A\sqrt{t}}^{t}e^{-\frac{2s^2}{t-s}+(\sqrt{2}K+\frac{3}{2}\log s)\frac{s+\gamma t}{t-s}}\mathrm{d}s.\end{aligned}$$

For t large enough, one has

$$\left(\sqrt{2}K+\frac{3}{2}\log s\right)\frac{s+\gamma t}{t-s}\le\begin{cases}\frac{s^2}{t-s}, & \text{if } s\in[\sqrt{t}\log t,t];\\ \frac{3\gamma}{2}\log s+2\sqrt{2}K+\frac{3\sqrt{2}(\log t)^2}{\sqrt{t}}, & \text{if } s\in[A\sqrt{t},\sqrt{t}\log t],\end{cases}$$

which implies that

$$U_{(A.33)a}\le c_5\frac{e^{-(1+\gamma^2)t}}{\sqrt{t}}\left(\int_{\sqrt{t}\log t}^{t}e^{-\frac{s^2}{t-s}}\mathrm{d}s+e^{2\sqrt{2}K}\int_{A\sqrt{t}}^{\sqrt{t}\log t}s^{3\gamma/2}e^{-\frac{2s^2}{t}}\mathrm{d}s\right)=o_t\left(\frac{e^{-(1+\gamma^2)t}}{\sqrt{t}}\right).$$

On the other hand, for s sufficiently large and $z<-K$, by similar reasonings as in Lemma A.4, we have for $\delta\in(0,1/2]$, $\eta=1-2\delta$, $\varepsilon<\frac{\gamma\eta}{1+2\gamma(1-\delta)}$,

$$\begin{aligned}U_{(A.33)b}\le&\,c_\delta^2\int_{A\sqrt{t}}^{\varepsilon t}\frac{\sqrt{t-s}e^{-(t-s)-\frac{(-\sqrt{2}\gamma t-m_s+K)^2}{2(t-s)}}}{-2s+\sqrt{2}\gamma\eta(t-s)}\mathrm{d}s\\&+c_\delta^2\int_{\varepsilon t}^{t}e^{-(t-s)(1-\gamma^2(1+\eta)^2)-\sqrt{2}\gamma(1+\eta)(m_s-\sqrt{2}\gamma t)}\mathrm{d}s\\=:&\,U_{(A.33)b1}+U_{(A.33)b2},\end{aligned}$$

that we bound separately.

Note that

$$U_{(\text{A.33})b2} = \int_\varepsilon^1 (ut)^{3\gamma(1+\eta/2)} t e^{-t[(1-u)(1-\gamma^2(1+\eta)^2)+2\gamma(1+\eta)(u+\gamma)]} \mathrm{d}u$$
$$\le t^{\frac{3\gamma}{2}(1+\eta)+1} e^{-t \min_{u\in[\varepsilon,1]}[(1-u)(1-\gamma^2(1+\eta)^2)+2\gamma(1+\eta)(u+\gamma)]}.$$

One can check that

$$\min_{u\in[\varepsilon,1]}[(1-u)(1-\gamma^2(1+\eta)^2)+2\gamma(1+\eta)(u+\gamma)]$$
$$= (1+\gamma^2)+\varepsilon(1+\gamma^2)\eta - \eta^2\gamma^2(1-\varepsilon).$$

Take $\varepsilon \in (\frac{\eta\gamma^2}{1+\gamma^2}, \frac{\gamma\eta}{1+2\gamma(1-\delta)})$ as $\eta = 1-2\delta \in (0,1)$. Then,

$$U_{(\text{A.33})b2} \le t^{\frac{3\gamma}{2}(1+\eta)+1} e^{-t(1+\gamma^2+\varepsilon\eta^2\gamma^2)} = o_t(1) t^{3\gamma/4} e^{-t(1+\gamma^2)}.$$

It remains to bound $U_{(\text{A.33})b1}$. Recalling (3.7), we observe that

$$U_{(\text{A.33})b1} \le \frac{C^{(7)}_{\delta,\varepsilon}}{\sqrt{t}} \int_{A\sqrt{t}}^{\varepsilon t} e^{-(t-s)-\frac{(-\sqrt{2}\gamma t - m_s + K)^2}{2(t-s)}} \mathrm{d}s$$
$$\le C^{(7)}_{\delta,\varepsilon} e^{C^{(8)}_{\delta,\varepsilon}K} \sqrt{t} \int_{\frac{A}{\sqrt{t}}}^{\varepsilon} e^{-tg_{-\gamma}(u)+\frac{3}{2}\log(ut)\frac{u+\gamma}{1-u}} \mathrm{d}u,$$

where we use the fact that for $s \in [A\sqrt{t}, \varepsilon t]$,

$$\frac{(-\sqrt{2}\gamma t - m_s + K)^2}{2(t-s)} = \frac{2(\gamma t + s)^2 + (\frac{3}{2\sqrt{2}}\log s + K)^2 - 2\sqrt{2}(\gamma t+s)(\frac{3}{2\sqrt{2}}\log s + K)}{2(t-s)}$$
$$\ge \frac{(\gamma t+s)^2}{t-s} - \frac{3(\gamma t + s)\log s}{2(t-s)} - \frac{\sqrt{2}\gamma K}{(1-\varepsilon)}. \tag{A.34}$$

Since $g_{-\gamma}(u) = 1+\gamma^2+2u^2+o(u^2)$, as $u \downarrow 0$, then

$$\sqrt{t}\int_{\frac{\log t}{\sqrt{t}}}^{\varepsilon} e^{-tg_{-\gamma}(u)+\frac{3}{2}\log(ut)\frac{u+\gamma}{1-u}} \mathrm{d}u \le \sqrt{t}\int_{\frac{\log t}{\sqrt{t}}}^{\varepsilon} (ut)^{3(\varepsilon+\gamma)/2} e^{-u^2t-(1+\gamma^2)t} \mathrm{d}u,$$

which is $o_t(1)t^{3\gamma/4}e^{-(1+\gamma^2)t}$. For $u \in [\frac{A}{\sqrt{t}}, \frac{\log t}{\sqrt{t}}]$, $\log(ut)\frac{u+\gamma}{1-u} = \gamma\log(ut) + o_t(1)$. Therefore,

$$\sqrt{t}\int_{\frac{A}{\sqrt{t}}}^{\frac{\log t}{\sqrt{t}}} e^{-tg_{-\gamma}(u)+\frac{3}{2}\log(ut)\frac{u+\gamma}{1-u}} \mathrm{d}u \le e\int_{\frac{A}{\sqrt{t}}}^{\frac{\log t}{\sqrt{t}}} (ut)^{3\gamma/2} e^{-t(1+\gamma^2)-u^2t} \mathrm{d}u$$
$$\le e t^{3\gamma/4} e^{-(1+\gamma^2)t} \int_A^{\log t} x^{3\gamma/2} e^{-2x^2} \mathrm{d}x = o_A(1) t^{3\gamma/4} e^{-(1+\gamma^2)t}.$$

We have completed the proof of (A.29).

Proof of (A.30): We have

$$\begin{aligned}
&U_2(-\sqrt{2}\gamma t, t, [0, \sqrt{t}/A]) \\
&\quad = U_2(-\sqrt{2}\gamma t, t, [0, \sqrt{t}/A], [-K, \infty)) + U_2(-\sqrt{2}\gamma t, t, [0, \sqrt{t}/A], [-\infty, -K]) \\
&\quad =: U_{(\mathrm{A}.35)a} + U_{(\mathrm{A}.35)b}.
\end{aligned} \tag{A.35}$$

As $u(m_s + a, s) \le 1$, applying (2.7) gives that for t large enough,

$$\begin{aligned}
U_{(\mathrm{A}.35)a} &\le \int_0^{\sqrt{t}/A} \frac{\sqrt{t-s}}{-K + m_s + \sqrt{2}\gamma t} e^{-(t-s) - \frac{(m_s + \sqrt{2}\gamma t - K)^2}{2(t-s)}} \mathrm{d}s \\
&\le c_7 e^{\sqrt{2}K} e^{-(1+\gamma^2)t} \int_0^{\sqrt{t}/A} s^{3\gamma/2} e^{-\frac{2s^2}{t}} \frac{\mathrm{d}s}{\sqrt{t}} \\
&= c_7 e^{\sqrt{2}K} t^{3\gamma/4} e^{-(1+\gamma^2)t} \int_0^{1/A} u^{3\gamma/2} e^{-2u^2} \mathrm{d}u,
\end{aligned}$$

which is $o_A(1) t^{3\gamma/4} e^{-(1+\gamma^2)t}$. Similarly as $U_{(\mathrm{A}.33)b1}$,

$$\begin{aligned}
U_{(\mathrm{A}.35)b} \le & C^{(12)}_{\delta,\varepsilon} e^{C^{(11)}_{\delta,\varepsilon} K} \sqrt{t} \int_0^{\frac{1}{A\sqrt{t}}} e^{-t g_{-\gamma}(u) + \frac{3\gamma}{2}\log(ut)} \frac{u+\gamma}{1-u} \mathrm{d}u \\
\le & C^{(1)}_{\delta,\varepsilon,K} t^{3\gamma/4} e^{-(1+\gamma^2)t} \int_0^{\frac{1}{A\sqrt{t}}} (u\sqrt{t})^{3\gamma/2} e^{-u^2 t} \sqrt{t} \mathrm{d}u \\
= & c_8 t^{3\gamma/4} e^{-(1+\gamma^2)t} \int_0^{1/A} u^{3\gamma/2} e^{-u^2} \mathrm{d}u = o_A(1) t^{3\gamma/4} e^{-(1+\gamma^2)t},
\end{aligned}$$

concluding (A.30). □

Proof of Lemma A.8. **Proof of (A.31):** Take $\delta \in (0, 1/3)$ and $\eta = 1 - 2\delta$. By similar reasoning as above, we have

$$\begin{aligned}
& \int_{\frac{\sqrt{t}}{A}}^{A\sqrt{t}} \mathrm{d}s \int_{-\infty}^{-K} \mathrm{d}z \frac{e^{-(t-s) - \frac{(z + m_s - \sqrt{2}ct)^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}} u^2(m_s + z, s) \\
\le & c_\delta^2 \int_{\frac{\sqrt{t}}{A}}^{A\sqrt{t}} \frac{\sqrt{t-s} e^{-(t-s) - \frac{(-\sqrt{2}\gamma t - m_s + K)^2}{2(t-s)} - \sqrt{2}\gamma(1+\eta)K}}{-2s + \sqrt{2}\gamma\eta(t-s)} \mathrm{d}s \\
\le & C^{(1)}_{\delta,\gamma,A} t^{3\gamma/4} e^{-K\sqrt{2}\gamma(1-3\delta)} \int_{\frac{\sqrt{t}}{A}}^{A\sqrt{t}} \frac{1}{\sqrt{t}} e^{-(t-s) - \frac{(s+\gamma t)^2}{t-s}} \mathrm{d}s \\
\le & C^{(2)}_{\delta,\gamma,A} e^{-K\sqrt{2}\gamma(1-2\delta)} t^{3\gamma/4} e^{-(1+\gamma^2)t},
\end{aligned}$$

where for the second inequality, we used the fact that for $s \in [\frac{\sqrt{t}}{A}, A\sqrt{t}]$,

$$\begin{aligned}
\frac{(-\sqrt{2}\gamma t - m_s + K)^2}{2(t-s)} &\ge \frac{(\gamma t + s)^2}{t-s} - \frac{3(\gamma t + s)\log s}{2(t-s)} - \frac{\sqrt{2}(\gamma t + s)\gamma K}{(t-s)} \\
&\ge \frac{(\gamma t + s)^2}{t-s} - \frac{3\gamma}{4}\log t - \sqrt{2}\gamma^2 K + o_t(1).
\end{aligned}$$

and the last inequality follows from the fact that $(t-s)+\frac{(s+\gamma t)^2}{t-s}=(1+\gamma^2)t+\frac{(1+\gamma)^2s^2}{t-s}$.

Proof of (A.32)**:** For $z\geq K$, using the fact $u(m_s+z,s)\leq 1$, we obtain that

$$U_2(-\sqrt{2}\gamma t,t,[\sqrt{t}/A,A\sqrt{t}],[K,\infty))\leq \int_{\frac{\sqrt{t}}{A}}^{A\sqrt{t}}\mathrm{d}s\int_K^\infty \mathrm{d}z\frac{e^{-(t-s)-\frac{(z+m_s+\sqrt{2}\gamma t)^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}}$$
$$=\int_{\frac{\sqrt{t}}{A}}^{A\sqrt{t}}e^{-(t-s)}\mathrm{d}s\int_{K+m_s+\sqrt{2}\gamma t}^\infty \mathrm{d}z\frac{e^{-\frac{z^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}},$$

which by (2.7) is less than

$$\int_{\frac{\sqrt{t}}{A}}^{A\sqrt{t}}e^{-(t-s)}\frac{\sqrt{t-s}}{K+m_s+\sqrt{2}\gamma t}e^{-\frac{(K+m_s+\sqrt{2}\gamma t)^2}{2(t-s)}}\mathrm{d}s\leq \frac{C_{\gamma,A}}{\sqrt{t}}\int_{\frac{\sqrt{t}}{A}}^{A\sqrt{t}}e^{-(t-s)-\frac{(m_s+\sqrt{2}\gamma t)^2}{2(t-s)}-K\frac{\sqrt{2}\gamma t+m_s}{t-s}}\mathrm{d}s.$$

Similarly as above, we end up with

$$U_2(-\sqrt{2}\gamma t,t,[\sqrt{t}/A,A\sqrt{t}],[K,\infty))\leq C_{\gamma,A}e^{-K\sqrt{2}\gamma}t^{3\gamma/4}e^{-(1+\gamma^2)t}.$$

This suffices to conclude (A.32). □

References

[ABBS13] E. Aïdékon, J. Berestycki, É. Brunet, and Z. Shi. Branching Brownian motion seen from its tip. *Probab. Theory Relat. Fields*, 157(1-2):405–451, 2013.

[ABK11] L.-P. Arguin, A. Bovier, and N. Kistler. Genealogy of extremal particles of branching Brownian motion. *Commun. Pure Appl. Math.*, 64(12):1647–1676, 2011.

[ABK13] L.-P. Arguin, A. Bovier, and N. Kistler. The extremal process of branching Brownian motion. *Probab. Theory Relat. Fields*, 157(3-4):535–574, 2013.

[BBCM18] Julien Berestycki, Éric Brunet, Aser Cortines, and Bastien Mallein. Extremes of branching Ornstein-Uhlenbeck processes. Submitted, oct 2018.

[BGMS14] Nathanaël Berestycki, Nina Gantert, Peter Mörters, and Nadia Sidorova. Galton-Watson trees with vanishing martingale limit. *J. Stat. Phys.*, 155(4):737–762, 2014.

[BM19] D. Buraczewski and M. Maślanka. Large deviation estimates for branching random walks. *ESAIM: Prob. Stats.*, 23:823–840, 2019.

[Bov17] A. Bovier. *Gaussian processes on trees. From spin glasses to branching Brownian motion*, volume 163. Cambridge: Cambridge University Press, 2017.

[Bra83] Maury Bramson. *Convergence of solutions of the Kolmogorov equation to travelling waves*, volume 285. Providence, RI: American Mathematical Society (AMS), 1983.

[CH20] X. Chen and H. He. Lower deviation and moderate deviation probabilities for maximum of a branching random walk. *Ann. Inst. Henri Poincaré Probab. Stat.*, to appear, 2020+.

[CR88] B. Chauvin and A. Rouault. KPP equation and supercritical branching Brownian motion in the subcritical speed area. Application to spatial trees. *Probab. Theory Relat. Fields*, 80(2):299–314, 1988.

[CR90] B. Chauvin and A. Rouault. Supercritical branching Brownian motion and K-P-P equation in the critical speed-area. *Math. Nachr.*, 149:41–59, 1990.

[DS17a] B. Derrida and Z. Shi. Slower deviations of the branching Brownian motion and of branching random walks. *J. Phys. A, Math. Theor.*, 50(34):13, 2017.

[DS17b] B. Derrida and Z. Shi. Large deviations for the rightmost position in a branching Brownian motion. In *Modern problems of stochastic analysis and statistics. Selected contributions in honor of Valentin Konakov's 70th birthday, Moscow, Russia, May 29 – June 2, 2016*, pages 303–312. Cham: Springer, 2017.

[DZ98] Amir Dembo and Ofer Zeitouni. *Large deviations techniques and applications. 2nd ed.*, volume 38. New York, NY: Springer, 2nd ed. edition, 1998.

[GH18] Nina Gantert and Thomas Höfelsauer. Large deviations for the maximum of a branching random walk. *Electron. Commun. Probab.*, 23:12, 2018. No 34.

[LS87] S. P. Lalley and T. Sellke. A Conditional Limit Theorem for the Frontier of a Branching Brownian Motion. *Ann. Probab.*, 15(3):1052–1061, 1987.

[McK75] H. P. McKean. Application of Brownian motion to the equation of Kolmogorov-Petrovskii- Piskunov. *Commun. Pure Appl. Math.*, 28:323–331, 1975.

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