Generating random alternating permutations in time $n \log n$

P. Marchal

Université Paris 13
Sorbonne Paris Cité
LAGA
CNRS (UMR 7539)
F-93430, Villetaneuse

marchal@math.univ-paris13.fr

Abstract

We introduce an algorithm generating uniformly distributed random alternating permutations of length n in time $n \log n$.

1 The main result

An alternating permutation σ of $\{1, 2, \dots N\}$ is a permutation such that

$$\sigma(1) > \sigma(2) < \sigma(3) > \sigma(4) \dots$$

Alternating permutations are a very classical topic in combinatorics. See for instance the surveys [KPP, ST] for numerous references and the link with increasing trees. The basic result, which dates back from the 19th century [A], states that if p_N is the probability that a permutation of $\{1, 2, \dots N\}$ chosen uniformly at random is alternating, then

$$\sum_{N \geq 0} p_N x^N = \sec x + \tan x$$

The integers $N! p_N$ are called Euler numbers. The goal of this paper is to introduce an algorithm generating random alternating permutations of $\{1, 2, \dots N\}$ in time $N \log N$.

An alternative way to generate random alternating permutations is to use Boltzmann sampling [DFLS, BRS]. In a general framework, this method constructs random combinatorial objects of approximate size in linear time. However, to get the exact size, one must use a rejection procedure and the complexity of the algorithm increases. In the context of permutations, the average time to run the algorithm is quadratic in the length of the permutation.

Before describing our algorithm, we begin by a simple remark. Let $Y = (Y_1, \dots Y_N)$ be a sequence of distinct reals in $[0, 1]$. We can construct from Y a permutation as follows. Let $k_1 \in \{1, 2, \dots N\}$ be the integer such that Y_{k_1} is minimal in $\{Y_1, \dots Y_N\}$ and put $\sigma(k_1) = 1$. Then, let $k_2 \in \{1, 2, \dots N\} - \{k_1\}$ be the integer such that Y_{k_2} is minimal in $\{Y_1, \dots Y_N\} - \{Y_{k_1}\}$, put $\sigma(k_2) = 2$ and so on. To recover σ from Y, one can use a sorting algorithm.

If the sequence is alternating, that is, if

$$Y_1 > Y_2 < Y_3 > Y_4 \dots$$

then σ is alternating. Besides, if Y is chosen according to the Lebesgue measure on $[0, 1]^N$, then σ is uniformly distributed over all permutations of $\{1, 2, \dots N\}$. As a consequence, if Y is a random, uniform alternating sequence, that is, if its density with respect to the Lebesgue measure on $[0, 1]^N$ is

$$(1/p_N)\mathbf{1}_{\{Y_1 > Y_2 < Y_3 > Y_4 \dots\}} \tag{1}$$

then σ is uniformly distributed over all alternating permutations. Therefore, we want to find an algorithm constructing a random, uniform alternating sequence. With probability 1, all the reals in the sequence will be distinct. Our method, although this does not appear in the proof, is based on the general theory of quasitationary Markov chains. Further comments on this point can be found in the concluding remarks of the paper.

Algorithm

Fix an integer N and let $U_1, U_2, \dots U_N$ be iid random variables, uniformly distributed on $[0, 1]$. First, define a random sequence $(X_1, \dots, X_N)$ as follows:

- $X_1 = U_1$
- for $n \in [1, N - 1]$,

$$X_{n+1} = 1 - \frac{2}{\pi} \arcsin\Big(U_{n+1} \sin(\frac{\pi}{2} X_n)\Big)$$

Next, put

$$\alpha_N = \frac{\sin(\frac{\pi}{2} X_N)}{\sin(\frac{\pi}{2} X_1)} \tag{2}$$

and define the sequence Y as follows

- With probability $1/(\alpha_N + \alpha_N^{-1})$, put

$$Y = (X_1, 1 - X_2, X_3, 1 - X_4 \dots)$$

- With probability $1/(\alpha_N + \alpha_N^{-1})$, put

$$Y = (X_N, 1 - X_{N-1}, X_{N-2}, 1 - X_{N-3} \dots)$$

- With probability $1 - 2/(\alpha_N + \alpha_N^{-1})$, start over.

Finally, recover a permutation from Y by randomized quicksort.

Theorem 1 *The algorithm described above yields a random alternating permutation of $\{1, 2, \dots N\}$, uniformly distributed over all alternating permutations. The rejection probability is bounded above, uniformly on N, by*

$$\mathbb{E}[1 - 2/(\alpha_N + \alpha_N^{-1})] \leq 1 - \frac{2}{3\pi}$$

As a consequence, the average complexity is $\Theta(N \log N)$.

The upper bound on the rejection probability is certainly not optimal, however, it is sufficient for our purpose. We now proceed to the proof of the theorem.

2 Proof of the result

First, let us observe that the random sequence X is a Markov chain. Let us compute its transition probabilities. Let $x \in [0, 1]$ and $n \geq 1$. With probability x, $U_n \leq 1 - x$ and therefore, conditionally on X_n, with probability x,

$$X_{n+1} \geq 1 - \frac{2}{\pi} \arcsin\Big((1 - x) \sin(\frac{\pi}{2} X_n)\Big)$$

In other words, the right-hand side of the inequality above is the inverse of F_{n+1}, where F_{n+1} stands for the cumulative distribution function of X_{n+1}, conditionally on X_n. Therefore $F_{n+1}(x) = 0$ for $x \leq 1 - X_n$ and for $x \geq 1 - X_n$,

$$F_{n+1}(x) = 1 - \frac{\sin(\frac{\pi}{2}(1 - x))}{\sin(\frac{\pi}{2} X_n)} = 1 - \frac{\cos(\frac{\pi}{2} x)}{\sin(\frac{\pi}{2} X_n)}$$

Differentiating, we find that the conditional density of X_{n+1} given X_n is f_{X_n}, with

$$f_x(y) = \mathbf{1}_{\{y \geq 1 - x\}} \left(\frac{\pi}{2}\right) \frac{\sin(\frac{\pi}{2} y)}{\sin(\frac{\pi}{2} x)}$$

To sum up, the transition probabilities of X are given by

$$\mathbb{P}_x(X_2 \in dy) = f_x(y) dy$$

where, as usual, $\mathbb{P}_x$ denotes the probability for the Markov chain started at x. As a consequence, the density of X with respect to the Lebesgue measure on $[0, 1]^N$ is

$$\begin{aligned} &\mathbf{1}_{\{X_2 \geq 1 - X_1, X_3 \geq 1 - X_2, \dots\}} \left(\frac{\pi}{2}\right)^{N-1} \frac{\sin(\frac{\pi}{2} X_N)}{\sin(\frac{\pi}{2} X_1)} \\ &= \mathbf{1}_{\{X_2 \geq 1 - X_1, X_3 \geq 1 - X_2, \dots\}} \left(\frac{\pi}{2}\right)^{N-1} \alpha_N \end{aligned}$$

with α_N defined in (2). Likewise, the density of $(X_N, X_{N-1}, X_{N-2}, X_{N-3}\dots)$ with respect to the Lebesgue measure on $[0,1]^N$ is

$$\mathbf{1}_{\{X_2 \geq 1-X_1, X_3 \geq 1-X_2, \dots\}} \left(\frac{\pi}{2}\right)^{N-1} \alpha_N^{-1}$$

Now define a random sequence Z as follows. With probability $1/(\alpha_N + \alpha_N^{-1})$, put

$$Z = (X_1, X_2, X_3 \dots)$$

With probability $1/(\alpha_N + \alpha_N^{-1})$, put

$$Z = (X_N, X_{N-1}, X_{N-2}, \dots)$$

Since Z is obtained either by keeping X or by taking its time reversal, its density is a convex combination of the density of X and of the density of the time-reversal of X. More precisely, the density of Z with respect to the Lebesgue measure on $[0,1]^N$ is given by

$$\begin{aligned}
&\mathbf{1}_{\{Z_2 \geq 1-Z_1, Z_3 \geq 1-Z_2, \dots\}} \left(\frac{\pi}{2}\right)^{N-1} \left[\frac{\alpha_N}{\alpha_N + \alpha_N^{-1}} + \frac{\alpha_N^{-1}}{\alpha_N + \alpha_N^{-1}}\right] \\
&= \mathbf{1}_{\{Z_2 \geq 1-Z_1, Z_3 \geq 1-Z_2, \dots\}} \left(\frac{\pi}{2}\right)^{N-1}
\end{aligned}$$

The total mass of the density is less than 1 since there is a positive probability that Z is not defined. On the event that Z is not defined, start over until the the process yields a sequence Z. Then almost surely, the procedure terminates and yields a random sequence Z with density

$$\frac{1}{Q_N} \left(\frac{\pi}{2}\right)^{N-1} \mathbf{1}_{\{Z_2 \geq 1-Z_1, Z_3 \geq 1-Z_2, \dots\}}$$

where Q_N is the non-rejection probability

$$Q_N = \mathbb{E}[2/(\alpha_N + \alpha_N^{-1})]$$

Finally, the sequence Y in the algorithm is given by

$$Y = (Z_1, 1 - Z_2, Z_3, 1 - Z_4 \dots)$$

and the density of Y is

$$\frac{1}{Q_N} \left(\frac{\pi}{2}\right)^{N-1} \mathbf{1}_{\{Y_1 \geq Y_2 \leq Y_3 \geq Y_4 \dots\}} \tag{3}$$

which is exactly what we were aiming at. Therefore, producing a random permutation from Y yields a random, uniform alternating permutation.

Let us prove the bound on the rejection probability. First, the invariant probability measure of the Markov chain X is

$$\mu(dy) = 2\sin^2(\frac{\pi}{2}y)dy$$

Indeed,

$$\int_0^1 \mu(dx)f_x(y) = 2\int_{1-y}^1 dx\frac{\pi}{2}\sin(\frac{\pi}{2}x)\sin(\frac{\pi}{2}y) = 2\sin^2(\frac{\pi}{2}y) = \mu(dy)$$

The non-rejection probability can be expressed as

$$Q_N = \mathbb{E}[2/(\alpha_N + \alpha_N^{-1})] = 2\mathbb{E}\frac{\sin(\frac{\pi}{2}X_N)\sin(\frac{\pi}{2}X_1)}{\sin^2(\frac{\pi}{2}X_N)+\sin^2(\frac{\pi}{2}X_1)} \geq \mathbb{E}\sin(\frac{\pi}{2}X_N)\sin(\frac{\pi}{2}X_1)$$

Write

$$\begin{aligned}
\mathbb{E}\sin(\frac{\pi}{2}X_N)\sin(\frac{\pi}{2}X_1) &= \int_0^1 dx\int_0^1 \mathbb{P}_x(X_N\in dy)\sin(\frac{\pi}{2}x)\sin(\frac{\pi}{2}y)\\
&\geq \int_0^1 dx\int_0^1 \mathbb{P}_x(X_N\in dy)\sin^2(\frac{\pi}{2}x)\sin(\frac{\pi}{2}y)\\
&= \frac{1}{2}\int_0^1 \mu(dx)\int_0^1 \mathbb{P}_x(X_N\in dy)\sin(\frac{\pi}{2}y)\\
&= \frac{1}{2}\mathbb{E}_\mu\sin(\frac{\pi}{2}X_N)
\end{aligned}$$

where $\mathbb{E}_\mu$ stands for the expectation for the Markov chain started with initial distribution μ. Since μ is the invariant measure, under $\mathbb{P}_\mu$, X_N is distributed according to μ. Hence

$$\mathbb{E}_\mu\sin(\frac{\pi}{2}X_N)) = \int_0^1 sin^3(\frac{\pi}{2}x)dx\frac{2}{\pi}\int_0^1 1-z^2dz = \frac{4}{3\pi}$$

This gives the bound in Theorem 1.

Finally, notice that using randomized quicksort, we get the same complexity on average as if we were using quicksort with a random, uniform permutation (whereas we are dealing here with a random, uniform alternating permutation). See [K].

3 Concluding remarks

As already mentioned in the first section, the method used here is based on the theory of quasistationary distributions for Markov chains. Indeed, a random sequence on $[0,1]$ can be viewed as a Markov chain, and a random alternating sequence is a submarkovian chain obtained from the initial Markov chain by forbidding some transitions. Transforming this submarkovian chain into a Markov chain leads to the definition of X. See [T] for a reference on quasistationary distributions on a continuous state space. The sequence $(X_1, 1-X_2, X_3, 1-X_4 \ldots)$ is, loosely speaking, a sequence conditioned to be alternating forever. If we want to obtain a finite-length alternating sequence, we get the bias α_N. To eliminate this bias, we need to use a rejection procedure.

When $N \to \infty$, the law of X_N converges to the invariant measure and X_N is asymptotically independent of X_1. Therefore, the rejection probability converges to

$$1 - 4\int_0^1 dx \int_0^1 dy \frac{\sin(\frac{\pi}{2}x)\sin^3(\frac{\pi}{2}y)}{\sin^2(\frac{\pi}{2}x)+\sin^2(\frac{\pi}{2}y)} = 1 - \frac{8}{\pi^2}$$

the integral being is easily computed by symmetrizing the integrand.

Alternatively, comparing (1) with (3), we see that $(2/\pi)^N Q_N = p_N$. Moreover, it is known that for large N,

$$p_N \sim \frac{4}{\pi}\left(\frac{2}{\pi}\right)^{N+1}$$

(see Example IV.35 in [FS]) whence $Q_N \to 8/\pi^2$ as $N \to \infty$.

This method inpired by quasistationary distributions may be adapted to generate more general types of permutations.

Acknowledgments. Christian Houdré brought my attention to this topic. I thank Axel Bacher, Cyril Banderier and Olivier Bodini for very interesting discussions.

References

[A] André, D. Développements de sec x et tan x. *Comptes Rendus Acad. Sci.*, Paris **88**, 965-967, 1879

[BRS] Bodini, Olivier; Roussel, Olivier; Soria, Michle; Boltzmann samplers for first-order differential specifications. *Discrete Appl. Math.* **160** (2012), no. 18, 25632572.

[DFLS] Duchon, Philippe; Flajolet, Philippe; Louchard, Guy; Schaeffer, Gilles. Boltzmann samplers for the random generation of combinatorial structures. *Combin. Probab. Comput.***13** (2004), no. 4-5, 577-625.

[FS] Flajolet, Philippe; Sedgewick, Robert. Analytic combinatorics. Cambridge University Press, Cambridge, 2009.

[K] Knuth, Donald E. *The art of computer programming.* Volume 3. Sorting and searching. Addison-Wesley Series in Computer Science and Information Processing. Addison-Wesley Publishing Co., 1973.

[KPP] Kuznetsov, A. G.; Pak, I. M.; Postnikov, A. E. Increasing trees and alternating permutations. *Uspekhi Mat. Nauk* **49** (1994), no. 6(300), 79–110.

[ST] Stanley, Richard P. A survey of alternating permutations. *Combinatorics and graphs*, 165-196, Contemp. Math., **531**, Amer. Math. Soc., Providence, RI, 2010.

[T] Tweedie, Richard L. R-theory for Markov chains on a general state space. I. Solidarity properties and R-recurrent chains. *Ann. Probability*, **2** (1974), 840864.