Positive polynomials and numerical approximation

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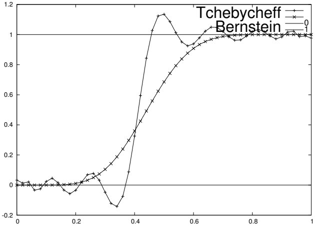
thanks M. Herda, F. Charles, M. Campos-Pinto

The main difficulty on an example

History: Godunov, VanLeer, Harten, Roe, Sweby, Shu-Osher ENO 88', P. Lax (Gibbs phenomena, 2006) ...
Modern: Shu, Bound-preserving high order accurate schemes 2013, ...

In Journal Computational Physics (27/10/2019), articles with : "maximum principle" (398), "TVD" (1016), "ENO" (643), ...

Approximation of the Heaviside step function $H(x - 0.43)$
with high order real polynomials $2 \leq \deg(p) < \infty$.



- Tchebycheff interpolation $\Longrightarrow$ huge oscillations
- Bernstein interpolation $\Longrightarrow$ no oscillation but low order approximation

The Lukacs(-Markov) Theorem

Introduction

Positive interpolation

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Take the convex set $P_n^+ \subset P_n$: $\quad P_n^+ = \{p \in P_n : p(x) \geq 0 \ \forall x \in [0,1]\}$.

- First case: $\mathbf{n} = \mathbf{2k}$. Then $p \in P_n^+$ **if and only if** there exists $(a, b) \in P_k \times P_{k-1}$ such that

$$p(x) = a(x)^2 + x(1-x)\, b(x)^2.$$

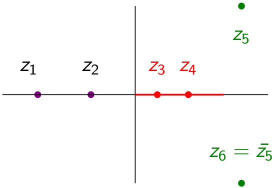
- Second case: $\mathbf{n} = \mathbf{2k} + \mathbf{1}$. Then $p \in P_n^+$ **if and only if** there exists $a, b \in P_k$ such that

$$p(x) = x\, a(x)^2 + (1-x)\, b(x)^2.$$

- Non-uniqueness $1 = 1^2 + x(1-x)0^2 = (1-2x)^2 + x(1-x)2^2$.

A simple direct proof

- $n = 2: \; p(x) = \left(\sqrt{p(0)}(1-x) - \sqrt{p(1)}x\right)^2 + x(1-x)\underbrace{b^2}_{\geq 0}.$
- $n \in 2\mathbb{N}$:
$$p(x) = \prod_{j=1}^{n/2} p_j(x), \text{ with } p_j \in P_2^+$$
$$= \prod_{j=1}^{n/2} \left| a_j(x) + \mathbf{i}\sqrt{x(1-x)}b_j \right|^2$$
$$= \left| \prod_{j=1}^{n/2} \left(a_j(x) + \mathbf{i}\sqrt{x(1-x)}b_j \right) \right|^2$$
$$= \left| a(x) + \mathbf{i}\sqrt{x(1-x)}b \right|^2 = a(x)^2 + x(1-x)b^2$$



- $n \in 2\mathbb{N}+1$:
$xp(x) = \hat{p}(x) = \hat{a}(x)^2 + x(1-x)\hat{b}^2 = (xa(x))^2 + x(1-x)b^2.$
Simplify by x.

Two bounds

$$U_{2n} \underset{def}{:=} \{p \in P_{2n}^{+} \mid 1 - p_n \in P_{2n}^{+}\}.$$

- $p \in P_{2n}^{+}$: the Lukacs theorem yields the representation

$$p = a^2 + b^2 w \tag{1}$$

where $w(x) = x(1-x)$ is the weight.

- $1 - p \in P_{2n}^{+}$: similarly

$$1 - p = c^2 + d^2 w \tag{2}$$

- Therefore $p \in U_{2n}$ iff there exists 4 polynomials (a, b, c, d) such that

$$1 = a^2 + b^2 w + c^2 + d^2 w.$$

$\boxed{\text{Set}}$

$$\mathcal{U}_n = \Big\{(a, b, c, d) \in P_n \times P_{n-1} \times P_n \times P_{n-1} \text{ such that } 1 = a^2 + b^2 w + c^2 + d^2 w\Big\}.$$

Basic properties

- Consider the algebra

$$\begin{cases} A &= a\alpha &+wb\beta &+c\gamma &+wd\delta, \\ B &= a\beta &-b\alpha &+c\delta &-d\gamma, \\ C &= a\gamma &-wb\delta &-c\alpha &+wd\beta, \\ D &= a\delta &+b\gamma &-c\beta &-d\alpha. \end{cases}$$

One has the weighted 4-squares Euler identity

$$A^2+B^2w+C^2+D^2w = \left(a^2+b^2w+c^2+d^2w\right)\left(\alpha^2+\beta^2w+\gamma^2+\delta^2w\right).$$

- Euler. Novi commentarii academiae scientiarum Petropolitanae, 1760.

- Take $(a,b,c,d) \in \mathcal{U}_n$ and $(\alpha,\beta,\gamma,\delta) \in \mathcal{U}_m$. Then $(A,B,C,D) \in \mathcal{U}_{n+m}$.

The usual quaternions are $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$. Writing

$$\begin{aligned} e &= a + \mathbf{i}b\sqrt{w} + \mathbf{j}c + \mathbf{k}d\sqrt{w} \in \mathcal{Q}_n, \\ \epsilon &= \alpha + \mathbf{i}\beta\sqrt{w} + \mathbf{j}\gamma + \mathbf{k}\delta\sqrt{w} \in \mathcal{Q}_m, \\ E &= A + \mathbf{i}B\sqrt{w} + \mathbf{j}C + \mathbf{k}D\sqrt{w} \in \mathcal{Q}_{n+m}, \end{aligned}$$

the algebra is rephrased as $E = \epsilon\overline{e}$ and $|E|^2 = |\epsilon|^2|e|^2$.

Introduction

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DEMONSTRATIO
THEOREMATIS FERMATIANI
OMNEM NVMERVM PRIMVM FORMAE $4n+1$ ESSE SVMMAM DVORVM QVADRATORVM.

AVCTORE *LEONARDO EVLERO*

§. 1.

Cum nuper eos eſſem contemplatus numeros, qui ex additione duorum quadratorum oriuntur, plures demonſtraui proprietates, quibus tales numeri ſunt praediti: neque tamen meas meditationes eo vsque perducere licuit, vt huius theorematis, quod Fermatius olim Geometris demonſtrandum propoſuit, veritatem ſolide oſtendere potuiſſem. Tentamen tamen demonſtrationis tum expoſui, vnde certitudo huius theorematis multo luculentius elucet, etiamſi criteriis rigidae demonſtrationis deſtituatur: neque dubitaui, quin iisdem veſtigiis inſiſtendo tandem demonſtratio deſiderata facilius obtineri poſſit; quod quidem ex eo tempore mihi ipſi vſu venit, ita, vt tentamen illud, ſi alia quaedam leuis conſideratio accedat, in rigidam demonſtrationem abeat. Nihil quidem noui in hac re me praeſtitiſſe gloriari poſſum, cum ipſe Fermatius iam demonſtrationem huius theorematis elicuiſſe ſe profiteatur; verum, quod eam nusquam publici iuris fecit, eius iactura perinde ac plurimorum aliorum egregiorum huius viri inuentorum efficit, vt, quae nunc demum de his deperditis rebus quaſi recuperamus, ea non immerito pro nouis inuentis habeantur. Cum enim nemo vnquam

A 2 tam

ſemper in quatuor quadrata ſaltem in fractis reſolui poteſt. Multiplicemus enim numeratorem et denominatorem per $pp+qq+rr+ss$, vt denominator fiat quadratus, erit quotus iſte $= \frac{(aa+bb+cc+dd)(pp+qq+rr+ss)}{(pp+qq+rr+ss)^2}$; quod ſi iam numerator in quatuor quadrata reſolui queat, ipſa fractio aequabitur aggregato quatuor quadratorum. At numerator pluribus modis in quatuor quadrata reſolui poteſt; ſi enim ponatur $(aa+bb+cc+dd)(pp+qq+rr+ss) = xx+yy+zz+vv$, erit

$$\left.\begin{aligned} x &= ap+bq+cr+ds \\ y &= aq-bp\pm cs\mp dr \\ z &= ar\mp bs-cp\pm dq \\ v &= as\pm br\mp cq-dp \end{aligned}\right\}$$

qui quatuor numeri, ſi ſinguli diuidantur per communem denominatorem $pp+qq+rr+ss$, dabunt radices quatuor quadratorum, quorum ſumma aequatur quoto propoſito.

Niſi igitur hi numeri x, y, z, et v ſint diuiſibiles per $pp+qq+rr+ss$, ſaltem in fractis aſſignari poſſunt quatuor quadrata, quorum ſumma aequalis eſt quoto $\frac{aa+bb+cc+dd}{pp+qq+rr+ss}$.

COROLL. 1.

94. Quae hic de quatuor quadratorum ſummis ſunt demonſtrata, etiam ad ſummas trium, vel etiam duorum patent, cum nihil impediat, quominus vnus, vel duo ex numeris a, b, c, d, et p, q, r, s ſint aequales nihilo.

COROLL. 2.

94. Si igitur ſumma trium quadratorum per ſummam quatuor, vel etiam trium quadratorum diuidatur, quotus certe erit ſumma quatuor quadratorum.

COROL.

A first result

- Then $e = a + \mathbf{i} b\sqrt{w} + \mathbf{j} c + \mathbf{k} d\sqrt{w} \in \mathcal{U}_1$ can be parametrized with 3 angles $(\theta, \varphi, \mu) \in \mathbb{R}^3$

$$\left\{ \begin{array}{cl} a(x) & = \cos\theta x + \cos\varphi(1-x), \\ b & = R\cos\mu, \\ c(x) & = \sin\theta x + \sin\varphi(1-x), \\ d & = R\sin\mu. \end{array} \right. \qquad R = 2\sin\left(\frac{\theta-\varphi}{2}\right)$$

In the sense of multiplication of quaternions, one has $(\mathcal{U}_1)^n \subset \mathcal{U}_n$. One shows any quadruplet $q \in \mathcal{U}_n$ admits a factorization in at most n elements in $\mathcal{U}_1$: $q = e_1 e_2 \dots e_n$ where all $e_{1,\dots} \in \mathcal{U}_1$. So $\mathcal{U}_n \subset (\mathcal{U}_1)^n$.

Theorem (D.+Herda 2017): Let $n = 2k$. There exists a smooth function from $\mathbb{R}^{3k}$ onto U_n. The smooth function is explicit and is 2π-periodic with respect to all its arguments.

Multivariate case (one lower bound only)

- Take a semi-algebraic set

$$\mathbb{K} = \left\{ \mathbf{x} \in \mathbb{R}^d \text{ such that } g_j(\mathbf{x}) \geq 0 \text{ for } g_j \in \mathrm{P}[\mathbf{X}], 1 \leq j \leq j_* \right\}.$$

- Take a SOS (Sum of Squares) real polynomial $p \in \mathbb{Q}(i_*)$

$$\mathbb{Q}(i_*) = \left\{ p = \sum_{j=1}^{j_*} g_j \left(\sum_{i=1}^{i_*} p_{ij}^2 \right) = \sum_{i=1}^{i_*} \left(\sum_{j=1}^{j_*} g_j p_{ij}^2 \right) = \sum_{j=1}^{j_*} \sum_{i=1}^{i_*} g_j p_{ij}^2 \right\}.$$

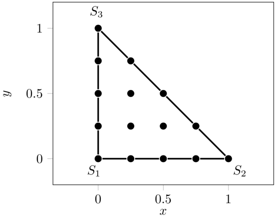

Figure 8: The simplex $\mathbb{K}$ and interpolation points for $n = 4$.

Exemple: $p(\mathbf{x}) = \sum_{j=1}^{3} g_j(\mathbf{x}) \sum_{i=1}^{i_*} p_{ij}^2(\mathbf{x})$.

Here $g_j = \lambda_j$. Also i_* and $\deg(p_{ij})$ "large enough".

Barycentric functions $\lambda_1(\mathbf{x}) = 1 - x - y$, $\lambda_2(\mathbf{x}) = x$ and $\lambda_3(\mathbf{x}) = y$.

Putinar Positivstellensatz

Theorem (Putinar 93', Lasserre 2010', Powers):
Assume there exists i_s and a polynomial $u \in \mathbb{Q}(i_s)$ such that $\{u(\mathbf{x}) \geq 0\}$ is a compact set. Assume $p \in \mathbb{R}[\mathbf{x}]$ is strictly positive on $\mathbb{K}$.
There exists i_* such that $p \in \mathbb{Q}(i_*)$.

• The Putinar Positivstellensatz holds for the triangle.
Consider the the barycentric functions and polynomial

$$u = \sum_{j=1}^{3} \lambda_j \left((1-\lambda_j)^2 + \sum_{1 \leq r \neq j}^{3} \lambda_r^2 \right) \in \mathbb{Q}(3).$$

One also has that

$$u = \sum_{j=1}^{3} \left(\lambda_j (1-\lambda_j)^2 + (1-\lambda_j)\lambda_j^2 \right) = \sum_{j=1}^{3} \lambda_j (1-\lambda_j) = 1 - \sum_{j=1}^{3} \lambda_j^2$$

• The real issue is the control of i_*.

References

- Classical "polynomial" proofs of the Lukacs Theorem

- Szego *Orthogonal polynomials, 38'* (complex algebra),
- Achiezer *The classical moment problem 65'* (real algebra).
- Krein-Nudelman, 1977 (uniqueness of Lukacs representation).

- Main references

(link with real algebraic geometry, comments the 17th Hilbert problem)
- Lasserre, **Moments, Positive Polynomials and Their Applications**, 2010
- Powers, Positive polynomial and SOS: theory and practice, 2017.

$\Longrightarrow$ algorithms SOLLYA, SOSTOOLS, GLOBTIPOLY

- Own production with M. Campos-Pinto, F. Charles and M. Herda:

$\Longrightarrow$ algorithmic development of the idea of positive interpolation.

Formulation

Consider the values of p at an **unisolvent** set of interpolation points $\mathbf{x}_r$

$$y_r = p(\mathbf{x}_r) \in \mathbb{R} \qquad 1 \leq r \leq r_*.$$

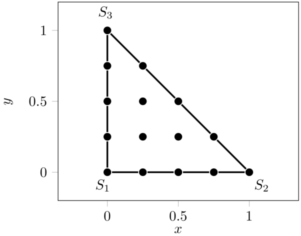

Figure 8: The simplex $\mathbb{K}$ and interpolation points for $n = 4$. **We take $i_* = r_*$**

Definition of Positive interpolation **(Charles, Campos-Pinto, D., SIAM, 2019) :**

From the sole knowledge of $(y_r, \mathbf{x}_r)_{1 \leq r \leq r_*}$, compute iteratively some polynomials $(p_{ij})_{ij}$ such that the SOS holds at the limit.

Parametrization

The $p_{ij} \in \mathrm{P}^{n_j}[\mathbf{X}]$ with $n_j = \left\lfloor \frac{r_* - \deg(g_j)}{2} \right\rfloor$ are the unknowns in

$$p = \sum_{i=1}^{r_*} \left(\sum_{j=1}^{j_*} g_j p_{ij}^2 \right) .$$

$$\begin{aligned}
&p_{ij}(\mathbf{X}) = \textstyle\sum_{|\alpha| \le n_j} c_\alpha^{ij} \mathbf{X}^\alpha, \quad \mathbf{X}^\alpha = X_1^{\alpha_1} \dots X_d^{\alpha_d},\\
&\mathbf{U}_i = \left(c^{i1}, c^{i2}, \dots, c^{ij_*}\right)^t \in \mathbb{R}^{r_*} \quad \text{where } r_* = \textstyle\sum_{j=1}^{j_*} r_j,\\
&D^{n_j}_{\alpha,\beta}(\mathbf{X}) = \mathbf{X}^\alpha \mathbf{X}^\beta, \qquad |\alpha|, |\beta| \le n_j,\\
&B(\mathbf{X}) = \operatorname{diag}\left(g_1(\mathbf{X}) D^{n_1}(\mathbf{X}), g_2(\mathbf{X}) D^{n_2}(\mathbf{X}), \dots, g_{j_*}(\mathbf{X}) D^{n_{j_*}}(\mathbf{X})\right),\\
&B_r = B(\mathbf{x}_r) \in \mathbb{R}^{r_* \times r_*}
\end{aligned}$$

$B_r = B(\mathbf{x}_r) \in \mathbb{R}^{r_* \times r_*}$ is a **localizing matrix** (Lasserre, Curto-Fialkow 2000).

It defines a real algebraic set, non convex and perhaps empty

$$\mathcal{U} = \{\mathbf{U} = (\mathbf{U}_1, \dots, \mathbf{U}_{r_*}) \in (\mathbb{R}^{r_*})^{r_*} \mid y_r = \sum_{i=1}^{r_*} \langle B_r \mathbf{U}_i, \mathbf{U}_i \rangle \text{ for } 1 \le r \le r_* \}.$$

The mathematical trick: duality

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Project an arbitrary starting point $\mathbf{V} = (\mathbf{V}_i)_{1\le i\le r_*} \in \mathbb{R}^{r_*\times r_*}$

$$\underset{\mathbf{U}\in\mathcal{U}}{\text{minimize}}\ \frac{1}{2}\sum_{i=1}^{r_*} \|\mathbf{U}_i - \mathbf{V}_i\|^2.$$

Consider a **Lagrange multiplier** $\lambda = (\lambda_1, \dots, \lambda_{r_*}) \in \mathbb{R}^{r_*}$ and

$$\mathcal{L}(\mathbf{U},\lambda) = \frac{1}{2}\sum_{i=1}^{r_*}\left(\|\mathbf{U}_i - \mathbf{V}_i\|^2 + \sum_{r=1}^{r_*}\lambda_r \langle B_r\mathbf{U}_i, \mathbf{U}_i\rangle\right) - \sum_{r=1}^{r_*}\lambda_r y_r.$$

The optimality relation yields

$$M(\lambda)\mathbf{U}_i = \mathbf{V}_i, \qquad M(\lambda) = I + \sum_{r=1}^{r_*}\lambda_r B_r.$$

Assuming invertibility of $M(\lambda)$ one gets

$$\mathbf{U}_i = \mathbf{U}_i(\lambda) := M(\lambda)^{-1}\mathbf{V}_i.$$

Nesterov-Nemirovskii, Interior-point polynomial algorithms in convex programming, 1994.

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Duality recovers convexity and coercivity

Take $\mathbf{V} = (\mathbf{V}_1, \dots, \mathbf{V}_{r_*}) = Id \in \mathcal{M}_{r_*}(\mathbb{R})$, eliminate $\mathbf{U}(\lambda) = M(\lambda)^{-1}$ and introduce the dual functional

$$G(\lambda) = \mathrm{tr}(M(\lambda)^{-1}) + \langle \lambda, \mathbf{y} \rangle .$$

The first and second variations of G are

$$\begin{aligned} \frac{\partial G}{\partial \lambda_r}(\lambda) &= -\textstyle\sum_{i=1}^{r_*} \langle \mathbf{U}_i(\lambda), B_r \mathbf{U}_i(\lambda) \rangle + y_r , \\ \frac{\partial^2 G}{\partial \lambda_r \partial \lambda_s}(\lambda) &= 2 \sum_{i=1}^{r_*} \left\langle B_r \mathbf{U}_i(\lambda), M(\lambda)^{-1} B_s \mathbf{U}_i(\lambda) \right\rangle . \end{aligned}$$

Lemma: G is convex over its domain $\mathcal{D} \underset{\mathrm{def}}{:=} \{\lambda \in \mathbb{R}^{r_*} \text{ such that } M(\lambda) > 0\}$.

Theorem: Take $p_n > 0$ over $[0, 1]$. Then G is coercive.

The general case: Assume the matrices B_r linearly indepedant. Take $p \in Q(i_*)(\mathbb{K})$. Same results hold for arbitrary small perturbations $p_\varepsilon > p$.

- D.+ Herda, Iterative Calculation of Sum Of Squares, accepted in SINUM.

Sketch of the proof

Hint of the proof: The asymptotic cone $C_\infty = \{\lambda \mid \sum_{r=1}^{r_*} \lambda_r B_r \geq 0\}$

Lemma (Equivalence between the following points)

- *For any $\lambda \in C_\infty$, one has $\langle \lambda, \mathbf{y} \rangle \geq 0$.*
- *There exists polynomials p_{ij} for $1 \leq j \leq j_*$ and $1 \leq i \leq r_*$ such that*

$$p(\mathbf{X}) = \sum_{i=1}^{r_*} \left(\sum_{j=1}^{j_*} g_j(\mathbf{X}) p_{ij}^2(\mathbf{X}) \right).$$

Proof: For $\mathbf{W} \in \mathbb{R}^{r_*}$, define the vector $s_{\mathbf{W}} = (\langle B_r \mathbf{W}, \mathbf{W} \rangle)_{1 \leq r \leq r_*} \in \mathbb{R}^{r_*}$

$$C_\infty = \{\lambda \in \mathbb{R}^{r_*} \text{ such that } \langle s_{\mathbf{W}}, \lambda \rangle \geq 0 \text{ for all } \mathbf{W} \in \mathbb{R}^{r_*}\}.$$

Use the **Generalized Farkas Theorem**.
The first assertion is equivalent to $\mathbf{y} = \sum_{i=1}^{r_*} \alpha_i s_{\mathbf{W}_i}$ where $\alpha_i \geq 0$ for all i, and r_* is "large enough".
It is rewritten as $\mathbf{y} = \sum_{i=1}^{r_*} s_{\mathbf{Z}_i}$ for $\mathbf{Z}_i = (\alpha_i)^{\frac{1}{2}} \mathbf{W}_i$.

Discrete descent methods (codes by Maxime)

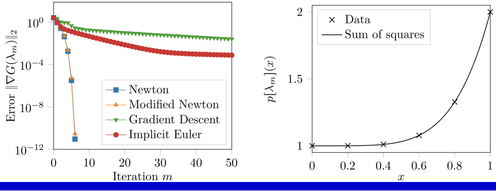
Take $H(t) = H(t)^* > 0$. Consider the ODE

$$\begin{cases} \lambda(0) = 0, \\ \lambda'(t) = -H(t)^{-1}\nabla G(\lambda(t)). \end{cases}$$

Discretization yields an **non constrained** descent method under the form

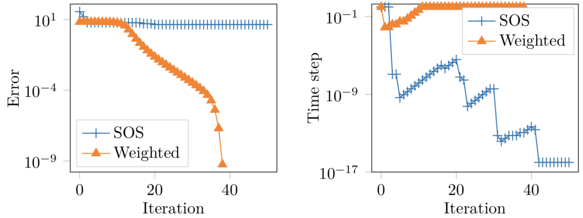
$$\lambda_{m+1} = \lambda_m - \tau_m H_m^{-1}\nabla G(\lambda_m), \qquad \lambda_0 = 0,$$

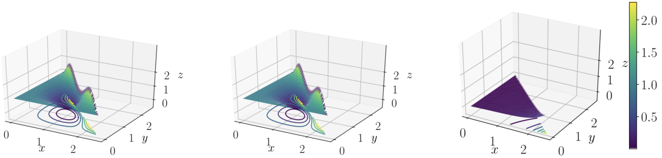
H_m	τ_m	name
I	adapted	Gradient
$I + \tau_m \nabla^2 G(\lambda_m)$	adapted	Backward
$\nabla^2 G(\lambda_m)$	≈ 1	Newton
$\alpha_m \nabla G(\lambda_m) \otimes \nabla G(\lambda_m) + \nabla^2 G(\lambda_m)$	≈ 1	**Mod. Newton**



$$\mathrm{Motzkin}(x,y) = x^2y^4 + y^2x^4 - 3x^2y^2 + 1$$

One has (67'): a) $\mathrm{Motzkin}(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^2$,
b) $\mathrm{Motzkin} \neq \sum_{l=1}^{L} p_l^2$ for all possible real polynomials p_l.





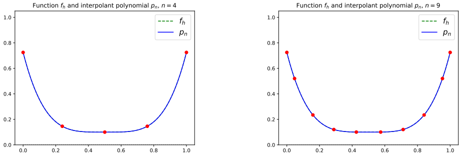
Left) $\mathrm{Motzkin}(1-x, 1-y)$.
Results with our algorithm: center) $g_j = \lambda_j$, right) $g_j = 1$

Typical application: a practical certificate of positivity

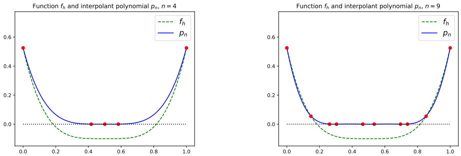
- Charles, Campos-Pinto, D.: Algorithms for positive polynomial approximation, SIAM J. Numer. Anal, 2019.

$q_\lambda(x) = 10(x - 1/2)^4 + \mu$ with optimal iteration number.

$\mu = 0.1 > 0$



$\mu = -0.1 < 0$



Optimal iteration number and h-convergence

- Charles, Campos-Pinto, D.: Algorithms for positive polynomial approximation, SIAM J. Numer. Anal, 2019.

Let $n = 2k + 1$ and $f \in W^{n+1,\infty}(0,1)$ be a positive function over $[0,1]$. Denote $f_h(\cdot) = f(\cdot h)$ for $0 \leq h \leq 1$.

A simplified Newton-Raphson scheme allows to compute a sequence of positive polynomials $p^m \in P_n^+$ for $m = 0, 1, \dots$ with oscillating polynomials $(a^m, b^m) \in P_k^2$ and the odd order Lukàcs representation

$$p^m(x) = xa^m(x)^2 + (1-x)b^m(x)^2.$$

Theorem

There exists $h_0 > 0$ such that for all $0 \leq h \leq h_0$, then

$$(a^m, b^m) \xrightarrow{m\to\infty} (a^\infty, b^\infty).$$

At iteration m, one has $\|p^m - f_h\|_{L^\infty(0,1)} \leq Ch^{\min(n+1, 2(m+1))}$.

The optimal number of iterations is $m = k$.

Similar results hold for $n = 2p$.

An application to PDE

Consider the advection equation

$$\begin{cases} \partial_t u + a\partial_x u = 0, & x \in \mathbb{R}, \quad t > 0, \quad a = 1, \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

Algo. for the update of u_j^{old}:

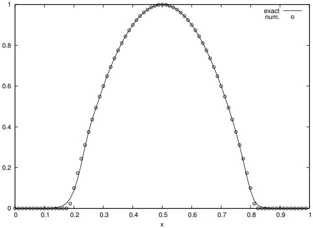
a) standard Lagrange interpolant $p_n^{\mathrm{lag.}}$ of u_r^{old} for $j - k_- \leq r \leq j + k_+$,
b) positive interpolation $p_n^{\mathrm{pos.pol.}} \longleftarrow p_n^{\mathrm{lag.}}$ with opt. num. of iterations,
c) update $u_j^{\mathrm{new}} = p_n^{\mathrm{pos.pol.}}(j\Delta x - a\Delta t)$.

Δx	$n = 1$	$n = 3$	$n = 5$	$n = 7$
20	0.195	0.0045767	0.00020966477	0.000016399842
40	0.109	0.0005707	0.00001083749	0.000000551818
80	0.058	0.0000725	0.00000039502	0.000000006955
160	0.029	0.0000091	0.00000001303.	0.000000000065
320	0.015	0.0000011	0.00000000041	ε_{mac}.
order	≈ 1	≈ 3	≈ 5	≈ 7

FV implementation

ENO-inspired method: cellwise to pointwise linear interpolation, linear Lagrange interpolation, positivity restoring using pos., interpolation, ...
Schemes are conservative, stable and full order of convergence is observed.

Δx	$n = 1$	$n = 3$	$n = 5$	$n = 7$
20	0.03791	0.000800990	0.000022346530	0.000001284228
40	0.00964	0.000052541	0.000000460477	0.000000023856
80	0.00242	0.000003355	0.000000008021	0.000000000168
160	0.00060	0.000000211	0.000000000132	$\varepsilon_{mac.}$
320	0.00015	0.000000013	$\varepsilon_{mac.}$	$\varepsilon_{mac.}$
order	≈ 2	≈ 4	≈ 6	≈ 8



Numerical result at time $T_{end} = 1$ for the numerical advection of $f(x) = \max\left(1 - 10 \times (x - 0.5)^2, 0\right)$ for $0 \le x \le 1$.

Wrap up

• This presentation described first steps to adapt distant descendants of the 17th Hilbert problem to numerical analysis and computational methods.

Combination of real algebraic geometry, convex analysis, numerical analysis and scientific computing.

• Natural developments: acceleration and optimal implementation of the new iterative algorithms. Couple with PDE solvers, DG methods, . . .

• Many open questions for positive interpolation with quaternion algebra for two bounds: the **quaternion algebra** yields unconditional control of the **Gibbs phenomenon**.