A class of special subordinators with nested ranges

P. Marchal

Université Paris 13
Sorbonne Paris Cité
LAGA
CNRS (UMR 7539)
F-93430, Villetaneuse

marchal@math.univ-paris13.fr

Abstract

We construct, on a single probability space, a class of regenerative sets $\mathcal{R}^{(\alpha)}$, indexed by all measurable functions $\alpha : [0,1] \to [0,1]$. For each function α, $\mathcal{R}^{(\alpha)}$, has the law of the range of a special subordinator. Constant functions correspond to stable subordinators. If $\alpha \leq \beta$, then $\mathcal{R}^{(\alpha)} \subset \mathcal{R}^{(\beta)}$. Other examples of special subordinators are given in the lattice case.

1 Introduction

Recall that a (possibly killed) subordinator $(S_t)_{t\geq 0}$ is a Lévy process on $\mathbb{R}_+$ with Laplace exponent given, for $\lambda \geq 0$, by

$$\phi(\lambda) = -\log \mathbb{E}[\exp(-\lambda S_1)] = a + b\lambda + \int_0^\infty \Pi(dx)(1 - e^{-\lambda x}) \tag{1}$$

The coefficient $b \geq 0$ is the drift, Π is the Lévy measure and $a \geq 0$ is a killing parameter. If $a > 0$, S is submarkovian. A function of the form (1) is called a *Bernstein function*.

The subordinator S is *special* if it admits a *dual* subordinator $(\widehat{S}_t)_{t\geq 0}$ with Laplace exponent $\widehat{\phi}$, such that for every $\lambda > 0$,

$$\phi(\lambda)\widehat{\phi}(\lambda) = \lambda \tag{2}$$

The canonical example is the case when S (resp. $\widehat{S}$) is the subordinator of the ascending (resp. descending) ladder times of a real-valued Lévy process X. In particular, if X drifts to $-\infty$, then S is a killed subordinator (that is, the parameter a in (1) is positive). If X is stable, then S and $\widehat{S}$ are stable, with

respective indices $\alpha = \mathbb{P}(X_1 > 0)$ and $1 - \alpha$. If X is symmetric and is not a compound Poisson process, then S and $\widehat{S}$ are stable with index $1/2$. See, among others, Bertoin [1], Doney, [3], Schilling et al. [8] for numerous references on subordinators, Bernstein functions and the connections with fluctuation theory for Lévy processes.

It turns out that, apart from the classical example of ladder times of a Lévy process, the class of special subordinators or special Bernstein functions is not known in detail. The main goal of this paper is to introduce a family of special subordinators indexed by all measurable functions $\alpha : [0,1] \to [0,1]$. A property of this family is that we can construct the ranges of all these subordinators on a single probability space, with the property that if $\alpha \leq \beta$, then the range of $S^{(\alpha)}$ is contained in the range of $S^{(\beta)}$. Here are the statements:

Theorem 1 *For every measurable function $\alpha : [0,1] \to [0,1]$, there exists a special subordinator $(S_t^{(\alpha)})_{t \geq 0}$ with Laplace exponent*

$$\phi^{(\alpha)}(\lambda) = -\log \mathbb{E}[\exp(-\lambda S_1^{(\alpha)})] = \exp \int_0^1 \frac{(\lambda - 1)\alpha(x)}{1 + (\lambda - 1)x} dx$$

for $\lambda \geq 0$. Its dual is the subordinator $(S_t^{(1-\alpha)})_{t \geq 0}$.

Note that when α is constant, $S^{(\alpha)}$ is stable with index α. Moreover, put $\psi^{(\alpha)}(\mu) = \phi^{(\alpha)}(\mu + 1)$. Then

$$\psi^{(\alpha)}(\mu) = \exp \int_0^1 \frac{\mu\alpha(x)}{1 + \mu x} dx = \exp \left(\sum_{n=1}^{\infty} (-1)^{n+1} \mu^n \int_0^1 [\alpha(x)]^n dx \right)$$

can be expanded as a power series in μ whose coefficients can be computed from the moments of the measure $\nu(dx) = \alpha(x)dx$ on $[0,1]$. Observe that the measure ν is characterized by its moments and that these moments determine the function $\psi^{(\alpha)}$, and thus also determine $\phi^{(\alpha)}$. It follows that if $\alpha(x) \neq \beta(x)$ for x in a set of positive Lebesgue measure, then $\phi^{(\alpha)} \neq \phi^{(\beta)}$.

Theorem 2 *One can construct, on a single probability space, a family of regenerative sets $\mathcal{R}^{(\alpha)}$ indexed by all measurable functions $\alpha : [0,1] \to [0,1]$, such that*

- *for every measurable function α,*

$$\mathcal{R}^{(\alpha)} \stackrel{law}{=} \overline{\{S_t^{(\alpha)}, t \geq 0\}}$$

- *if α, β are two measurable functions such that for every $x \in [0,1]$, $\alpha(x) \leq \beta(x)$, then*

$$\mathcal{R}^{(\alpha)} \subset \mathcal{R}^{(\beta)}$$

The properties of a subordinator can be read from its Laplace exponent. In turn, the properties of this exponent can be deduced from the function α, see Proposition 1 in Section 3.

Our construction generalizes a former construction for stable processes. This was used to construct Ruelle cascades, using nested stable regenerative sets obtained by subordination [6]. Other constructions of regenerative sets can be found in [4, 5, 7, 9].

We first explain, in Section 2, a similar construction in the lattice case, that is, in the framework of integer-valued regenerative sets. We use it to prove Theorems 1 and 2 in Section 3. In the lattice case, an extension is given in Section 4. In particular, this extension includes a lattice version of the special subordinators described in [10]. It should be possible to give a continuous version of the construction described in Section 4, however, we shall not handle this question here.

2 The lattice case

The lattice equivalent of a subordinator is a random walk on $\mathbb{N} \cup \{\infty\}$ (we include here the possibility of killing the random walk by sending it to ∞). Such a random walk S has a generating function $\psi(t) = \mathbb{E}(t^{S_1})$, defined for $t < 1$. The dual of S, if it exists, is the random walk $\widehat{S}$ with generating function $\widehat{\psi}$ such that

$$(1 - \psi(t))(1 - \widehat{\psi}(t)) = 1 - t \tag{3}$$

which is a discrete version of (2). A lattice regenerative set is the range of a random walk on $\mathbb{N}$ started at 0.

For instance, the set of strong ladder times of a discrete time real-valued random walk X is a lattice regenerative set. This regenerative set has a dual, namely the set of weak ladder times of $-X$.

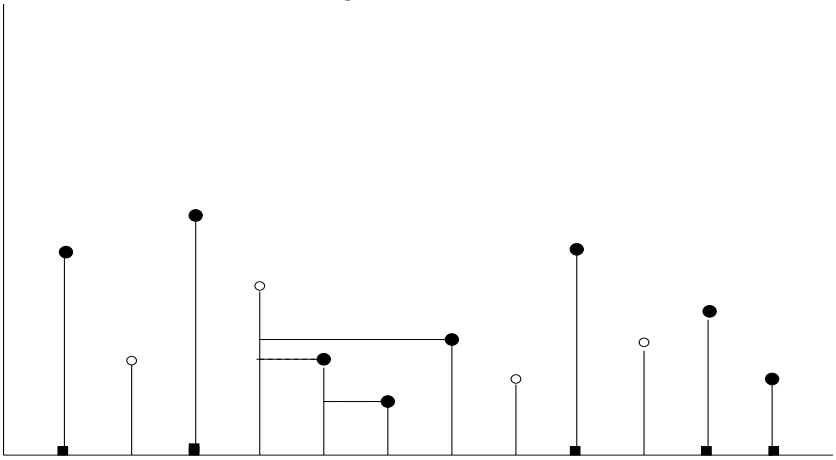
It is a classical fact that a random subset R of $\mathbb{N} \cup \{\infty\}$ is a lattice regenerative set if and only if it contains 0 and satisfies the regenerative property: for every $n \in \mathbb{N}$, conditionally on the event that $n \in R$, the set $R \cap [n, \infty]$ is independent of $R \cap [0, n]$ and has the same law as $R + n$.

We construct a family of random walks on $\mathbb{N}$, indexed by measurable functions α as in Theorem 1.

Construction 1.

Fix a measurable function $\alpha : [0, 1] \to [0, 1]$. Let $(X_n, n \geq 1)$ be iid random variables, uniformly distributed on $[0, 1]^2$. We denote $X_n = (h_n, U_n)$. One should view h as a height and U as a parameter. Say that X_n is α-green if $U_n \leq \alpha(h_n)$, and α-red otherwise. Say that an integer $k \in [1, n]$ is n-visible if $h_k \geq h_m$ for all integers $m \in [k, n]$. Finally, say that n percolates for α if, for every $k \leq n$ such that k is n-visible, X_k is α-green. Let $R^{(\alpha)}$ be the set of integers that percolate for α (by convention, 0 percolates for α).

Figure 1: Construction 1



See Figure 1. Green points are represented by black circles, red points by white circles and the black squares stand for the integers that percolate. The horizontal lines express the fact that the red point at 4 prevents 5, 6 and 7 from percolating.

Remark that if α is a constant, then the X_n are green or red with probability α (resp. $1-\alpha$), independently of the height. This is a discrete version of the construction given in [6].

Theorem 3 *The set $R^{(\alpha)}$ defined by Construction 1 is a lattice regenerative set. It can be viewed as the image of a random walk $(S_n^{(\alpha)}, n \geq 0)$, where $S_n^{(\alpha)} = Y_1^{(\alpha)} + \ldots + Y_n^{(\alpha)}$, the $Y_i^{(\alpha)}$ being iid random variables taking values in $\mathbb{N} \cup \{\infty\}$, with generating function*

$$\psi^{(\alpha)}(t) = \mathbb{E}(t^{Y_1^{(\alpha)}}) = 1 - \exp\left(-\int_0^1 \frac{t\alpha(x)}{1-tx} dx\right)$$

Moreover, $R^{(\alpha)}$ has a dual, namely $R^{(1-\alpha)}$.

From the very definition, the nested property of the sets $R^{(\alpha)}$ is obvious: if $\alpha \leq \beta$ and if X_n is α-green, then it is also β-green. Therefore $R^{(\alpha)} \subset R^{(\beta)}$. So we have immediately:

Theorem 4 *One can construct, on a single probability space, the sets $R^{(\alpha)}$ for all measurable functions $\alpha : [0,1] \to [0,1]$, with the property that if α, β are two measurable functions satisfying $\alpha \leq \beta$, then*

$$R^{(\alpha)} \subset R^{(\beta)}$$

Proof of Theorem 3.

Let $n \in \mathbb{N}$ and let E_n be the event that n percolates. Conditionally on E_n, all the n-visible points are green. Moreover, for every $N \geq n$ and every $k \leq n$, if k is N-visible, then k is also n-visible. Therefore, for every $N \geq n$, conditionally on E_n, N percolates if and only if all N-visible points in $[n+1, N]$ are α-green. This is independent of $(X_i, i \in [1, n])$ and has the same probability as the probability that $N - n$ is α-green. Hence $R^{(\alpha)}$ satisfies the regenerative property.

Let us compute the probability that $n \in R^{(\alpha)}$. If n is α-green, then there is a left-most n-visible point, say n_1, with height $x_1 = h_{n_1}$. Then n_1 has to be green, which occurs with probability $\alpha(x_1)$, and for all $i \in [1, n_1 - 1]$, $h_i \leq x_1$, which occurs with probability $x_1^{n_1-1}$. If $n_1 \neq n$, then there is second left-most n-visible point, say $n_1 + n_2$, and so on. So we have

$$\begin{aligned}\mathbb{P}(n \in R^{(\alpha)}) &= \sum_k \sum_{n_1+\ldots+n_k=n} \\ &\quad \int_0^1 dx_1 \int_0^{x_1} dx_2 \ldots \int_0^{x_{k-1}} dx_k \alpha(x_1) x_1^{n_1-1} \ldots \alpha(x_k) x_k^{n_k-1}\end{aligned}$$

By symmetrization,

$$\mathbb{P}(n \in R^{(\alpha)}) = \sum_k \frac{1}{k!} \sum_{n_1+\ldots+n_k=n} \int_0^1 dx_1 \ldots \int_0^1 dx_k \alpha(x_1) x_1^{n_1-1} \ldots \alpha(x_k) x_k^{n_k-1}$$

Summing over n,

$$\sum_n \mathbb{P}(n \in R^{(\alpha)}) t^n = \exp\left(\int_0^1 \frac{t\alpha(x)}{1-tx} dx \right)$$

On the other hand,

$$\sum_n \mathbb{P}(n \in R^{(\alpha)}) t^n = \sum_i \mathbb{P}(S_i^{(\alpha)} = n) t^n = \sum_i \mathbb{E}(t^{S_i^{(\alpha)}}) = \frac{1}{1 - \mathbb{E}(t^{Y_1^{(\alpha)}})}$$

Finally, the duality property follows from a straighforward computation:

$$(1 - \psi^{(\alpha)}(t))(1 - \psi^{(1-\alpha)}(t)) = \exp\left(-\int_0^1 \frac{t}{1-tx} dx \right) = 1 - t$$

3 From the lattice case to the continuous case

3.1 Proof of Theorem 1

We first state a lemma:

Lemma 1 *Let $\beta : [0,1] \to [0,1]$ be a measurable function and $F : [0,1] \to [0,1]$ be a Lipschitz, nondecreasing function such that $F(0) = 0$, $F(1) = 1$. Let $q > 0$, $\theta \geq 0$ be two reals. Then the the function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ defined by*

$$\phi(\lambda) = \theta \exp\left(-\int_0^1 \frac{q\beta(F(x))}{\lambda + q - qF(x)} F'(x) dx \right)$$

is a Bernstein function.

Proof of Lemma 1

First, the case $\theta = 0$ corresponds to a subordinator which is constantly 0. Assume now $\theta > 0$.

From Construction 1 we derive a continuous process. Consider the random walk $(S_n^{(\beta)}, n \geq 0)$. First, let $e_1, e_2, \ldots$ be iid exponential random variables with parameter $q > 0$, independent of $S^{(\beta)}$. We get a discrete-time, continuous-state random walk $(Z_n, n \geq 0)$ by setting

$$Z_n = e_1 + \ldots + e_{S_n^{(\beta)}}$$

Next, let $(N_t, t \geq 0)$ be a Poisson process with parameter $\theta > 0$, independent of $S^{(\beta)}$ and of the random variables $(e_n, n \geq 1)$. For $t \geq 0$, put

$$X_t = Z_{N_t}$$

Then $(X_t, t \geq 0)$ is a subordinator, more specifically a compound Poisson process, whose range is the same as the range of Z, the only difference between X and Z being the time parametrization. For every $\lambda > 0$,

$$\mathbb{E}(\exp(-\lambda X_1)) = \sum_{n \geq 0} \frac{\theta^n}{n!} e^{-\theta} (\psi^{(\beta)}(q/(q+\lambda))^n = \exp[-\theta(1 - \psi^{(\beta)}(q/(q+\lambda)))]$$

Thus the Laplace exponent of X is

$$\phi(\lambda) = -\log \mathbb{E}(\exp(-\lambda X_1)) = \theta[1 - \psi^{(\beta)}(q/(q+\lambda))]$$

That is,

$$\phi(\lambda) = \theta \exp\left(- \int_0^1 \frac{q\beta(x)}{\lambda + q - qx} dx \right)$$

Remark that by a change of variable,

$$\phi(\lambda) = \theta \exp\left(- \int_0^1 \frac{q\beta(F(x))}{\lambda + q - qF(x)} F'(x) dx \right)$$

which proves Lemma 1. □

Proof of Theorem 1

Fix a measurable function $\alpha : [0,1] \to [0,1]$. For an integer $m \geq 2$ define

$$\begin{aligned} q_m &= m - 1 \\ F_m(x) &= \frac{\mathbf{1}_{\{x \in [1/m, 1]\}}(mx - 1)}{(m-1)x} \\ \beta_m(x) &= \mathbf{1}_{\{x > 0\}} \alpha\left(\frac{1}{m - (m-1)x} \right) \end{aligned} \tag{4}$$

and

$$\theta_m = \exp\left(\int_{1/m}^1 \frac{\beta_m(F_m(x))}{x}dx\right)$$

Remark that q_m, β_m, F_m and θ_m satisfy the assumptions of Lemma 1, and that

$$F_m'(x) = \frac{\mathbf{1}_{\{x\in[1/m,1]\}}}{(m-1)x^2}$$

If $x \in [1/m, 1]$,

$$\lambda + q_m - q_m F_m(x) = \lambda + \frac{1-x}{x} = \frac{1+(\lambda-1)x}{x}$$

and thus

$$\frac{q_m F_m'(x)}{\lambda + q_m - q_m F_m(x)} = \frac{1}{x[1+(\lambda-1)x]}$$

Applying Lemma 1, we find that the following function

$$\begin{aligned}\phi^{(\alpha,m)}(\lambda) &= \exp\left(-\int_{1/m}^1 \frac{\beta_m(F_m(x))}{x[1+(\lambda-1)x]}dx + \int_{1/m}^1 \frac{\beta_m(F_m(x))}{x}dx\right)\\ &= \exp\left(\int_{1/m}^1 \frac{(\lambda-1)\beta_m(F_m(x))}{1+(\lambda-1)x}dx\right)\end{aligned}$$

is a Bernstein function. Moreover,

$$\beta_m(F_m(x)) = \alpha(x)\mathbf{1}_{\{x\in(1/m,1]\}}$$

whence

$$\phi^{(\alpha,m)}(\lambda) = \exp\left(\int_{1/m}^1 \frac{(\lambda-1)\alpha(x)}{1+(\lambda-1)x}dx\right)$$

So any function of this form is a Bernstein function and it is known [8] that every limit of Bernstein functions is a Bernstein function. Therefore, letting $m \to \infty$, we get that

$$\phi^{(\alpha)}(\lambda) = \exp\left(\int_0^1 \frac{(\lambda-1)\alpha(x)}{1+(\lambda-1)x}dx\right)$$

is the Laplace exponent of a subordinator. Likewise, the function $\phi^{(1-\alpha)}$ is a Bernstein function and the duality relation follows from the equality

$$\lambda = \exp\int_0^1 \frac{\lambda-1}{1+(\lambda-1)x}dx \tag{5}$$

3.2 Some properties

The basic properties of a subordinator can be read easily from the asymptotic behaviour of its Laplace exponent. It turns out that the small-time properties of $S^{(\alpha)}$ depend on the behaviour of α near 0, while the large-time properties depend on the behaviour of α near 1. More precisely,

Proposition 1 *Let $\mathcal{R}^{(\alpha)}$ be as in Theorem 2.*

(i) If

$$\int_{1/2}^{1} \frac{\alpha(x)}{1-x} dx < \infty$$

then $\mathcal{R}^{(\alpha)}$ is bounded almost surely. Otherwise, $\mathcal{R}^{(\alpha)}$ is unbounded almost surely.

(ii) If

$$\int_{0}^{1/2} \frac{1-\alpha(x)}{x} dx < \infty$$

then the Lebesgue measure of $\mathcal{R}^{(\alpha)}$ is positive almost surely. Otherwise, this Lebesgue measure is almost surely 0.

(iii) If $\alpha(x) \to \beta$ as $x \to 0$, then the Hausdorff dimension of $\mathcal{R}^{(\alpha)}$ is β almost surely.

Proof

We use here classical results on subordinators, which can be found for instance in [1], Chapter 1. First, if the killing rate of a subordinator is positive, then its range is bounded almost surely. Otherwise, the range is unbounded almost surely. The killing rate of $S^{(\alpha)}$ is

$$\phi^{(\alpha)}(0) = \exp\left(-\int_0^1 \frac{\alpha(x)}{1-x} dx\right)$$

which easily gives (i).

Next, recall that if the drift of a subordinator is positive, then the Lebesgue measure of its range is positive almost surely. If this drift is zero, then the Lebesgue measure of the range is zero almost surely. Moreover the drift is given by $\lim_{\lambda\to\infty} \phi(\lambda)/\lambda$. Using (5), we find

$$\frac{\phi^{(\alpha)}(\lambda)}{\lambda} = \exp \int_0^1 \frac{\alpha(x)-1}{x + [1/(\lambda-1)]} dx$$

By monotone convergence, this ratio has a finite limit as $\lambda \to \infty$ if and only if

$$\int_{0}^{1/2} \frac{1-\alpha(x)}{x} dx < \infty$$

which proves (ii).

Finally, recall that index of the exponent $\phi^{(\alpha)}$ is given by

$$I = \lim_{\lambda\to\infty} \frac{\log \phi^{(\alpha)}(\lambda)}{\log \lambda}$$

if this limit exists. If so, the Hausdorff dimension of the range is equal to the index. See [2], or [1], Chapter 5. It is easy to check that that if $\alpha(x) \to \beta$ as $x \to 0$, then the index of the exponent is β.

3.3 Proof of Theorem 2

Consider a Poisson Point process $\mathcal{N}$ on $\mathbb{R}_+ \times [0,1] \times [0,1]$ with intensity $dx \otimes y^{-2}dy \otimes dz$. Given a measurable function $\alpha : [0,1] \to [0,1]$, we can define an analogue of Construction 1 as follows.

Construction 1'.

Say that a point $X = (t,h,U)$ of $\mathcal{N}$ is α-green if $U \leq \alpha(h)$, and α-red otherwise. Say that another point $X' = (t',h',U')$ of $\mathcal{N}$ is visible for X if $t' \leq t$ and if, for all points of $\mathcal{N}$ of the form $X'' = (t'',h'',u'')$ with $t' \leq t'' \leq t$, we have $h' \geq h''$. Finally, say that X percolates for α if, for every X' such that X' is visible for X, X' is α-green. By convention, 0 percolates for α. We denote by $\mathcal{R}_1^{(\alpha)}$ the set of first coordinates of percolating points, and we set

$$\mathcal{R}^{(\alpha)} = \overline{\mathcal{R}_1^{(\alpha)}}$$

For every point $X = (t,h,U)$ of $\mathcal{N}$, let $U(X)$ be the set of points of $\mathcal{N}$ of the form $X' = (t',h',u')$ with $t' \leq t$ and $h' \geq h$. Then almost surely, $U(X)$ is finite, since almost surely, every strip of the form $[0,t] \times [h,\infty] \times [0,1]$ with $h > 0$ contains a finite number of points of $\mathcal{N}$. Moreover, determining whether X percolates only depends on $U(X)$, and therefore Construction 1' is well-defined.

Alternatively, one can define $\mathcal{R}^{(\alpha)}$ as follows. For $m \geq 2$ an integer, consider the restriction $\mathcal{N}^{(m)}$ of $\mathcal{N}$ to the subset $\mathbb{R}_+ \times [1/m,1] \times [0,1]$. Let $(X_n^{(m)}, n \geq 1)$ be the set of points of $\mathcal{N}^{(m)}$, ranked by increasing x-coordinate. Denote, for each $n \geq 1$,

$$X_n^{(m)} = (t_n^{(m)}, h_n^{(m)}, U_n^{(m)})$$

Consider the functions F_m, β_m and the constant θ_m as in the proof of Theorem 1. Then we can define the sequence $(Y_n^{(m)}, n \geq 1)$ by

$$Y_n^{(m)} = (F_m(h_n^{(m)}), U_n^{(m)})$$

Note that $(F_m(h_n^{(m)}), n \geq 1)$ is a sequence of iid, uniform random variables on $[0,1]$. Therefore, using Construction 1, we can define a lattice regenerative set $\mathcal{S}^{(\alpha,m)}$ from the the sequence $(Y_n^{(m)}, n \geq 1)$ and the function β_m. As proved in Theorem 3, $\mathcal{S}^{(\alpha,m)}$ can be viewed as the range of a random walk $(T_n^{(\alpha,m)}, n \geq 0)$ with generating function

$$\mathbb{E}t^{T_1^{(\alpha,m)}} = 1 - \exp\left(-\int_0^1 \frac{t\beta_m(x)}{1-tx}dx\right)$$

Next, observe that the family $(t_{n+1}^{(m)} - t_n^{(m)}, n \geq 0)$ is a family of iid, exponential random variables with parameter $m-1$, and that these random variables are independent of $(Y_n^{(m)}, n \geq 1)$. Therefore, we can do as in the proof of Lemma 1 and transform the lattice regenerative set $\mathcal{S}^{(\alpha,m)}$ into a continuous-state regenerative set $\mathcal{R}^{(\alpha,m)}$. To do so, we put

$$Z_n^{(\alpha,m)} = \sum_{k=0}^{T_n^{(\alpha,m)}-1} [t_{k+1}^{(m)} - t_k^{(m)}]$$

and we define $\mathcal{R}^{(\alpha,m)}$ as the range of $Z^{(\alpha,m)}$.

By construction, one checks that if $m < n$,

$$\mathcal{R}^{(\alpha,m)} \subset \mathcal{R}^{(\alpha,n)} \tag{6}$$

and if $\alpha \leq \gamma$,

$$\mathcal{R}^{(\alpha,m)} \subset \mathcal{R}^{(\gamma,m)} \tag{7}$$

Finally, we define

$$\mathcal{R}^{(\alpha)} = \overline{\cup_{m>0} \mathcal{R}^{(\alpha,m)}}$$

and it is easy to see that this definition coincides with Construction 1'. Note that the nesting property of the sets $\mathcal{R}^{(\alpha)}$ as stated in Theorem 2 follows from (7), or directly from Construction 1'.

It remains to show that for every measurable function α, $\mathcal{R}^{(\alpha)}$ is a regenerative set with the Laplace exponent given in Theorem 1. From now on the measurable function α is fixed.

Using the proof of Lemma 1, we get that for every integer $m \geq 2$, $\mathcal{R}^{(\alpha,m)}$ can be viewed as the image of a subordinator $(S_t^{(m)}, t \geq 0)$ with Laplace exponent

$$\phi^{(\alpha,m)}(\lambda) = \exp\left(\int_{1/m}^{1} \frac{(\lambda-1)\alpha(x)}{1+(\lambda-1)x} dx \right)$$

So it is possible to construct, on a single probability space, a family of subordinators $(S_t^{(m)}, t \geq 0)$, for all integers $m \geq 2$ with respective ranges $\mathcal{R}^{(\alpha,m)}$ and respective Laplace exponent $\phi^{(\alpha,m)}$.

The convergence of the Laplace exponents $\phi_m^{(\alpha)}$ to $\phi^{(\alpha)}$ as $m \to \infty$ entails that the processes $(S_s^{(m)}, s \geq 0)$ converge in law to a subordinator with Laplace exponent $\phi^{(\alpha)}$. Moreover, for each integer $m \geq 2$ and each real $s > 0$, the law of $S_s^{(m)}$ is diffuse, as the law of a sum of independent exponential random variables. Therefore, there exists a subsequence $(u_n, n \geq 0)$ such that $S_1^{(u_n)}$ converges almost surely as $n \to \infty$. From this subsequence, one can extract a subsequence $(v_n, n \geq 0)$ such that $S_{1/2}^{(v_n)}$, $S_1^{(v_n)}$, $S_{3/2}^{(v_n)}$ and $S_2^{(v_n)}$ converge almost surely as $n \to \infty$. Iterating the procedure and using diagonal extraction, we can find a subsequence $(w_n, n \geq 0)$ such that $S_s^{(w_n)}$ converges almost surely as $n \to \infty$, for all dyadic $s \geq 0$. Let S_s denote the limit.

The reals S_s are defined for all dyadic $s \geq 0$. We extend the definition by setting, for every $t \geq 0$,

$$S_t = \inf_{\{s \geq t,\ s \text{ dyadic}\}} S_s$$

Since the marginals of $(S_t, t \geq 0)$ are the marginals of a subordinator with Laplace exponent $\phi^{(\alpha)}$ for all dyadic t and since S is cadlag, S is a subordinator with Laplace exponent $\phi^{(\alpha)}$.

Let R be the the range of S. Using the inclusion property (6), the definition of $\mathcal{R}^{(\alpha)}$ and of S, we see that $R \subset \mathcal{R}^{(\alpha)}$. Moreover, since $S^{(w_n)}$ converges to S in the Skorokhod topology, it is easy to check that the Hausdorff distance

$$d(R \cap [0,T], \mathcal{R}^{(\alpha,w_n)} \cap [0,T])$$

converges to 0 as $n \to \infty$, for every $T > 0$. It follows that $R = \mathcal{R}^{(\alpha)}$, and thus $\mathcal{R}^{(\alpha)}$ is a regenerative set with Laplace exponent $\phi^{(\alpha)}$.

4 A generalization in the lattice case

Construction 2.

Take two arbitrary probability distributions ν, $\widehat{\nu}$ on $\mathbb{R}$. Let $(S_n, n \geq 0)$ be a real-valued random walk started at 0, with increments $(X_n, n \geq 1)$. Let $(H_n, n \geq 1)$ be iid real-valued random variables with law ν and $(\widehat{H}_n, n \geq 0)$ be iid real-valued random variables with law $\widehat{\nu}$. Assume that the X_n, H_n and $\widehat{H}_n$ are independent.

For $n \geq 1$, say that an integer $k \in [0, n-1]$ is an n-obstacle if, for every $m \in [k+1, n]$,

$$S_m + H_m < S_k + \widehat{H}_k \tag{8}$$

Say that $n \geq 1$ percolates if, for every $k \in [0, n-1]$, k is not an n-obstacle. By convention, say that 0 percolates. Let $\mathcal{R}$ be the set of integers that percolate.

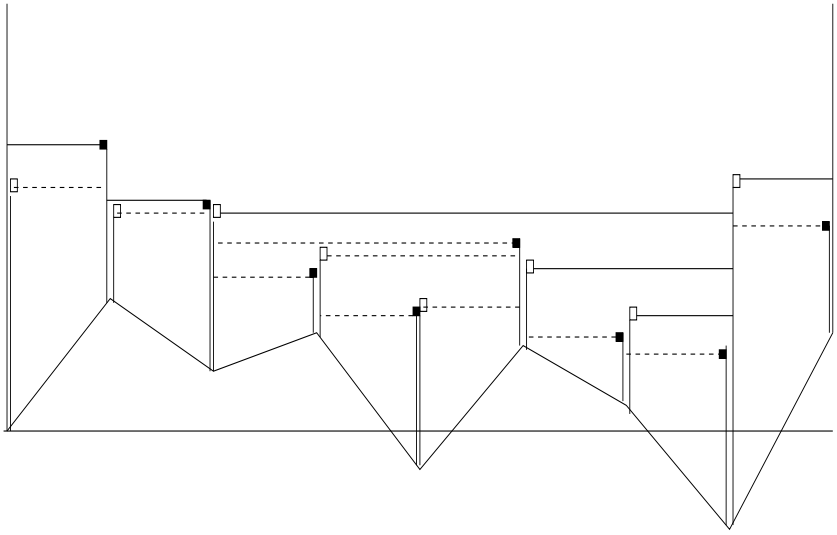
Theorem 5 *The random set $\mathcal{R}$ defined by Construction 2 is a lattice regenerative set. Its dual is obtained by replacing the random walk $(S_n, n \geq 0)$ with $(-S_n, n \geq 0)$ and exchanging the role of the random variables (H_n) and $(\widehat{H}_n)$.*

Proof

The regenerative property is established by the same argument as for Theorem 3.

To show the duality, consider a regenerative set $\mathcal{R}'$ defined as in Construction 2, using independent random variables $(X'_n, n \geq 1)$, $(H'_n, n \geq 1)$ and $(\widehat{H}'_n, n \geq 0)$, where S'_1 has the same law as $-S_1$, H'_1 has the same law as $\widehat{H}_1$, and $\widehat{H}'_1$ has the same law as H_1. The only difference is that we define an obstacle using a large inequality, in contrast to the strict inequality in (8). To avoid any ambiguity, we shall use the terms *dual obstacle*, *dual-percolate* for the construction of $\mathcal{R}'$.

Figure 2: Construction 2



One can construct the sets $\mathcal{R} \cap [0, N]$ and $\mathcal{R}' \cap [0, N]$ on the same probability space, using the random variables X_n, $n \in [0, N]$, H_m, $m \in [1, N]$, $\widetilde{H}_l$, $l \in [0, N-1]$, by putting

$$S'_n = S_{N-n} - S_n, \quad n \in [0, N]$$

$$H'_n = \widetilde{H}_{N-n}, \quad n \in [1, N]$$

$$\widetilde{H}'_n = H_{N-n}, \quad n \in [0, N-1]$$

See Figure 2. The black squares stand for the variables $S_n + H_n$, the white squares for the variables $S_n + \widehat{H}_n$. Black squares "look to the left", white squares "look to the right". The horizontal dashed lines express the fact that they see an obstacle when looking to the left, or a dual obstacle when looking to the right. In turn, the plain lines express the fact that they see no obstacle or dual obstacle and, therefore, percolate or dual-percolate.

Let $G_N = \max(\mathcal{R} \cap [0, N])$, $G'_N = \max(\mathcal{R}' \cap [0, N])$. We claim that

(i) $N - G_N$ dual-percolates.

(ii) For every $n \in [N - G_N + 1, N]$, n does not dual-percolate.

To show (i), suppose that $N - G_N$ does not dual-percolate. Let k be the largest integer that is a dual obstacle for $N - G_N$. Then from the definition of a dual obstacle, there exists no $(N-k)$-obstacle in $[G_N, k-1]$. Moreover, from the definition of G_N, there is no $(N-k)$-obstacle either in $[0, G_N - 1]$. Therefore, k percolates, but this contradicts the definition of G_N. This proves (i). One proves (ii) by similar arguments. As a consequence, $G'_N = N - G_N$. This being true for every $N \geq 1$, we find that

(iii) For every $N > 0$, $N - G_N$ and G'_N have the same law.

It is then standard to check that (iii) is equivalent to the analytical property (3).

Some examples

1. If both ν and $\widehat{\nu}$ are the Dirac mass at 0, then $\mathcal{R}$ is the set of strict ascending ladder times of the random walk S, that is, the set of integers n such that $S_n > \max_{k\leq n-1} S_k$. On the other hand, $\mathcal{R}'$ is the set of weak descending ladder times of S, ie the set of integers n such that $S_n \leq \min_{k\leq n-1} S_k$.

2. Suppose that $\widehat{\nu}$ is the Dirac mass at 0 and that

$$\nu(dx) = (1-r)\delta_0 + r\delta_{-\infty}$$

for some fixed $r \in [0,1]$. Then the event that $T_1 > n$ is the event that for every every ladder time $k \leq n$, $H_k = -\infty$. Therefore,

$$\phi(t) = \sum_{n=1}^{\infty} t^n \mathbb{E}(r^{L_{n-1}} - r^{L_n})$$

where L_n is the number of ladder times between time 1 and time n. Put

$$\psi(t) = \mathbb{E}(t^\tau)$$

where τ is the first ladder time. Then by standard computations, we find

$$\phi(t) = \frac{(1-r)\psi(t)}{1-r\psi(t)}$$

3. Suppose that $\widehat{\nu}$ is the Dirac mass at 0 and that

$$\nu(dx) = c\exp(-c|x|)\mathbf{1}_{\{x<0\}}dx$$

Then the event that $T_1 > n$ is the event that for every every ladder time $k \leq n$,

$$|H_k| \geq S_k - \sup_{i<k} S_i$$

Conditionnally on S_k and $\sup_{i<k} S_i$, the latter event has probability $\exp[-c(S_k - \sup_{i<k} S_i)]$. Therefore,

$$\mathbb{P}(T_1 > n) = \mathbb{E}\exp[-c\sup_{i\leq n} S_i]$$

By time reversal, we have:

$$\mathbb{P}(\widehat{T}_1 > n) = \int_0^\infty ce^{-cx}\mathbb{P}(\forall k \in [1,n], S_k \geq x)dx$$

Note that in the limit $c \to \infty$, we recover the first example.

4. Let S be deterministic, $S_n = -n$. Also, suppose that $\widehat{\nu}$ is the Dirac mass at 0. Then the event that $T_1 > n$ is the intersection of the events $\{H_1 \leq 1\}$, $\{H_2 \leq 2\}$, $\ldots \{H_n \leq n\}$, all these events being independent. Therefore,

$$\mathbb{P}(T_1 > n) = \prod_{i=1}^{n} \nu([0, n])$$

In particular, the sequence of ratios

$$\frac{\mathbb{P}(T_1 > n+1)}{\mathbb{P}(T_1 > n)} = \nu([0, n+1])$$

can be chosen to be any nondecreasing sequence of reals $\in [0, 1]$. If we consider the dual process, we see that

$$\mathbb{P}(n \in \widehat{\mathcal{R}}) = \prod_{i=1}^{n} \nu([0, n])$$

This is the lattice equivalent of Corollary 2.5 in [10]. In particular, if the support of ν is bounded, say $supp(\nu) \subset [0, A]$, then $\mathbb{P}(n \in \widehat{\mathcal{R}})$ is constant for $n \geq A$. This corresponds to the examples given in Section 3 in [10].

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