Blood-flow modelling along and trough braided multi-layer metallic stents

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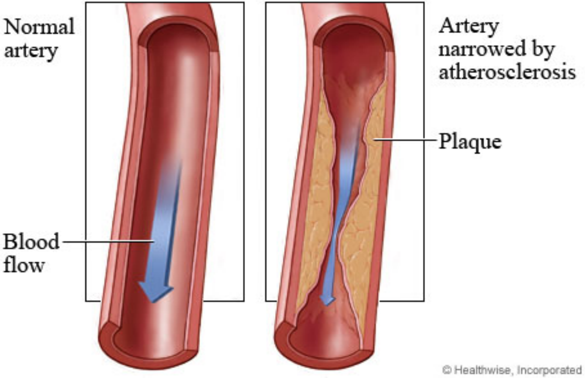
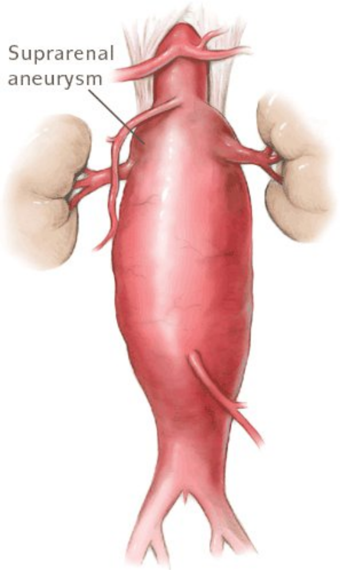
Two common pathologies of the cardio-vascular system





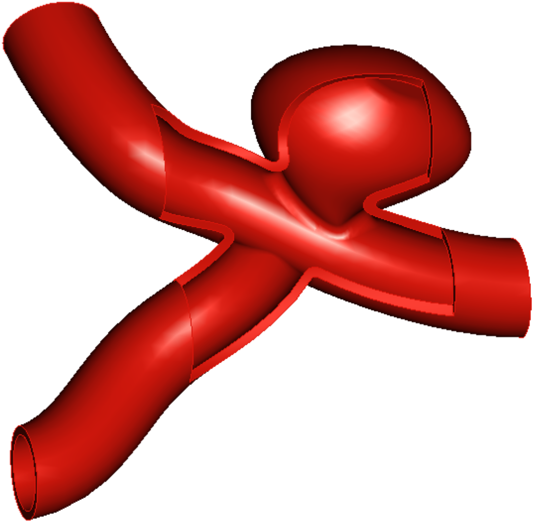
Industrial context

Cardiatis®: conception and comercialisation of metallic wired multi-layer stents ▸ Image 3D

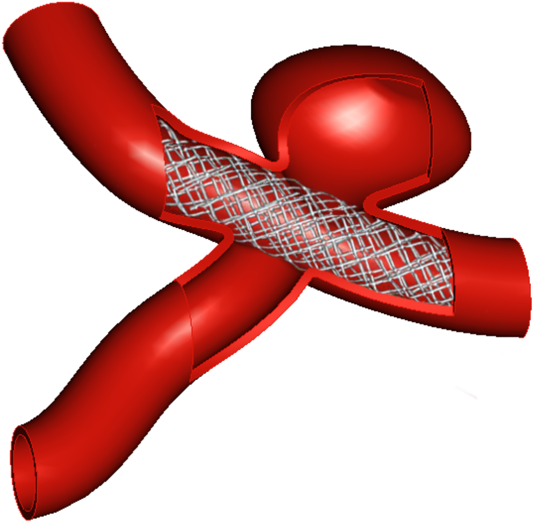
- A new technology
  One controls
  - The # of layers
  - Their connectivity
- In vivo experiments
  1. on mini-pigs show :no thrombus up to 6 months ▸ Dissection pictures
  2. on humans : ▸ Microscopy pictures
- Multi-Scale phenomenon lying on:
  - Hemodynamics
  - Chemical reactions between blood flow and the surrounding wires and tissues

Theoretical & numerical study of hemodynamics

Where does the stent apply ?



Where does the stent apply ?



Problem description

- Geometrical properties
  - Femoral artery diameter: $\emptyset_{\mathrm{A}} = 6mm$
  - Total thickness of the stent : $\epsilon = 0.25mm$
  - Thickness of a single wire: $\epsilon = 0.04mm$
  - Red blood cell diameter: $\emptyset_{\mathrm{RC}} = 0.008mm$

$$\frac{\epsilon}{\emptyset_A} = \frac{0.25}{6} \sim 4\%$$

stent $\sim$ periodic rugous wall in a straight cilindrical geometry

- The blood flow is a pressure driven flow composed of
  - Steady state part: geometrical perturbation of a Poiseuille profile
  - Plus a pulsatile periodic perturbation: Womersley profile

Objectives and references

-We aim to

- understand the dynamics of flows in rugous possibly complex geomteries
  $\Longrightarrow$ Boundary layer correctors
- quantify macroscopic averaged effect $\Longrightarrow$ Homogenization
- find more general implicit relationships independent on the geometry of the pbm
  $\Longrightarrow$ Implicit interface conditions

Use of **assymptotic expansions** adapted for the **perturbed boundaries**.

References

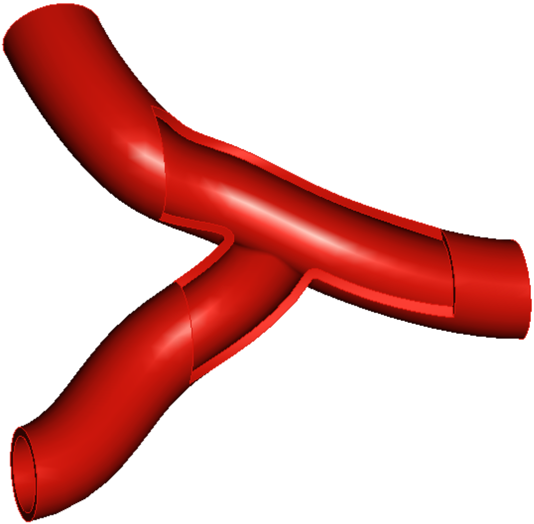
N. Neuss, M. Neuss-Radu, and A. Mikelić.
Effective laws for the poisson equation on domains with curved oscillating boundaries.
Applicable Analysis, 2006.

G. Allaire
Homogenization of the navier-stokes equations in open sets perforated with tiny holes. ii. noncritical sizes of the holes for a volume distribution and a surface distribution of holes.
Arch. Rational Mech. Anal., 1991.

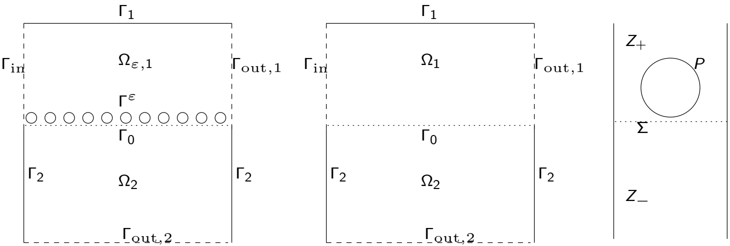
Main results

- Stented collateral artery
  1. explicit first order averaged velocity profile
  2. explicit flow-rate
  3. implicit interface conditions

- Stented anevrismal sac
  1. explicit averaged zero order pressure in the sac
  2. first order averaged velocity profile
  3. the presence of the wires inverses the vertex



Notations and Methodology



We denote:

- $P = \{y \in \mathbb{R}^2 \text{ s.t. } y_= y(t),\ t \in [0,1]\}$,
- Ω the "smooth domain", Γ^0 the fictitious interface,
- x the slow space variable , $y = \frac{x}{\epsilon}$ the fast one.

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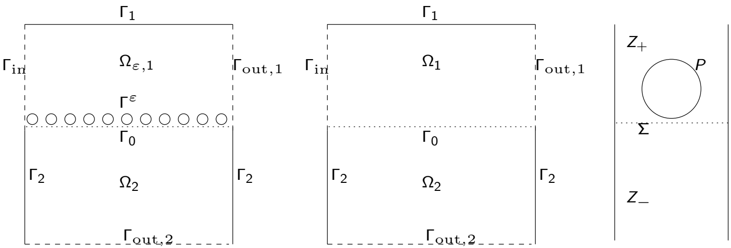

Our method: transform the rough boundary problem in Ω_ε into a fictitious interface problem in Ω^0

1. Construction of a complete boundary layer corrector: Ω_ε
2. Averaging the oscillations
3. Generlize to other geometries
   => derivation of implicit interface conditions

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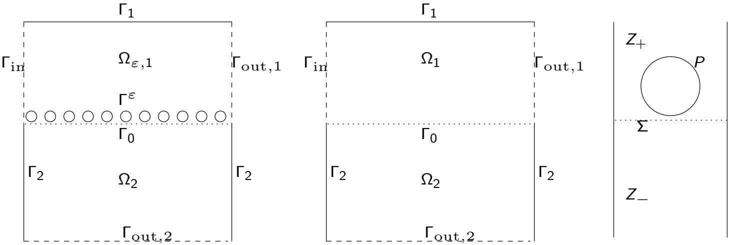

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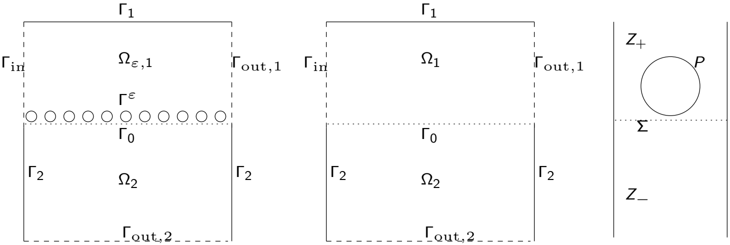

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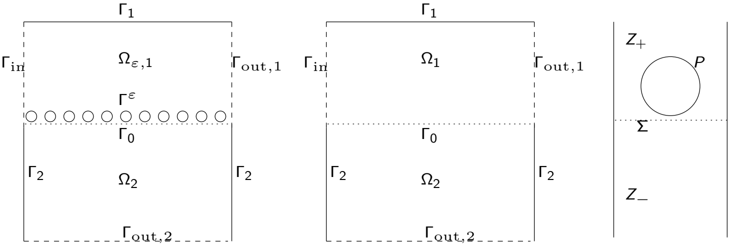

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The problem

Let

- $\mathbf{u}_\varepsilon$ be the velocity vector,
- p_ε the pressure

unknown variables describing the motion of the fluid.

- Consider a laminar pressure driven flow

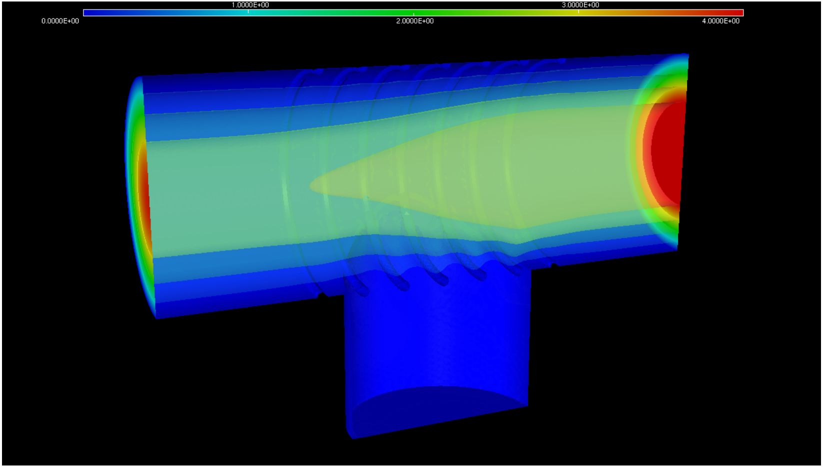
$$\begin{cases} -\Delta \mathbf{u}_\varepsilon + \nabla p_\varepsilon = 0 \text{ in } \Omega_\varepsilon \\ \mathrm{div}\, \mathbf{u}_\varepsilon = 0 \\ \mathbf{u}_\varepsilon = 0 \text{ on } \Gamma_1 \cup \Gamma_2 \cup \Gamma^\varepsilon \\ \mathbf{u}_\varepsilon \cdot \boldsymbol{\tau} = 0 \text{ on } \Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out},1} \cup \Gamma_{\mathrm{out},2} \\ p_\varepsilon = p_{\mathrm{in}} \text{ on } \Gamma_{\mathrm{in}}, \quad p_\varepsilon = p_{\mathrm{out},1} \text{ on } \Gamma_{\mathrm{out},1}, p_\varepsilon = p_{\mathrm{out},2} \text{ on } \Gamma_{\mathrm{out},2}, \end{cases}$$

- Pressure driven $\neq$ Dirichlet velocity as in

C. Conca.
Étude d'un fluide traversant une paroi perforée. I & II.
J. Math. Pures Appl. (9), 66(1):1–70, 1987.

Expected behaviour

In 3D in a fiew days/weeks of direct simulation



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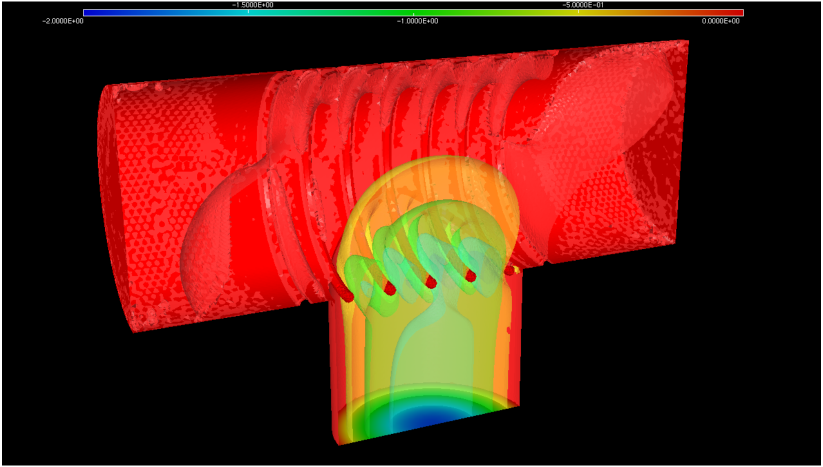

Limit solution when $\epsilon \to 0$

The limiting regime corresponds to

- $\#$ of holes goes to ∞ ($N = 1/\epsilon \to \infty$)
- size of the holes vanishes ($\epsilon \to 0$)

the "Density" of adherence conditions on Γ^ε increases:
Γ^ε becomes a rigid non-slip wall

- When $\epsilon = 0$ the solution is the Poiseuille flow

$$\begin{cases} \mathbf{u}_0(x) = \dfrac{p_{\mathrm{in}}}{2}(1 - x_2)x_2 \mathbf{e}_1 1_{\Omega_1}, & \forall x \in \Omega \\ p_0(x) = p_{\mathrm{in}}(1 - x_1)1_{\Omega_1} \end{cases}$$

- Error estimates

Theorem

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{\mathbf{L}^2(\Omega_\varepsilon)^2} + \|p_\varepsilon - p_0\|_{H^{-1}(\Omega_\varepsilon)} \leq k\varepsilon$$

the constant k does not depend on ϵ

Limit solution when $\epsilon \to 0$

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- In Ω_ε it solves

$$\begin{cases} -\Delta \mathbf{u}_0 + \nabla p_0 = [\sigma_{\mathbf{u}_0, p_0}] \cdot \mathbf{n}\, \delta_{\Gamma_0} \text{ in } \Omega_\varepsilon \\ \operatorname{div} \mathbf{u}_0 = 0 \\ \mathbf{u}_0 = 0 \text{ on } \Gamma_1 \cup \Gamma_2 \\ \mathbf{u}_0 \cdot \boldsymbol{\tau} = 0 \text{ on } \Gamma_{\rm in} \cup \Gamma_{{\rm out},1} \cup \Gamma_{{\rm out},2} \\ p_0 = p_{\rm in} \text{ on } \Gamma_{\rm in}, \quad p_0 = 0 \text{ on } \Gamma_{{\rm out},1} \cup \Gamma_{{\rm out},2} \\ \mathbf{u}_0 \neq 0 \text{ on } \Gamma^\varepsilon \end{cases}$$

where $[\sigma_{\mathbf{u}_0, p_0}] \cdot \mathbf{n} = [\nabla \mathbf{u}_0 - p_0 \mathrm{Id}_2] \cdot \mathbf{n}$ is the jump across Γ_0 of the stress tensor

- Error estimates

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the constant k does not depend on ϵ.

Higher order approximation ?

- No flow at zeroth order through $\Gamma_{\mathrm{out},2}$!
- Errors treefold
  1. Dirichlet non homogeneous on Γ^ε
  2. Jumps at Γ_0 of $\frac{\partial u_{0,1}}{\partial x_2}$
  3. Jumps at Γ_0 of p_0
- Correction : zoom on roughness
  $\sim$ study two kind microscopic boundary layers
  1. verical
  2. horizontal

Dirichlet correction

- Microscopic corrector *à la Pironneau*

$$\begin{cases} -\Delta\boldsymbol{\beta} + \nabla\pi = 0 \text{ in } S \\ \operatorname{div}\boldsymbol{\beta} = 0 \\ \boldsymbol{\beta} = -y_2\mathbf{e}_1 \text{ on } P \\ \beta_2 \to 0, \quad |y_2| \to \infty \\ \boldsymbol{\beta} \text{ is } y_1 - \text{periodic} \end{cases}$$

- Properties

Theorem

$\exists!(\boldsymbol{\beta},\pi)$, π defined up to a constant, s.t.

$$\nabla\boldsymbol{\beta} \in L^2(S)^4, (\boldsymbol{\beta} - \overline{\beta}) \in \mathbf{L}^2(S), \quad \pi \in L^2_{\mathrm{loc}}(S)$$

Moreover, one has:

$$\boldsymbol{\beta}(y) \to \overline{\beta}(\pm\infty)\mathbf{e}_1, \quad y_2 \to \pm\infty, \quad \overline{\beta}(\cdot) := \int_0^1 \boldsymbol{\beta}(y_1,\cdot)\,dy_1$$

and

$$\begin{cases} \overline{\beta}_2(y_2) = 0, & \forall y_2 \in \mathbb{R} \\ \overline{\beta}_1(y_2) = -\mu(Q) - |\nabla\boldsymbol{\beta}|^2_{L^2(S)} + \overline{\beta}_1(0), & y_2 > y_{2,P}, \\ \overline{\beta}_1(y_2) = \overline{\beta}_1(0), & y_2 < 0 \\ \overline{\pi}(y_2) = 0, & y_2 > y_{2,P} \text{ and } y_2 < 0 \end{cases}$$

where $y_{2,P} := max_{y \in P} y_2$ and $\mu(Q)$ is the volume of Q

Normal derivative horizontal velocity correction

- Microscopic corrector *à la Mikelić*

$$\begin{cases} -\Delta\Upsilon + \nabla\varpi = -\delta_{\Sigma}\mathbf{e}_1 \text{ in } S \\ \operatorname{div}\Upsilon = 0 \\ \Upsilon = 0 \text{ on } P \\ \Upsilon_2 \to 0, \quad |y_2| \to \infty \end{cases}$$

- Properties

Theorem

$\exists!(\Upsilon,\varpi)$, ϖ defined up to a constant, s.t.

$$\nabla\boldsymbol{\Upsilon} \in L^2(S)^4, \quad (\boldsymbol{\Upsilon} - \overline{\Upsilon}) \in \mathbf{L}^2(S), \quad \varpi \in L^2_{\mathrm{loc}}(S)$$

Moreover, one has:

$$\Upsilon(y) \to \overline{\Upsilon}_{\pm}\mathbf{e}_1, \quad y_2 \to \pm\infty$$

and

$$\begin{cases} \overline{\Upsilon}_2(y_2) = 0, & \forall y_2 \in \mathbb{R} \\ \overline{\Upsilon}_1(y_2) = \overline{\Upsilon}_1(0) + \overline{\beta}_1(0), & y_2 > y_{2,P}, \\ \overline{\Upsilon}_1(y_2) = \overline{\Upsilon}_1(0), & y_2 < 0 \\ \overline{\varpi}(y_2) = 0, & y_2 > y_{2,P} \text{ and } y_2 < 0 \end{cases}$$

where $y_{2,P} := max_{y\in P} y_2$.

Vertical correctors

- Microscopic corrector *à la Conca*

$$\begin{cases} -\Delta\chi + \nabla\eta = 0 \text{ in } S \\ \operatorname{div}\chi = 0 \\ \chi = 0 \text{ on } P \\ \chi_2 \to -1, \quad |y_2| \to \infty \end{cases}$$

- Properties

Theorem

$\exists!(\boldsymbol{\chi},\eta)$, η defined up to a constant, s.t.

$$\nabla\boldsymbol{\chi} \in L^2(S)^4, \quad (\boldsymbol{\chi} - \overline{\chi}) \in \mathbf{L}^2(S), \quad \eta \in L^2_{\mathrm{loc}}(S)$$

Moreover, one has:

$$\boldsymbol{\chi}(y) \to -\overline{\chi}_{\pm}\mathbf{e}_2, \quad \eta(y) \to \overline{\eta}_{\pm}, \quad y_2 \to \pm\infty$$

and

$$\begin{cases} \eta(y) = \overline{\eta}^{\pm}, \quad \forall y_2 \in \mathbb{R}_- \times]y_{2,P}, +\infty[, \\ |\nabla\boldsymbol{\chi}|^2_{L^2(S)} = [\overline{\eta}]^+_- \end{cases}$$

where $y_{2,P} := max_{y\in P} y_2$.

First order approximation

$$\mathcal{U}_\varepsilon := \mathbf{u}_0 + \epsilon \left\{ \frac{\partial u_{0,1}}{\partial x_2} (\boldsymbol{\beta}_\varepsilon - \overline{\overline{\boldsymbol{\beta}}}) + \left[\frac{\partial u_{0,1}}{\partial x_2} \right] (\boldsymbol{\Upsilon}_\epsilon - \overline{\overline{\boldsymbol{\Upsilon}}}) + \frac{[p_0]}{[\overline{\overline{\eta}}]} (\boldsymbol{\chi}_\varepsilon - \overline{\overline{\boldsymbol{\chi}}}) \right\} + \epsilon \mathbf{u}_1$$

$$\mathcal{P}_\varepsilon := p_0 + \left\{ \frac{\partial u_{0,1}}{\partial x_2} \pi_\epsilon + \left[\frac{\partial u_{0,1}}{\partial x_2} \right] \varpi_\epsilon + \frac{[p_0]}{[\overline{\overline{\eta}}]} (\eta_\epsilon - \overline{\overline{\eta}}) \right\} + \epsilon p_1$$

- multi-scale version of boundary layer correctors $\forall x \in \Omega_\varepsilon$:

$$\boldsymbol{\beta}_\varepsilon(x) := \boldsymbol{\beta}\left(\frac{x}{\varepsilon}\right), \quad \boldsymbol{\Upsilon}_\epsilon(x) := \boldsymbol{\Upsilon}\left(\frac{x}{\varepsilon}\right), \quad \boldsymbol{\chi}_\varepsilon(x) := \boldsymbol{\chi}\left(\frac{x}{\varepsilon}\right), \ldots$$

Higher order macroscopic correctors

$$
(*) \begin{cases}
-\Delta \mathbf{u}_1 + \nabla p_1 = 0 \text{ in } \Omega_1 \cup \Omega_2 \\
\operatorname{div} \mathbf{u}_1 = 0 \\
\mathbf{u}_1 = 0 \text{ on } \Gamma_1 \cup \Gamma_2 \\
\left.\begin{array}{l} \mathbf{u}_1 \cdot \boldsymbol{\tau} = 0, \\ p_1 = 0 \end{array}\right\} \text{ on } \Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out},1} \cup \Gamma_{\mathrm{out},2} \\
\mathbf{u}_1 = \left\{ \frac{\partial u_{0,1}}{\partial x_2} \overline{\overline{\beta}}^{\pm} + \left[\frac{\partial u_{0,1}}{\partial x_2} \right] \overline{\overline{\Upsilon}}^{\pm} \right\} \mathbf{e}_1 + \frac{[p_0]}{[\overline{\overline{\eta}}]} \overline{\overline{\chi}} \mathbf{e}_2 \text{ on } {\Gamma_0}^{\pm}
\end{cases}
$$

$\mathbf{u}_1$ discontinuous at corners: $(\mathbf{u}_1, p_1)$ only very weak solution

Theorem

$\exists!(\mathbf{u}_1, p_1) \in \mathbf{L}^2(\Omega_1 \cup \Omega_2) \times H^{-1}(\Omega_1 \cup \Omega_2)$ *solving* $(*)$

$(\mathcal{U}_\varepsilon, \mathcal{P}_\varepsilon)$ cannot belong to $\mathbf{H}^1(\Omega_\varepsilon) \times L^2(\Omega_\varepsilon)$

Main convergence results

Theorem

In the framework of very weak solutions a la *Conca*

$$\|\mathbf{u}_\varepsilon - \mathcal{U}_\varepsilon\|_{L^2(\Omega_1 \cup \Omega_2)} + \|p_\varepsilon - \mathcal{P}_\varepsilon\|_{H^{-1}(\Omega_1' \cup B_\epsilon \cup \Omega_2)} \leq k \epsilon^{\frac{3}{2}^-}$$

At this point the approximation $(\mathcal{V}_\varepsilon, \mathcal{P}_\varepsilon)$ is multi-scale

Main ingredients of the proof

Main difficulties

1. Restriction operator $R_\epsilon \mathbf{u}$ s.t.

$$\|\nabla R_\epsilon \mathbf{u}\|_{L^2(\Omega_\varepsilon)^4} \leq \frac{1}{\varepsilon}\|\mathbf{u}\|_{\mathbf{L}^2(\Omega)} + \|\nabla \mathbf{u}\|_{L^2(\Omega)^4}$$

consequence divergence surjective

$$\forall f \in L^2 \quad \exists \mathbf{v} \text{ s.t. } \operatorname{div} \mathbf{v} = f \text{ but } \|\nabla \mathbf{v}\|_{L^2(\Omega_\varepsilon)^4} \leq \frac{1}{\sqrt{\varepsilon}}\|f\|_{L^2(\Omega_\varepsilon)}$$

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2. Spurious oscillations on latteral boundaries of $\boldsymbol{\beta}_\varepsilon, \boldsymbol{\chi}_\varepsilon, \ldots$
   - microscopic vertical boundary layer correctors $(\mathbf{w}(y), \theta(y))$: Well-posedness in microscopic Weighted Sobolev spaces

$$W^{m,p}_\alpha(\Pi) := \left\{ v \in \mathcal{D}'(\Pi) \text{ s.t. } |D^\lambda v| \rho^{\alpha+|\lambda|-m} \in L^p(\Pi),\, 0 \leq |\lambda| \leq m \right\},$$

   equivalent to macroscopic decay rates in terms of ε.

$$\|\varepsilon \mathbf{w}(\cdot/\varepsilon)\|_{\mathbf{L}^2(\Omega)} + \|\theta(\cdot/\varepsilon)\|_{H^{-1}(\Omega)} \leq \varepsilon^{\frac{3}{2}^-}$$

Main ingredients of the proof

Main difficulties

3. Lack of continuity of $(\mathcal{U}_\varepsilon, \mathcal{P}_\varepsilon)$
   - Special regularizing cut-of function

$$\lambda_\delta := 1_{B(x,\delta)}(z)\frac{z_1 - x_1}{\delta} + 1_{\{\Omega_1 \setminus B(x,\delta)\}}(z)\frac{z_1 - x_1}{|x - z|}, \quad \forall z \in \Omega_1.$$

s.t.

$$\|\lambda_\varepsilon\|_{H^1(\Omega_1)} \leq k\{|\log(\varepsilon)|^{\frac{1}{2}} + 1\},$$

Bibliographic perspective

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  - Roughness (Mikelić Neuss)
  - Sieve : direct estimates in $\mathbf{L}^2(\Omega_i) \times H^{-1}(\Omega_i)$ (Conca)
- Here : improved Conca's approach

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- Here : improved Conca's approach
  1. Regularize $\mathcal{U}_\varepsilon \Rightarrow \mathcal{U}_\varepsilon^\lambda$
  2. Obtain $\mathbf{H}^1(\Omega_{\varepsilon,1})$ estimates
  3. compute again very weak norms on $\Omega_{\varepsilon,1}$ and Ω_2 using Poincaré
  4. compare $\mathcal{U}_\varepsilon$ and $\mathcal{U}_\varepsilon^\lambda$ in the very weak setting

Averaged macroscopic approximation

Suppress the oscillating boundary layer, keep the rest:

$$\overline{\mathbf{u}}_\varepsilon := \mathbf{u}_0 + \epsilon \mathbf{u}_1$$
$$\overline{p}_\varepsilon := p_0 + \epsilon p_1$$

It solves

$$\begin{cases} -\Delta \overline{\mathbf{u}}_\varepsilon + \nabla \overline{p}_\varepsilon = 0 \text{ in } \Omega_1 \cup \Omega_2 \\ \operatorname{div} \overline{\mathbf{u}}_\varepsilon = 0 \\ \overline{\mathbf{u}}_\varepsilon = 0 \text{ on } \Gamma_1 \cup \Gamma_2 \\ \overline{\mathbf{u}}_\varepsilon \cdot \boldsymbol{\tau} = 0 \text{ on } \Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out},1} \cup \Gamma_{\mathrm{out},2}, \\ \overline{p}_\varepsilon = p_{\mathrm{in}} \text{ on } \Gamma_{\mathrm{in}}, \quad \overline{p}_\varepsilon = 0 \text{ on } \Gamma_{\mathrm{out},1} \cup \Gamma_{\mathrm{out},2} \\ \overline{\mathbf{u}}_\varepsilon = \epsilon \left\{ \dfrac{\partial u_{0,1}}{\partial x_2} \overline{\overline{\beta}}_1^{\pm} + \left[\dfrac{\partial u_{0,1}}{\partial x_2} \right] \overline{\overline{\Upsilon}}_1^{\pm} \right\} \mathbf{e}_1 + \epsilon \dfrac{[p_0]}{[\overline{\overline{\eta}}]} \overline{\overline{\chi}} \mathbf{e}_2 \text{ on } {\Gamma_0}^{\pm} \end{cases}$$

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Theorem

Very weak solutions à la Conca

$$\|\mathbf{u}_\varepsilon - \overline{\mathbf{u}}_\varepsilon\|_{L^2(\Omega_1 \cup \Omega_2)} + \|p_\varepsilon - \overline{p}_\varepsilon\|_{H^{-1}(\Omega_1' \cup B_\epsilon \cup \Omega_2)} \leq k \epsilon^{\frac{3}{2}^-}$$

Idea of the proof

$$\|\mathbf{u}_\varepsilon - \overline{\mathbf{u}}_\varepsilon\|_{L^2(\Omega_1 \cup \Omega_2)} \leq \|\mathbf{u}_\varepsilon - \mathcal{U}_\varepsilon\|_{L^2(\Omega_1 \cup \Omega_2)} + \|\mathcal{U}_\varepsilon - \overline{\mathbf{u}}_\varepsilon\|_{L^2(\Omega_1 \cup \Omega_2)}$$

we obtain pressure estimates in H^{-1} as well.

Compute the first order flow rate

- velocity profile normal direction to Γ_0

$$\mathcal{U}_{\epsilon,2}(x) = (u_{0,2} + \epsilon u_{1,2})(x) \equiv -\epsilon \frac{[p_0]}{[\overline{\overline{\eta}}]}(x)$$

- Solve with a computer a cell problem (cheap even in 3D):

$$[\overline{\overline{\eta}}] = \overline{\overline{\eta}}^{+} - \overline{\overline{\eta}}^{-}$$

- First order flow rate

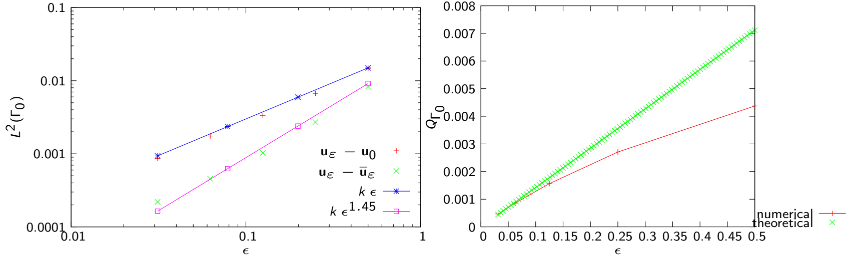
$$Q_{\Gamma_0} = \frac{\epsilon}{[\overline{\overline{\eta}}]} \int_a^b [p_0]\, dx_1$$

Numerical evidence

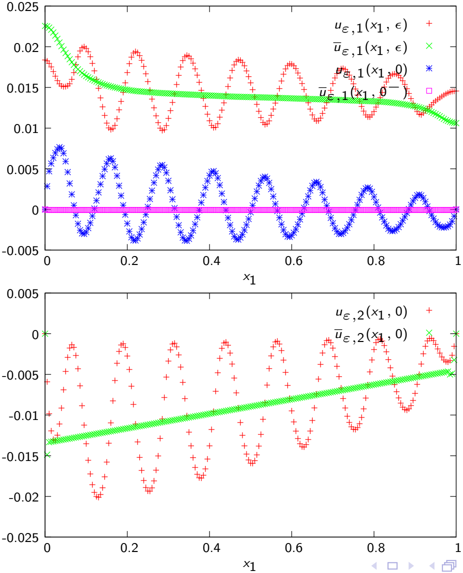
- Compute the exact problem
- Compute boundary layers for a single # cell
  - Extract the constants at infinity

$$[\overline{\overline{\eta}}] = 52.6961$$

- Compute the flow-rate



Normal velocity profile



Averaged implicit macroscopic approximation

Take again

$$\overline{\mathbf{u}}_\varepsilon := \mathbf{u}_0 + \epsilon \mathbf{u}_1$$

$$\overline{p}_\varepsilon := p_0 + \epsilon p_1$$

On Γ_0 $\overline{\mathbf{u}}_\varepsilon$ satisfies

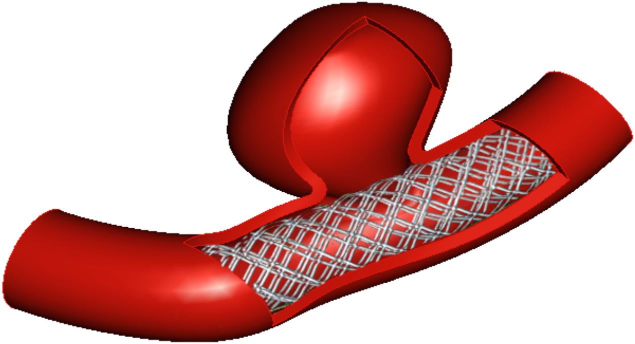
$$\overline{\mathbf{u}}_\varepsilon^\pm = \epsilon \left\{ \frac{\partial u_{0,1}}{\partial x_2} \overline{\overline{\beta}}_1^\pm + \left[\frac{\partial u_{0,1}}{\partial x_2} \right] \overline{\overline{\Upsilon}}_1^\pm \right\} \mathbf{e}_1 + \epsilon \frac{[p_0]}{[\overline{\overline{\eta}}]} \overline{\overline{\chi}} \mathbf{e}_2$$

Impliciting the interface conditions:

$$\begin{cases} \overline{\mathbf{u}}_\varepsilon^+ \cdot \mathbf{n} = \overline{\mathbf{u}}_\varepsilon^- \cdot \mathbf{n}, \quad [\overline{\overline{\eta}}] \overline{\mathbf{u}}_\varepsilon \cdot \mathbf{n} = -\epsilon \left(\left[\sigma_{\overline{\mathbf{u}}_\varepsilon, \overline{p}_\varepsilon} \right] \cdot \mathbf{n}, \mathbf{n} \right) \\ \overline{\mathbf{u}}_\varepsilon^\pm \cdot \boldsymbol{\tau} = \epsilon \left(\overline{\overline{\beta}}_1^\pm (\sigma^+_{\overline{\mathbf{u}}_\varepsilon, \overline{p}_\varepsilon} \cdot \mathbf{n}, \boldsymbol{\tau}) + \overline{\overline{\Upsilon}}_1^\pm \left[\sigma_{\overline{\mathbf{u}}_\varepsilon, \overline{p}_\varepsilon} \cdot \mathbf{n} \right] \cdot \boldsymbol{\tau} \right) \end{cases}$$

gives:

- a new "Strange" term in the normal direction
- "Wall law" expression in the tangential one

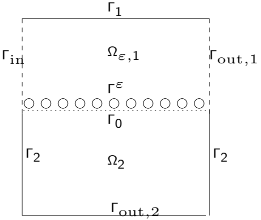


Same problem

but . . .

No output allowed trough $\Gamma_{\mathrm{out},2}$

$$\begin{cases} -\Delta \mathbf{u}_\varepsilon + \nabla p_\varepsilon = 0 \text{ in } \Omega_\varepsilon \\ \operatorname{div} \mathbf{u}_\varepsilon = 0 \\ \mathbf{u}_\varepsilon = 0 \text{ on } \Gamma_1 \cup \Gamma_2 \cup \Gamma^\varepsilon \\ \mathbf{u}_\varepsilon \cdot \boldsymbol{\tau} = 0 \text{ on } \Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out},1} \\ \mathbf{u}_\varepsilon = 0 \text{ on } \Gamma_{\mathrm{out},2} \\ p_\varepsilon = p_{\mathrm{in}} \text{ on } \Gamma_{\mathrm{in}}, \quad p_\varepsilon = p_{\mathrm{out},1} \text{ on } \Gamma_{\mathrm{out},1}, \end{cases}$$



Pressure imposed at inlet and outlet but not at $\Gamma_{\mathrm{out},2}$

When ϵ goes to 0

- The Poiseuille flow

$$\begin{cases} -\Delta \mathbf{u}_0 + \nabla p_0 = [\sigma_{\mathbf{u}_0,p_0}] \cdot \mathbf{n}\, \delta_{\Gamma_0} \text{ in } \Omega \\ \operatorname{div} \mathbf{u}_0 = 0 \\ \mathbf{u}_0 = 0 \text{ on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_{\mathrm{out},2} \\ \mathbf{u}_0 \cdot \boldsymbol{\tau} = 0 \text{ on } \Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out},1} \\ p_0 = p_{\mathrm{in}} \text{ on } \Gamma_{\mathrm{in}}, \quad p_0 = 0 \text{ on } \Gamma_{\mathrm{out},1} \\ \mathbf{u}_0 \neq 0 \text{ on } \Gamma^{\varepsilon} \end{cases}$$

- $(\mathbf{u}_0, p_0)$ is explicit and reads:

$$\begin{cases} \mathbf{u}_0(x) = \frac{p_{\mathrm{in}}}{2}(1 - x_2)x_2 \mathbf{e}_1 1_{\Omega_1}, \quad \forall x \in \Omega \\ p_0(x) = p_{\mathrm{in}}(1 - x_1) 1_{\Omega_1} + p_0^- 1_{\Omega_2}, \quad \forall p_0^- \in \mathbb{R} \end{cases}$$

First order approximation

Again the same trick

$$\mathcal{V}_\varepsilon := \mathbf{u}_0 + \epsilon \left\{ \frac{\partial u_{0,1}}{\partial x_2}(\boldsymbol{\beta}_\varepsilon - \overline{\overline{\boldsymbol{\beta}}}) + \left[\frac{\partial u_{0,1}}{\partial x_2} \right] (\Upsilon_\epsilon - \overline{\overline{\boldsymbol{\Upsilon}}}) + \frac{[p_0]}{[\overline{\overline{\eta}}]}(\boldsymbol{\chi}_\varepsilon - \overline{\overline{\boldsymbol{\chi}}}) + \mathbf{u}_1 \right\}$$

$$\mathcal{P}_\varepsilon := p_0 + \left\{ \frac{\partial u_{0,1}}{\partial x_2}\pi_\epsilon + \left[\frac{\partial u_{0,1}}{\partial x_2} \right] \varpi_\epsilon + \frac{[p_0]}{[\overline{\overline{\eta}}]}(\eta_\epsilon - \overline{\overline{\eta}}) + \epsilon p_1 \right\}$$

- multi-scale version of boundary layer correctors:

$$\boldsymbol{\beta}_\varepsilon(x) := \boldsymbol{\beta}\left(\frac{x}{\varepsilon}\right), \quad \boldsymbol{\Upsilon}_\varepsilon(x) := \boldsymbol{\Upsilon}\left(\frac{x}{\varepsilon}\right), \quad \boldsymbol{\chi}_\varepsilon(x) := \boldsymbol{\chi}\left(\frac{x}{\varepsilon}\right), \dots$$

The pressure in the sac

$$\begin{cases} -\Delta \mathbf{u}_1 + \nabla p_1 = 0 \text{ in } \Omega_1 \cup \Omega_2 \\ \operatorname{div} \mathbf{u}_1 = 0 \\ \mathbf{u}_1 = 0 \text{ on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_{\mathrm{out},2} \\ \mathbf{u}_1 \cdot \boldsymbol{\tau} = 0 \text{ on } \Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out},1}, \\ p_1 = 0 \text{ on } \Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out},1} \\ \mathbf{u}_1 = \left\{ \frac{\partial u_{0,1}}{\partial x_2} \overline{\overline{\beta}}^{\pm} + \left[\frac{\partial u_{0,1}}{\partial x_2} \right] \overline{\overline{\Upsilon}}^{\pm} \right\} \mathbf{e}_1 + \frac{[p_0]}{[\overline{\overline{\eta}}]} \overline{\overline{\chi}} \mathbf{e}_2 \text{ on } {\Gamma_0}^{\pm} \end{cases}$$

divergence condition and normal veolicty imposed on Γ_0:

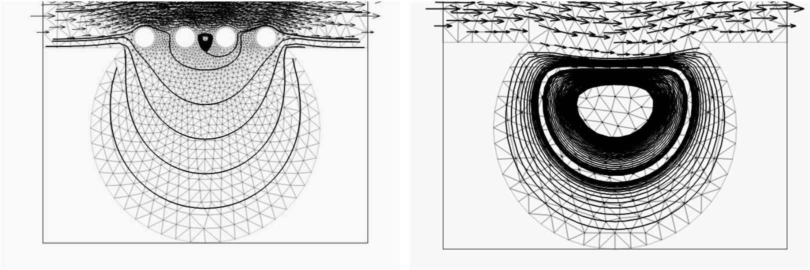
$$\int_{\Omega_2} \operatorname{div} \mathbf{u}_1 dx = \int_{\partial \Omega_2} \mathbf{u}_1 \cdot \mathbf{n} d\sigma = \int_{\Gamma_0} \mathbf{u}_1 \cdot \mathbf{n} d\sigma = 0$$

gives in the sac

$$p_0^- = \frac{1}{|\Gamma_0|} \int_{\Gamma_0} p_0^+(x_1, 0) dx_1$$

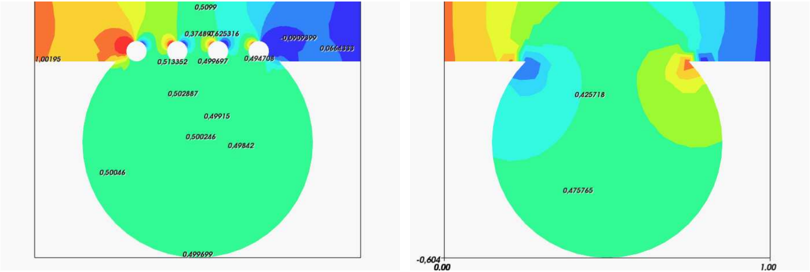
"Pathological" vs "Stented" cases

Velocities & streamlines



"Pathological" vs "Stented" cases

Pressures



The full time-dependent non-linear system

$$\begin{cases} \rho(\partial_t \mathbf{u}_\varepsilon + (\mathbf{u}_\varepsilon \cdot \nabla)\mathbf{u}_\varepsilon) - \nu \Delta \mathbf{u}_\varepsilon + \nabla p_\varepsilon = 0, \text{ in } \Omega_\varepsilon \\ \operatorname{div} \mathbf{u}_\varepsilon = 0 \\ \mathbf{u}_\varepsilon = 0 \text{ on } \Gamma_1 \cup \Gamma_2 \cup \Gamma^\varepsilon \\ \mathbf{u}_\varepsilon \cdot \boldsymbol{\tau} = 0 \text{ on } \Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out},1} \cup \Gamma_{\mathrm{out},2} \\ p_\varepsilon = p_{\mathrm{in}}(t) \text{ on } \Gamma_{\mathrm{in}}, \quad p_\varepsilon = p_{\mathrm{out},1} \text{ on } \Gamma_{\mathrm{out},1}, p_\varepsilon = p_{\mathrm{out},2} \text{ on } \Gamma_{\mathrm{out},2}, \end{cases}$$

where for instance

$$p_{\mathrm{in}} \in L^2(0, T)$$

Theoretical investigation

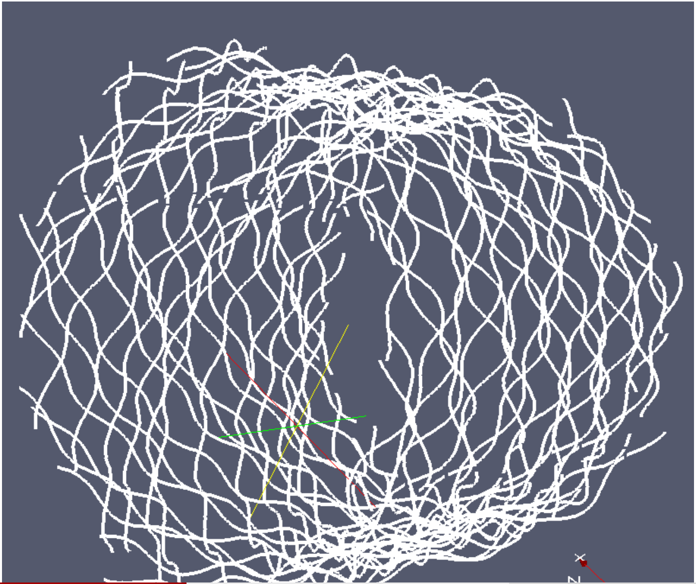
Joint work with A. Rambaud (Phd Ens Lyon) on a simplified problem

$$\begin{cases} \partial_t u_\varepsilon - \Delta u_\varepsilon = f(t) & \text{in } \Omega_\varepsilon \\ u_\varepsilon = 0 & \text{on } \Gamma^\varepsilon \cup \Gamma_1 \\ u_\varepsilon \text{ is } x_1 - \text{periodic} & \text{on } \Gamma_{\text{in}} \cup \Gamma_{\text{out}} \\ u_\varepsilon = u_{0,\epsilon}(x) & \text{in } \Omega_\varepsilon \end{cases}$$

Fourier analysis: time$\times$space domain => frequency$\times$space domain

the microscopic boundary layer is steady

The real geometry



The real geometry

- from 48 up to 80 wires !!
- diameter single wire $\ll$ thikness of the stent

Direct computation is impossible for the time being.

Homogenization

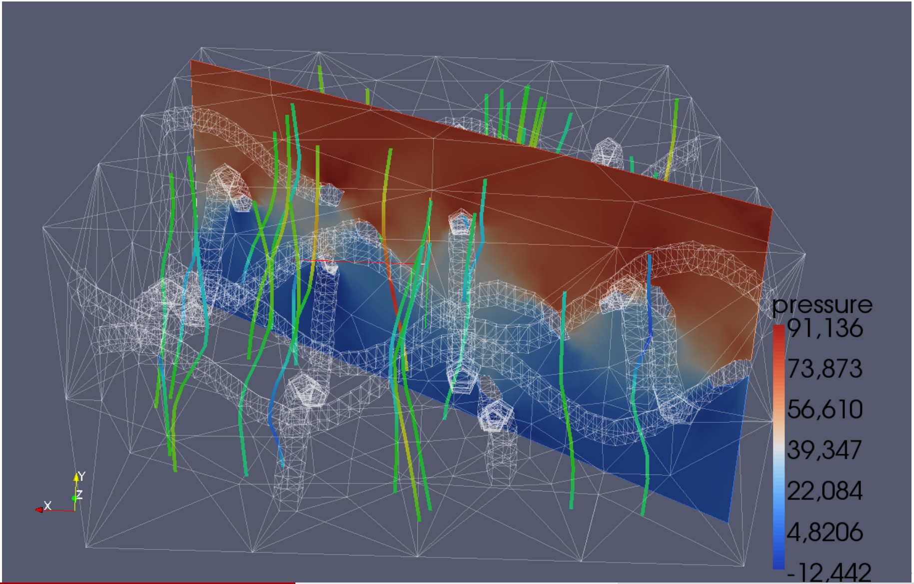
It is the only way to get an idea of the flow.

Strategy is then:

- Find the minimal periodic cell
- Build the fluid mesh ($\neq$ periodic)
- Run a Stokes solver for periodic infinite strips
  We use the Uzawa Conjugate Gradient Algorythm coupled together with a domain decomposition method in order to catch:
  - bahaviour at $|y_3| \to \infty$
  - periodicity on lateral sides

  Numerical behaviour:
  - (microscopic level $\sim$ 50 000 dof)
  - Single Uzawa step $\sim$ 10 min
  - Stabilization / Penalty diverges when coupled to DD method.
- extract meaningfull values (for inst. $\overline{\overline{\eta}}$)
- Use them in a macro computation
- Compare to experimental values

a single computation



a single computation

One of the 3D homogenized quantities of interest:

$$\overline{\overline{\eta}} = 188$$

To be used in computations

- explicit by hand computations (idelized geometry)
- macroscopic numerical computations

Work in progress !!

Conclusions & Perspectives

Conclusions

- Clear understanding in the “mean”
  - First order agreement of the flow rate in stented arteries, both cases
  - Pressure in the sac computed accurately
    - thoeretícaly: first order
    - numerically: second order

Perspectives

- curved boundaries
- error estimates for Navier-Stokes
- homogenized drug delivery
- cell growth

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  - Cardiatis
    see www.cardiatis.com