Blood-flow modelling along and trough a braided multi-layer metallic stent

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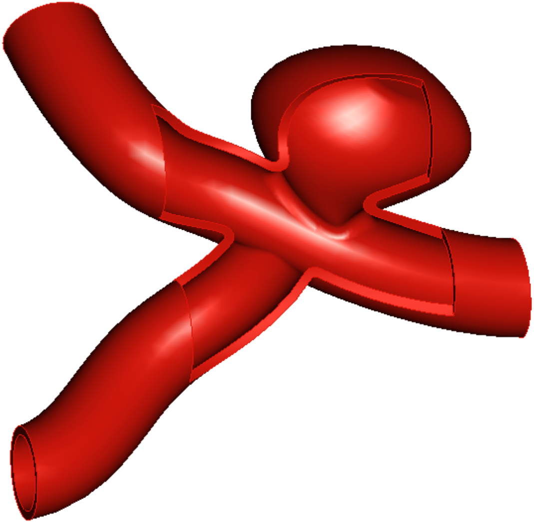
Industrial context

Cardiatis®: conception and comercialisation of metallic wired multi-layer stents ▸ Image 3D

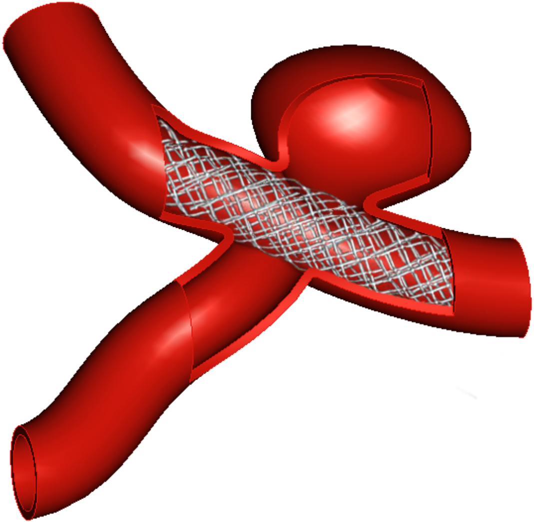
- A new technology
  One controls
  - The # of layers
  - Their connectivity
- In vivo experiments
  1. on mini-pigs show :no thrombus up to 6 months ▸ Dissection pictures
  2. on humans : ▸ Microscopy pictures
- Multi-Scale phenomenon lying on:
  - Hemodynamics
  - Chemical reactions between blood flow and the surrounding wires and tissues

Theoretical & numerical study of hemodynamics

Where does the stent apply ?



Where does the stent apply ?



Problem description

- Geometrical properties
  - Femoral artery diameter:$\emptyset_{\mathrm{A}} = 6mm$
  - Total thickness of the stent : $\epsilon = 0.25mm$
  - Thickness of a single wire: $\epsilon = 0.04mm$
  - Red blood cell diameter: $\emptyset_{\mathrm{RC}} = 0.008mm$

$$\frac{\epsilon}{\emptyset_A} = \frac{0.25}{6} \sim 4\%$$

  stent $\sim$ periodic rugous wall in a straight cilindrical geometry
- The blood flow is a pressure driven flow composed of
  - Steady state part: geometrical perturbation of a Poiseuille profile
  - Plus a pulsatile periodic perturbation: Womersley profile
- We consider here the steady part

Methods and references

-We aim to

- understand the dynamics of flows in rugous possibly complex geomteries
  $\Longrightarrow$ Boundary layer correctors
- Avoid heavy discretisations related to the rugous wall
  $\Longrightarrow$ Wall laws
- Include the micro scales in the macro Poiseuille profile
  $\Longrightarrow$ Multi-scale aspects

Use of **assymptotic expansions** adapted for the **perturbed boundaries**.

References

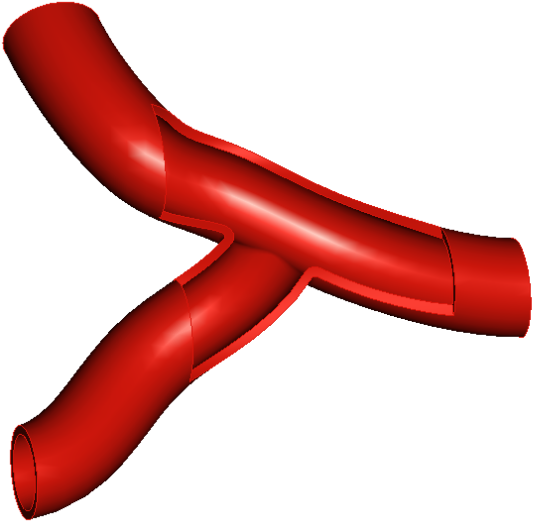
N. Neuss, M. Neuss-Radu, and A. Mikelić.
Effective laws for the poisson equation on domains with curved oscillating boundaries.
Applicable Analysis, 2006.

Y. Achdou, P. Le Tallec, F. Valentin, and O. Pironneau,
Constructing wall laws with domain decomposition or asymptotic expansion techniques
Comput. Methods Appl. Mech. Eng. 1998

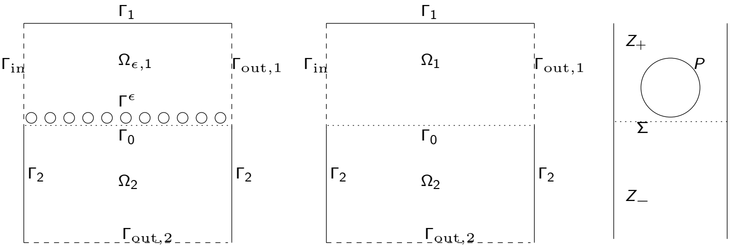
Main results

- Stented collateral artery
  1. explicit first order averaged velocity profile
  2. explicit flow-rate
  3. implicit interface conditions

- Stented anevrismal sac
  1. explicit averaged zero order pressure in the sac
  2. first order averaged velocity profile
  3. the presence of the wires inverses the vertex



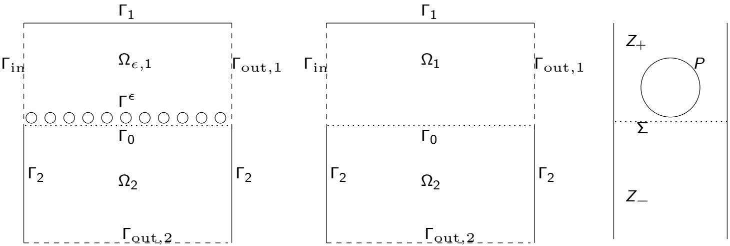
Notations and Methodology



We denote:

- $P = \{y \in \mathbb{R}^2$ s.t. $y_= r(\cos(\theta), \sin(\theta))\}$,
- Ω the "smooth domain", Γ^0 the fictitious interface,
- x the slow space variable , $y = \frac{x}{\epsilon}$ the fast one.

Notations and Methodology



Our method: transform the rough boundary problem in Ω^ϵ
into a fictitious interface problem in Ω^0

1. Construction of a complete boundary layer corrector: Ω^ϵ
2. Averaging the oscillations
3. Generlize to other geometries
   => derivation of implicit interface conditions

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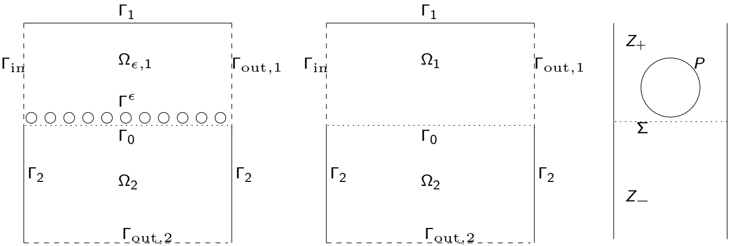

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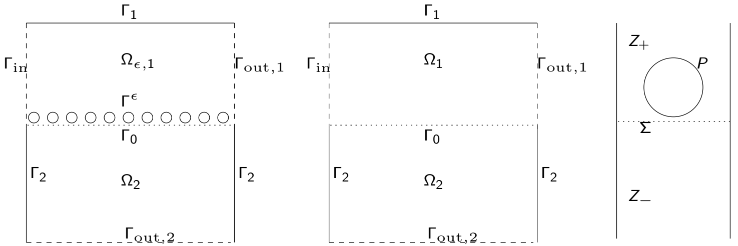

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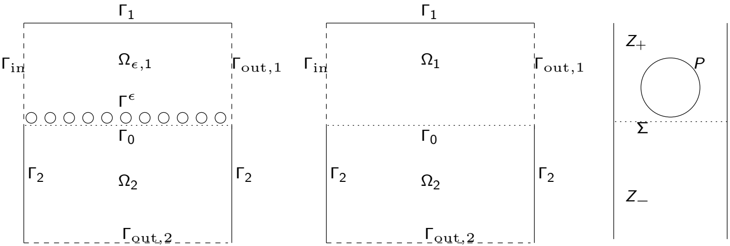

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The problem

- The flow is laminar

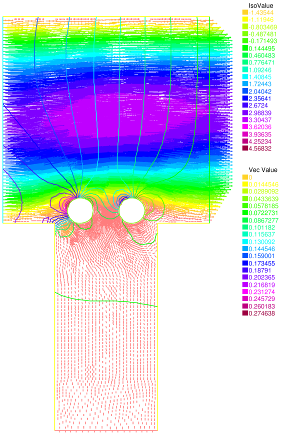
$$
\begin{cases}
-\Delta \mathbf{u}_\epsilon + \nabla p_\epsilon = 0 \text{ in } \Omega^\epsilon \\
\operatorname{div} \mathbf{u}_\epsilon = 0 \\
\mathbf{u}_\epsilon = 0 \text{ on } \Gamma_1 \cup \Gamma_2 \cup \Gamma^\epsilon \\
\mathbf{u}_\epsilon \cdot \boldsymbol{\tau} = 0 \text{ on } \Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out},1} \cup \Gamma_{\mathrm{out},2} \\
p_\epsilon = p_{\mathrm{in}} \text{ on } \Gamma_{\mathrm{in}}, \quad p_\epsilon = p_{\mathrm{out},1} \text{ on } \Gamma_{\mathrm{out},1}, p_\epsilon = p_{\mathrm{out},2} \text{ on } \Gamma_{\mathrm{out},2},
\end{cases}
$$

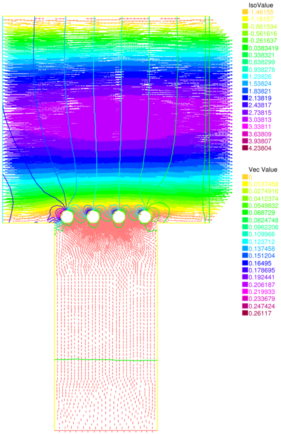
- Pressure driven $\neq$ Dirichlet velocity as in

C. Conca.
Étude d'un fluide traversant une paroi perforée. I & II.
J. Math. Pures Appl. (9), 66(1):1–70, 1987.

Expected behaviour I

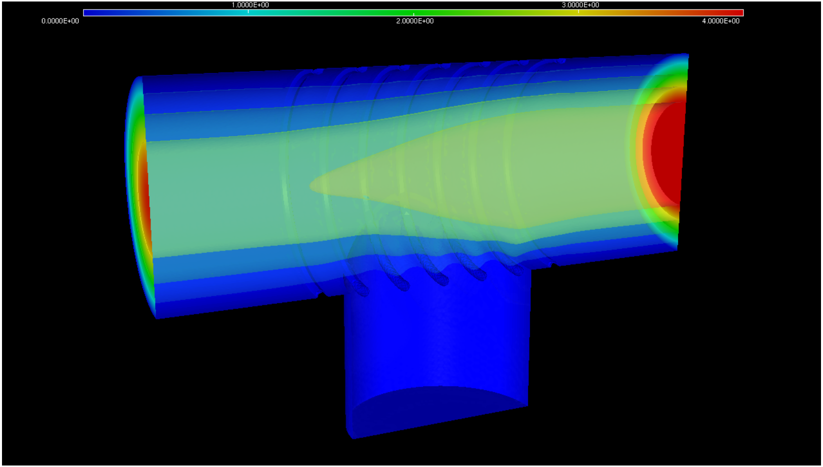
In 2D in a fiew seconds/minutes of direct simulation





Expected behaviour II

In 3D in a fiew days/weeks of direct simulation



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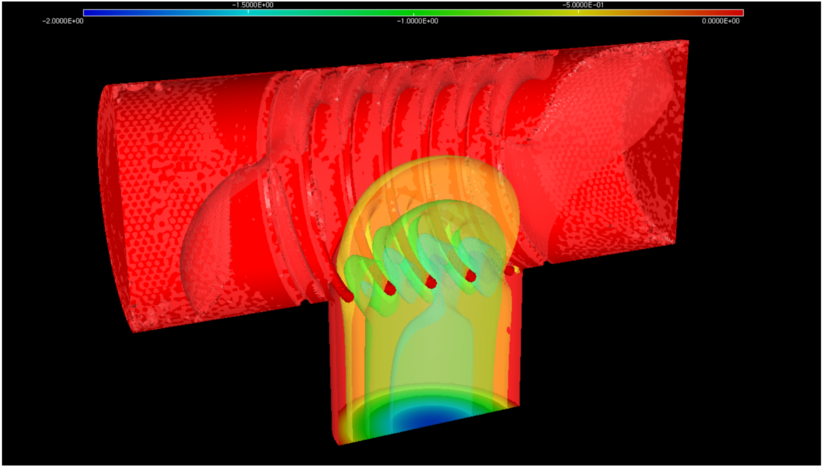

Limit solution when $\epsilon \to 0$

- The Poiseuille flow

$$\begin{cases} \mathbf{u}_0(x) = \frac{p_{\text{in}}}{2}(1 - x_2)x_2\mathbf{e}_1 1_{\Omega_1}, & \forall x \in \Omega \\ p_0(x) = p_{\text{in}}(1 - x_1)1_{\Omega_1} \end{cases}$$

Limit solution when $\epsilon \to 0$

- In Ω^ϵ it solves

$$\begin{cases} -\Delta \mathbf{u}_0 + \nabla p_0 = [\sigma_{\mathbf{u}_0,p_0}] \cdot \mathbf{n}\, \delta_{\Gamma_0} \text{ in } \Omega^\epsilon \\ \operatorname{div} \mathbf{u}_0 = 0 \\ \mathbf{u}_0 = 0 \text{ on } \Gamma_1 \cup \Gamma_2 \\ u_{0,2} = 0 \text{ on } \Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out},1}, \quad u_{0,1} = 0 \text{ on } \Gamma_{\mathrm{out},2} \\ p_0 = p_{\mathrm{in}} \text{ on } \Gamma_{\mathrm{in}}, \quad p_0 = 0 \text{ on } \Gamma_{\mathrm{out},1} \cup \Gamma_{\mathrm{out},2} \\ \mathbf{u}_0 \neq 0 \text{ on } \Gamma^\epsilon \end{cases}$$

where $[\sigma_{\mathbf{u}_0,p_0}] \cdot \mathbf{n}$ is the jump across Γ_0 of the stress tensor

Limit solution when $\epsilon \to 0$

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Theorem

$$\|\mathbf{u}_\epsilon - \mathbf{u}_0\|_{H^1(\Omega^\epsilon)^2} + \|p_\epsilon - p_0\|_{L^2(\Omega^\epsilon)} \leq k\sqrt{\epsilon}$$

the constant k does not depend on ϵ.

Higher order approximation ?

- No flow at zeroth order through $\Gamma_{\mathrm{out},2}$!
- Errors treefold
  1. Dirichlet non homogeneous on Γ^ϵ
  2. Jumps at Γ_0 of $\frac{\partial u_{0,1}}{\partial x_2}$
  3. Jumps at Γ_0 of p_0
- Use of two kind boundary layers
  1. verical
  2. horizontal

Dirichlet correction

- Microscopic corrector *à la Pironneau*

$$\begin{cases} -\Delta\boldsymbol{\beta} + \nabla\pi = 0 \text{ in } S \\ \operatorname{div}\boldsymbol{\beta} = 0 \\ \boldsymbol{\beta} = -y_2\mathbf{e}_1 \text{ on } P \\ \beta_2 \to 0, \quad |y_2| \to \infty \\ \boldsymbol{\beta} \text{ is } y_1 - \text{periodic} \end{cases}$$

- Properties

Theorem

$\exists!(\boldsymbol{\beta},\pi)$, π defined up to a constant, s.t.

$$\nabla\boldsymbol{\beta} \in L^2(S)^4, (\boldsymbol{\beta} - \overline{\beta}) \in \mathbf{L}^2(S), \quad \pi \in L^2_{\mathrm{loc}}(S)$$

Moreover, one has:

$$\boldsymbol{\beta}(y) \to \overline{\beta}(\pm\infty)\mathbf{e}_1, \quad y_2 \to \pm\infty, \quad \overline{\beta}(\cdot) := \int_0^1 \boldsymbol{\beta}(y_1,\cdot)\,dy_1$$

and

$$\begin{cases} \overline{\beta}_2(y_2) = 0, & \forall y_2 \in \mathbb{R} \\ \overline{\beta}_1(y_2) = -\mu(Q) - |\nabla\boldsymbol{\beta}|^2_{L^2(S)} + \overline{\beta}_1(0), & y_2 > y_{2,P}, \\ \overline{\beta}_1(y_2) = \overline{\beta}_1(0), & y_2 < 0 \\ \overline{\pi}(y_2) = 0, & y_2 > y_{2,P} \text{ and } y_2 < 0 \end{cases}$$

where $y_{2,P} := max_{y\in P} y_2$ and $\mu(Q)$ is the volume of Q

Normal derivative horizontal velocity correction

- Microscopic corrector *à la Mikelić*

$$\begin{cases} -\Delta\Upsilon + \nabla\varpi = -\delta_{\Sigma}\mathbf{e}_1 \text{ in } S \\ \operatorname{div}\Upsilon = 0 \\ \Upsilon = 0 \text{ on } P \\ \Upsilon_2 \to 0, \quad |y_2| \to \infty \end{cases}$$

- Properties

Theorem

$\exists!(\Upsilon,\varpi)$, ϖ defined up to a constant, s.t.

$$\nabla\boldsymbol{\Upsilon} \in L^2(S)^4, \quad (\boldsymbol{\Upsilon} - \overline{\Upsilon}) \in \mathbf{L}^2(S), \quad \varpi \in L^2_{\mathrm{loc}}(S)$$

Moreover, one has:

$$\Upsilon(y) \to \overline{\Upsilon}_{\pm}\mathbf{e}_1, \quad y_2 \to \pm\infty$$

and

$$\begin{cases} \overline{\Upsilon}_2(y_2) = 0, & \forall y_2 \in \mathbb{R} \\ \overline{\Upsilon}_1(y_2) = \overline{\Upsilon}_1(0) + \overline{\beta}_1(0), & y_2 > y_{2,P}, \\ \overline{\Upsilon}_1(y_2) = \overline{\Upsilon}_1(0), & y_2 < 0 \\ \overline{\varpi}(y_2) = 0, & y_2 > y_{2,P} \text{ and } y_2 < 0 \end{cases}$$

where $y_{2,P} := max_{y\in P} y_2$.

Vertical correctors

- Microscopic corrector *à la Conca*

$$\begin{cases} -\Delta \boldsymbol{\chi} + \nabla \eta = 0 \text{ in } S \\ \operatorname{div} \boldsymbol{\chi} = 0 \\ \boldsymbol{\chi} = 0 \text{ on } P \\ \chi_2 \to -1, \quad |y_2| \to \infty \end{cases}$$

- Properties

Theorem

$\exists!(\boldsymbol{\chi}, \eta)$, η defined up to a constant, s.t.

$$\nabla \boldsymbol{\chi} \in L^2(S)^4, \quad (\boldsymbol{\chi} - \overline{\boldsymbol{\chi}}) \in \mathbf{L}^2(S), \quad \eta \in L^2_{\mathrm{loc}}(S)$$

Moreover, one has:

$$\boldsymbol{\chi}(y) \to -\overline{\chi}_\pm \mathbf{e}_2, \quad \eta(y) \to \overline{\eta}_\pm, \quad y_2 \to \pm\infty$$

and

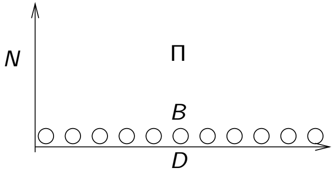
$$\begin{cases} \eta(y) = \overline{\eta}^\pm, \quad \forall y_2 \in \mathbb{R}_- \times]y_{2,P}, +\infty[, \\ |\nabla \boldsymbol{\chi}|^2_{L^2(S)} = [\overline{\eta}]^+_- \end{cases}$$

where $y_{2,P} := max_{y \in P} y_2$.

Vertical correctors

- Microscopic corrector

$$\begin{cases} -\Delta w_\beta + \nabla \theta_\beta = 0 \text{ in } \Pi \\ \operatorname{div} w_\beta = 0 \\ w_\beta = 0 \text{ on } D \cup B \\ \left.\begin{array}{l} w_\beta \wedge \mathbf{n} = \beta_2 \\ \theta_\beta = \pi \end{array}\right\} \text{ on } N \end{cases}$$



- the usual weighted Sobolev space :

$$W^{m,p}_\alpha(\Omega) := \left\{ v \in \mathcal{D}'(\Omega) \text{ s.t. } |D^\lambda v|(1+\rho^2)^{\frac{\alpha+|\lambda|-m}{2}} \in L^p(\Omega),\ 0 \le |\lambda| \le m \right\}$$

- Properties

Theorem

$$\exists! \ (\mathbf{w}, \theta) \in \mathbf{W}^{1,2}_\alpha(\Pi)^2 \times W^{0,2}_\alpha(\Pi) \ \textit{if} \ \alpha < 1$$

Localizing

1. The corner cut-of Set $\psi_1 := \overline{\psi}(x)$ and $\psi_2 := \overline{\psi}(x-(0,1))$, where $\overline{\psi}$ s.t.

$$\overline{\psi} := \begin{cases} 1 \text{ if } |x| \leq \dfrac{1}{3} \\ 0 \text{ if } |x| \geq \dfrac{2}{3} \end{cases}$$

NB: $\partial_{\mathbf{n}} \overline{\psi} = 0$ on $\Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out},1}$.

2. The "far from the corner" cut-off
Φ complementary on $\Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out},1} \cup \Gamma_2$ s.t.

$$\begin{cases} \psi + \Phi = 1 \\ \partial_{\mathbf{n}} \Phi = 0, \end{cases} \quad \text{on } \Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out},1} \cup \Gamma_2$$

for instance $\Phi(x) := 1 - \psi(0, x_2)$ for all $x \in \Omega$.

Macro corrector

$$
\begin{cases}
\left.\begin{array}{l} \Delta \mathbf{W} + \nabla Z = 0 \\ \operatorname{div} \mathbf{W} = 0 \end{array}\right., \quad \text{in } \Omega^\epsilon \\
\left.\begin{array}{r} \mathbf{W} \cdot \boldsymbol{\tau} = \epsilon \left\{ \dfrac{\partial u_{0,1}}{\partial x_2} (\boldsymbol{\beta}_\epsilon - \overline{\overline{\beta}}) \cdot \boldsymbol{\tau} \right\} \Phi, \\ Z = \left\{ \dfrac{\partial u_{0,1}}{\partial x_2} \boldsymbol{\beta} \pi \right\} \Phi \end{array}\right\} \text{ on } \Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out},1} \cup \Gamma_{\mathrm{out},2} \\
\mathbf{W} = 0 \text{ on } \Gamma^\epsilon \\
\mathbf{W} = \epsilon \left\{ \dfrac{\partial u_{0,1}}{\partial x_2} (\boldsymbol{\beta}_\epsilon - \overline{\beta}) \right\} \Phi \text{ on } \Gamma_1 \cup \Gamma_2
\end{cases}
$$

Theorem

$\exists\,!$ *solution* $(\mathbf{W}, Z) \in \mathbf{H}^1(\Omega^\epsilon) \times L^2(\Omega^\epsilon)$, *moreover:*

$$
\|\mathbf{W}\|_{\mathbf{H}^1(\Omega^\epsilon)} + \|Z\|_{L^2(\Omega^\epsilon)} \leq k e^{-\frac{\gamma}{\epsilon}}
$$

rate γ *and constant* k *do not depend on* ϵ.

Macro corrector

$$
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\mathbf{W} = 0 \text{ on } \Gamma^\epsilon \\
\mathbf{W} = \epsilon \left\{ \dfrac{\partial u_{0,1}}{\partial x_2} (\boldsymbol{\beta}_\epsilon - \overline{\beta}) \right\} \Phi \text{ on } \Gamma_1 \cup \Gamma_2
\end{cases}
$$

define

$$
\mathcal{W}_\epsilon(x) := \epsilon \left\{ \psi_1(x) \mathbf{w} \left(\frac{x}{\epsilon} \right) + \psi_2((1,0) - x) \mathbf{w} \left(\frac{(1,0) - x}{\epsilon} \right) \right\} + \mathbf{W}(x),
$$

$$
\mathcal{Z}_\epsilon(x) := \left\{ \psi_1(x) \theta \left(\frac{x}{\epsilon} \right) + \psi_2((1,0) - x) \theta \left(\frac{(1,0) - x}{\epsilon} \right) \right\} + Z(x),
$$

First order approximation

$$
\begin{aligned}
\mathcal{U}_\epsilon := \mathbf{u}_0 + \epsilon \left\{ \frac{\partial u_{0,1}}{\partial x_2} (\boldsymbol{\beta}_\epsilon - \overline{\overline{\beta}}) + \left[\frac{\partial u_{0,1}}{\partial x_2} \right] (\boldsymbol{\Upsilon}_\epsilon - \overline{\overline{\Upsilon}}) + \frac{[p_0]}{[\overline{\overline{\eta}}]} (\boldsymbol{\chi}_\epsilon - \overline{\overline{\chi}}) + \mathbf{u}_1 \right\} \\
+ \epsilon^2 \left\{ p_{\text{in}} (\varkappa_\epsilon - \overline{\overline{\varkappa}}) + \mathbf{u}_2 \right\} + \mathcal{W}_\epsilon \\
\mathcal{P}_\epsilon := p_0 + \left\{ \frac{\partial u_{0,1}}{\partial x_2} \pi_\epsilon + \left[\frac{\partial u_{0,1}}{\partial x_2} \right] \varpi_\epsilon + \frac{[p_0]}{[\overline{\overline{\eta}}]} (\eta_\epsilon - \overline{\overline{\eta}}) + \epsilon p_1 \right\} \\
+ \epsilon p_{\text{in}} (\mu_\epsilon - \overline{\overline{\mu}}) + \epsilon^2 p_2 + \mathcal{Z}_\epsilon
\end{aligned}
$$

- multi-scale version of boundary layer correctors $\forall x \in \Omega^\epsilon$:

$$
\boldsymbol{\beta}_\epsilon(x) := \boldsymbol{\beta}\left(\frac{x}{\epsilon}\right), \quad \boldsymbol{\Upsilon}_\epsilon(x) := \boldsymbol{\Upsilon}\left(\frac{x}{\epsilon}\right), \quad \boldsymbol{\chi}_\epsilon(x) := \boldsymbol{\chi}\left(\frac{x}{\epsilon}\right), \quad \varkappa_\epsilon(x) := \varkappa\left(\frac{x}{\epsilon}\right)
$$

Higher order macroscopic correctors

$$
(*) \begin{cases}
-\Delta \mathbf{u}_1 + \nabla p_1 = 0 \text{ in } \Omega_1 \cup \Omega_2 \\
\operatorname{div} \mathbf{u}_1 = 0 \\
\mathbf{u}_1 = 0 \text{ on } \Gamma_1 \cup \Gamma_2 \\
u_{1,2} = 0 \text{ on } \Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out},1}, \\
u_{1,1} = 0, \text{ on } \Gamma_{\mathrm{out},2} \\
p_1 = 0 \text{ on } \Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out},1} \cup \Gamma_{\mathrm{out},2} \\
\mathbf{u}_1 = \left\{ \frac{\partial u_{0,1}}{\partial x_2} \overline{\overline{\beta}}^{\pm} + \left[\frac{\partial u_{0,1}}{\partial x_2} \right] \overline{\overline{\Upsilon}}^{\pm} \right\} \mathbf{e}_1 + \frac{[p_0]}{[\overline{\overline{\eta}}]} \overline{\overline{\chi}} \mathbf{e}_2 \text{ on } {\Gamma_0}^{\pm}
\end{cases}
$$

Theorem

$\exists ! (\mathbf{u}_1, p_1) \in (\mathbf{H}^1(\Omega_1) \times L^2(\Omega_1)) \cup (\mathbf{L}^2(\Omega_2) \times H^{-1}(\Omega_2))$ *solving* $(*)$

$(\mathcal{U}_\epsilon, \mathcal{P}_\epsilon)$ cannot belong to $\mathbf{H}^1(\Omega^\epsilon) \times L^2(\Omega^\epsilon)$

Error estimates

How to estimate $\mathbf{u}_\epsilon - \mathcal{U}_\epsilon$ and $p_\epsilon - \mathcal{P}_\epsilon$?

- In the litterature
  - Roughness (Mikelić Neuss)
  - Sieve : direct estimates in $\mathbf{L}^2(\Omega_i) \times H^{-1}(\Omega_i)$ (Conca)
- Here : improved Conca's approach

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    1. $\mathbf{H}^{-1}(\Omega^\epsilon) \times L^2(\Omega^\epsilon)$
    2. very weak solutions $\mathbf{L}^2(\Omega^0) \times H^{-1}(\Omega^0)$ depends on $\mathbf{L}^2(\Gamma_0)$
    3. Poincaré on the rough layer + use of 1 : conclude
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    1. Method of energy : weak convergence on Γ_0
    2. Specific square $\mathbf{H}^1(\Omega_i)$ estimates : strong convergence in $L^2(\Gamma_0)$
    3. Very weak solutions : conclude
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    1. Method of energy : weak convergence on Γ_0
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    3. Very weak solutions : conclude
- Here : improved Conca's approach
  1. Split $\mathcal{U}_\epsilon$ into regular and singular terms
  2. Obtain $\mathbf{H}^1(\Omega_{\epsilon,1})$ estimates for the regular ones
  3. compute again very weak norms using Poincaré on regular terms

Behaviour on Γ_0

- We need to characterize $\mathbf{u}_\epsilon - \mathcal{U}_\epsilon$ on Γ_0

$$\mathcal{V}_\epsilon := \mathbf{u}_\epsilon - \mathcal{U}_\epsilon = \mathbf{u}_\epsilon - \mathbf{u}_0 - \epsilon \left\{ \frac{\partial u_{0,1}}{\partial x_2} \boldsymbol{\beta}_\epsilon + \left[\frac{\partial u_{0,1}}{\partial x_2} \right] \boldsymbol{\Upsilon}_\epsilon + \frac{[p_0]}{[\overline{\overline{\eta}}]} \boldsymbol{\chi}_\epsilon \right\} - O(\epsilon^2)$$

  this is the regular part of the corrector.
  Localize this in a fixed neighborhood of Γ_0
- Problem : near the corner
- Solution : define some more vertical correctors

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Theorem

$$\|\mathcal{V}_\epsilon\|_{\mathbf{H}^{-1}} + \|\mathcal{Q}_\epsilon\|_{L^2} \leq k\epsilon^{-}$$

Behaviour on Γ_0

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Theorem

$$\|\mathcal{V}_\epsilon\|_{\mathbf{H}^{-1}} + \|\mathcal{Q}_\epsilon\|_{L^2} \leq k\epsilon^{-}$$

Main ingredient of the proof

microscopic Sobolev norms of the vertical correctors $\implies$ ϵ^{-} error

Main convergence results

Theorem

In the framework of very weak solutions a la *Conca*

$$\|\mathbf{u}_\epsilon - \mathcal{U}_\epsilon\|_{L^2(\Omega_1 \cup \Omega_2)} + \|p_\epsilon - \mathcal{P}_\epsilon\|_{H^{-1}(\Omega_1' \cup B_\epsilon \cup \Omega_2)/\mathbb{R}} \leq k \epsilon^{\frac{3}{2}^-}$$

At this point the approximation $(\mathcal{V}_\epsilon, \mathcal{P}_\epsilon)$ is multi-scale

Averaged macroscopic approximation

Suppress the oscillating boundary layer, keep the rest:

$$\overline{\mathbf{u}}_\epsilon := \mathbf{u}_0 + \epsilon \mathbf{u}_1$$
$$\overline{p}_\epsilon := p_0 + \epsilon p_1$$

It solves

$$\begin{cases} -\Delta \overline{\mathbf{u}}_\epsilon + \nabla \overline{p}_\epsilon = 0 \text{ in } \Omega_1 \cup \Omega_2 \\ \operatorname{div} \overline{\mathbf{u}}_\epsilon = 0 \\ \overline{\mathbf{u}}_\epsilon = 0 \text{ on } \Gamma_1 \cup \Gamma_2 \\ \overline{\mathbf{u}}_\epsilon \wedge \mathbf{n} = 0 \text{ on } \Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out},1} \cup \Gamma_{\mathrm{out},2}, \\ \overline{p}_\epsilon = p_{\mathrm{in}} \text{ on } \Gamma_{\mathrm{in}}, \quad \overline{p}_\epsilon = 0 \text{ on } \Gamma_{\mathrm{out},1} \cup \Gamma_{\mathrm{out},2} \\ \overline{\mathbf{u}}_\epsilon = \epsilon \left\{ \frac{\partial u_{0,1}}{\partial x_2} \overline{\overline{\beta}}^{\pm} + \left[\frac{\partial u_{0,1}}{\partial x_2} \right] \overline{\overline{\Upsilon}}^{\pm} \right\} \mathbf{e}_1 + \epsilon \frac{[p_0]}{[\overline{\overline{\eta}}]} \overline{\overline{\chi}} \mathbf{e}_2 \text{ on } {\Gamma_0}^{\pm} \end{cases}$$

Averaged macroscopic approximation

Suppress the oscillating boundary layer, keep the rest:

$$\overline{\mathbf{u}}_\epsilon := \mathbf{u}_0 + \epsilon \mathbf{u}_1$$

$$\overline{p}_\epsilon := p_0 + \epsilon p_1$$

Theorem

Very weak solutions à la Conca

$$\|\mathbf{u}_\epsilon - \overline{\mathbf{u}}_\epsilon\|_{L^2(\Omega_1 \cup \Omega_2)} + \sqrt{\epsilon}\|p_\epsilon - \overline{p}_\epsilon\|_{H^{-1}(\Omega_1' \cup B_\epsilon \cup \Omega_2)/\mathbb{R}} \leq k\epsilon^{\frac{3}{2}}$$

Proof.

Use the triangular inequality

$$\|\mathbf{u}_\epsilon - \overline{\mathbf{u}}_\epsilon\|_{L^2(\Omega_1 \cup \Omega_2)} \leq \|\mathbf{u}_\epsilon - \mathcal{U}_\epsilon\|_{L^2(\Omega_1 \cup \Omega_2)} + \|\mathcal{U}_\epsilon - \overline{\mathbf{u}}_\epsilon\|_{L^2(\Omega_1 \cup \Omega_2)}$$

mostly it remains only to zero-mean estimate oscillations and vertical correctors $\sim O(\frac{3}{2})$ □

Compute the first order flow rate

- velocity profile normal direction to Γ_0

$$\mathcal{U}_{\epsilon,2}(x) = (u_{0,2} + \epsilon u_{1,2})(x) \equiv -\epsilon \frac{[p_0]}{[\overline{\overline{\eta}}]}(x)$$

- Solve with a computer a cell problem (cheap even in 3D):

$$[\overline{\overline{\eta}}] = \overline{\overline{\eta}}^{+} - \overline{\overline{\eta}}^{-}$$

- First order flow rate

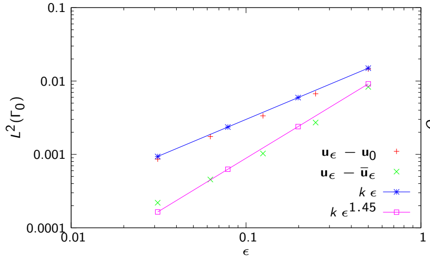
$$Q_{\Gamma_0} = \frac{\epsilon}{[\overline{\overline{\eta}}]} \int_a^b [p_0] \, dx_1$$

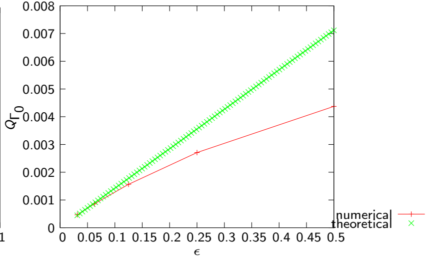
Numerical evidence

- Compute the exact problem
- Compute boundary layers for a single # cell
  - Extract the constants at infinity

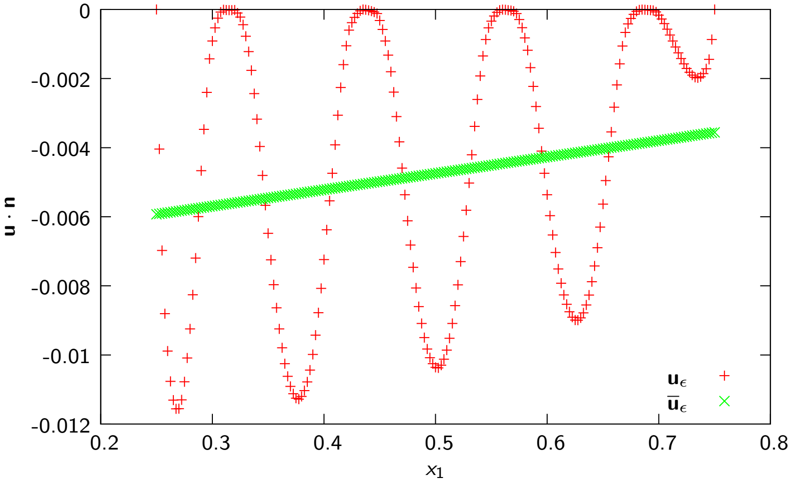
$$[\overline{\overline{\eta}}] = 52.6961$$

- Compute the flow-rate





Normal velocity profile



Averaged implicit macroscopic approximation

Take again

$$\overline{\mathbf{u}}_\epsilon := \mathbf{u}_0 + \epsilon \mathbf{u}_1$$
$$\overline{p}_\epsilon := p_0 + \epsilon p_1$$

On Γ_0 $\overline{\mathbf{u}}_\epsilon$ satisfies

$$\overline{\mathbf{u}}_\epsilon^{\pm} = \epsilon \left\{ \frac{\partial u_{0,1}}{\partial x_2} \overline{\overline{\beta}}^{\pm} + \left[\frac{\partial u_{0,1}}{\partial x_2} \right] \overline{\overline{\Upsilon}}^{\pm} \right\} \mathbf{e}_1 + \epsilon \frac{[p_0]}{[\overline{\overline{\eta}}]} \overline{\overline{\chi}} \mathbf{e}_2$$

Impliciting the interface conditions:

$$\begin{cases} \overline{\mathbf{u}}_\epsilon^{+} \cdot \mathbf{n} = \overline{\mathbf{u}}_\epsilon^{-} \cdot \mathbf{n}, \quad [\overline{\overline{\eta}}] \overline{\mathbf{u}}_\epsilon \cdot \mathbf{n} = \epsilon \left[\sigma_{\overline{\mathbf{u}}_\epsilon, \overline{p}_\epsilon} \right] \cdot \mathbf{n} \\ \overline{\mathbf{u}}_\epsilon^{\pm} \cdot \boldsymbol{\tau} = \epsilon \left(\overline{\overline{\beta}}^{\pm} (\sigma^{+}_{\overline{\mathbf{u}}_\epsilon, \overline{p}_\epsilon} \cdot \mathbf{n}, \boldsymbol{\tau}) + \overline{\overline{\Upsilon}}^{\pm} \left[\sigma_{\overline{\mathbf{u}}_\epsilon, \overline{p}_\epsilon} \cdot \mathbf{n} \right] \cdot \boldsymbol{\tau} \right) \end{cases}$$

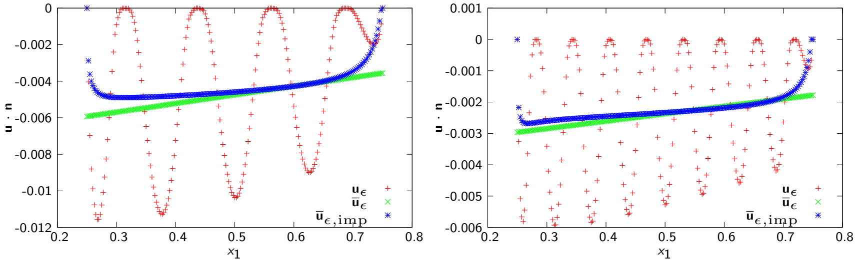
gives:

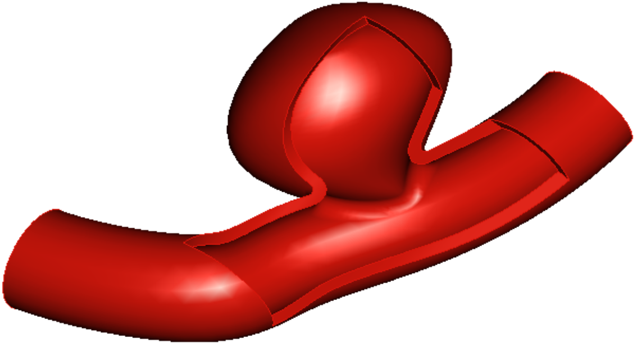
- a new "Strange" term in the normal direction
- "Wall law" expression in the tangential one

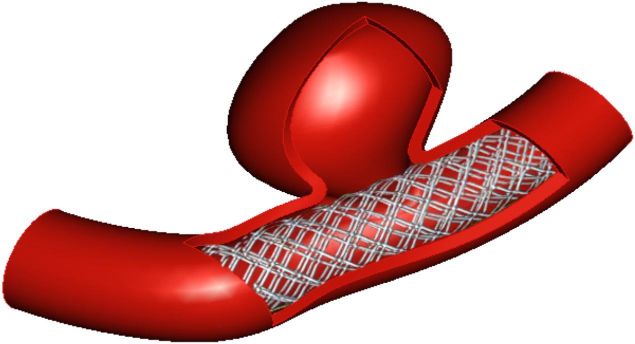
Numerical illustration

implicit interface conditions

Domain Decomposition (Dirichlet-Neuman) on the interface



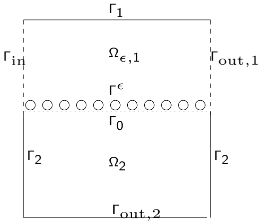




Same problem

but . . .

$$\begin{cases} -\Delta \mathbf{u}_\epsilon + \nabla p_\epsilon = 0 \text{ in } \Omega^\epsilon \\ \operatorname{div} \mathbf{u}_\epsilon = 0 \\ \mathbf{u}_\epsilon = 0 \text{ on } \Gamma_1 \cup \Gamma_2 \cup \Gamma^\epsilon \\ u_{\epsilon,1} = 0 \text{ on } \Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out},1} \\ \mathbf{u}_\epsilon = 0 \text{ on } \Gamma_{\mathrm{out},2} \\ p_\epsilon = p_{\mathrm{in}} \text{ on } \Gamma_{\mathrm{in}}, \quad p_\epsilon = p_{\mathrm{out},1} \text{ on } \Gamma_{\mathrm{out},1}, \end{cases}$$



Pressure imposed at inlet and outlet but not at $\Gamma_{\mathrm{out},2}$

When ϵ goes to 0

- The Poiseuille flow

$$\begin{cases} -\Delta \mathbf{u}_0 + \nabla p_0 = [\sigma_{\mathbf{u}_0,p_0}] \cdot \mathbf{n}\, \delta_{\Gamma_0} \text{ in } \Omega \\ \operatorname{div} \mathbf{u}_0 = 0 \\ \mathbf{u}_0 = 0 \text{ on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_{\mathrm{out},2} \\ \mathbf{u}_0 \wedge \mathbf{n} = 0 \text{ on } \Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out},1} \\ p_0 = p_{\mathrm{in}} \text{ on } \Gamma_{\mathrm{in}}, \quad p_0 = 0 \text{ on } \Gamma_{\mathrm{out},1} \\ \mathbf{u}_0 \neq 0 \text{ on } \Gamma^\epsilon \end{cases}$$

- $(\mathbf{u}_0, p_0)$ is explicit and reads:

$$\begin{cases} \mathbf{u}_0(x) = \dfrac{p_{\mathrm{in}}}{2}(1 - x_2)x_2 \mathbf{e}_1 1_{\Omega_1}, \quad \forall x \in \Omega \\ p_0(x) = p_{\mathrm{in}}(1 - x_1) 1_{\Omega_1} + p_0^- 1_{\Omega_2}, \quad \forall p_0^- \in \mathbb{R} \end{cases}$$

Theorem

$$\|\mathbf{u}_\epsilon - \mathbf{u}_0\|_{H^1(\Omega^\epsilon)^2} + \|p_\epsilon - p_0\|_{L^2(\Omega^\epsilon)} \leq k\sqrt{\epsilon}$$

where the constant k does not depend on ϵ.

First order approximation

Again the same trick

$$\mathcal{V}_\epsilon := \mathbf{u}_0 + \epsilon \left\{ \frac{\partial u_{0,1}}{\partial x_2} (\boldsymbol{\beta}_\epsilon - \overline{\overline{\beta}}) + \left[\frac{\partial u_{0,1}}{\partial x_2} \right] (\Upsilon_\epsilon - \overline{\overline{\Upsilon}}) + \frac{[p_0]}{[\overline{\overline{\eta}}]} (\boldsymbol{\chi}_\epsilon - \overline{\overline{\chi}}) + \mathbf{u}_1 \right\}$$
$$+ \epsilon^2 \left\{ p_{\mathrm{in}} (\varkappa_\epsilon - \overline{\overline{\varkappa}} \mathbf{u}_2 \right\} + \mathcal{W}_\epsilon$$
$$\mathcal{P}_\epsilon := p_0 + \left\{ \frac{\partial u_{0,1}}{\partial x_2} \pi_\epsilon + \left[\frac{\partial u_{0,1}}{\partial x_2} \right] \varpi_\epsilon + \frac{[p_0]}{[\overline{\overline{\eta}}]} (\eta_\epsilon - \overline{\overline{\eta}}) + \epsilon p_1 \right\}$$
$$+ \epsilon p_{\mathrm{in}} (\mu_\epsilon - \overline{\overline{\mu}}) + \epsilon^2 p_2 + \mathcal{Z}_\epsilon$$

- multi-scale version of boundary layer correctors:

$$\boldsymbol{\beta}_\epsilon(x) := \beta\left(\frac{x}{\epsilon}\right), \quad \Upsilon_\epsilon(x) := \Upsilon\left(\frac{x}{\epsilon}\right), \quad \boldsymbol{\chi}_\epsilon(x) := \chi\left(\frac{x}{\epsilon}\right), \quad \varkappa_\epsilon(x) := \varkappa\left(\frac{x}{\epsilon}\right),$$

The pressure in the sac

$$\begin{cases} -\Delta \mathbf{u}_1 + \nabla p_1 = 0 \text{ in } \Omega_1 \cup \Omega_2 \\ \operatorname{div} \mathbf{u}_1 = 0 \\ \mathbf{u}_1 = 0 \text{ on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_{\mathrm{out},2} \\ \mathbf{u}_1 \wedge \mathbf{n} = 0 \text{ on } \Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out},1}, \\ p_1 = 0 \text{ on } \Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out},1} \\ \mathbf{u}_1 = \left\{ \frac{\partial u_{0,1}}{\partial x_2} \overline{\overline{\beta}}^{\pm} + \left[\frac{\partial u_{0,1}}{\partial x_2} \right] \overline{\overline{\Upsilon}}^{\pm} \right\} \mathbf{e}_1 + \frac{[p_0]}{[\overline{\overline{\eta}}]} \overline{\overline{\chi}} \mathbf{e}_2 \text{ on } {\Gamma_0}^{\pm} \end{cases}$$

divergence condition and normal veolicty imposed on Γ_0:

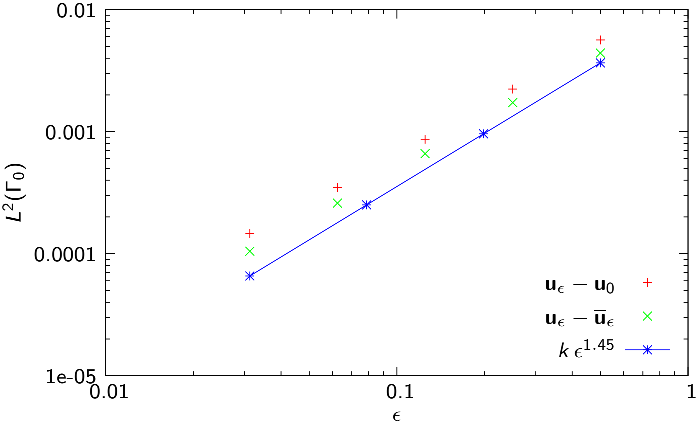
$$\int_{\Omega_2} \operatorname{div} \mathbf{u}_1 dx = \int_{\partial\Omega_2} \mathbf{u}_1 \cdot \mathbf{n} d\sigma = \int_{\Gamma_0} \mathbf{u}_1 \cdot \mathbf{n} d\sigma = 0$$

gives in the sac

$$p_0^- = \frac{1}{|\Gamma_0|} \int_{\Gamma_0} p_0^+(x_1, 0) dx_1$$

Error estimates

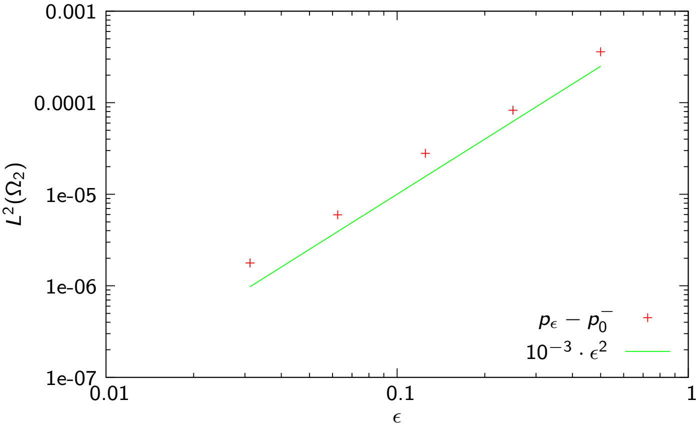
- Flow-rate accros Γ_0



- Pressure in Ω_2

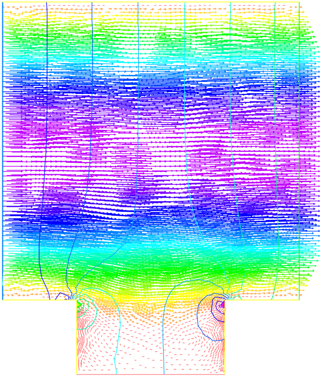
Error estimates

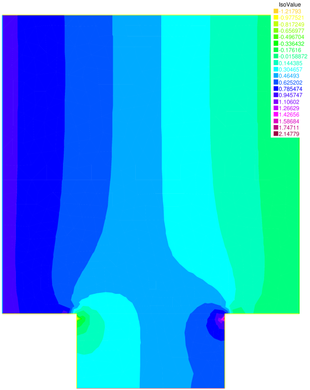
- Flow-rate accros Γ_0
- Pressure in Ω_2



Numerics

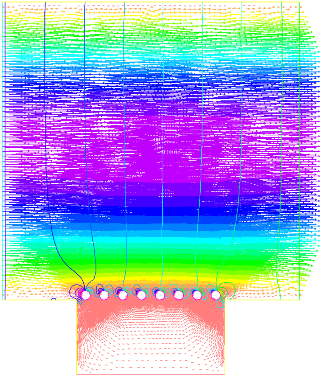
"Pathological" case

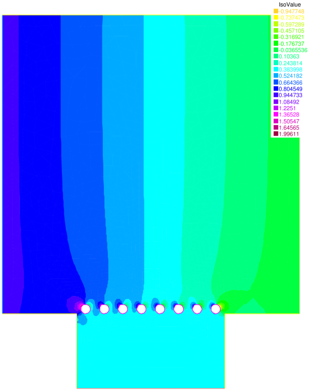




Numerics

"Stented" case





The full time-dependent non-linear system

$$\begin{cases} \rho(\partial_t \mathbf{u}_\epsilon + (\mathbf{u}_\epsilon \cdot \nabla)\mathbf{u}_\epsilon) - \nu \Delta \mathbf{u}_\epsilon + \nabla p_\epsilon = 0, \text{ in } \Omega^\epsilon \\ \operatorname{div} \mathbf{u}_\epsilon = 0 \\ \mathbf{u}_\epsilon = 0 \text{ on } \Gamma_1 \cup \Gamma_2 \cup \Gamma^\epsilon \\ \mathbf{u}_\epsilon \cdot \boldsymbol{\tau} = 0 \text{ on } \Gamma_{\text{in}} \cup \Gamma_{\text{out},1} \cup \Gamma_{\text{out},2} \\ p_\epsilon = p_{\text{in}}(t) \text{ on } \Gamma_{\text{in}}, \quad p_\epsilon = p_{\text{out},1} \text{ on } \Gamma_{\text{out},1}, p_\epsilon = p_{\text{out},2} \text{ on } \Gamma_{\text{out},2}, \end{cases}$$

where for instance

$$p_{\text{in}} \in L^2(0, T)$$

Theoretical investigation

Joint work with A. Rambaud (Phd Ens Lyon) on a simplified problem

$$\begin{cases} \partial_t u_\epsilon - \Delta u_\epsilon = f(t) & \text{in } \Omega^\epsilon \\ u_\epsilon = 0 & \text{on } \Gamma^\epsilon \cup \Gamma_1 \\ u_\epsilon \text{ is } x_1 - \text{periodic} & \text{on } \Gamma_{\text{in}} \cup \Gamma_{\text{out}} \\ u_\epsilon = u_{0,\epsilon}(x) & \text{in } \Omega^\epsilon \end{cases}$$

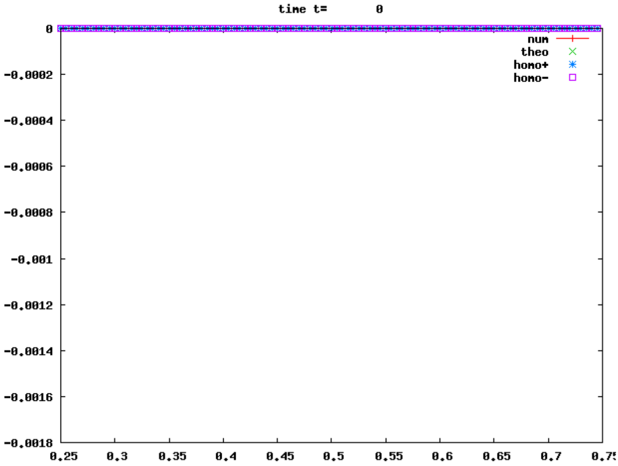
Fourier analysis: time$\times$space domain $=>$ frequency$\times$space domain

the boundary layer is steady

A numerical test

Unsteady Stokes system in a collateral artery

Normal velocity profile in time



Conclusions & Perspectives

Conclusions

- Theory: improved understanding of the fluid mechanics in
  - the collateral artery
  - the anevrismal sac
- Numerical evidence
  - First order agreement of the flow rate in both cases
  - Pressure in the sac computed accurately up to second order

Perspectives

- curved boundaries
- homogenized drug delivery
- error estimates for Navier-Stokes
- cell growth

Collaborators

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  - R. Dir. CNRS Didier Bresch
    LAMA, UMR 5127, Chambéry
  - Phd. Kirill Pinchon Gostaf
    LJLL, UMR 7598, Paris VI
  - Phd Amélie Rambaud,
    Ens Lyon, ICJ, UMR 5208, Lyon I
- Industrial
  - Cardiatis
    see `www.cardiatis.com`