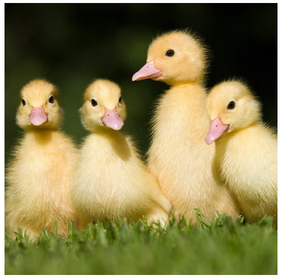
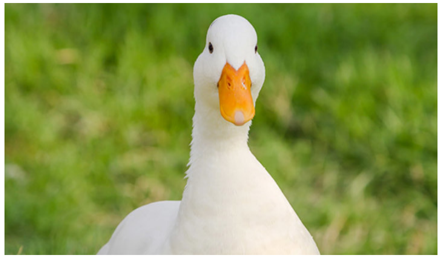
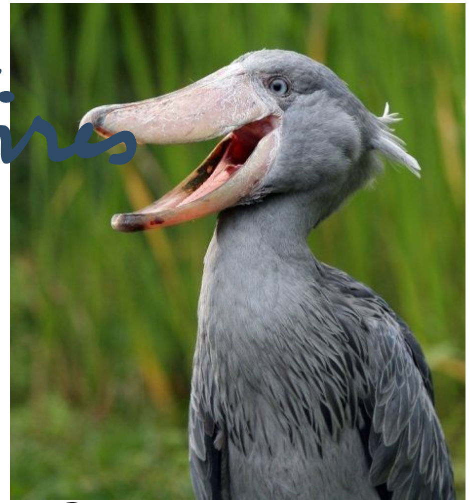





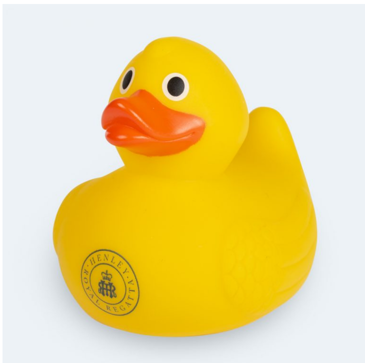
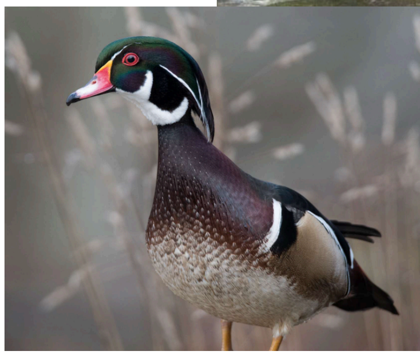

Géométrie & Spectres

des Surfaces Hyperboli Aléatoires



Aussois

12, 2025


Laplacian Δ

(M, m) Riem mfld. $f \in C^\infty(M)$

$$\Delta f = \operatorname{div}(\operatorname{grad} f)$$

measures how average of f over small spheres around p differs from $f(p)$

$$M = \mathcal{H} = \{x + yi \in \mathbb{C} \mid y > 0\}. \quad m = \frac{dx^2 + dy^2}{y^2}$$

$$\Delta_{\mathcal{H}} f = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) f$$

$$\operatorname{Isom}(\mathcal{H}) = PSL_2\mathbb{R}$$

Δ commutes with all isometries

discrete

$$\Rightarrow \Delta \text{ descends to all } X = \Gamma \backslash \mathcal{H}$$

$\mathrm{Spec}(\Delta)$

X hyperbolic surfaces

$\Delta_X : \mathcal{C}_c^\infty(X) \longrightarrow \mathcal{C}_c^\infty(X)$ extends uniquely to a densely def self-adjoint. non-negative unbounded linear operator

$$\mathrm{domain}(\Delta_X) \to L^2(X)$$

$$\mathrm{Spec}(\Delta_X) = \{\lambda \in \mathbb{C} \mid \Delta_X - \lambda \text{ is not invertible}\}$$

If X is closed

0 if $\mathrm{area}(X) < \infty$: $f \equiv \mathrm{cst}$ is eigenfct

$$\mathrm{spec}(\Delta_X) : \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \longrightarrow \infty$$

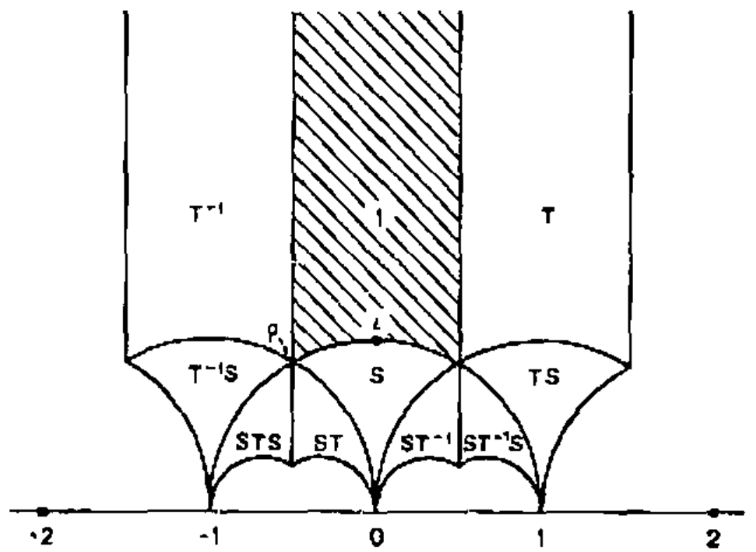
$0 < \lambda_1$ if X is connected

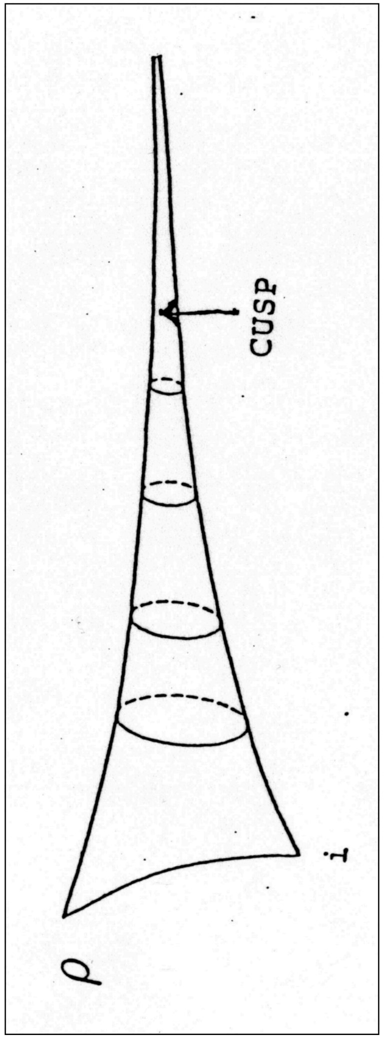
NO Example there is no hyp surf for which the entire spec is known

except $\mathcal{H}$: $\mathrm{spec}(\mathcal{H}) = [1/4, +\infty)$

$X(1) = SL_2\mathbb{Z} \backslash \mathcal{H}$: Nothing exact is known

1.2. *Domaine fondamental du groupe modulaire.*





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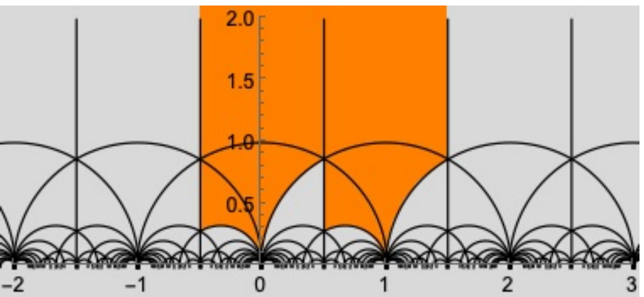
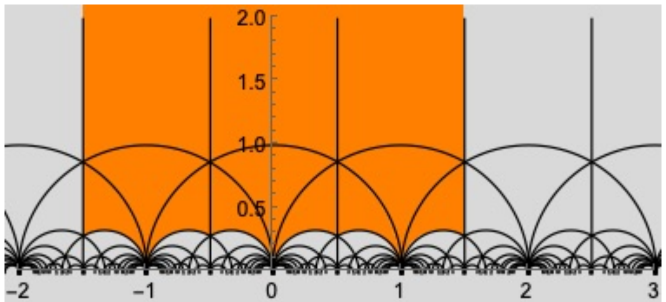
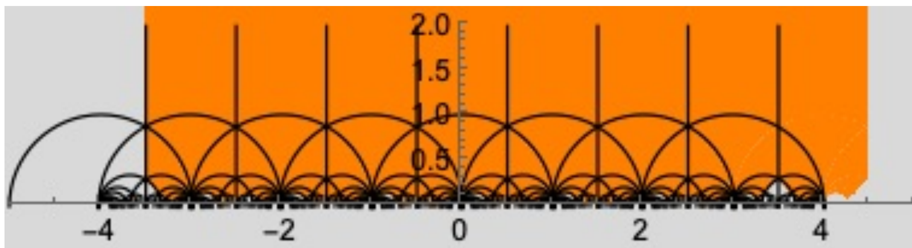
$X(N) = \Gamma(N) \backslash \mathcal{H}$, $\Gamma(N) = \left\{ m \in SL_2\mathbb{Z} \;\middle|\; m \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N \right\}$



$X(2)$ $g=0$, $n=3$



$X(3)$ $g=0$, $n=4$



$X(8)$ $g=5$, $n=24$

NO Example there is no hyp surf for which the entire spec is known

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Maass

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Maass

For $X(8)$: $\forall k \geq 1$, $\frac{1}{4} + \left(\frac{\pi k}{\log(1+\sqrt{2})} \right)^2$ is an eigenvalue

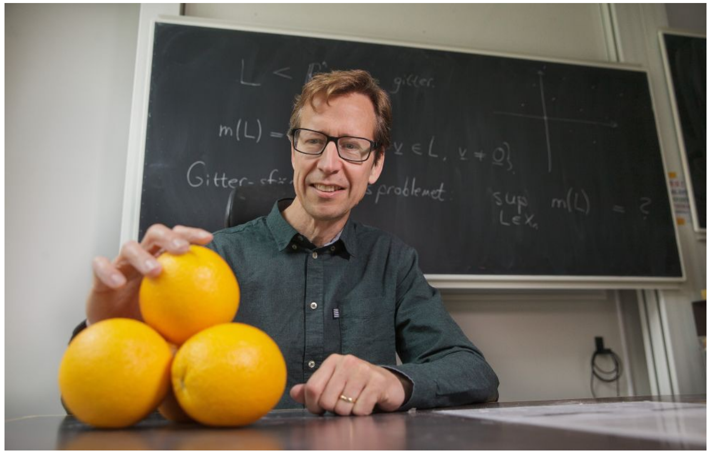
nothing is known for any closed surface 

Numerics

$$X(1) = SL_2\mathbb{Z} \backslash \mathcal{H}$$

A. BOOKER, A. STRÖMBERGSSON, A. VENKATESH

λ_1	91.141345336355278081809773807120545991693970815690902477965700195954242389565124727562896228809629116 6...
λ_2	148.4321316727207372163463002609961395007402473023059718429528711664752493611551782064774884270840759058...
λ_3	190.1315473199346475424535570151849406811358942831288385576731615371323200677917820840283441119667596429...
λ_4	206.4167955859576408693025699827412140869821744440892457353424774780566562633717682316760582937054960721...
λ_5	260.6874056893668544964087866187684342867051298054341025160896652698127129961088063918703474378409525572...


Numerics

$$X(1) = SL_2\mathbb{Z} \backslash \mathcal{H}$$

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λ_1	91.14134533635527808180977380712054599169397081569090 24779657001959542423895651247275628962288096291166 . . .
λ_2	148.43213167272073721634630026099613950074024730230597 18429528711664752493611551782064774884270840759058 . . .
λ_3	190.13154731993464754245355701518494068113589428312883 85576731615371323200677917820840283441119667596429 . . .
λ_4	206.41679558595764086930256998274121408698217444408924 57353424774780566562633717682316760582937054960721 . . .
λ_5	260.68740568936685449640878661876843428670512980543410 25160896652698127129961088063918703474378409525572 . . .

Bolza surface : the most symmetric surface $\in \mathcal{M}_2$

$$|\mathrm{Aut}| = 48$$

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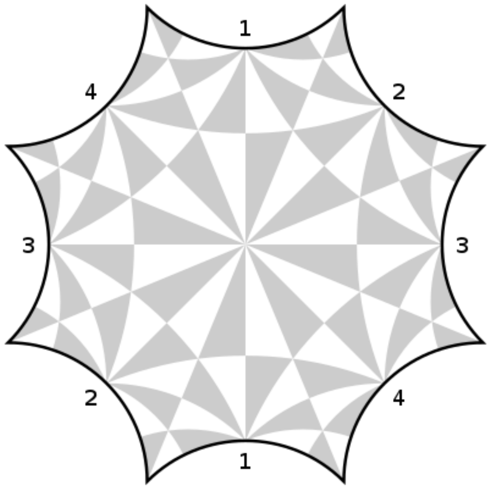

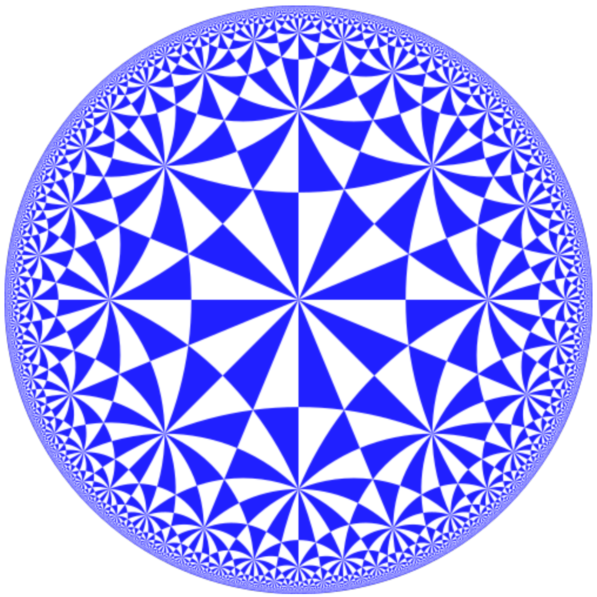

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λ_1	91.14134533635527808180977380712054599169397081569090 2477965700195954242389565124727562896228809629 1166 . . .
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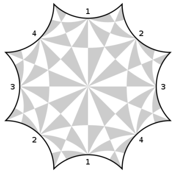

Bolza surface : the most symmetric surface $\in \mathcal{M}_2$

Numerical computations of the first ten positive eigenvalues of the Bolza surface

Eigenvalue	Numerical value	Multiplicity
λ_0	0	1
λ_1	3.8388872588421995185866224504354645970819150157	3
λ_2	5.3536013411890504109180483110314463763573 72198	4
λ_3	8.2495548152006581218901064506824565683905 78132	2
λ_4	14.726216787788832041289318442184835983733 84446932	4
λ_5	15.048916133267048746181584340258811275704 52711372	3

$$|\mathrm{Aut}| = 48$$

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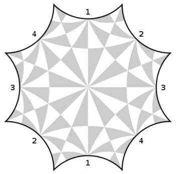
λ_2	148.43213167272073721634630026099613950074024730230597 1842952871166475249361155178206477488427084075905 8…
λ_3	190.13154731993464754245355701518494068113589428312883 8557673161537132320067791782084028344111966759642 9…
λ_4	206.41679558595764086930256998274121408698217444408924 5735342477478056656263371768231676058293705496072 1…
λ_5	260.68740568936685449640878661876843428670512980543410 2516089665269812712996108806391870347437840952557 2…



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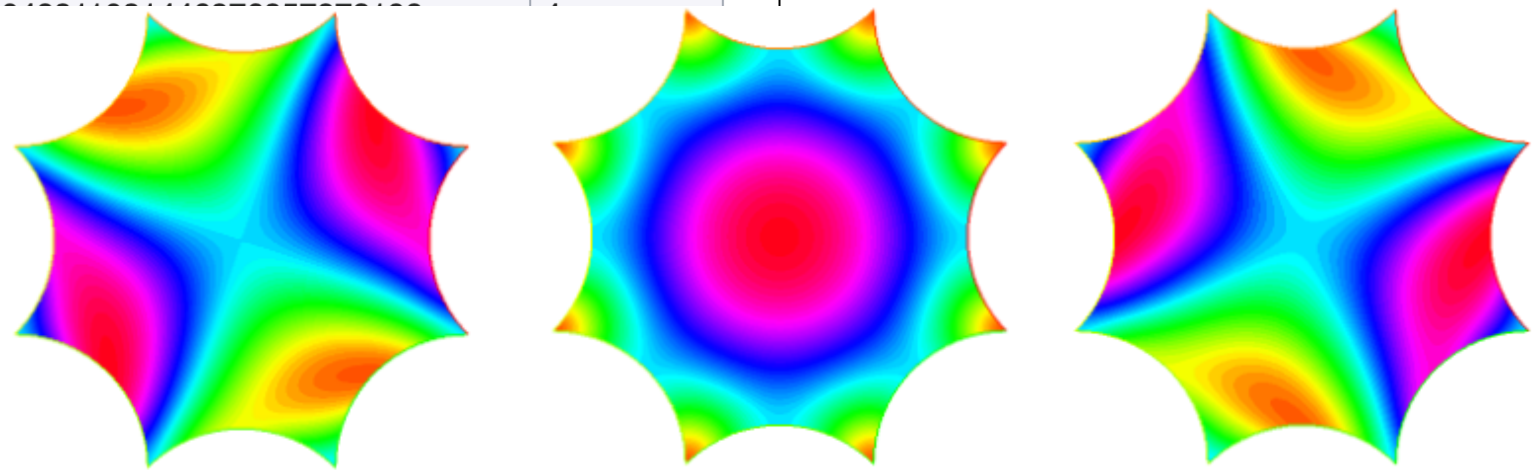
λ_3	8.2495548152006581218 9[illegible]	[illegible]
λ_4	14.7262167877888320412[illegible]	[illegible]
λ_5	15.048916133267048746 1[illegible]	[illegible]



Length Spectrum

$t_0 + \sqrt{D}\, u_0$ where $t_0, u_0 > 0$ and form a fundamental solution to the Pell equation $t^2 - Du^2 = 4$

$X(1) = \mathcal{H} \backslash SL_2\mathbb{Z}$

Proposition 4.22 *The lengths of closed oriented geodesics in the modular surface $\Gamma \backslash \mathcal{H}$ are the numbers*

$$\{2 \log \varepsilon_D \mid D \in \mathbf{N},\ D \equiv 0, 1 \pmod 4,\ \sqrt{D} \notin \mathbf{N}^2\}.$$

Each length appears with finite non-zero multiplicity equal to $h(D)$, the number of $\mathrm{SL}(2, \mathbf{Z})$*-classes of primitive integral quadratic forms of discriminant D.*

The n^{th} element ℓ_n of the length spectrum for the Bolza surface is given by

$$\ell_n = 2 \operatorname{arcosh}(m + n\sqrt{2}),$$

where n runs through the positive integers (but omitting 4, 24, 48, 72, 140, and various higher values) (Aurich, Bogomolny & Steiner 1991) and where m is the unique odd integer that minimizes

$$|m - n\sqrt{2}|.$$

$$X = \Gamma \backslash \mathcal{H} \quad \text{closed hyp surf}$$

$$\pi_1(X) \simeq \Gamma < PSL_2\mathbb{R}$$

$$\{\text{conjugacy classes of } \pi_1(X)\} \xleftrightarrow{\text{bij}} \{\text{free homotopy classes of closed curves on } X\}$$

$$\xleftrightarrow{\text{bij}} \{\text{closed geodesics on } X\}$$

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \Rightarrow \ell(\gamma_M) = 2\cosh^{-1}\left(\frac{|\mathrm{tr}\, M|}{2}\right)$$

Spectral gap $= \lambda_1 - \lambda_0 = \lambda_1$ Ratner '87

Why? Diffusion ∘ mixing speed of the geod flow φ

$$\sigma = \sqrt{1 - \max(1 - 4\lambda_1, 0)}$$

$\exists\, C > 0$ s.t. $\forall f, g \in C^1(T^1X)$

$$\left| \int_{T^1X} f(v)\, g(\varphi_t(v))\, dv - \left(\int_{T^1X} f(v)\, dv\right)\left(\int_{T^1X} g(v)\, dv\right) \right|$$

$$\leq C \|f\|_{C^1} \|g\|_{C^1}\, t e^{-\sigma t}$$

Spectral gap $=\lambda_1-\lambda_0=\lambda_1$

Why? Diffusion
- mixing speed of the geod flow φ
- counting of geodesics

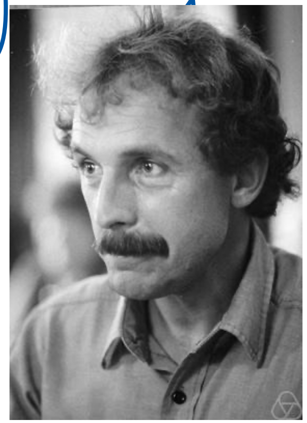
$$\#\{\gamma \text{ primitive geod} \mid e^{\ell(\gamma)}\leqslant N\} =$$

$$= \mathrm{li}(N)+\sum_{0<\lambda_j<\frac{3}{16}} \mathrm{li}\left(N^{\frac{1}{2}+\sqrt{\frac{1}{4}-\lambda_j}}\right)+O\left(\frac{N^{3/4}}{\log N}\right)$$

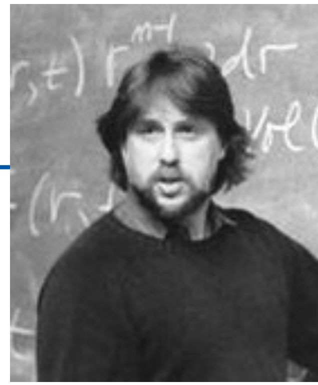
$$\int_2^N \frac{dt}{\log t} \underset{N\to\infty}{\sim} \frac{N}{\log N}$$

Spectral gap $= \lambda_1 - \lambda_0 = \lambda_1$

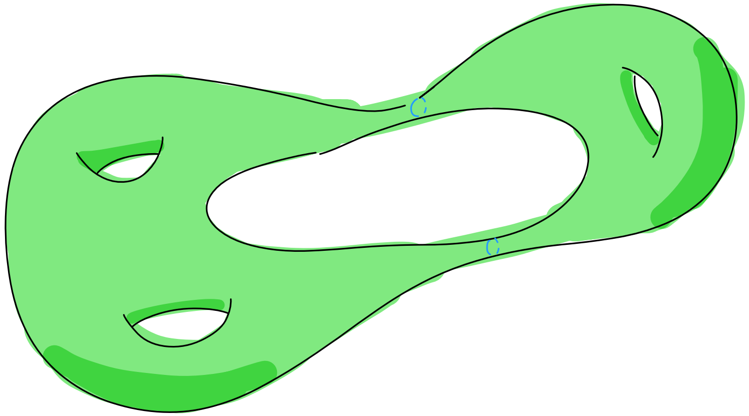
Why? Diffusion ∘ mixing speed of the geod flow ∘ counting of geodesics



Connectivity ∘ Cheeger constant



X closed $\quad h(X) = \inf\limits_{\substack{\gamma \\ X\setminus\gamma = A \sqcup B}} \dfrac{\ell(\gamma)}{\min(\text{area}\, A, \text{area}\, B)}$



$$\frac{\sqrt{1+10\lambda_1}-1}{10} \leqslant h(X) \leqslant 2\sqrt{\lambda_1}$$

Buser $\qquad$ Cheeger

Can λ_1 be very small?

yes! <u>Randol '74</u>

λ_1 can be arbitarily small

Can λ_1 be very big?

<u>Huber '74</u>

$$\lambda_1 \leq \frac{1}{4} + O\left(\frac{1}{\log^2 g}\right) \qquad \mathrm{spec}(\mathcal{H}) = [1/4, +\infty)$$

How big can λ_1 be?

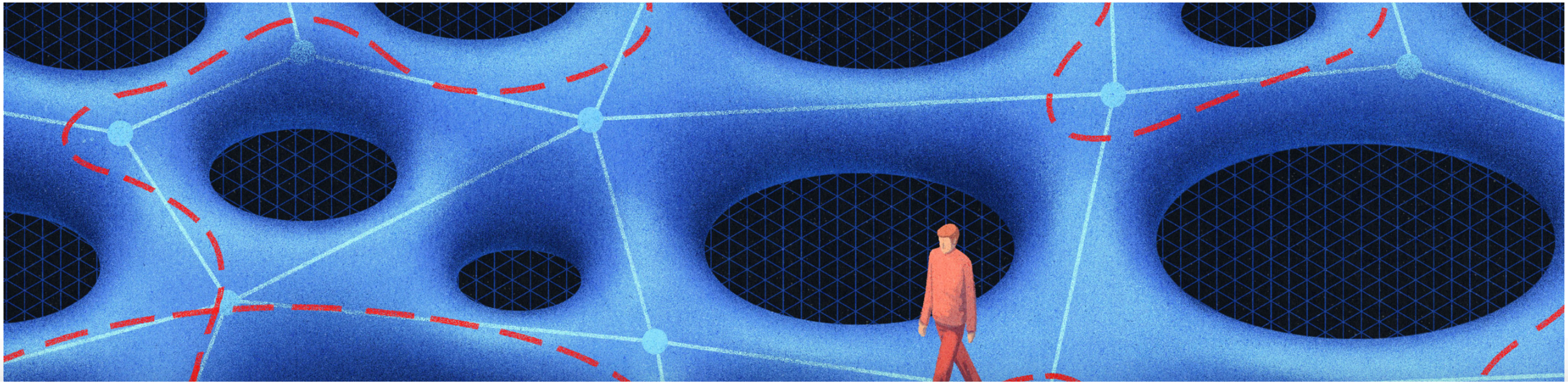
<u>Conjecture (Buser '84)</u> $\exists\, (X_i)_i$ closed hyp surf s.t. $g(X_i) \to \infty$ and $\lambda_1(X_i) \to 1/4$

TOPOLOGY

Surfaces Beyond Imagination Are Discovered After Decades-Long Search

19

Using ideas borrowed from graph theory, two mathematicians have shown that extremely complex surfaces are easy to traverse.



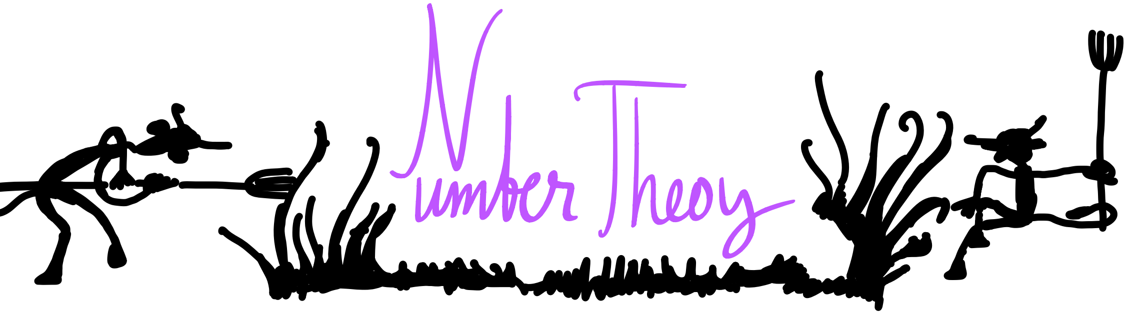
Number Theory

$$\Gamma(N) = \left\{ g \in SL_2\mathbb{Z} \;\middle|\; g \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N \right\}$$

$$X(N) = \Gamma(N) \backslash \mathcal{H}$$

Conjecture (Selberg '65)

$$\lambda_1(X(N)) \geqslant \frac{1}{4}, \quad \forall N$$



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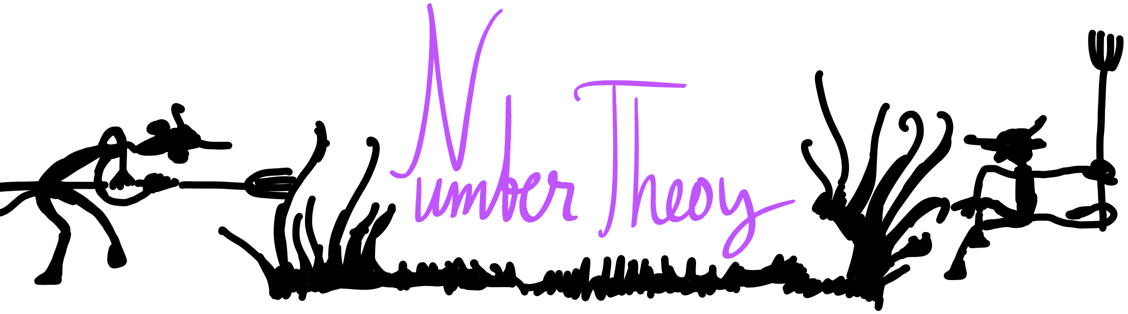
<u>Conjecture</u> (<u>Selberg '65</u>) $\lambda_1(X(N)) \geq \frac{1}{4}, \ \forall N$

- true if replace $1/4$ by $3/16$ (Selberg, same paper)
- true for $N \leq 856$ (Booker-Strömbergsson '07)
- analogue of Riemann Hypothesis for $X(N)$
- implied by Langlands functoriality

Record: Kim-Sarnak '03: $\frac{975}{4096} = \frac{1}{4} - \left(\frac{7}{64}\right)^2 \approx 0.238$

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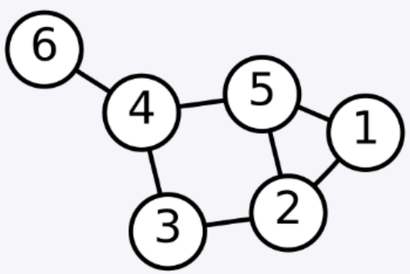
Closed surfaces? Jacquet-Langlands correspondence

Graphs $G=(V,E)$, $f: V \to \mathbb{R}$

d-reg

$$(\Delta f)(v) := \sum_{x \sim v} \big(f(v) - f(x)\big) = \big[(\cancelto{d}{D} - A) f\big](v)$$

Labelled graph	Degree matrix	Adjacency matrix	Laplacian matrix
	$\begin{pmatrix} 2&0&0&0&0&0\\ 0&3&0&0&0&0\\ 0&0&2&0&0&0\\ 0&0&0&3&0&0\\ 0&0&0&0&3&0\\ 0&0&0&0&0&1 \end{pmatrix}$	$\begin{pmatrix} 0&1&0&0&1&0\\ 1&0&1&0&1&0\\ 0&1&0&1&0&0\\ 0&0&1&0&1&1\\ 1&1&0&1&0&0\\ 0&0&0&1&0&0 \end{pmatrix}$	$\begin{pmatrix} 2&-1&0&0&-1&0\\ -1&3&-1&0&-1&0\\ 0&-1&2&-1&0&0\\ 0&0&-1&3&-1&-1\\ -1&-1&0&-1&3&0\\ 0&0&0&-1&0&1 \end{pmatrix}$

$\lambda_0 = 0$

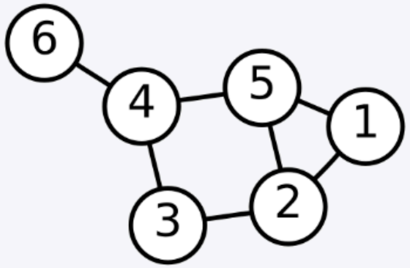
$\lambda_1 > 0$ iff G conn

Alon-Boppana '85 $\quad \lambda_1(G) \leqslant d - \sqrt{d-1} + O\left(\frac{1}{\log|V|}\right)$

Graphs

$G = (V, E)$, $f : V \to \mathbb{R}$ 　　 d-reg

$$(\Delta f)(v) := \sum_{x \sim v} \big(f(v) - f(x) \big) = \big[(\overset{d}{\cancel{D}} - A) f \big](v)$$

Labelled graph	Degree matrix	Adjacency matrix	Laplacian matrix
	$\begin{pmatrix} 2&0&0&0&0&0\\0&3&0&0&0&0\\0&0&2&0&0&0\\0&0&0&3&0&0\\0&0&0&0&3&0\\0&0&0&0&0&1 \end{pmatrix}$	$\begin{pmatrix} 0&1&0&0&1&0\\1&0&1&0&1&0\\0&1&0&1&0&0\\0&0&1&0&1&1\\1&1&0&1&0&0\\0&0&0&1&0&0 \end{pmatrix}$	$\begin{pmatrix} 2&-1&0&0&-1&0\\-1&3&-1&0&-1&0\\0&-1&2&-1&0&0\\0&0&-1&3&-1&-1\\-1&-1&0&-1&3&0\\0&0&0&-1&0&1 \end{pmatrix}$

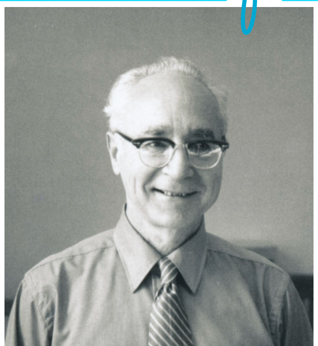
$\lambda_0 = 0$

$\lambda_1 > 0$

iff G conn

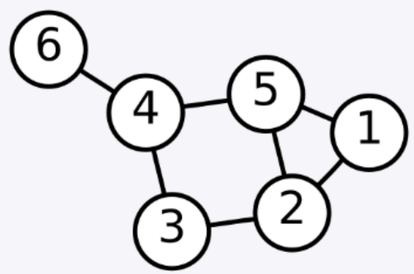
Alon-Boppana '85 　 $\lambda_1(G) \leq d - \sqrt{d-1} + O\left(\frac{1}{\log|V|} \right)$

Lubotzky-Phillips-Sarnak, Margulis '86



Graphs $G = (V, E)$, $f : V \to \mathbb{R}$ d-reg

$$(\Delta f)(v) := \sum_{x \sim v} \big(f(v) - f(x) \big) = \big[(\overset{d}{\cancel{D}} - A) f \big](v)$$

Labelled graph	Degree matrix	Adjacency matrix	Laplacian matrix
	$\begin{pmatrix} 2&0&0&0&0&0\\ 0&3&0&0&0&0\\ 0&0&2&0&0&0\\ 0&0&0&3&0&0\\ 0&0&0&0&3&0\\ 0&0&0&0&0&1 \end{pmatrix}$	$\begin{pmatrix} 0&1&0&0&1&0\\ 1&0&1&0&1&0\\ 0&1&0&1&0&0\\ 0&0&1&0&1&1\\ 1&1&0&1&0&0\\ 0&0&0&1&0&0 \end{pmatrix}$	$\begin{pmatrix} 2&-1&0&0&-1&0\\ -1&3&-1&0&-1&0\\ 0&-1&2&-1&0&0\\ 0&0&-1&3&-1&-1\\ -1&-1&0&-1&3&0\\ 0&0&0&-1&0&1 \end{pmatrix}$

$\lambda_0 = 0$

$\lambda_1 > 0$ iff G conn

Alon-Boppana '85 $\quad \lambda_1(G) \leqslant d - \sqrt{d-1} + O\left(\frac{1}{\log|V|}\right)$

Lubotzky-Phillips-Sarnak, Margulis '86

d-reg graphs with $\lambda_1 \geqslant d - \sqrt{d-1}$

Cayley graphs of $PSL_2(\mathbb{Z}/q\mathbb{Z})$

$d = p+1$

$p \equiv 1 \bmod 4$ prime

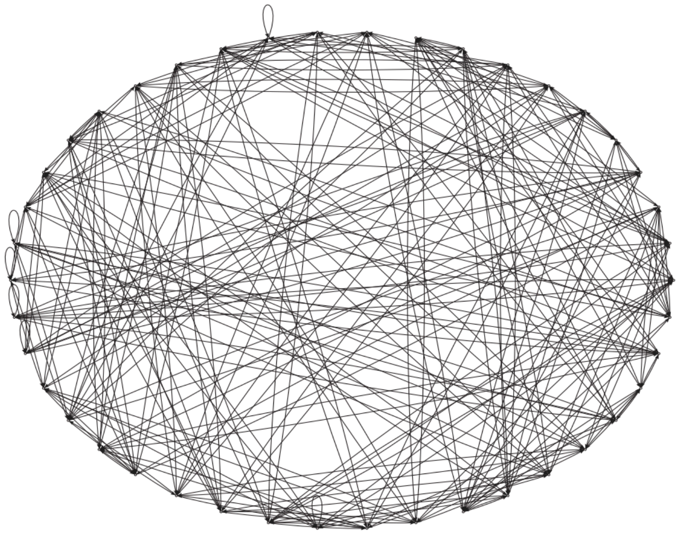


Fig. 1. Non-bipartite LPS Ramanujan graph with number of vertices $N = 42$ and degree $d = 6$ (figure generated using software Pajek.)

E. LUBETZKY AND Y. PERES

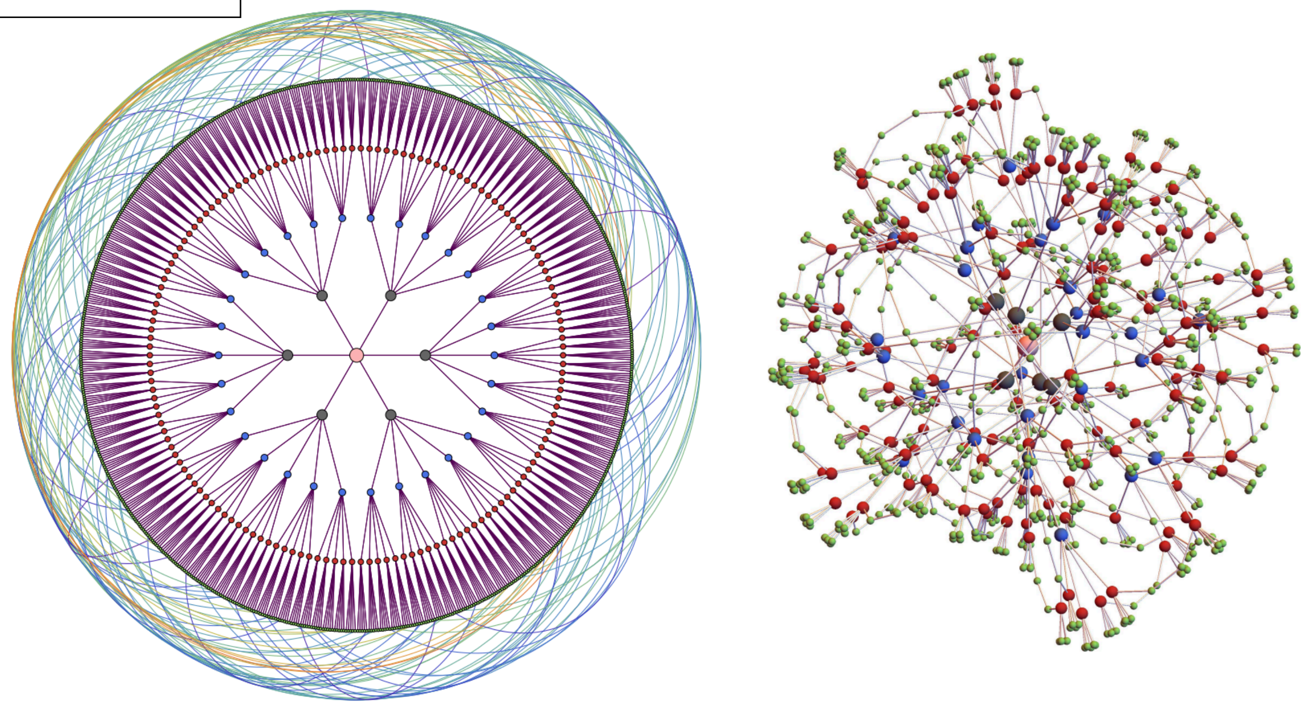


FIGURE 1. A ball of radius 4 in the Lubotzky–Phillips–Sarnak 6-regular Ramanujan graph on $n = 12180$ vertices via $\mathrm{PSL}(2, \mathbb{F}_{29})$.

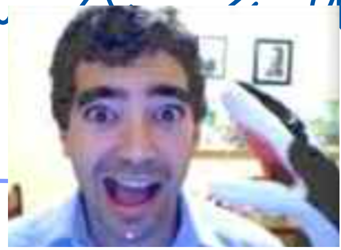
Graphs $G=(V,E)$, $f : V \to \mathbb{R}$ d-reg

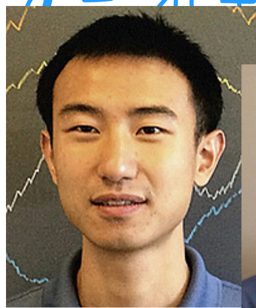
$$(\Delta f)(v) := \sum_{x\sim v}\big(f(v)-f(x)\big) = \big[(\overset{d}{\cancel{D}} - A)f\big](v)$$

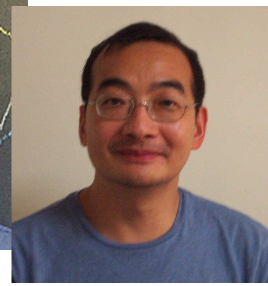
Alon-Boppana '85 $\lambda_1(G) \leq d-\sqrt{d-1} + O\left(\frac{1}{\log|V|}\right)$

Lubotzky-Phillips-Sarnak, Margulis '86 $d =$

d-reg graphs with $\lambda_1 \geq d -$







Cayley graphs of PSL $\mathbb{Z}$ $\equiv$

Conjecture (Alon '86) Friedman '08. Bordenave '17. Huang-Yau...

$$\forall \varepsilon > 0,\ \mathbb{P}\big(\lambda_1(G_n) \geq d-\sqrt{d-1}-\varepsilon\big) \xrightarrow[n\to\infty]{} 1$$

Weil-Petersson random surfaces

Theorem (Mirzakhani '10) X_g une WP random surface

$$\mathbb{P}\left(\lambda_1(X_g) > 0.002 \right) \xrightarrow[g\to\infty]{} 1$$

Wu-Xue, Lipnowski-Wright '21 : $\frac{3}{16} - \varepsilon$

Anantharaman-Monk '23 : $\frac{2}{9} - \varepsilon$

'25 : $\frac{1}{4} - \varepsilon$

Hide-Macera-Thomas '25 : $\frac{1}{4} - O\left(\frac{1}{g^c}\right)$

Selberg trace formula

Let $h:\mathbb{R}\to\mathbb{R}$ be a C_c^∞ even function

X be a closed hyperbolic surface of genus g

We have

Fourier transform

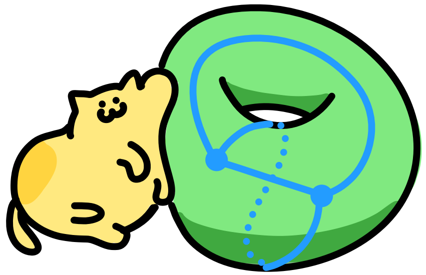
$$\sum_{j=0}^{\infty} \hat{h}(r_j(X)) = (g-1)\int_{\mathbb{R}} \hat{h}(r)\tanh(\pi r) r\, dr + \sum_{\gamma \text{ prim geod}} \sum_{k=1}^{\infty} \frac{\ell(\gamma) h(k\cdot \ell(\gamma))}{2\sinh\left(\frac{k\ell(\gamma)}{2}\right)}$$

$r_j \in \mathbb{R} \cup i[-\frac{1}{2},\frac{1}{2}]$

is a solution of

$\lambda_j = \frac{1}{4} + r_j^2$

Thank you!



Merci!