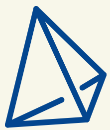
ABOUT THE LENGTH SPECTRUM OF RANDOM HYPERBOLIC 3-MANIFOLDS



Ana Roig Sanchis

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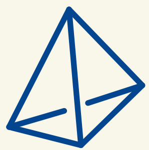
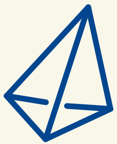


SMaRT - Luxembourg - March 19, 2024



A reason to study hyperbolic 3-manifolds...

<u>Thurston Geometrisation Conjecture</u> (Perelman, 2006)

Every closed orientable M^3 can be decomposed into canonical pieces, each of them admitting some geometric structure, among:

$$\mathbb{R}^3,\ \mathbb{S}^3,\ \mathbb{H}^3,\ \mathbb{S}^2\times\mathbb{R},\ \mathbb{H}^2\times\mathbb{R},\ \mathrm{Nil},\ \mathrm{Sol},\ \widetilde{SL}(2,\mathbb{R})$$

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$\oplus$ vaste

$\ominus$ known

Why random manifolds?

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- $\longrightarrow$ Study typical behaviours
- $\longrightarrow$ Approach extremal problems (Ex: $\min\{\mathrm{diam}(M^3) : \mathrm{vol}(M^3) \geq n\} \xrightarrow{n\to\infty} ?$)
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About the length spectrum...

<u>Theorem</u> (R-S): The length spectrum of random hyperbolic 3-manifolds with boundary converges in distribution to a Poisson Point Process (P.P.P.) on $\mathbb{R}_{\geq 0}$ with computable intensity λ.

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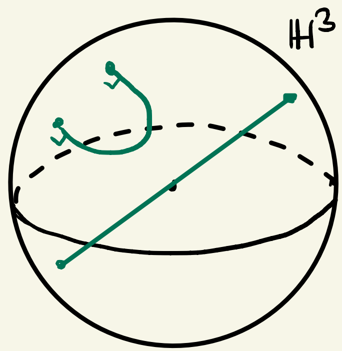
Model of $\mathbb{H}^3$: Poincaré ball

$$\mathbb{D}^3 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 < 1\}$$

with the metric: $$ds^2 = 4 \cdot \frac{dx^2+dy^2+dz^2}{(1-(x^2+y^2+z^2))^2}.$$



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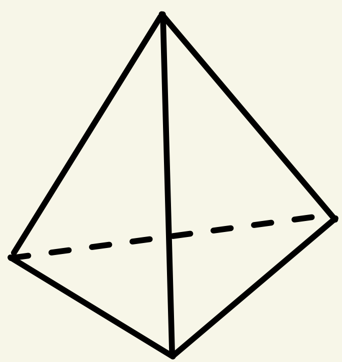
RANDOM MANIFOLDS

↳ Model: Random triangulation ← Introduced by Bram Petri and Jean Raimbault, 2020.

GENERAL IDEA: To construct manifolds by randomly gluing polyhedra together along their faces.

1st attempt:



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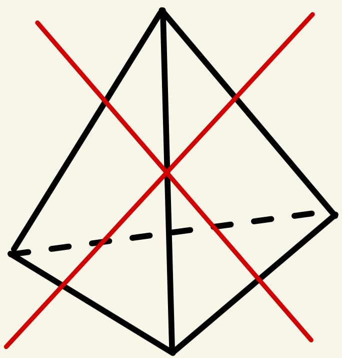

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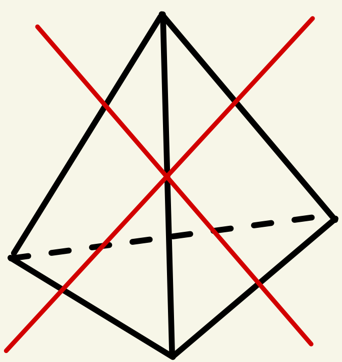

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$\Longrightarrow$ $\mathbb{P}[X_n \text{ is a manifold}] \xrightarrow[n\to\infty]{} 0$.

[Dunfield-Thurston]

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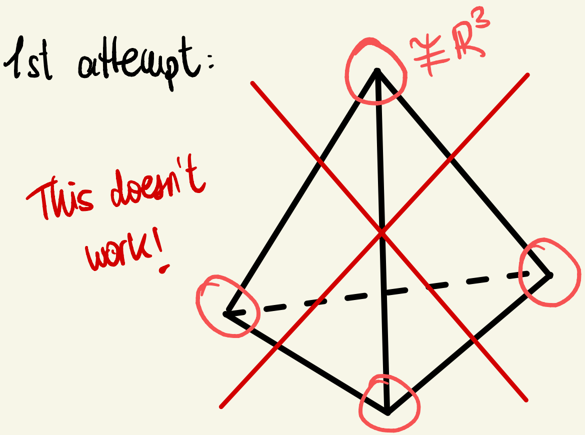

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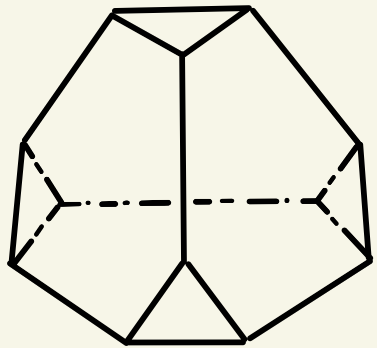

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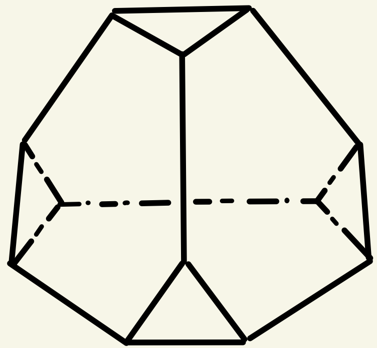

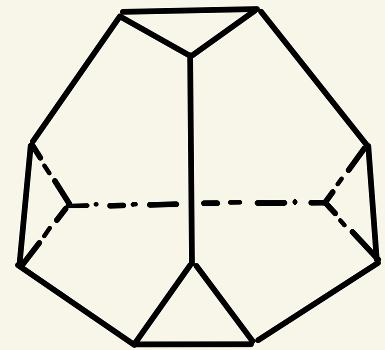

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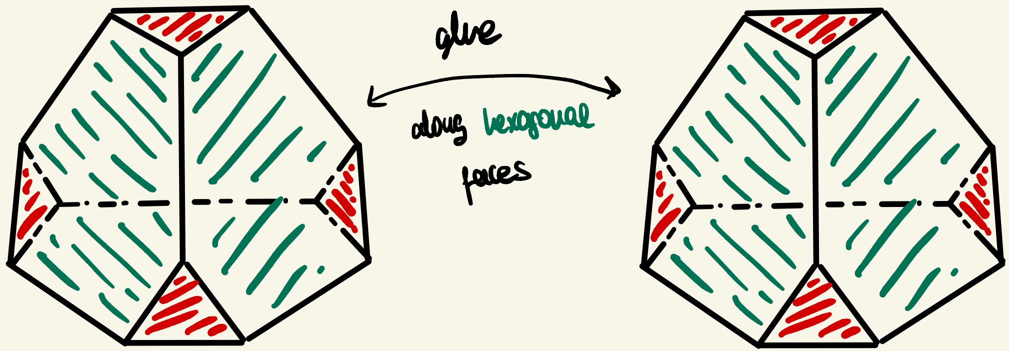

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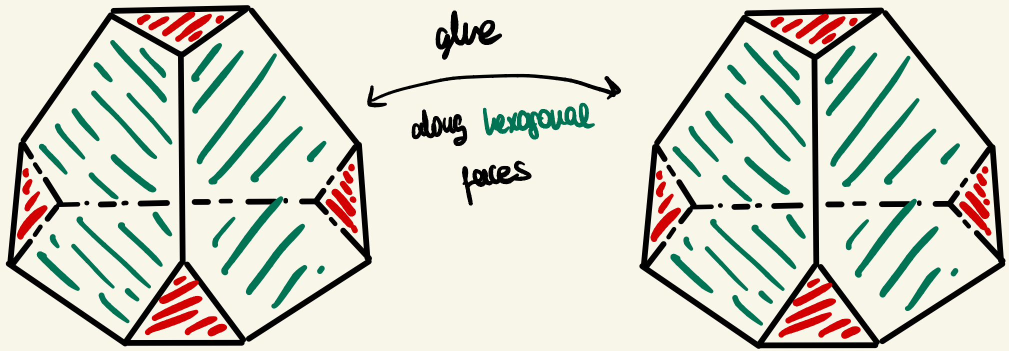

random compact 3-manifold with boundary

$$\shortparallel$$

M_n

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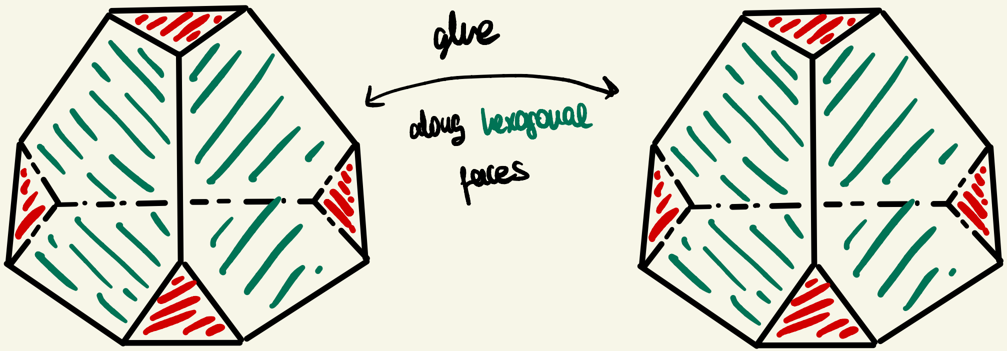

Because of the gluing!

random compact 3-manifold with boundary

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Hyperbolicity of M_n

<u>Theorem</u> (Petri–Raimbault, '20)

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$\Downarrow$ Mostow Rigidity

This metric is unique (up to isometry)!

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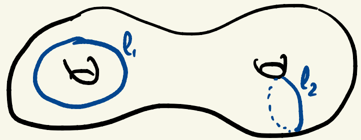
THE LENGTH SPECTRUM

- <u>Def</u>: Let M be a hyperbolic manifold.

  $\mathcal{L}(M) := \{$ (multi)-set of lengths of closed geodesics in $M \}$.



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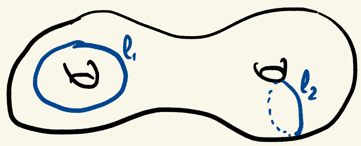

$\downarrow$

$\mathcal{L}(M)$ is a <u>countable</u> set!

Consequence of:

FACT: In every free-homotopy class of essential closed curves, there is a unique closed geodesic.

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POISSON POINT PROCESS

- **Def:** Let (X, μ) be a measurable space.

  A Poisson point process is a random countable subset $\mathcal{P} \subset X$ that satisfies:

  * $\forall A_1, \ldots, A_k$ countable, pairwise disjoint subsets of X,

    if $N_{A_i} := \#(\mathcal{P} \cap A_i)$, then $(N_{A_1}, \ldots, N_{A_k})$ is a vector of indep. r.v.

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    A r.v. $N : \Omega \longrightarrow \mathbb{N}$ is Poisson distributed with parameter $\lambda \in \mathbb{R}$ if $\forall k \in \mathbb{N}$:

$$\mathbb{P}(N = k) = \frac{\lambda^k e^{-\lambda}}{k!}.$$

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SOME NUMBER

"Limiting probability that M_n has k closed geodesics of length $\in [a,b]$"

$$\parallel$$

$$\lim_{n\to\infty} \mathbb{P}\Big[\underbrace{C_{[a,b]}(M_n)}_{\substack{\parallel \\ \#\{\text{closed geodesics in } M_n \text{ of lengths } \in [a,b]\}}} = k\Big] = \frac{e^{-\lambda}\lambda^k}{k!}.$$

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"Limiting probability that M_n has 1 closed geodesic of length $\in [2,3]$"

$$\shortparallel$$

$$\lim_{n\to\infty} \mathbb{P}[\, C_{[2,3]}(M_n) = 1 \,] = \frac{e^{-d} d^{1}}{1!} \overset{(d=4)}{\simeq} 0.073 .$$

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Getting There

THANK YOU !

MERCI !