Exact curve counting of given word-length and self-intersection on the once-punctured torus $\Sigma_{1,1}$

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A splash of history

Theorem (Huber, '59). For any closed hyperbolic surface X,

$$\#\{\gamma \text{ curve} \mid \ell_X(\gamma) \leq L\} \sim \frac{e^L}{L}.$$

We will do the exact counting for ω-l, and get $\sim \dfrac{3^{L+1}}{2L}$.

This was conjectured by (Chas-Phillips, '18).

Theorem (McShane-Rivin, '95). Let X be a hyperbolic structure on the once-punctured torus. Then, $\exists c > 0$,

$$\#\{\gamma \text{ primitive simple closed curve} \mid \ell_X(\gamma) \leq L\} \sim c \cdot L^2.$$

We will do the exact counting for ω-l, and get $\sim \dfrac{12}{\pi^2} L^2$.

__Theorem__ (Mirzakhani, '04). Given a closed hyperbolic surface of signature (g,r), and γ_0 a simple multicurve,

$$\#\{\gamma\sim\gamma_0 \mid \ell_X(\gamma)\leq L\}\sim C(X,\gamma_0)\cdot L^{6g-6+2r}.$$

__Theorem__ (Erlandsson-Souto, '22?) For $\Sigma_{g,r}$, $(g,r)\neq(0,3)$, $G<\mathrm{Map}(\Sigma\setminus\partial\Sigma)$ f.i., $F:\mathcal{C}_c(\Sigma)\to\mathbb{R}_{\geq 0}$, positive, homogeneous and continuous on compact subsets. Let γ_0 be a Ker_G-invariant multicurve,

$$\#\{\gamma\in G\cdot\gamma_0 \mid F(\gamma)\leq L\}\sim c^G(\gamma_0)/b^G_{g,r}\cdot m_{Th}(\{F(\cdot)\leq 1\})\cdot L^{6g-6+2r}.$$

compactly supported geodesic currents.

$$m_{Th}(\{\mu\in D\subseteq\mathcal{C}_{\gamma_0} \mid \iota(\mu,\gamma_0)\leq 1\})$$

$$\sum_{\gamma_0\in G\backslash\mathcal{ML}_{\mathbb{Z}}} c^G(\gamma_0)$$

Our problem

Background:

We call closed curves the free-homotopy classes of closed curves.

<u>Definition</u>: Given a curve $c = [\gamma : S^1 \to \Sigma_{1,1}]$, define its self-intersection

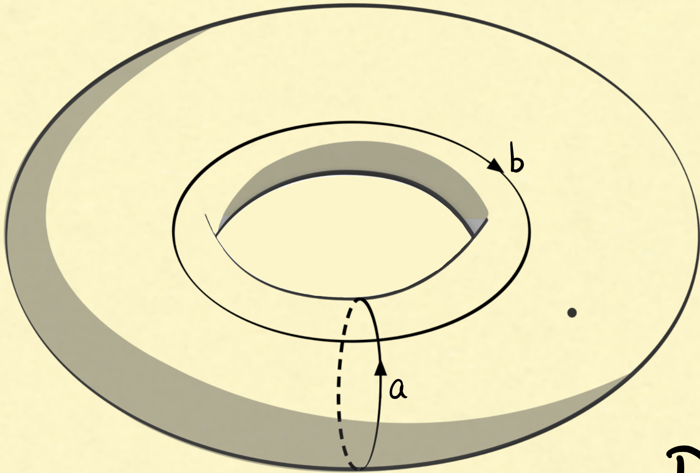
$$i(c) := \frac{1}{2} \min_{\gamma' \sim \gamma} \#\{(t,t') \in S^1 \times S^1 \mid t \neq t' \text{ and } \gamma'(t) = \gamma'(t')\}.$$

We call simple a curve c with $i(c) = 0$.

<u>Remark</u>: with this definition, all simple curves are primitive.

(not powers of other curves)

Fix canonical generators $\{a,b\}$, then any curve corresponds naturally to a conjugacy class in $F_{\{a,b\}}$. Therefore, we inherit a word-length on $\pi_1(\Sigma_{1,1})$.



Denote it by $\ell_w : \pi_1(\Sigma_{1,1})/_\sim \longrightarrow \mathbb{Z}_{>0}$.

<u>Problem</u>: Given $L \in \mathbb{Z}_{>0}$, $k \in \mathbb{Z}_{\geq 0}$, count

$$\#\{c \text{ curves} \mid \ell_w(c) = L,\ i(c) = k\}.$$

<u>Solution</u>: will discuss it for $k=0$, $k=1$, and without the intersection condition.

Simple curves 1: Buser-Semmler chriterion

Theorem (Buser-Semmler, '88): Every non-trivial simple closed curve can be represented, after renaming the generators, by one of the following:

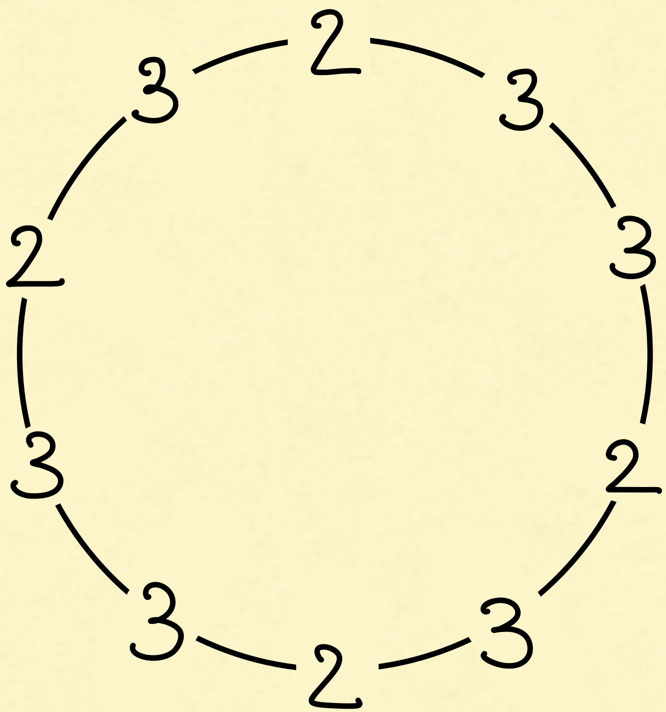
- a
- $aba^{-1}b^{-1}$
- $ab^{n_1}ab^{n_2}\cdots ab^{n_r}$

where $[n_1,\dots,n_r]$ has small variation,

i.e. $\forall s \leq r,\ \forall i_1, i_2 \in \{1,\dots,r\},\ \left|\sum_{j=1}^{s} n_{i_1+j} - \sum_{j=1}^{s} n_{i_2+j}\right| \leq 1.$

Conversely, each of these words is homotopic to a power of a simple closed curve.

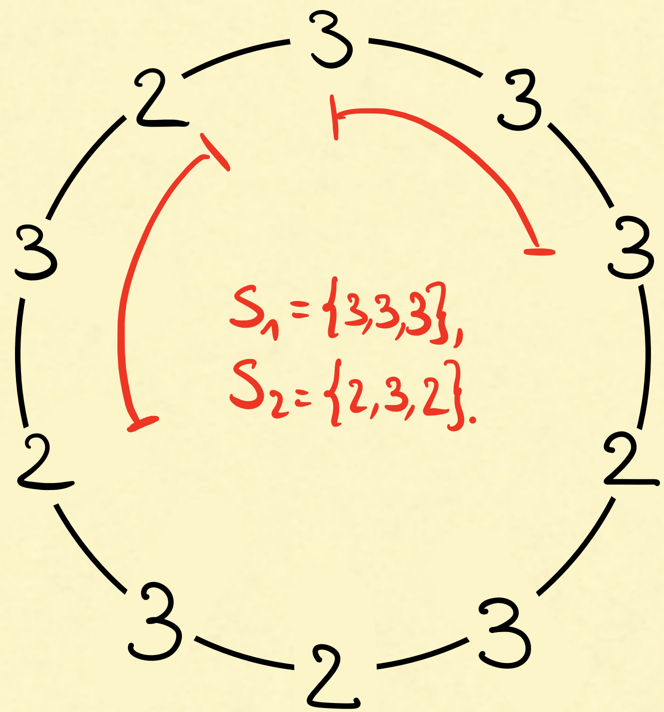
small variation



$[a^2ba^3ba^3ba^2ba^3ba^2ba^3ba^3ba^2ba^3b]$

is simple,

not small variation



$[a^2ba^3ba^3ba^3ba^2ba^3ba^2ba^3ba^2ba^3b]$

has self-intersection 3.

Simple curves 2: Counting - Sturmian & more

Method: Study the rigidity of the necklaces with small variation.

__Remark__: $[n_1, \ldots, n_r]$ has small variation iff $ab^{n_1} \cdots ab^{n_r}$ is sturmian.

__Remark__: $[n_1, \ldots, n_r]$ has small variation $\Rightarrow \exists m \geq 1$ s.t. $n_i \in \{m, m+1\}$ $\forall i$.

__Proposition__: For any $x, y \in \mathbb{Z}_{\geq 0}$, for any $m \in \mathbb{Z}_{\geq 1}$,

there is a unique necklace with small variation

$w = [n_1, \ldots, n_r]$ with $n_i \in \{m, m+1\}$, $|w|_m = x$, $|w|_{m+1} = y$.

this is an analog to the correspondence:

sturmian words $\longleftrightarrow$ cutting words.

Theorem: $\#\{\gamma \text{ simple closed curve} \mid \ell_\omega(\gamma) \leq L\} = 4\,\Phi(L) + 2$.

where Φ is the summation of Euler totient's function.

Confirming the Chas-Phillips conjecture that for

$p = 2n+1$ prime there are $8n$ simple prim. with $\ell_\omega = p$.

Theorem: $\#\{\gamma \text{ simple multicurve} \mid \ell_\omega(\gamma) \leq L\} = 2\left(L^2 + L + \lfloor L/4 \rfloor\right)$.

Self-intersection 1 : Characterization

<u>Theorem</u>: A primitive curve has self-intersection 1 if and only if up to renaming the generators can be rewritten as one of:

- $ab^{-1}a^{-1}b\,a^{n_1}b \cdots a^{n_r}b$ or $ab^{-1}a^{-1}b\,\bar{a}^{n_1}b \cdots \bar{a}^{n_r}b$

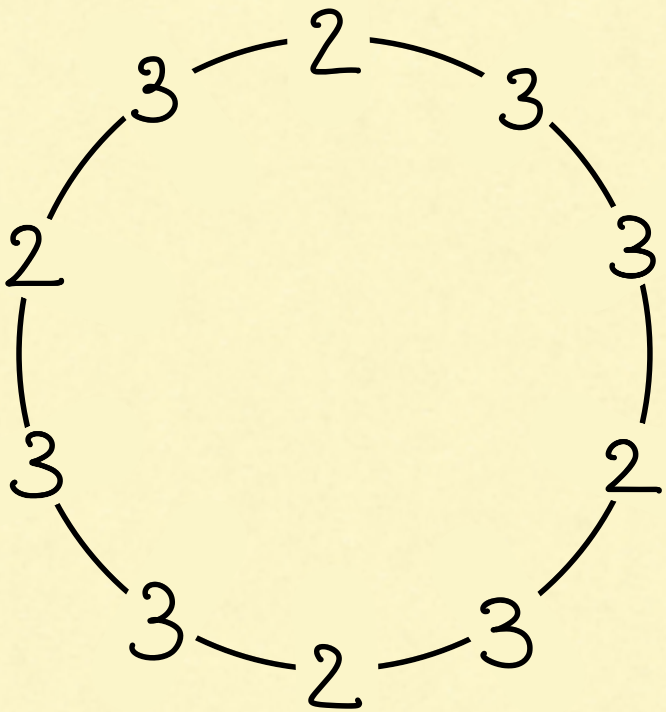
  where $a^{n_1}b \cdots a^{n_r}b$ and $\bar{a}^{n_1}b \cdots \bar{a}^{n_r}b$ are uniquely determined representatives of primitive simple closed curves,

- $a^{n_1}b \cdots a^{n_r}b$,

  where $[n_1, \dots, n_r]$ is a necklace with 2-variation, i.e. there is a unique pair of subsets $S_1, S_2 \subseteq \{n_1, \dots, n_r\}$ being cyclically consecutive such that $\left|\sum_{S_1} n_i - \sum_{S_2} n_j\right| = 2$, with $|S_1| = |S_2|$.

- short exceptional cases a^2b^2, $aba^{-1}b$, $ab^{-1}a^{-1}b^2$.

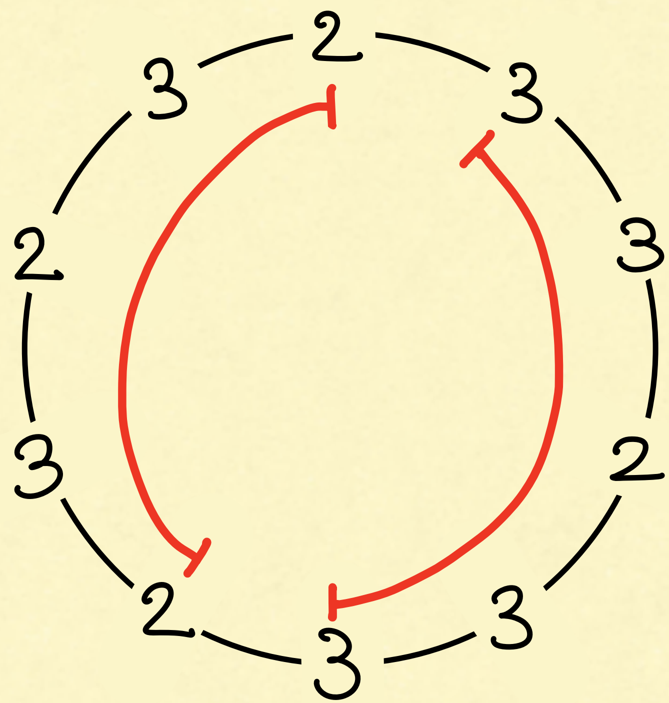
small variation



$[a^2ba^3ba^3ba^2ba^3ba^2ba^3ba^3ba^2ba^3b]$

is simple,

2-variation



$[a^2ba^3ba^3ba^2ba^3ba^3ba^2ba^3ba^2ba^3b]$

has self-intersection 1.

Self-intersection 1: Counting

This time, we study the rigidity of the 2-variation necklaces.

<u>Remark</u>: $[n_1, \ldots, n_r]$ has 2-variation implies:

- $[n_1, \ldots, n_r] = [m, m+2]$ for some $m \geq 1$, or
- $\exists\, m \geq 1$ such that $n_i \in \{m, m+1\}$ for all $i \in \{1, \ldots, r\}$.

<u>Proposition</u>: For all $m \in \mathbb{Z}_{\geq 1}$, for all $x, y \in \mathbb{Z}_{\geq 1}$ with $\gcd(x,y) = 2$, there is a unique necklace $w = [n_1, \ldots, n_r]$ with $n_i \in \{m, m+1\}$, $|w|_m = x$, $|w|_{m+1} = y$ with 2-variation. If $\gcd(x,y) \neq 2$, there is none.

Theorem: For $L > 4$,

$$\#\{\gamma \text{ primitive closed curve} \mid i(\gamma)=1,\ \ell_w(\gamma)=L\} =$$

$$= 8\cdot\left(\varphi(L-4) + \varphi(L/2)/2 \cdot \delta_{2\mathbb{Z}}\right).$$

Answering negatively to Chas-Phillips computational conjecture that $\sim 4(L-2)$

Remark: $\#\{\gamma \text{ primitive closed curve} \mid i(\gamma)=0,\ \ell_w(\gamma)\leq L\} \sim \frac{12}{\pi^2}L^2,$

$$\#\{\gamma \text{ primitive closed curve} \mid i(\gamma)=1,\ \ell_w(\gamma)\leq L\} \sim \frac{27}{\pi^2}L^2.$$

Higher self-intersection

- Open question,
- Recursivity,
- Orders of self-intersection,
- Possible?,
- Growth constants?,
- Distribution of intersections?.

Any self-intersection

With different methods, via analytic combinatorics and generating functions, we also prove:

Rmk: These are purely algebraic statements on conjugacy classes of a free group

Theorem: $\#\{\gamma \text{ closed primitive curve} \mid \ell_w(\gamma) = L\} =$

$$\frac{1}{n}\sum_{d \mid n} \mu(d) \cdot 3^{n/d} = \#\{\text{aperiodic necklaces with } L \text{ beads and } 3 \text{ colors}\} + \delta_{\{1,2\}}(L)$$

where $\delta_{\{1,2\}}(L) = 1$ if $L = 1, 2$, and $\delta_{\{1,2\}}(L) = 0$ otherwise

Theorem: $\#\{\gamma \text{ closed curve} \mid \ell_w(\gamma) = L\} =$

$$\frac{1}{n}\sum_{d \mid n} \varphi(d) \cdot 3^{n/d} = \#\{\text{necklaces with } L \text{ beads and } 3 \text{ colors}\} + \varepsilon(L)$$

where $\varepsilon(L) = 1$ if L is odd, and $\varepsilon(L) = 2$ if even.

Question: what is the nature of these bijections?

Thank you!