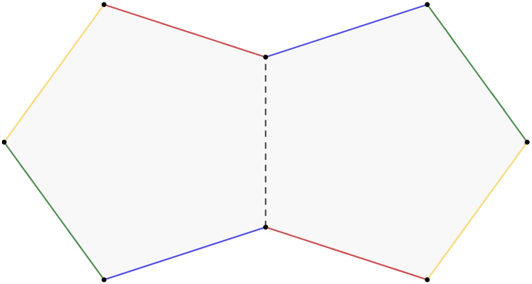
Algebraic intersections in regular polygons

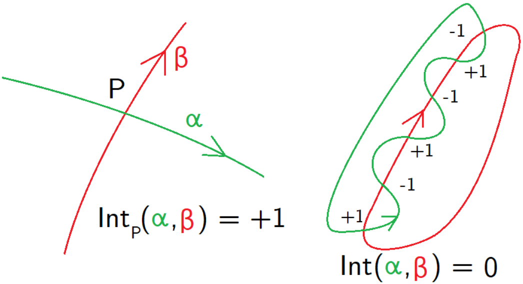
Julien Boulanger
SMaRT, Luxembourg, March 19, 2024



Algebraic intersection

Let X be a closed oriented surface with a Riemannian metric, possibly with singularities.

Given two oriented closed curves α and β on X, consider the algebraic intersection $\text{Int}(\alpha, \beta)$.

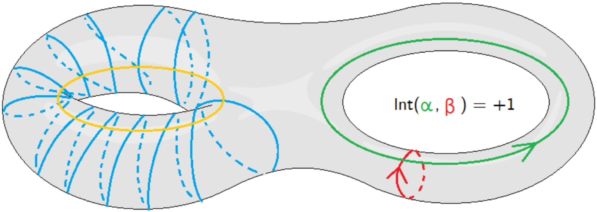


The algebraic intersection $\mathrm{Int}(\cdot, \cdot)$ is a bilinear symplectic form in homology.

Algebraic intersection

Question

How many times can two closed curves of a given length intersect ?



Definition [(algebraic) interaction strength]

$$\mathrm{KVol}(X) := \mathrm{Vol}(X) \cdot \sup_{\alpha,\beta \text{ closed curves}} \frac{\mathrm{Int}(\alpha,\beta)}{l(\alpha)l(\beta)}$$

Remark : Multiplying by the volume makes KVol scalar invariant.

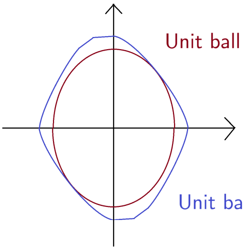
Outline

1. The algebraic interaction strength KVol
   - History and motivations
   - Example: flat tori
   - KVol on hyperbolic surfaces
   - KVol on translation surfaces
2. Translation surfaces and their Veech groups
   - Definition
   - Teichmüller space and moduli space
   - $SL_2(\mathbb{R})$-action on the moduli space, Veech group
   - Example : The Golden L
3. A few geometric ideas
   - KVol on the golden L
   - The action of a twist
   - KVol on the double pentagon

History and motivations

$\rightarrow$ In D. Massart's thesis (1996), KVol arises as a comparison constant between the *stable norm* $\|\cdot\|_s$ and the *Hodge norm* $\|\cdot\|_2$ in homology, namely we have for all $h \in H_1(X,\mathbb{R})$,

$$\frac{1}{\sqrt{\mathrm{Vol}(X)}}\|h\|_s \leq \|h\|_2 \leq \mathrm{KVol}(X)\frac{1}{\sqrt{\mathrm{Vol}(X)}}\|h\|_s.$$



The Hodge norm (coming from the L^2 norm in cohomology) is euclidean: its unit ball is an ellipse.

The stable norm depends on the metric, its unit ball can be very complicated (e.g. polyhedral with an infinite number of cells)

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Theorem (Massart, Muetzel, 2014)

For every Riemannian surface X of genus $g \geq 1$, we have:

$$KVol(X) \geq 1$$

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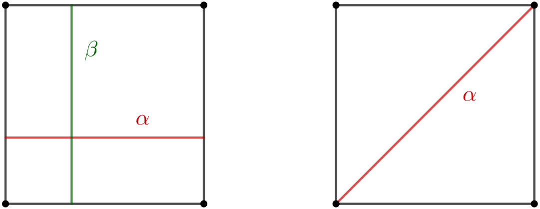

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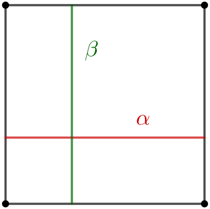

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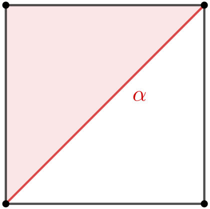

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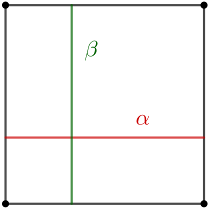

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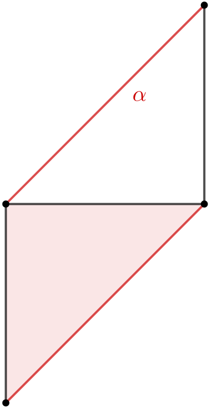

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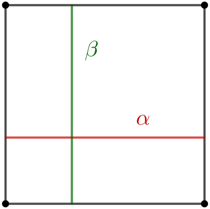
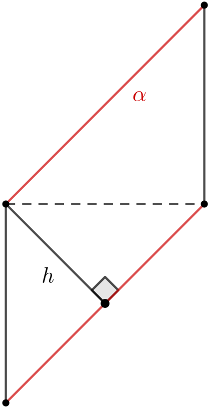


$$\frac{\mathrm{Int}(\alpha,\beta)}{l(\alpha)l(\beta)} = 1$$

$$l(\alpha) \cdot h = 1$$



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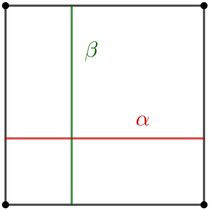

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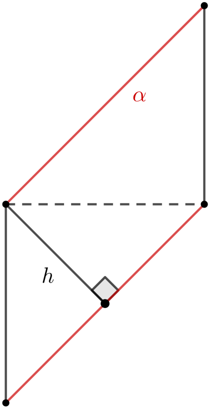

$$l(\alpha) \cdot h = 1$$

$$\text{Hence, } \forall\beta, \frac{\mathrm{Int}(\alpha,\beta)}{l(\alpha)l(\beta)} \leq \frac{1}{l(\alpha)h} = 1$$

KVol on hyperbolic surfaces

→ In 2014, D. Massart and B. Muetzel studied the behaviour of KVol(X) as X goes towards the boundary of the moduli space of hyperbolic surfaces, and gave geometric bounds on KVol, namely for any Riemannian surface:

$$\frac{\mathrm{Vol}(X)}{2Dl_0} \leq \mathrm{KVol}(X) \leq 9\frac{\mathrm{Vol}(X)}{l_0^2} = 9 \cdot \mathrm{SysVol}(X),$$

where D is the diameter and l_0 the homological systolic length of X.

Theorem (Consequence of Balacheff, Karam, Parlier, 2021)

There exist $c > 0$ such that for any hyperbolic surface X of genus $g \geq 2$, we have

$$KVol(X) \geq c\frac{g}{(log(g))^2}$$

This growth rate is optimal, see [Buser, Sarnak 1994].

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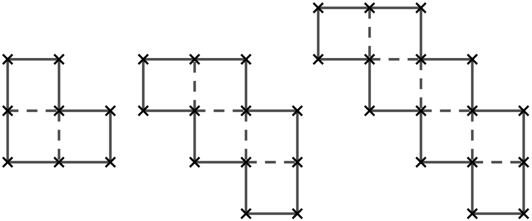

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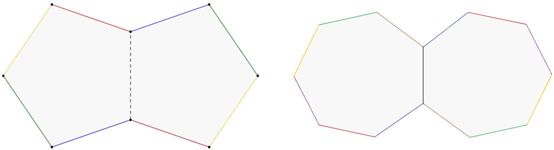
- In 2022, in a joint work with E.Lanneau and D.Massart we compute KVol on the $SL_2(\mathbb{R})$-orbit of the double regular n-gons for odd n.



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- In 2022, we extend this method with E.Lanneau and D.Massart to compute KVol on the $SL_2(\mathbb{R})$-orbit of the double regular n-gons for odd n.
- In 2023, we deal with the case of the regular n-gon for even n.
- We then generalize the method with I. Pasquinelli to the case of Bouw-Möller surfaces with a single singularity.

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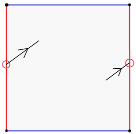
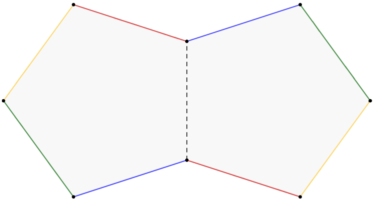
Translation surfaces

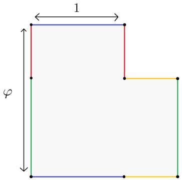
Definition

A translation surface is a surface obtained from a collection of euclidean polygon, by identifying pairs of parallel opposite sides of the same length (by translation).



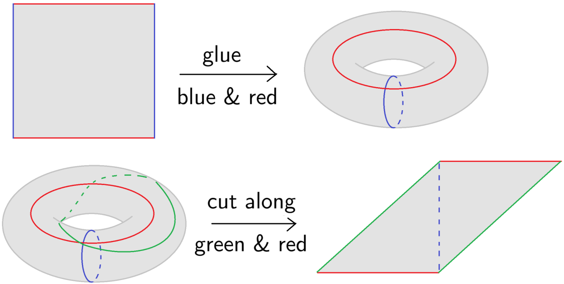
Double pentagon



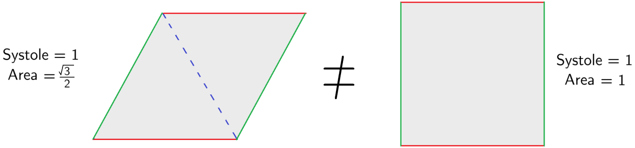


Golden L φ is the golden ratio

Teichmüller space and moduli space



The two polygonal models above give what we want to consider as the same resulting surface, whereas we have below two polygonal models of surfaces with different properties.

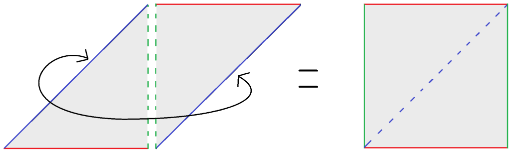


Definition

The **moduli space** $\Omega\mathcal{M}_g$ of translation surfaces of genus g is the set:

$$\Omega\mathcal{M}_g = \left\{ \begin{array}{c} \text{Collection of polygons with} \\ \text{identifications of parallel sides of} \\ \text{the same length and genus } g \end{array} \right\} / \text{cut and paste}$$

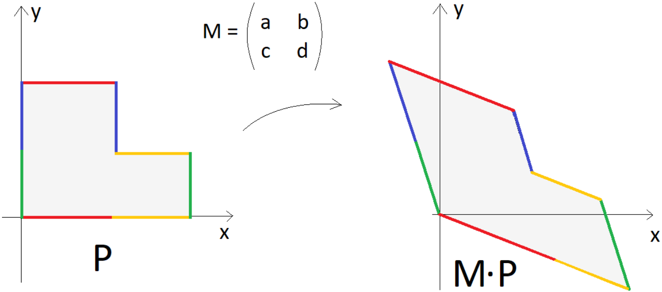
The **Teichmüller space** $\Omega\mathcal{T}_g$ of translation surfaces can be seen as the space of (X, φ) where $X \in \Omega\mathcal{M}_g$ and φ is a *marking* of a homology basis.



We have $\Omega\mathcal{M}_g = \Omega\mathcal{T}_g / MCG(g)$.

$SL_2(\mathbb{R})$-action on the moduli space

Given a translation surface X described by a collection of polygons and $M \in GL_2^+(\mathbb{R})$, we can construct the translation surface $M \cdot X$.

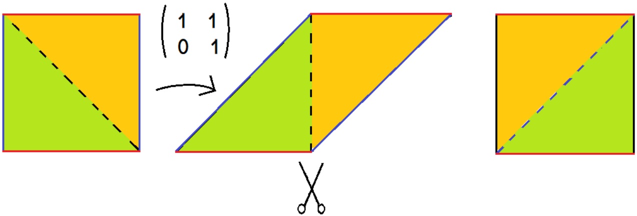


It is often convenient to consider the action of $SL_2(\mathbb{R})$ instead of $GL_2^+(\mathbb{R})$ as it preserves the area.

Veech group

Definition

The Veech group of a translation surface X is the stabilizer of X (in the moduli space) under the action of $SL_2(\mathbb{R})$. We denote it by $SL(X)$.



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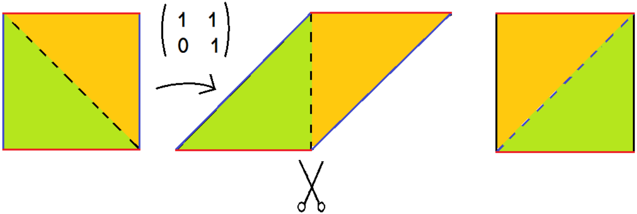

Proposition

The Veech group of any flat torus is conjugated to $SL_2(\mathbb{Z})$.

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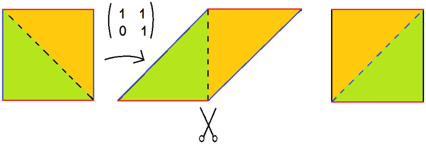

Theorem (W.Veech, 1989)

For any translation surface X, $SL(X)$ is a discrete subgroup of $SL_2(\mathbb{R})$.

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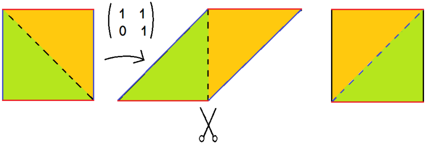


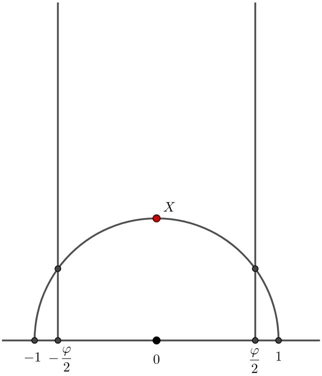
Theorem (W.Veech, 1989)

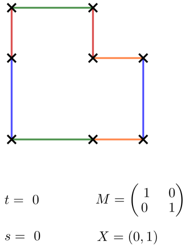
For any translation surface X, $SL(X)$ is a discrete subgroup of $SL_2(\mathbb{R})$.

Consequence: If we quotient by the action of the rotations $SO_2(\mathbb{R})$, the orbit of X under the action of $SL_2(\mathbb{R})$ can be identified with $\mathbb{H}^2/SL(X)$.

Example : The golden L

We choose a base surface S in the orbit. If $X = M \cdot S$ with $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$, we map the surface X to the point $\frac{di+b}{ci+a} \in \mathbb{H}^2$.





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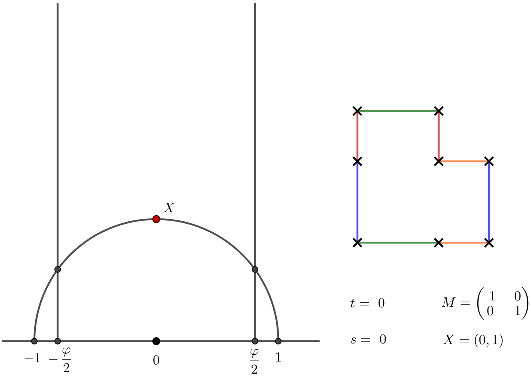

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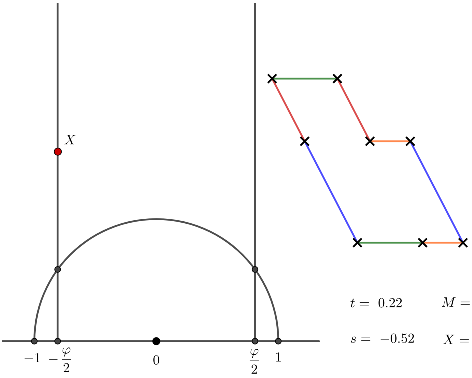

$t = 0.22$ $\quad$ $M = \begin{pmatrix} 0.8 & -0.42 \\ 0 & 1.25 \end{pmatrix}$

$s = -0.52$ $\quad$ $X = (-0.81, 1.55)$

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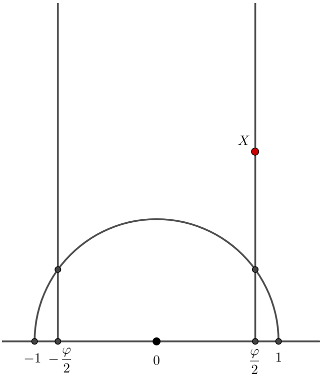

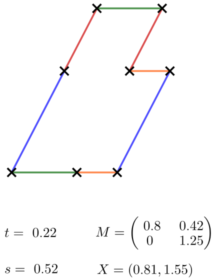

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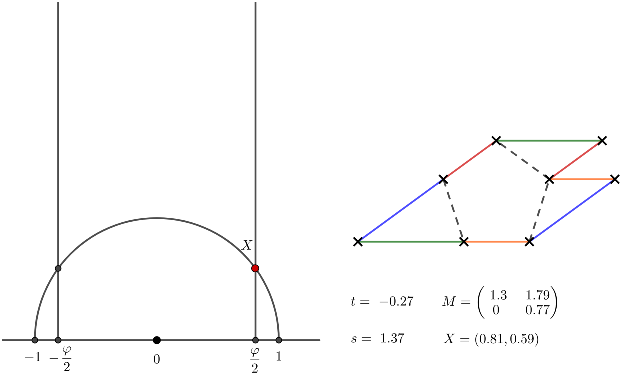

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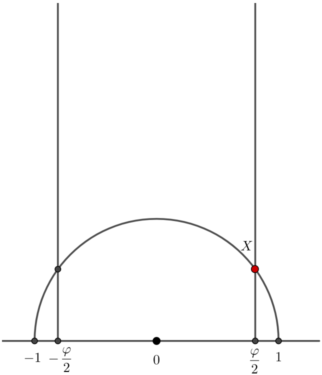
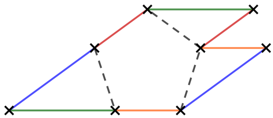




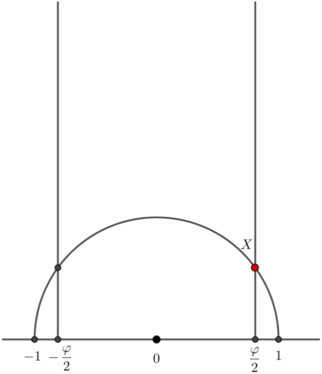
$t = -0.27$ $M = \begin{pmatrix} 1.3 & 1.79 \\ 0 & 0.77 \end{pmatrix}$

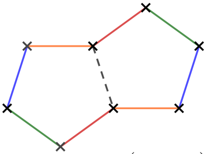
$s = 1.37$ $X = (0.81, 0.59)$

Example : The golden L

Proposition

The golden L and the double pentagon belong to the same $GL_2^+(\mathbb{R})$-orbit.
The Veech group of the golden L is the triangle group $\Delta^+(2,5,\infty)$.





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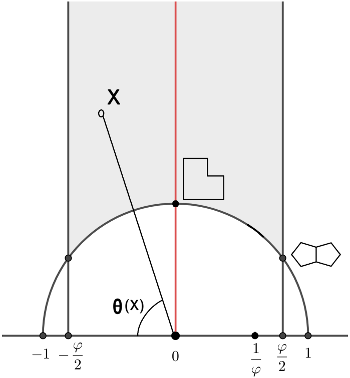
Theorem (B.-Lanneau-Massart, 2022)

Let $n \geq 5$ odd. For every X in the $SL_2(\mathbb{R})$-orbit of the double regular n-gon, represented as a point in the fundamental domain $\mathbb{H}^2/\Delta^+(2,5,\infty)$. Then

$$KVol(X) = \frac{n}{2}\cot\frac{\pi}{n} \cdot \frac{1}{\sin\frac{\pi}{n}} \sin\theta(X).$$



For $n = 5$:

$$KVol(X) = \frac{2\varphi - 1}{(\varphi - 1)^2} \sin\theta(X).$$

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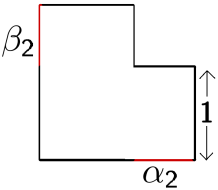
How to understand this result

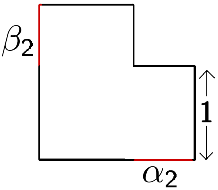
We start with the golden L and consider the curve α_2.



- Any saddle connection β intersecting *non-singularly* α_2 must have a length at least 2.
- More generally, any saddle connection β intersecting *non-singularly* α_2 K times must have a length at least $K+1$.
- The curve β_2 intersects α_2 once while having a length $\varphi - 1 \simeq 0.61$.

How to understand this result

We start with the golden L and consider the curve α_2.



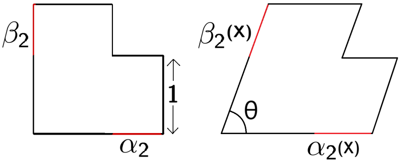
- Any saddle connection β intersecting *non-singularly* α_2 must have a length at least 2.
- More generally, any saddle connection β intersecting *non-singularly* α_2 K times must have a length at least $K+1$.
- The curve β_2 intersects α_2 once while having a length $\varphi - 1 \simeq 0.61$.

Proposition

- *KVol on the golden L is achieved uniquely by* (α_2, β_2).
- *For any other pair of closed curves* (α, β), *we have*

$$\frac{Int(\alpha_2, \beta)}{l(\alpha_2)l(\beta)} \leq \frac{1}{\varphi - 1}.$$

On the surface obtained from the Golden L by a twist of angle θ, the length of β_2 is multiplied by $\frac{1}{\sin\theta}$.



In particular,

$$\forall X,\ \frac{\mathrm{Int}(\alpha_2(X),\beta_2(X))}{l(\alpha_2(X))l(\beta_2(X))} = \frac{1}{(\varphi-1)^2}\sin\theta(X).$$

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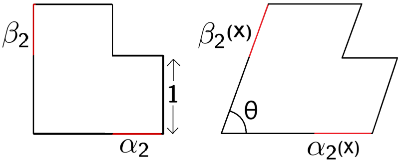

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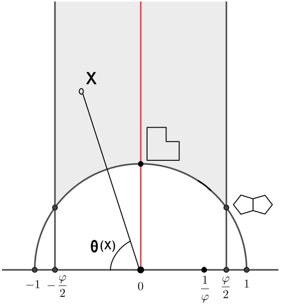
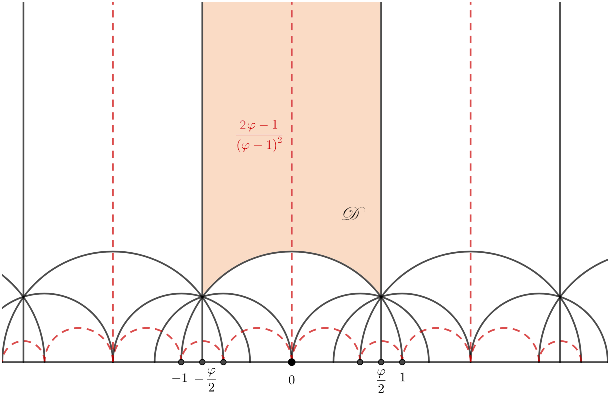
Lemma

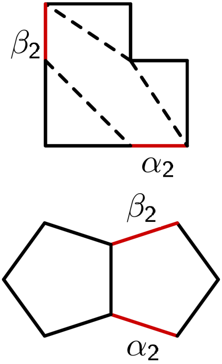
The angle θ corresponds to the angle $\theta(X)$ between the horizontal and the segment OX.

Theorem (Refinement of the previous result)

For every X in the fundamental domain, the supremum in the definition of KVol is achieved by the pair (α_2, β_2).







$\rightarrow$ Every red geodesic corresponds to the image of α_2, β_2 by the action of an element of the affine group.
$\rightarrow$ On the interior of $\mathcal{D}$, the pair (α_2, β_2) is the only pair achieving the supremum in the definition of KVol.
$\rightarrow$ KVol on the double pentagon is achieved uniquely by pairs of distinct sides.

The three main steps of the proof

1. Show that KVol is achieved by the curves α_2 and β_2 on the right-angled L surfaces of the orbit.
   $\rightarrow$ Same argument as for the torus (uses cylinder decomposition).
   $\rightarrow$ The method generalizes to so-called "Veech surfaces" for which we have finitely many cylinder decompositions up to the action of the Veech group.

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2. Show that KVol is achived by the (images of the) curves α_2 and β_2 on the double pentagon.
   $\to$ "Subdivision method" : decompose curves α, β into smaller segments for which we can control both the length and the intersections.
   $\to$ With work, it can be generalised to surfaces made with convex polygons having obtuse angles + a non-self-identification condition on the polygons (work in progress with I.Pasquinelli, out soon!).

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3. Use properties of KVol as a function in the $SL_2(\mathbb{R})$-orbit to interpolate between the right-angled staircases and the double pentagon.
   $\rightarrow$ It requires estimates on cylinder decompositions that are easy to obtain for the double $(2n+1)$-gon but are difficult to obtain in general.

Thanks for your attention