Length spectra of random metric maps of large g

A Teichmüller theory approach

Marseille 2.10.2023

joint work with

Simon Barazer

Alessandro Giacchetto

What is a map?

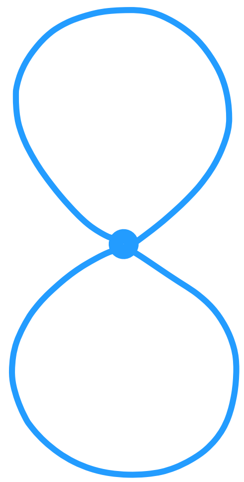
What is a map?

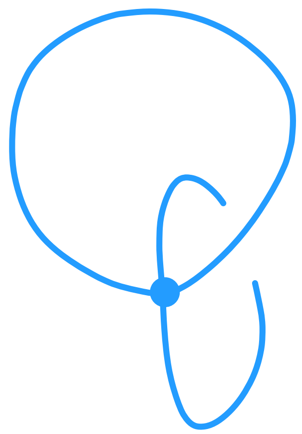
A map is a graph G embedded into a surface S such that $S \setminus G$ is a disjoint union of disks

What is a map?

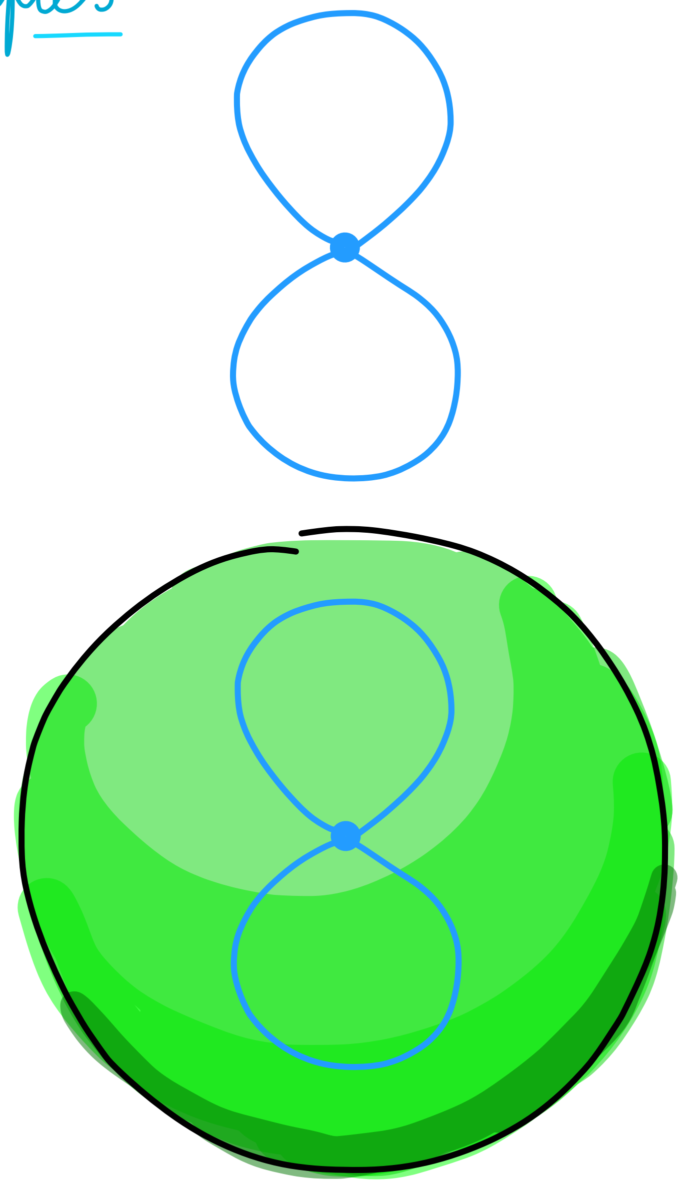
A map is a graph G embedded into a surface S such that $S \setminus G$ is a disjoint union of disks called faces

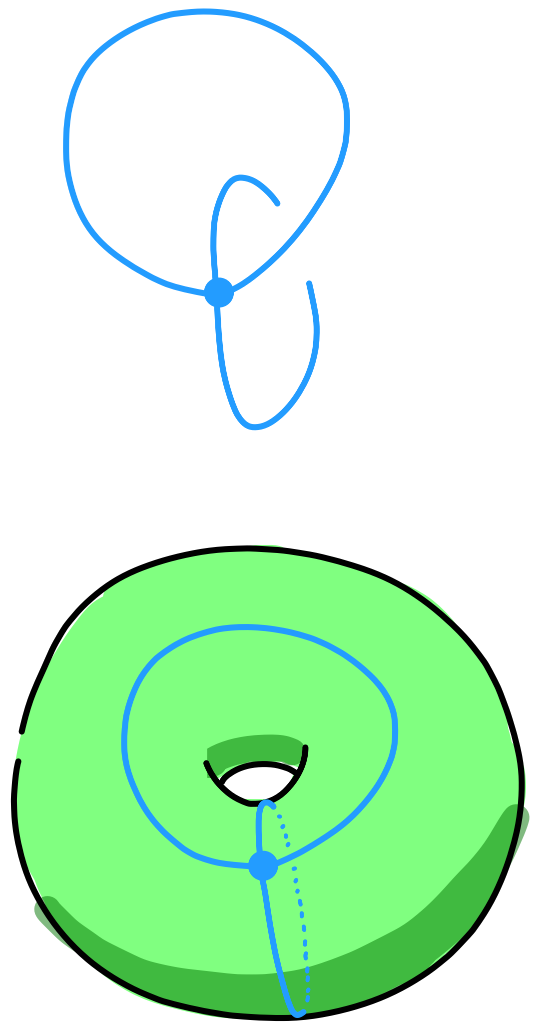
Examples



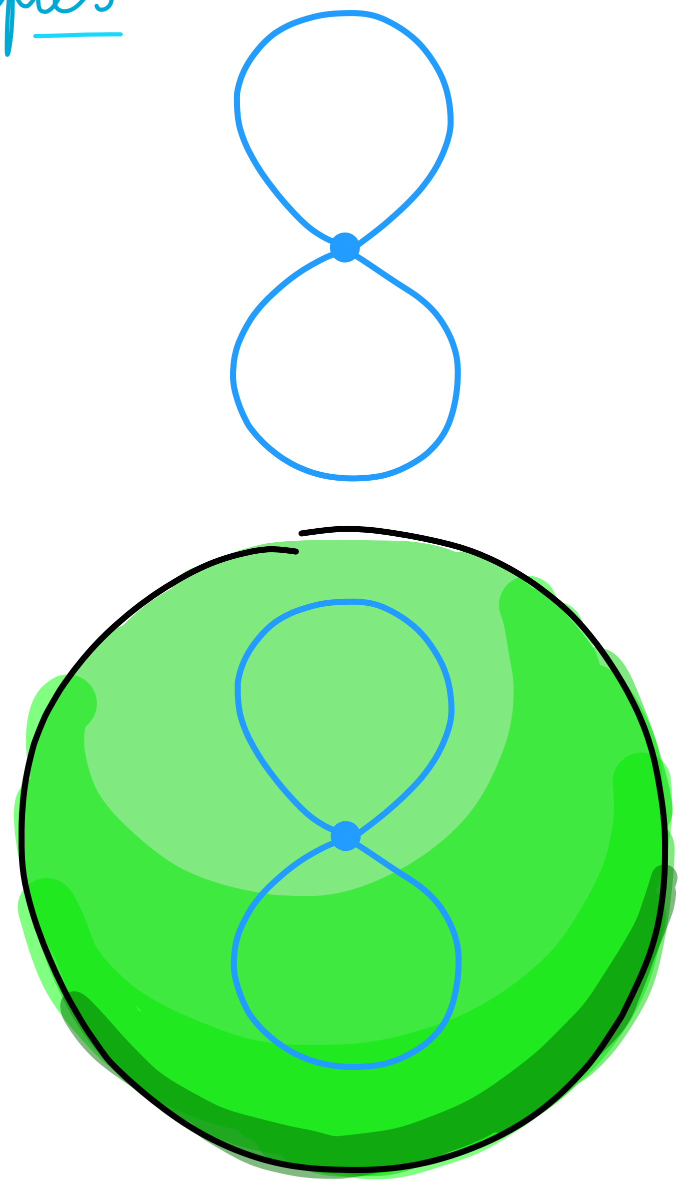


Examples

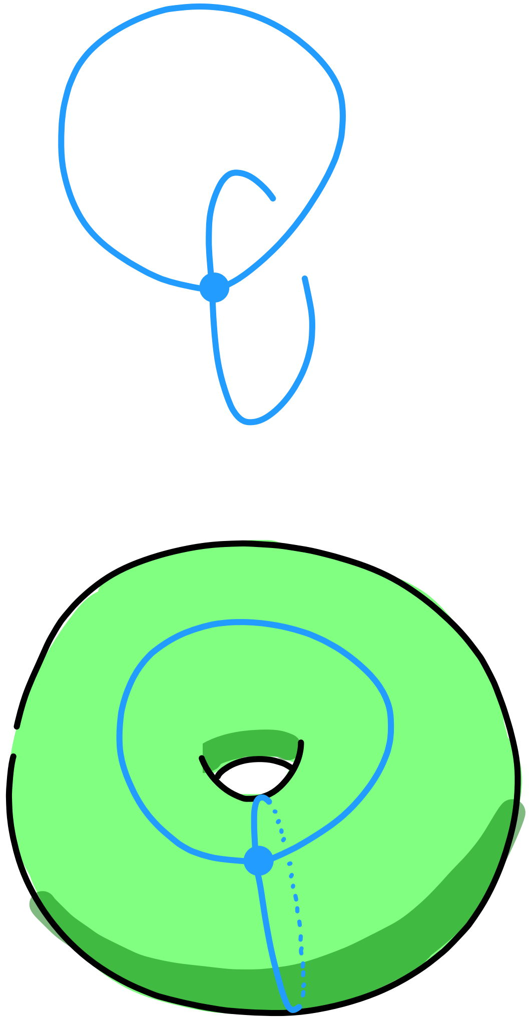




Examples

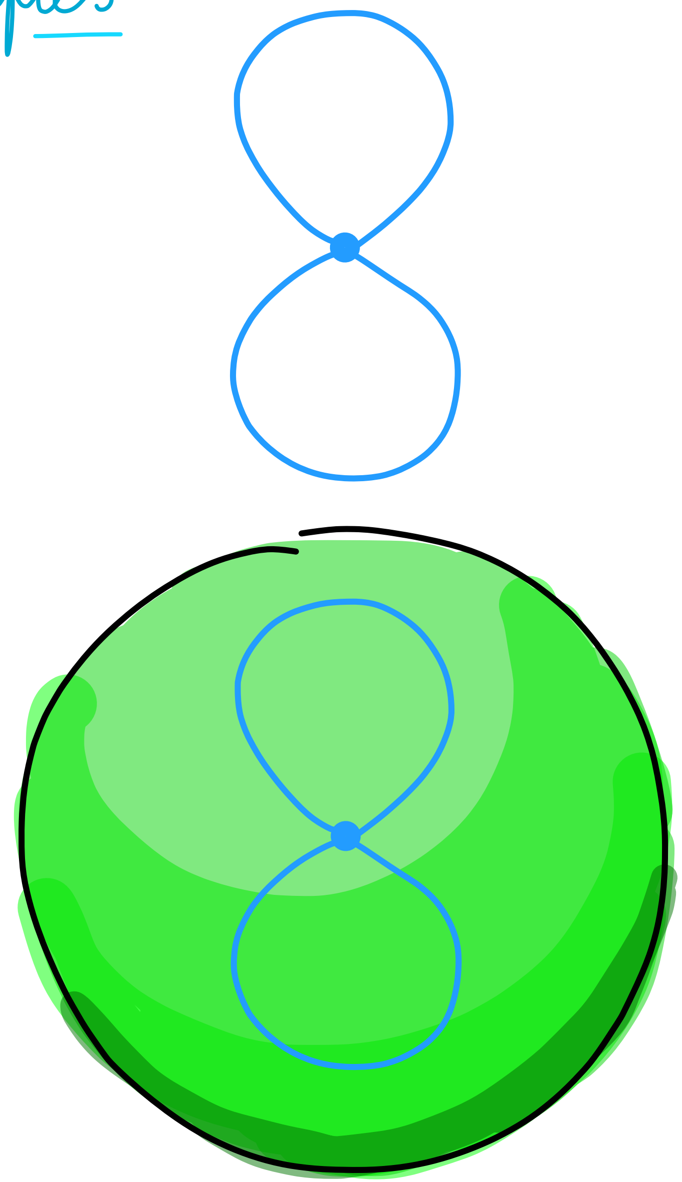


$g = 0,\ n = 3$

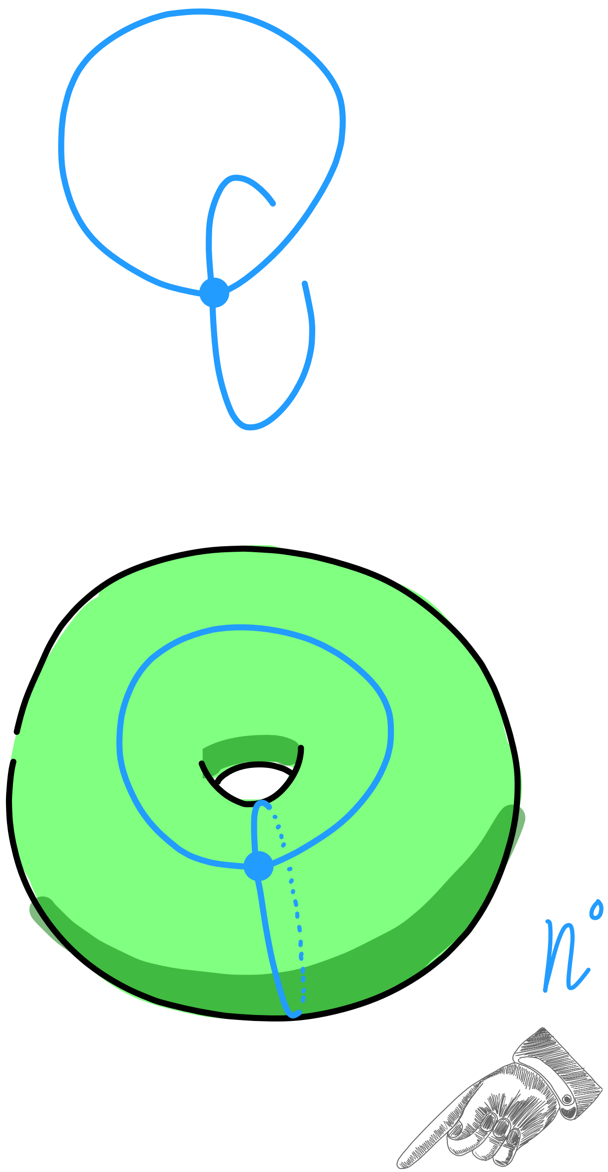


$g = 1,\ n = 1$

Examples



$g = 0, \; n = 3$

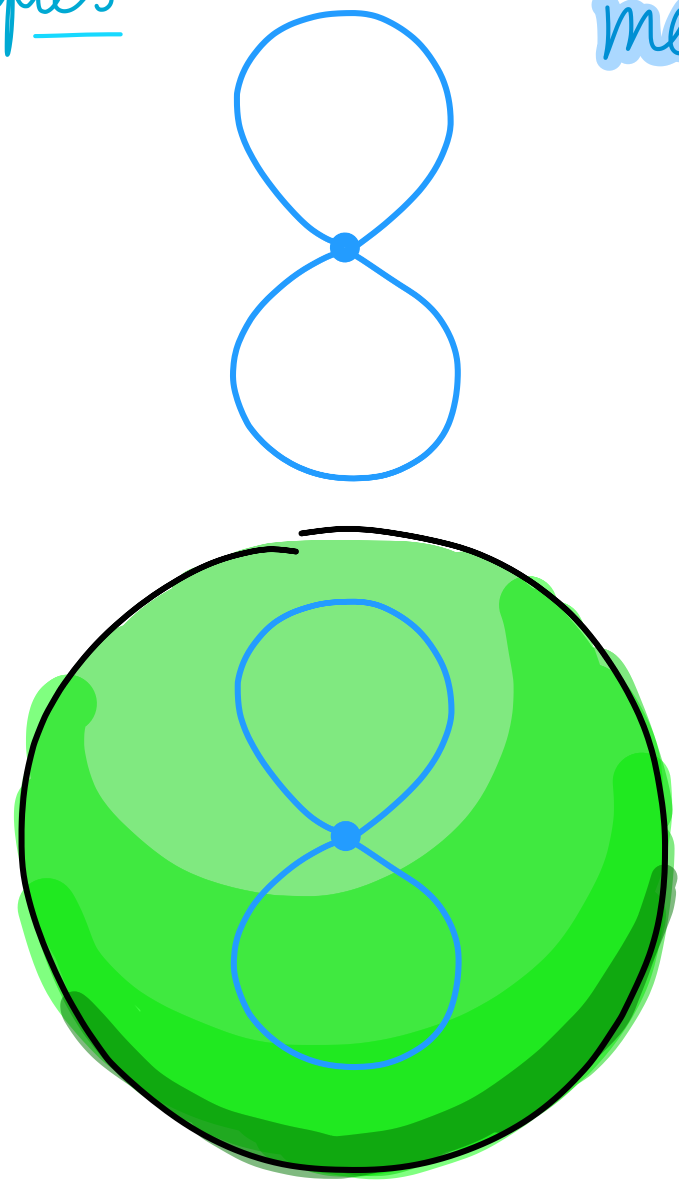


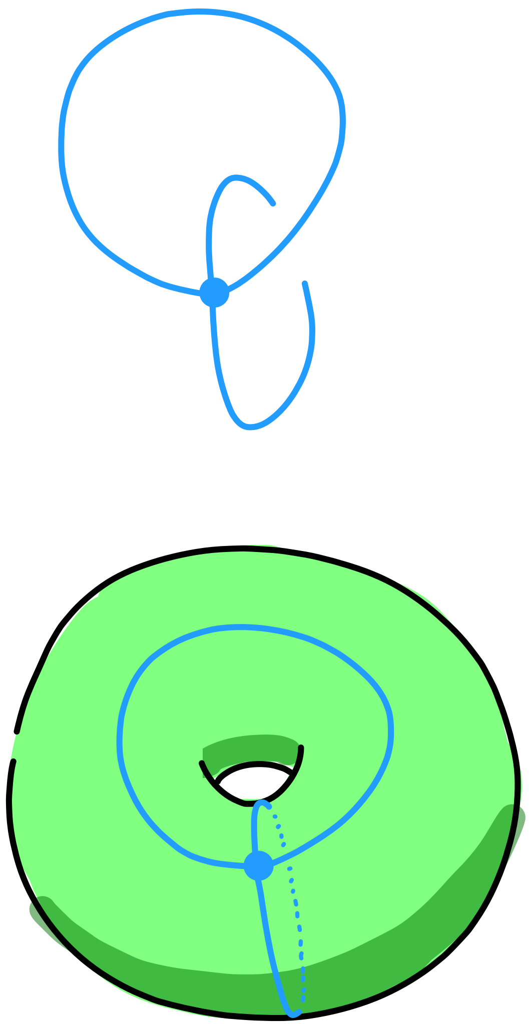
n° of faces

$g = 1, \; n = 1$

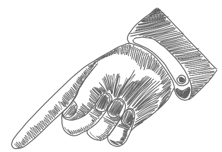
Examples

metric map?





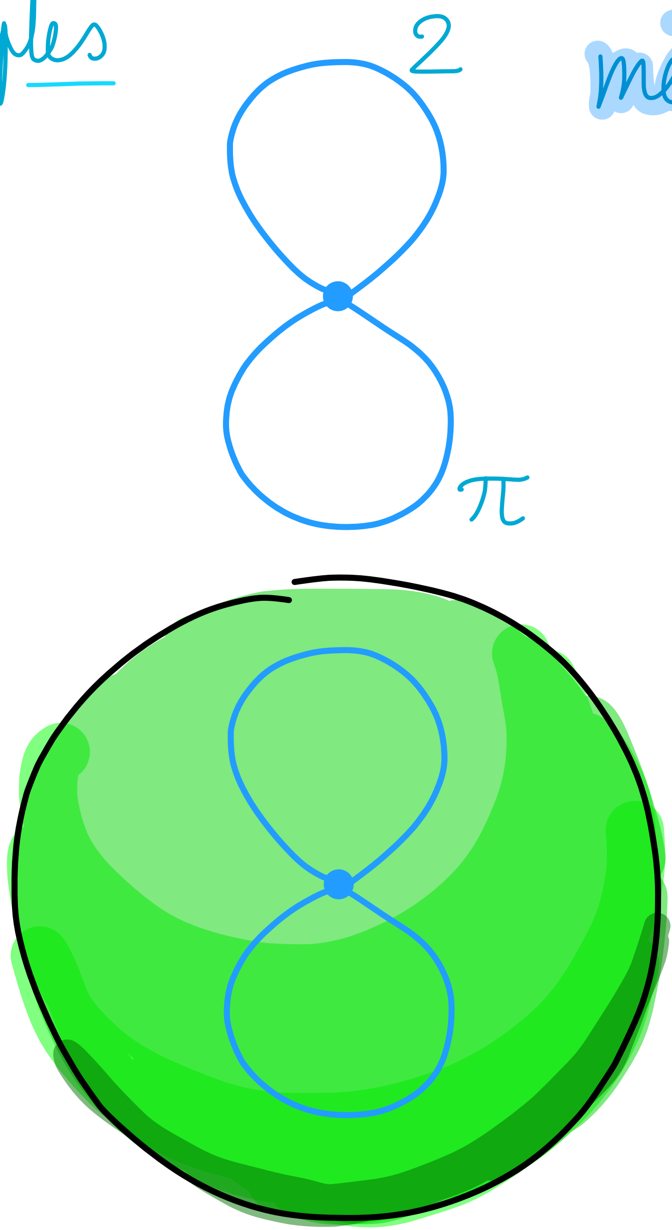
n° of faces

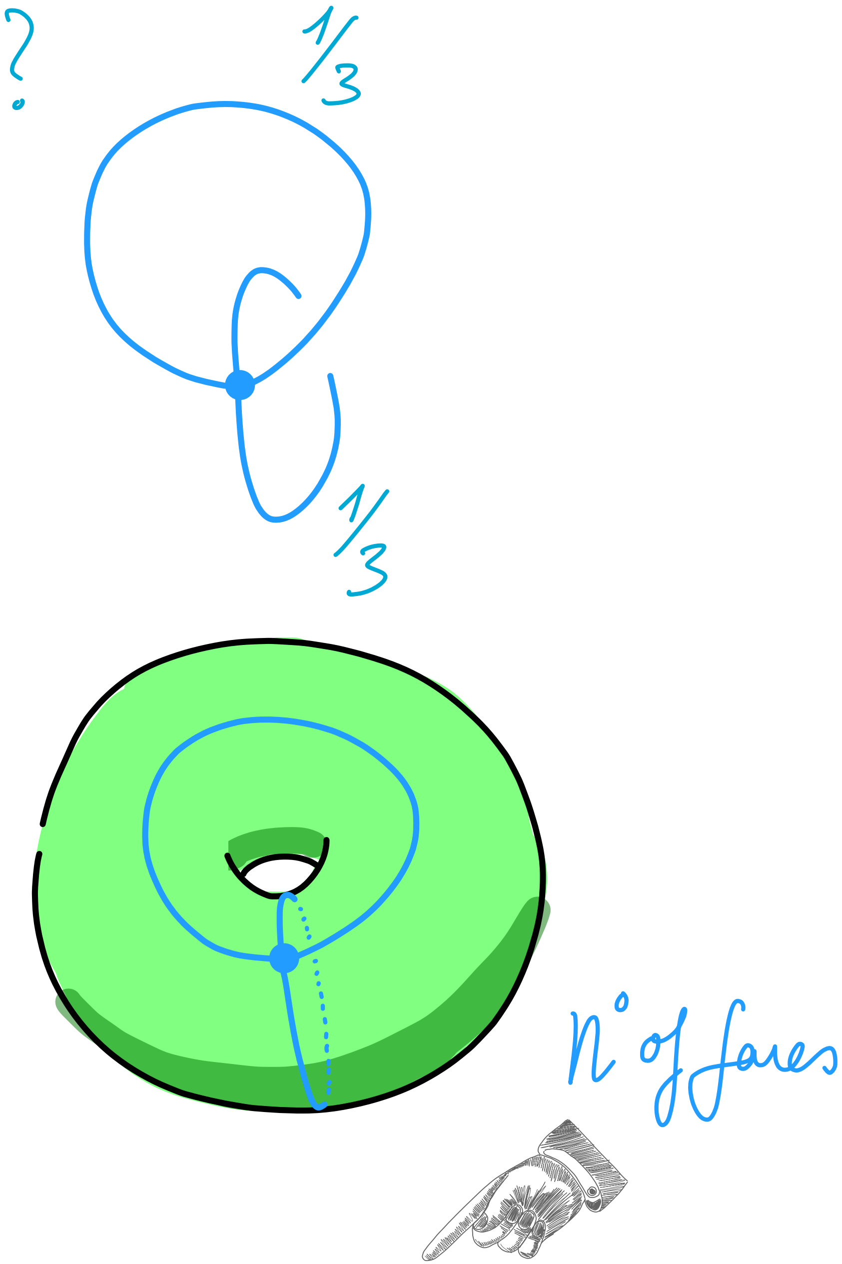


$g = 0, \ n = 3$ $\qquad$ $g = 1, \ n = 1$

Examples

metric map?





$g = 0, \ n = 3$ $\qquad$ $g = 1, \ n = 1$

What do we care?

What do we care?

Fix $0 \leq a < b$

What do we care?

Fix $0 \leq a < b$

$N_{[a,b]}(G) :=$ Number of primitive Cycles in G with length in $[a, b]$

What do we care?

Fix $0 \leq a < b$

$$N_{[a,b]}(G) := \text{number of primitive cycles in } G \text{ with length in } [a,b]$$

$$N_{g,[a,b]} : \mathcal{M}^{comb}_{g,n}(L_1, \dots, L_n) \longrightarrow \mathbb{Z}_{\geq 0}$$

What do we care? Fix $0 \leq a < b$

$N_{[a,b]}(G) :=$ number of primitive cycles in G with length in $[a, b]$

$$N_{g,[a,b]} : \mathcal{M}^{comb}_{g,n}(L_1, \ldots, L_n) \longrightarrow \mathbb{Z}_{\geq 0}$$

$$\mathcal{M}^{comb}_{g,n}(L_1, \ldots, L_n) = \{\text{metric maps of genus } g \text{ with } n \text{ faces of length } L_1, \ldots, L_n. \text{ Valence} \geq 3\}$$

What do we care? Fix $0 \leq a < b$

$$N_{[a,b]}(G) := \text{number of primitive cycles in } G \text{ with length in } [a,b]$$

$$N_{g,[a,b]} : \mathcal{M}^{comb}_{g,n}(L_1, \dots, L_n) \longrightarrow \mathbb{Z}_{\geq 0}$$

Theorem If $L_1 + \dots + L_n \sim 12g$ as $g \to \infty$

What do we care? Fix $0 \leq a < b$

$N_{[a,b]}(G) :=$ number of primitive cycles in G with length in $[a, b]$

$N_{g,[a,b]} : \mathcal{M}^{comb}_{g,n}(L_1, \dots, L_n) \longrightarrow \mathbb{Z}_{\geq 0}$

Theorem If $L_1 + \dots + L_n \sim 12g$ as $g \to \infty$

$$N_{g,[a,b]} \xrightarrow[g \to \infty]{d} \text{Poisson}$$

What do we care?

Fix $0 \leq a < b$

$N_{[a,b]}(G) :=$ number of primitive cycles in G with length in $[a, b]$

$$N_{g,[a,b]} : \mathcal{M}^{comb}_{g,n}(L_1, \ldots, L_n) \longrightarrow \mathbb{Z}_{\geq 0}$$

Theorem If $L_1 + \cdots + L_n \sim 12g$ as $g \to \infty$

$$N_{g,[a,b]} \xrightarrow[g \to \infty]{d} \mathrm{Poisson}\left(\int_a^b \frac{\cosh(x) - 1}{x} \, dx \right)$$

What do we care? Fix $0 \leq a < b$

$$N_{[a,b]}(G) := \text{number of primitive cycles in } G \text{ with length in } [a,b]$$

$$\mathcal{M}^{comb}_{g,n}(L_1, \dots, L_n) \longrightarrow \mathbb{Z}_{\geq 0}$$

$L_1 + \dots + L_n \dots g \to \infty$

17:15 - 18:15

Combinatorial maps and hyperbolic surfaces in high genus

Baptiste Louf (Abstract)

$$N_{g,[a,b]} \xrightarrow[g \to \infty]{} \text{Poisson}\left(\int_a^b \frac{\cosh(x) - 1}{x} \, dx \right)$$

What do we care?

Fix $0 \leq a$

$N_{[a,b]}(G) :=$ number of primitiv

with length in [

$\mathcal{M}^{\text{comb}}_{g,n}(L_1, \dots, L_n) \longrightarrow \mathbb{Z}_{\geq 0}$

$L_1 + \dots$ ∞

17:15 - 18:15

Combinatorial maps and hyperbolic surfaces in high genus

Baptiste Louf (Abstract)

$$N_{g,[a,b]} \xrightarrow[g \to \infty]{} \text{Poisson}\left(\int_a^b \frac{\cosh(x) - 1}{x} \, dx \right)$$

Why Teichmuller theory?

Why Teichmüller theory?

genus 1, 1 face of perimeter 1

Why Teichmüller theory?

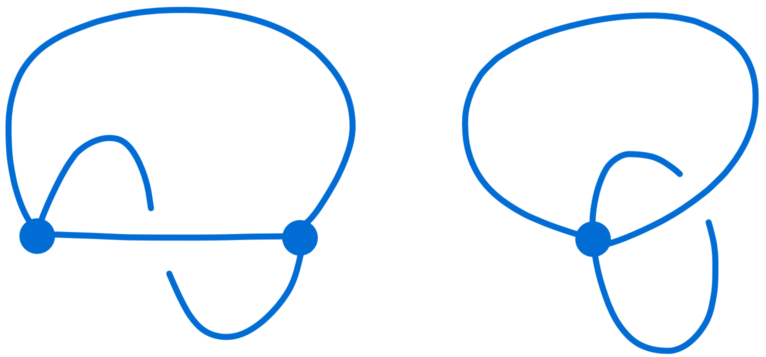
$\mathcal{M}^{\mathrm{comb}}_{1,1}(1)$

genus 1, 1 face of perimeter 1

Why Teichmüller theory?

$\mathcal{M}^{\mathrm{comb}}_{1,1}(1)$

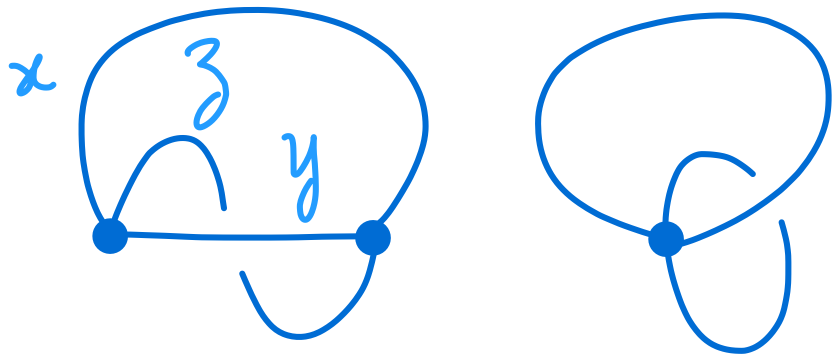
genus 1, 1 face of perimeter 1



Why Teichmüller theory?

$\mathcal{M}^{\mathrm{comb}}_{1,1}(1)$

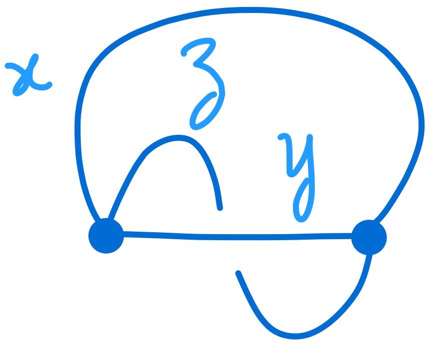
genus 1, 1 face of perimeter 1

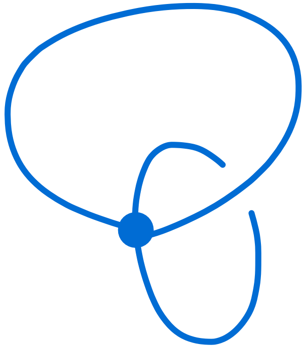


Why Teichmüller theory?

$$\mathcal{M}^{\mathrm{comb}}_{1,1}(1)$$

genus 1, 1 face of perimeter 1



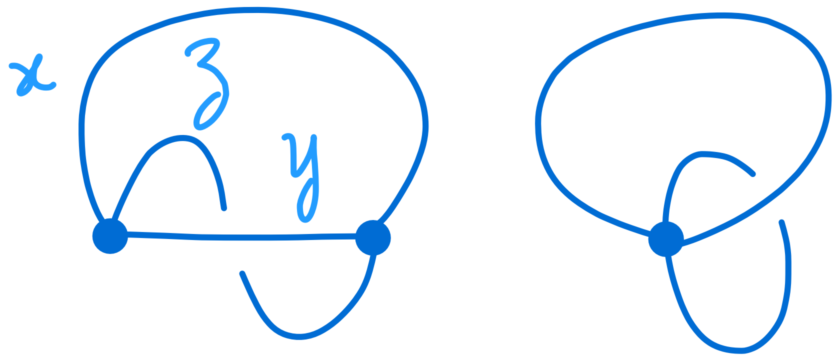


$$x+y+z=1$$

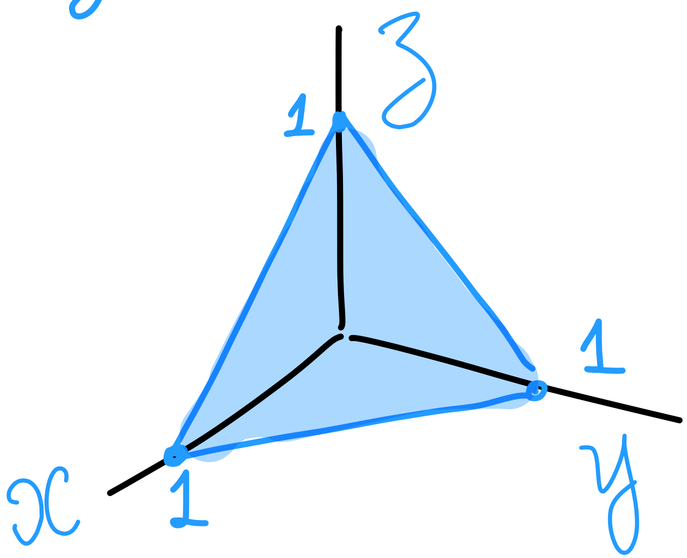
Why Teichmüller theory?

$\mathcal{M}_{1,1}^{comb}(1)$

genus 1, 1 face of perimeter 1



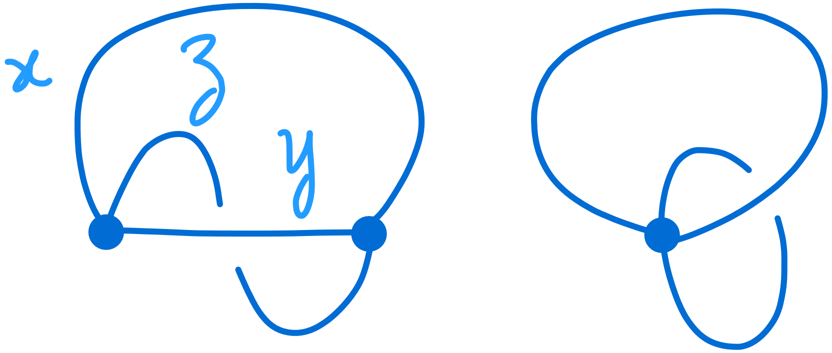
$x+y+z=1$



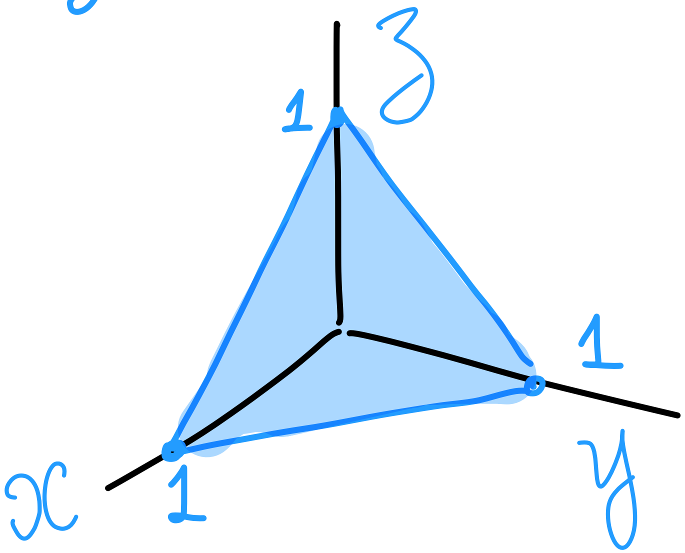
Why Teichmüller theory?

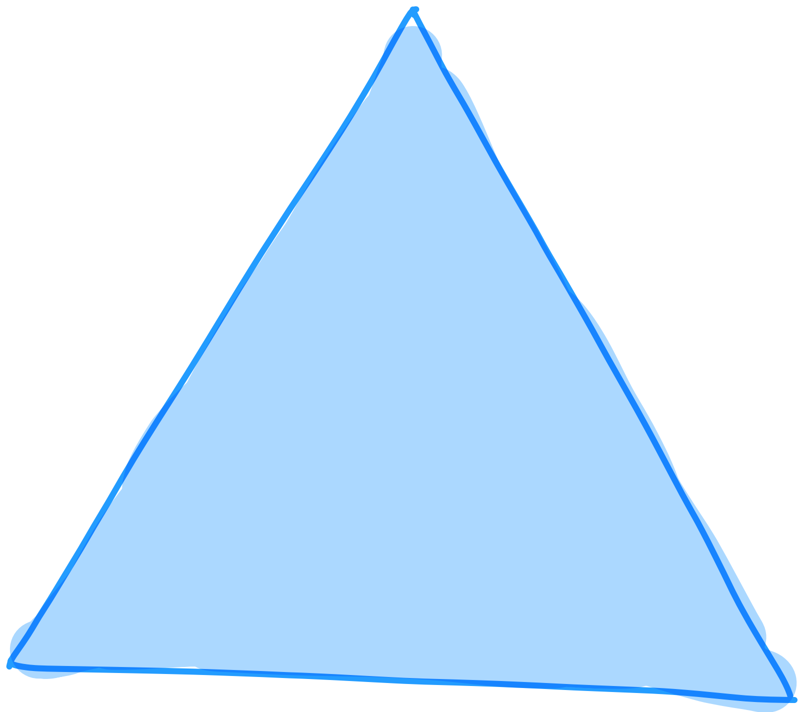
$\mathcal{M}^{comb}_{1,1}(1)$

genus 1, 1 face of perimeter 1



$x+y+z=1$

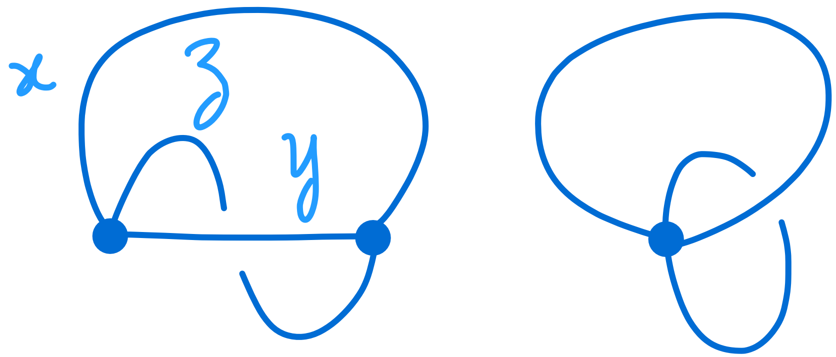




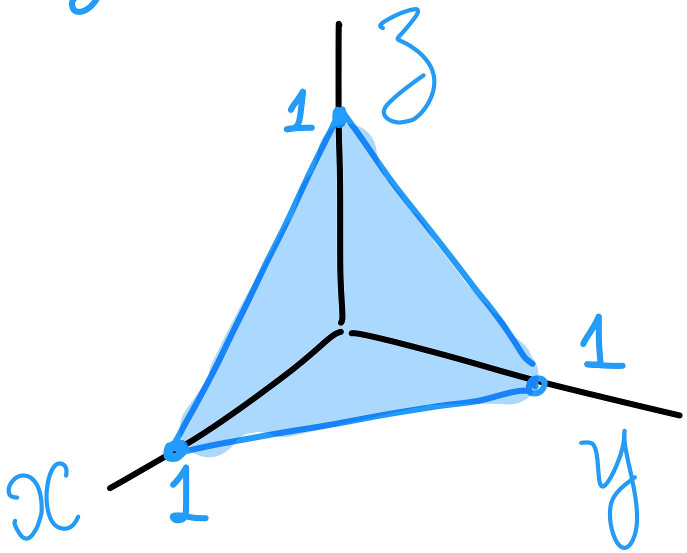
Why Teichmüller theory?

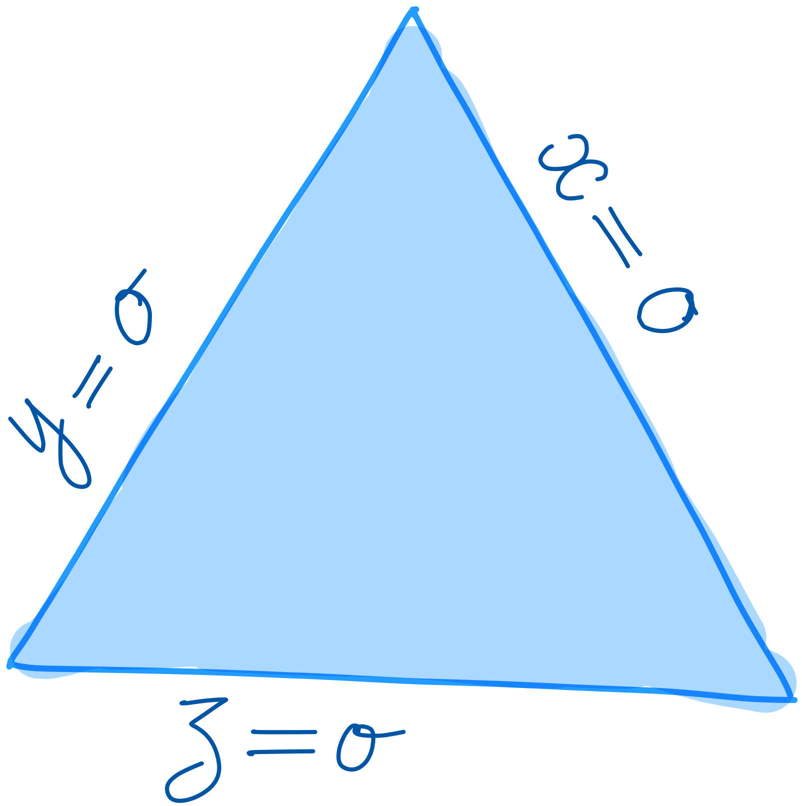
$\mathcal{M}^{comb}_{1,1}(1)$

genus 1, 1 face of perimeter 1



$x + y + z = 1$

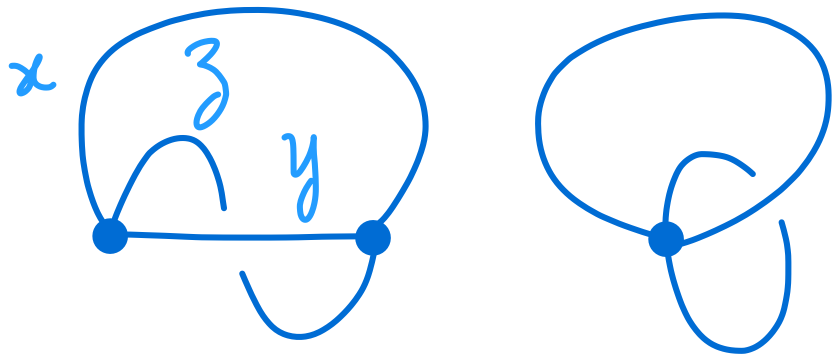




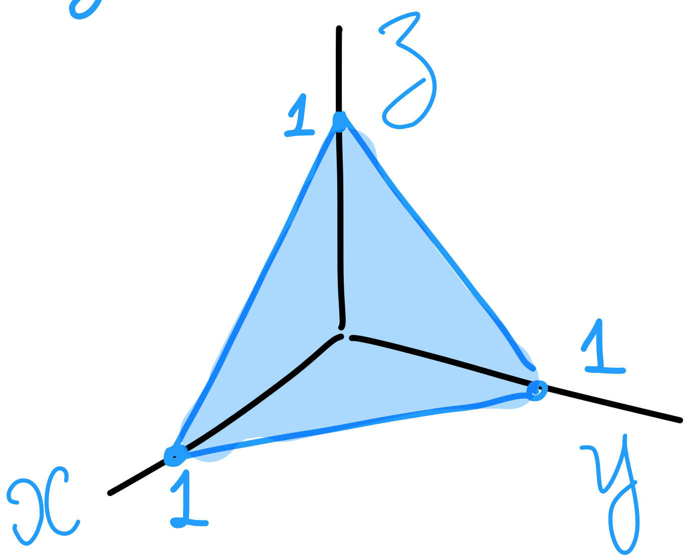
Why Teichmüller theory?

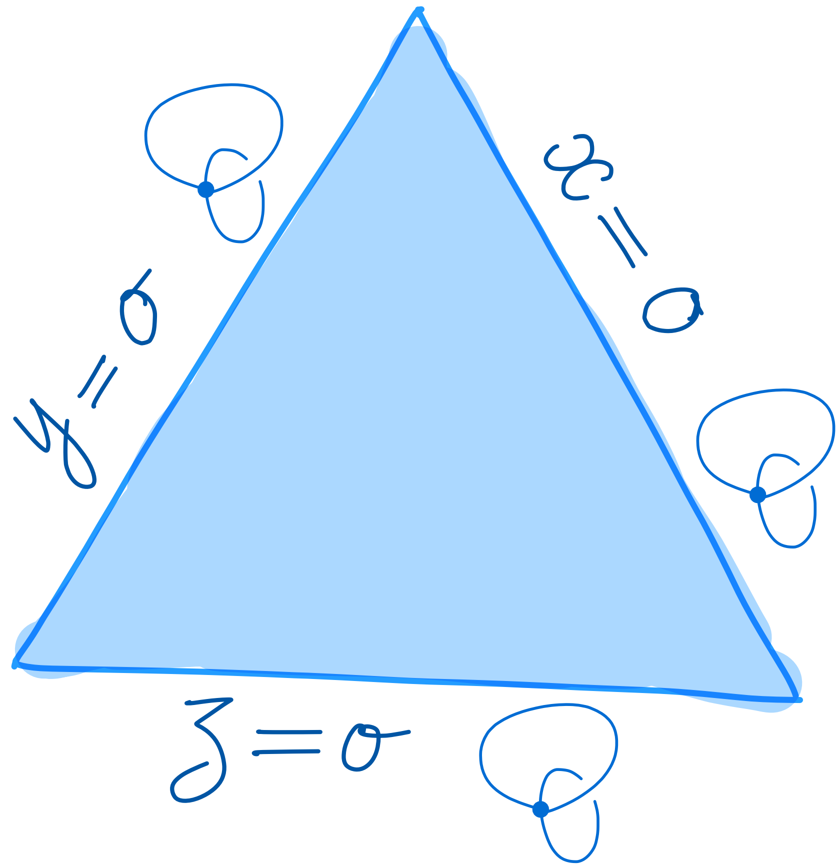
$\mathcal{M}^{comb}_{1,1}(1)$

genus 1, 1 face of perimeter 1



$x+y+z=1$

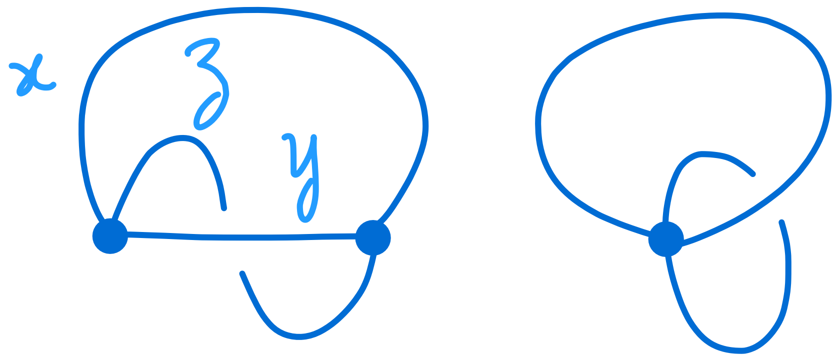




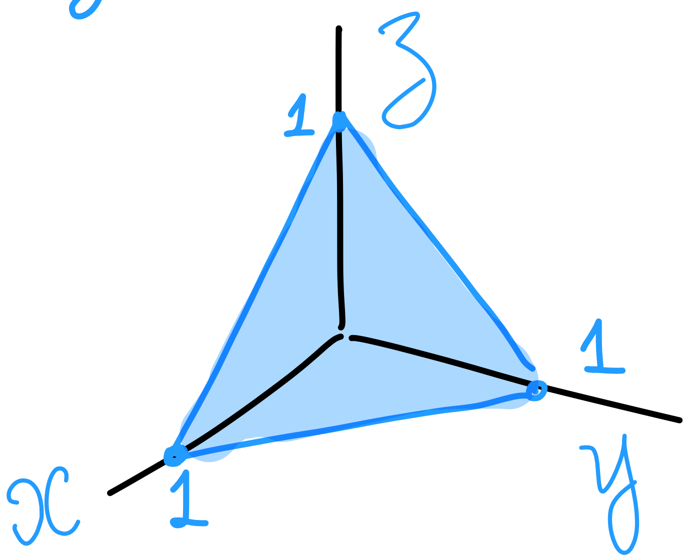
Why Teichmüller theory?

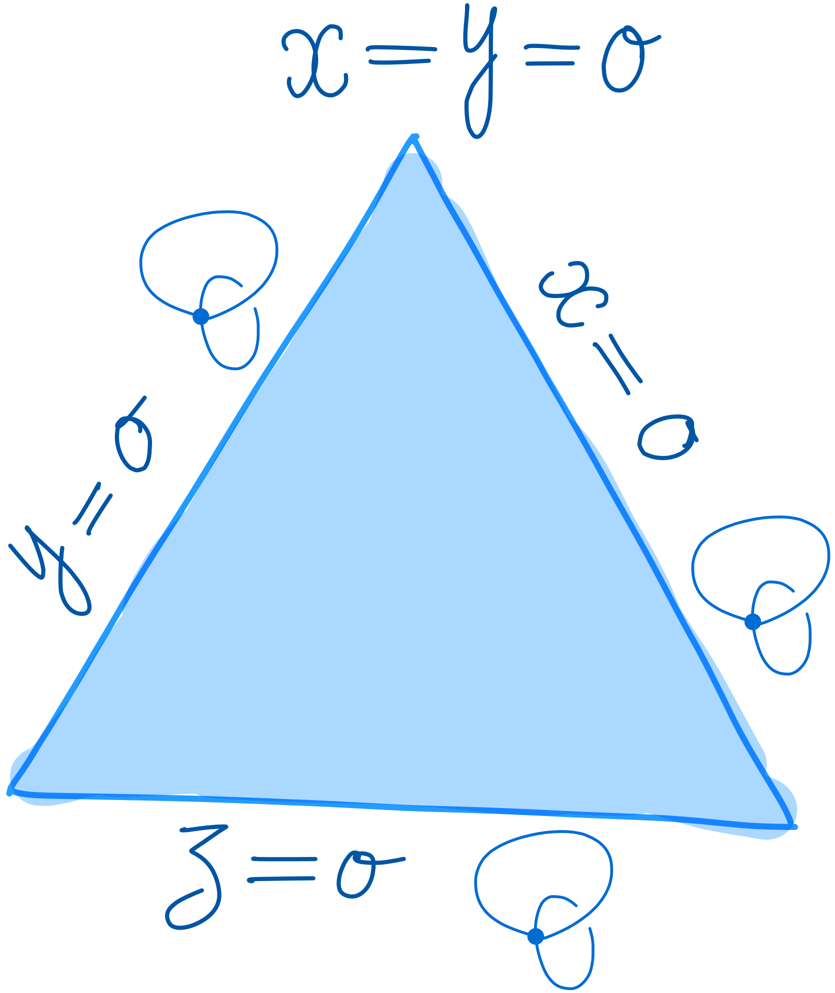
$\mathcal{M}^{comb}_{1,1}(1)$

genus 1, 1 face of perimeter 1



$x+y+z=1$

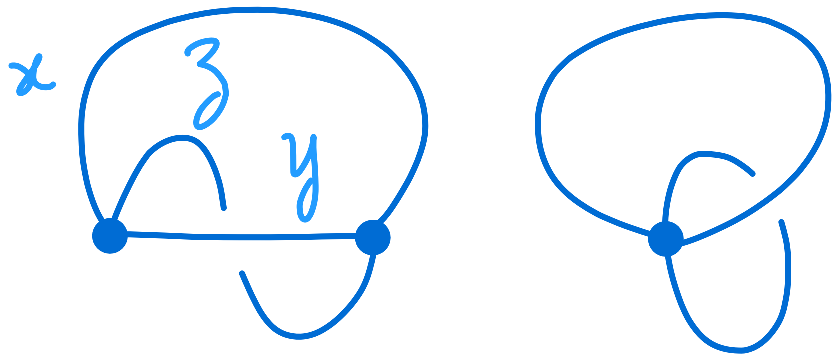




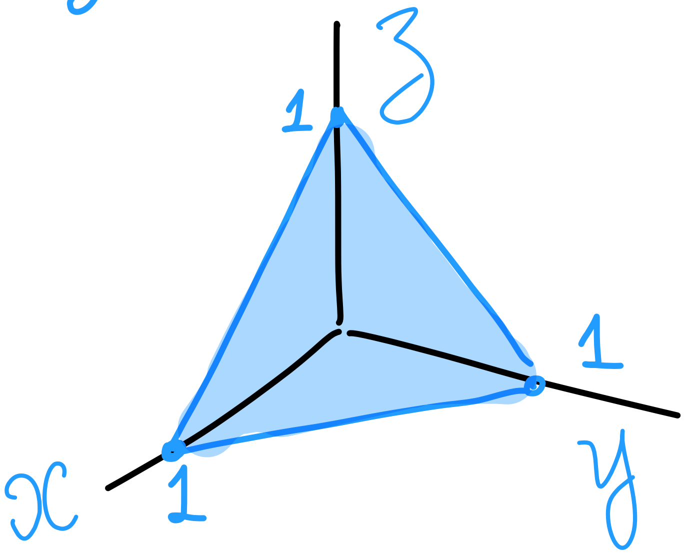
Why Teichmüller theory?

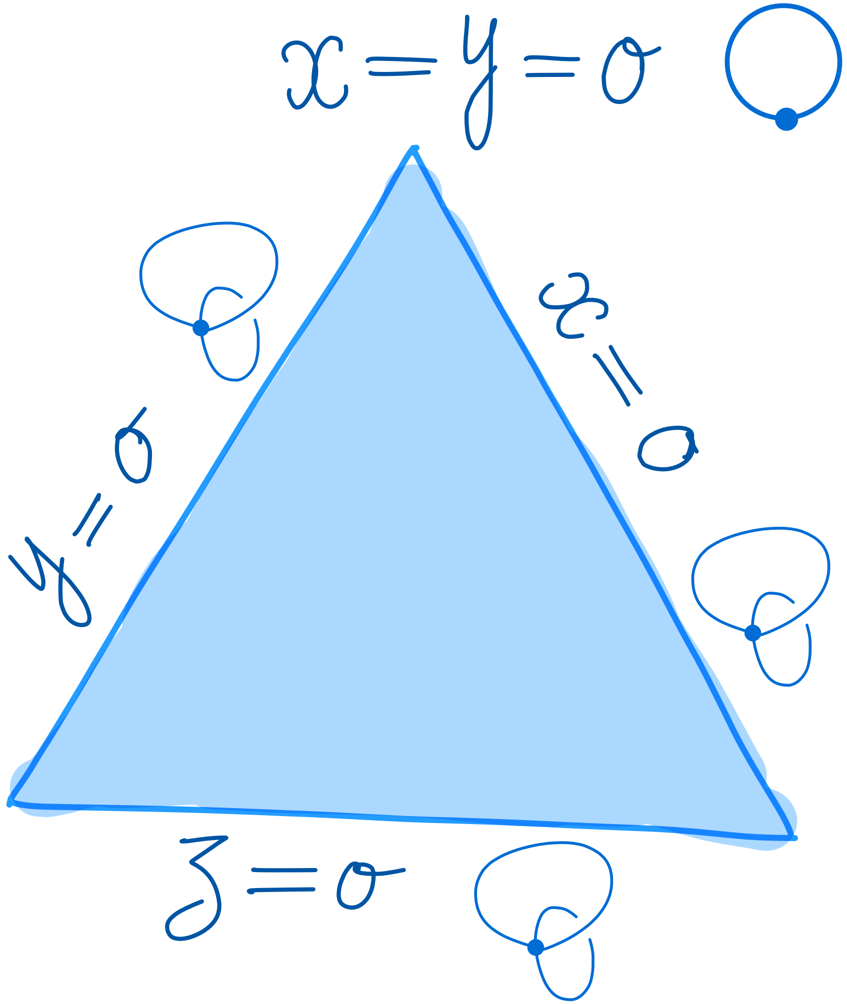
$\mathcal{M}^{comb}_{1,1}(1)$

genus 1, 1 face of perimeter 1



$x+y+z=1$

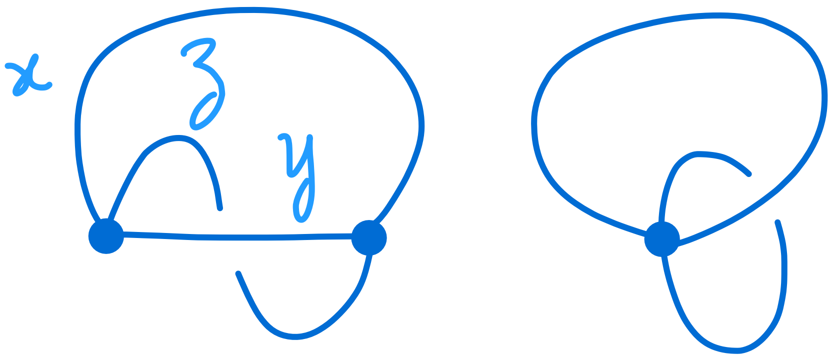




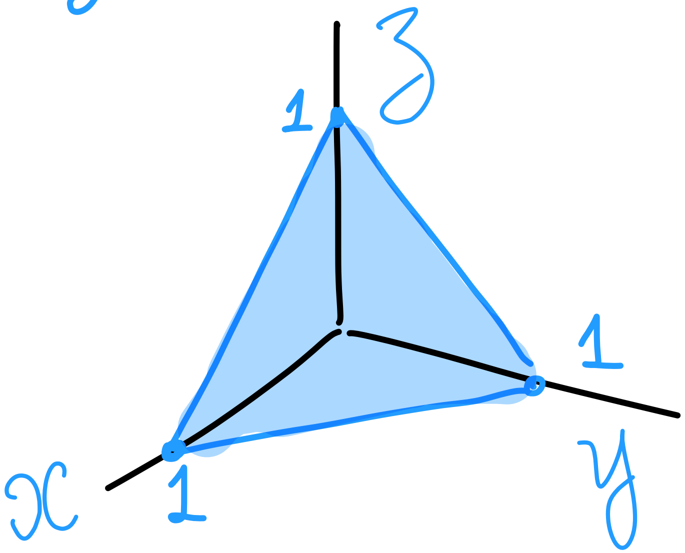
Why Teichmüller theory?

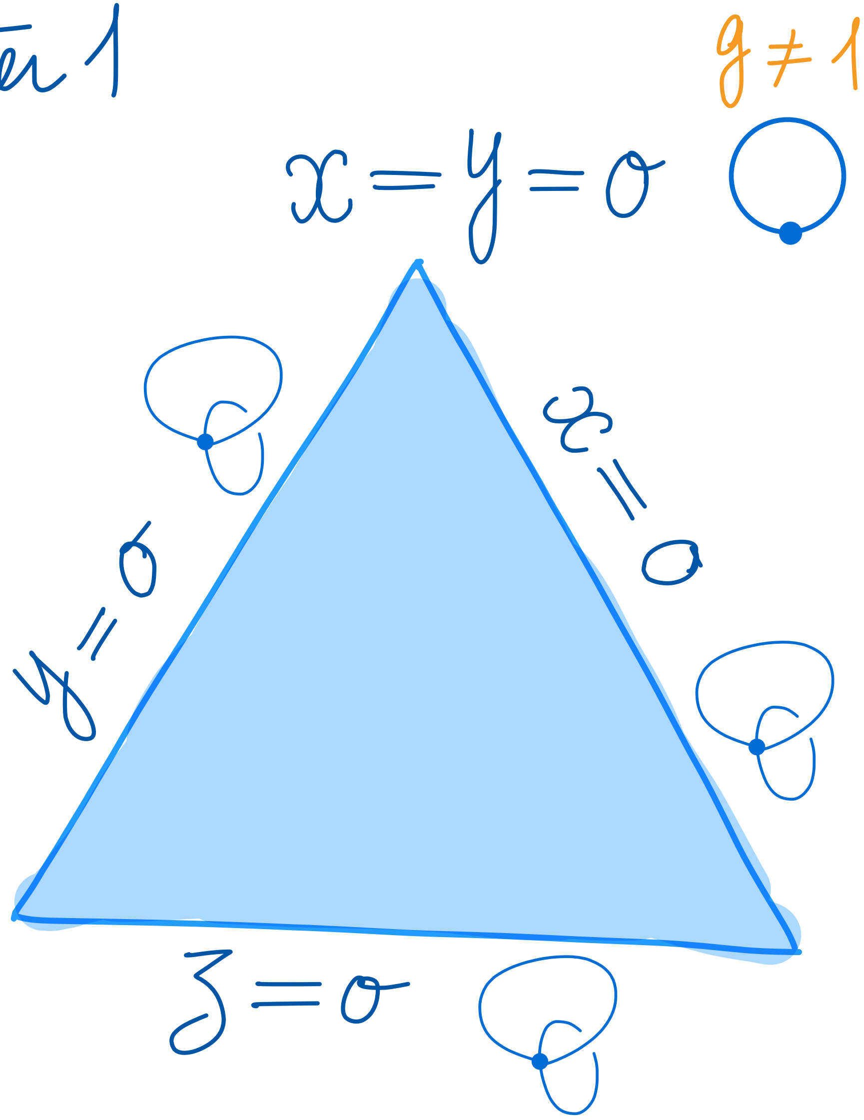
$$\mathcal{M}^{comb}_{1,1}(1)$$

genus 1, 1 face of perimeter 1



$$x+y+z=1$$



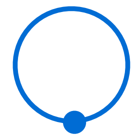


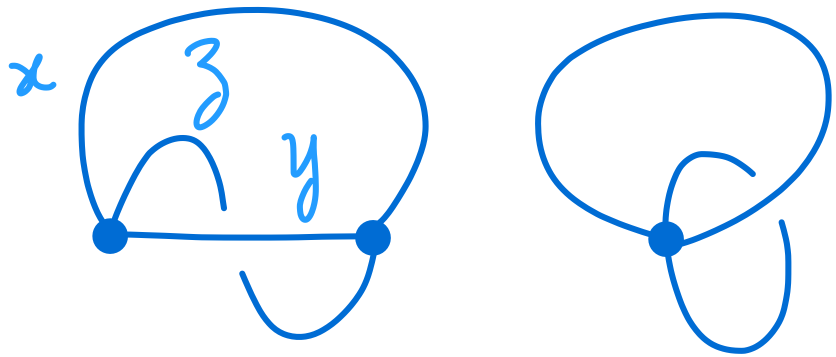
Why Teichmüller theory?

$\mathcal{M}^{comb}_{1,1}(1)$

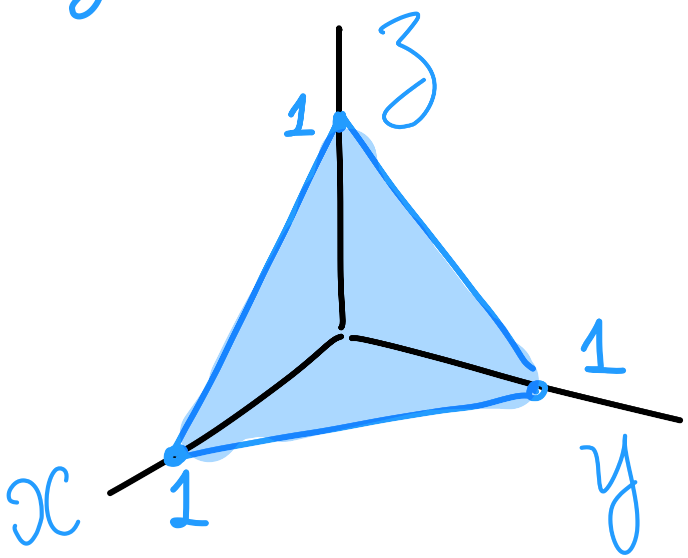
genus 1, 1 face of perimeter 1

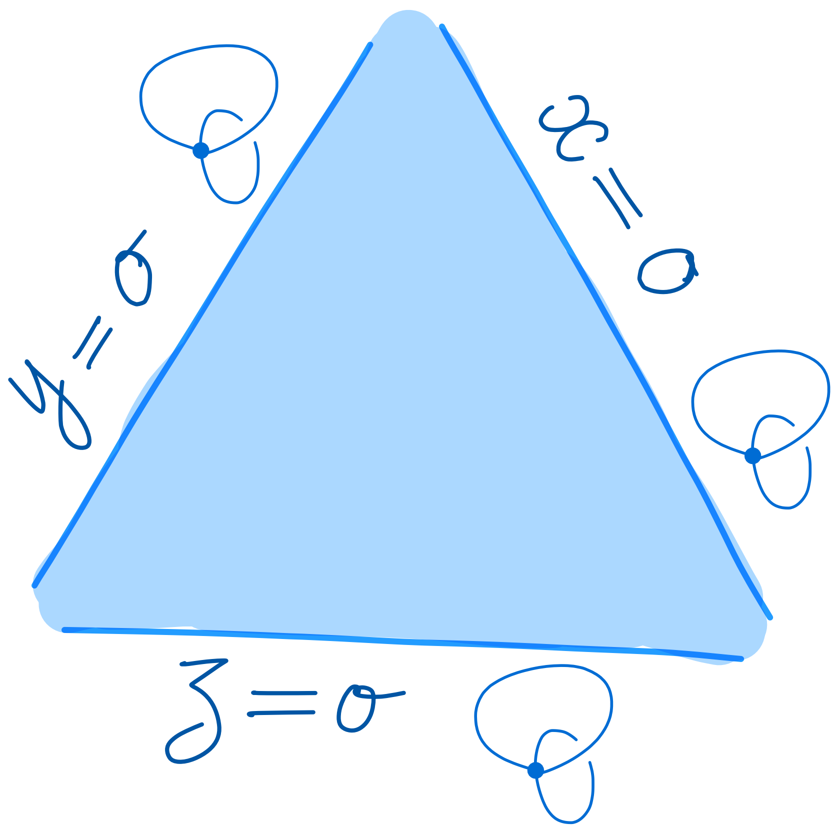
$g \neq 1$

$x = y = 0$ 



$x + y + z = 1$

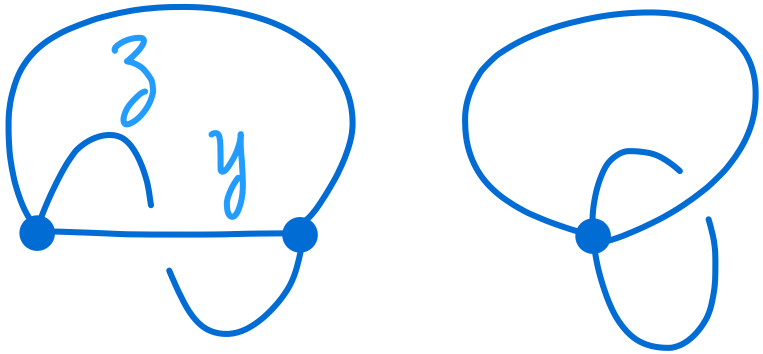




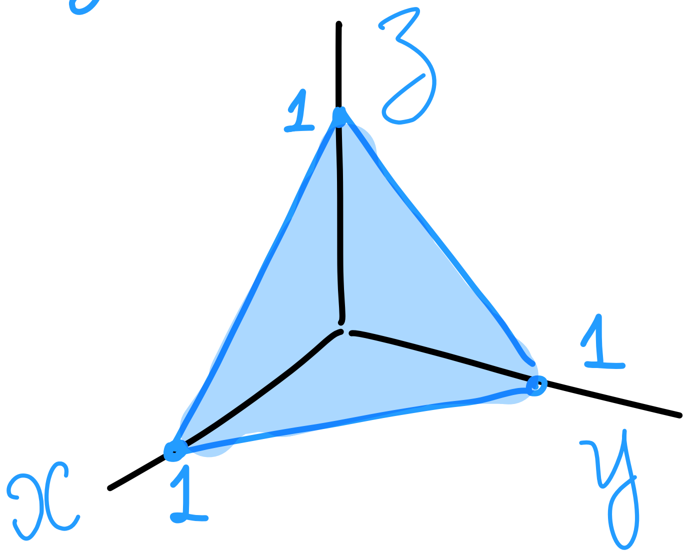
Why Teichmüller theory?

$\mathcal{M}^{comb}_{1,1}(1)$

genus 1, 1 face of perimeter 1

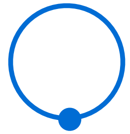


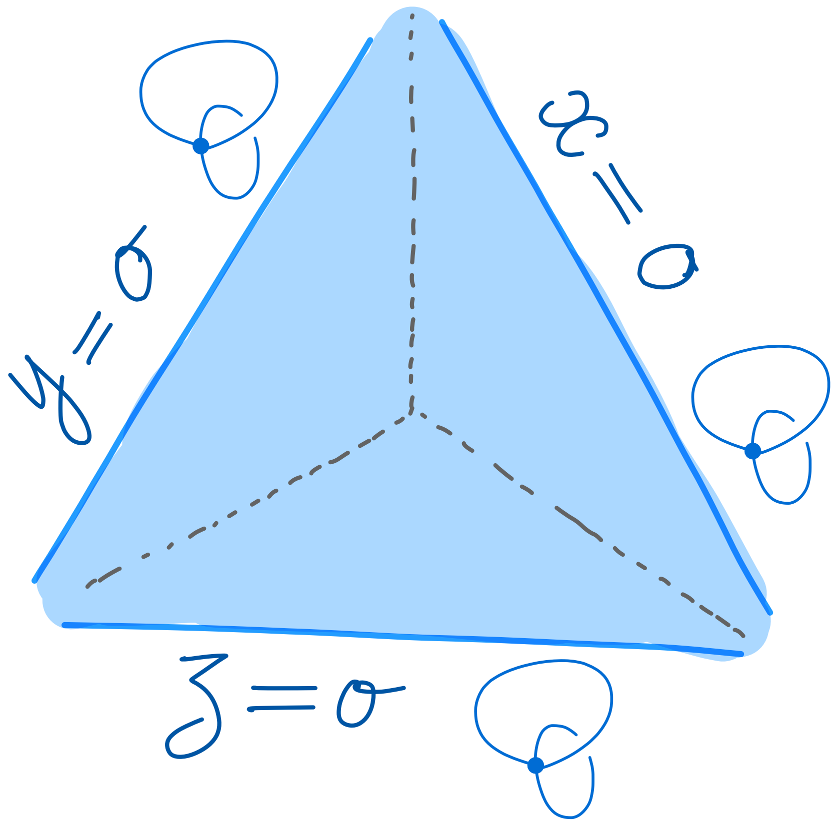
$x+y+z=1$



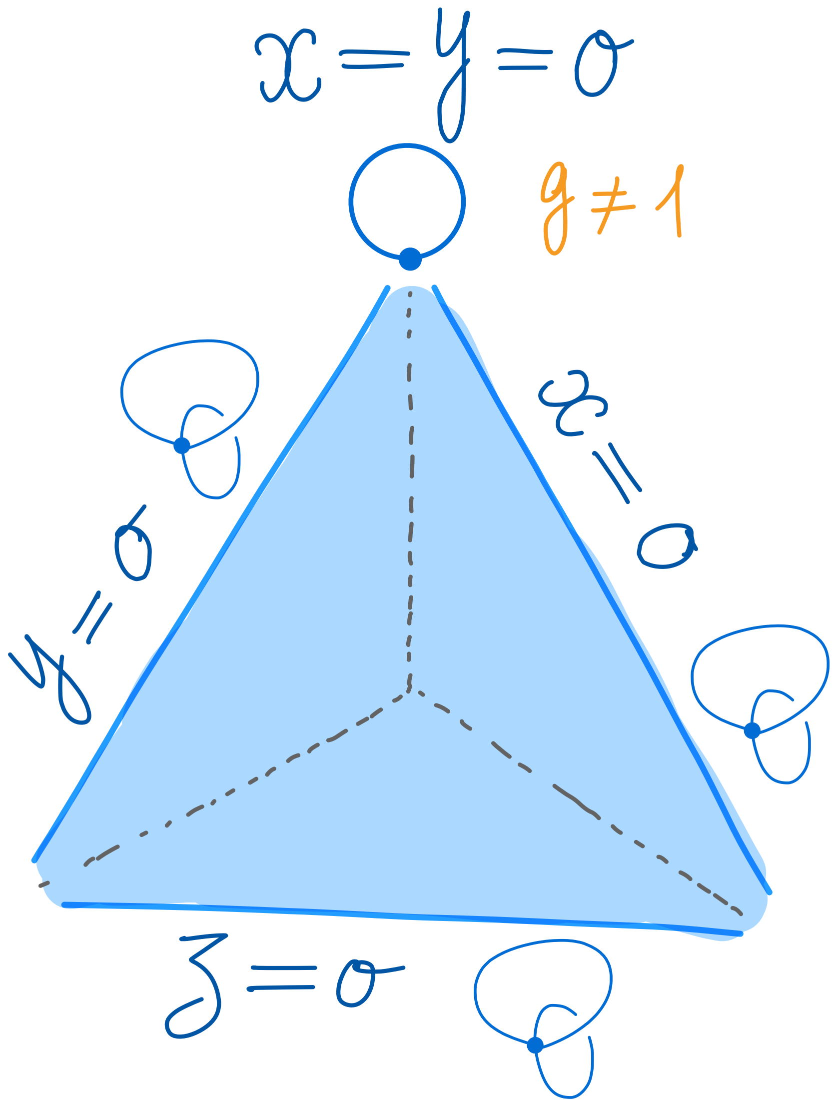
$g \neq 1$

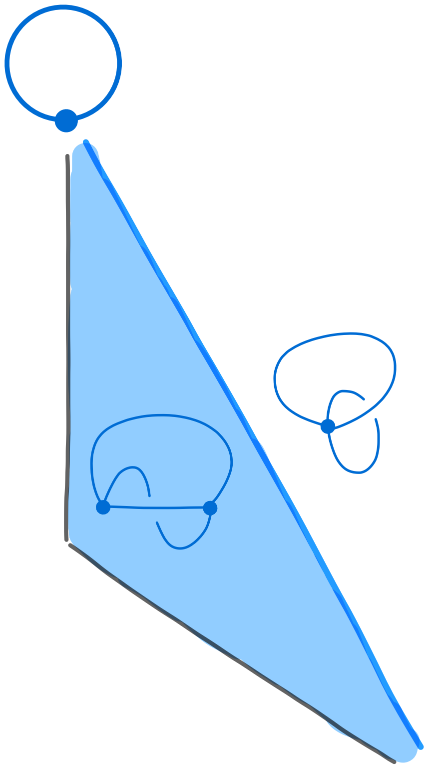
$x=y=0$



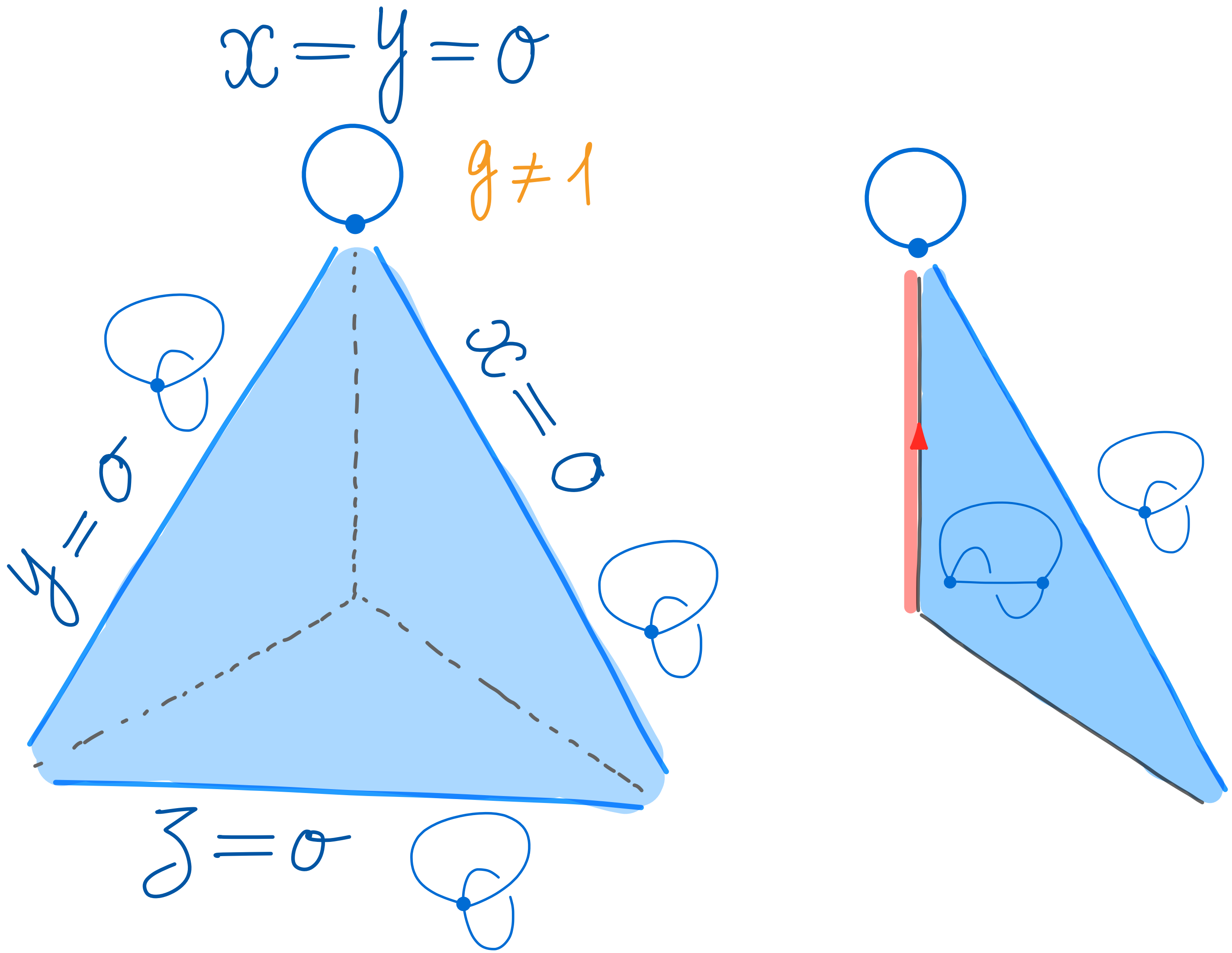


Why Teichmuller theory? $\mathcal{M}_{1,1}^{comb}(1)$



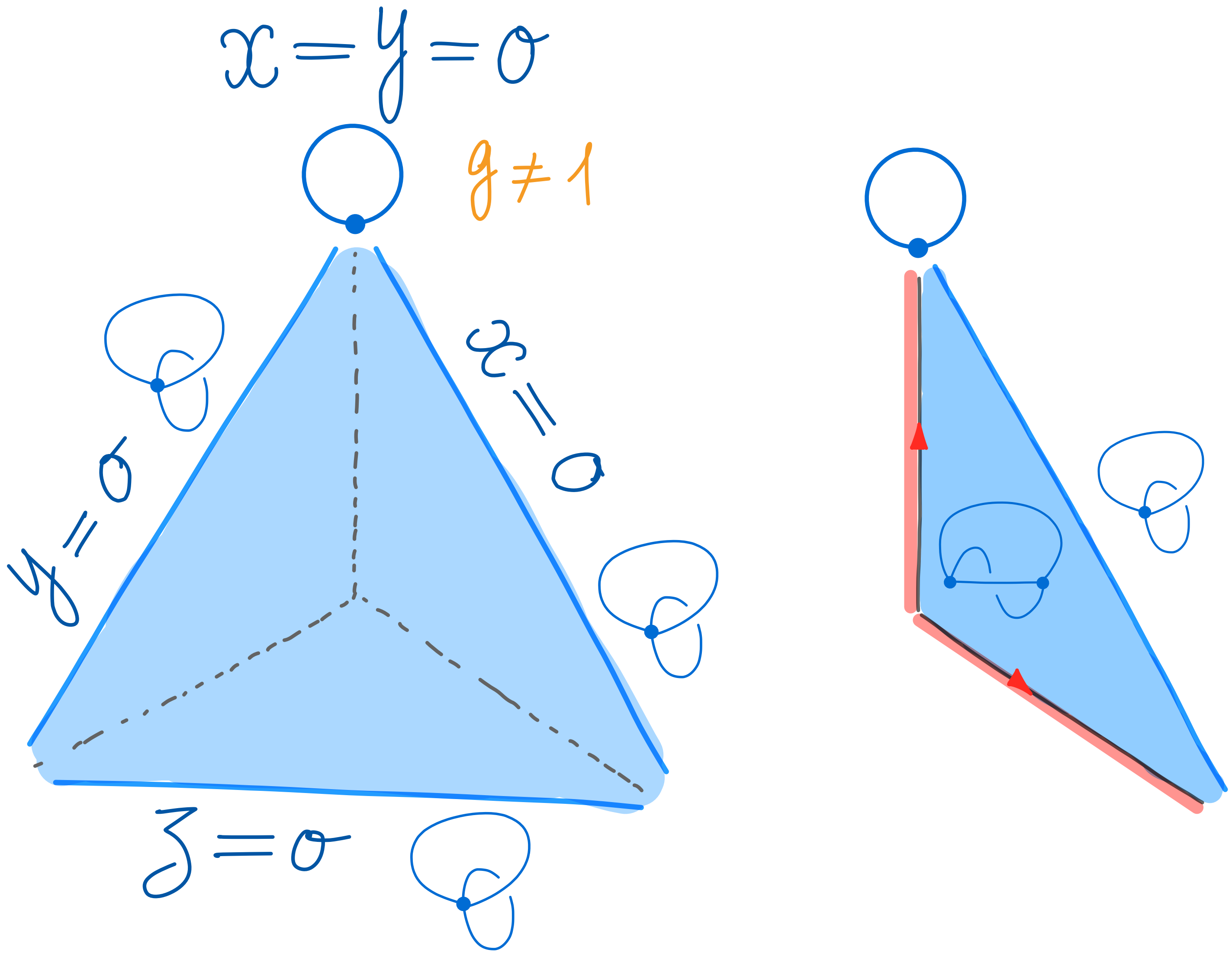


Why Teichmuller theory? $\mathcal{M}_{1,1}^{comb}(1)$

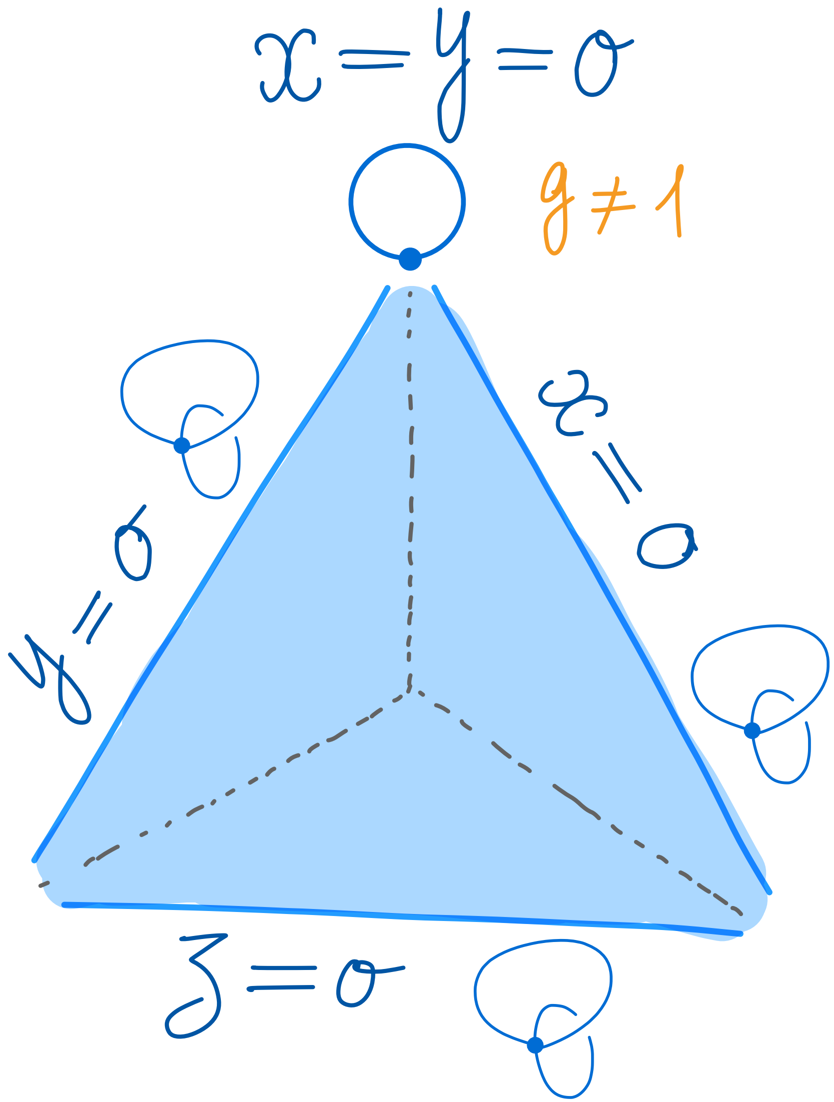


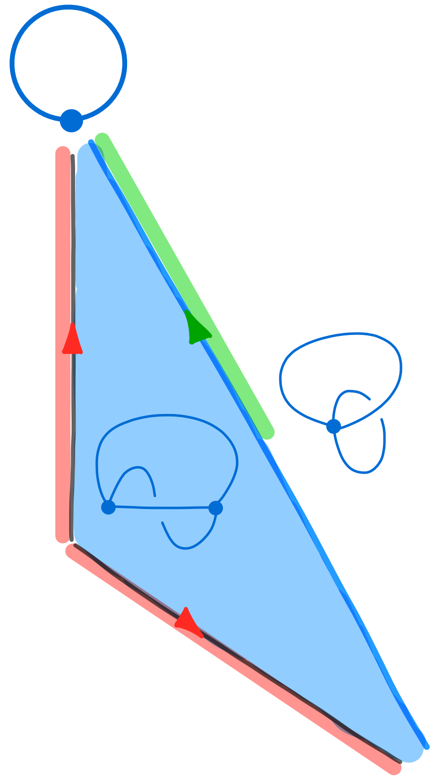
Why Teichmuller theory?

$\mathcal{M}_{1,1}^{comb}(1)$



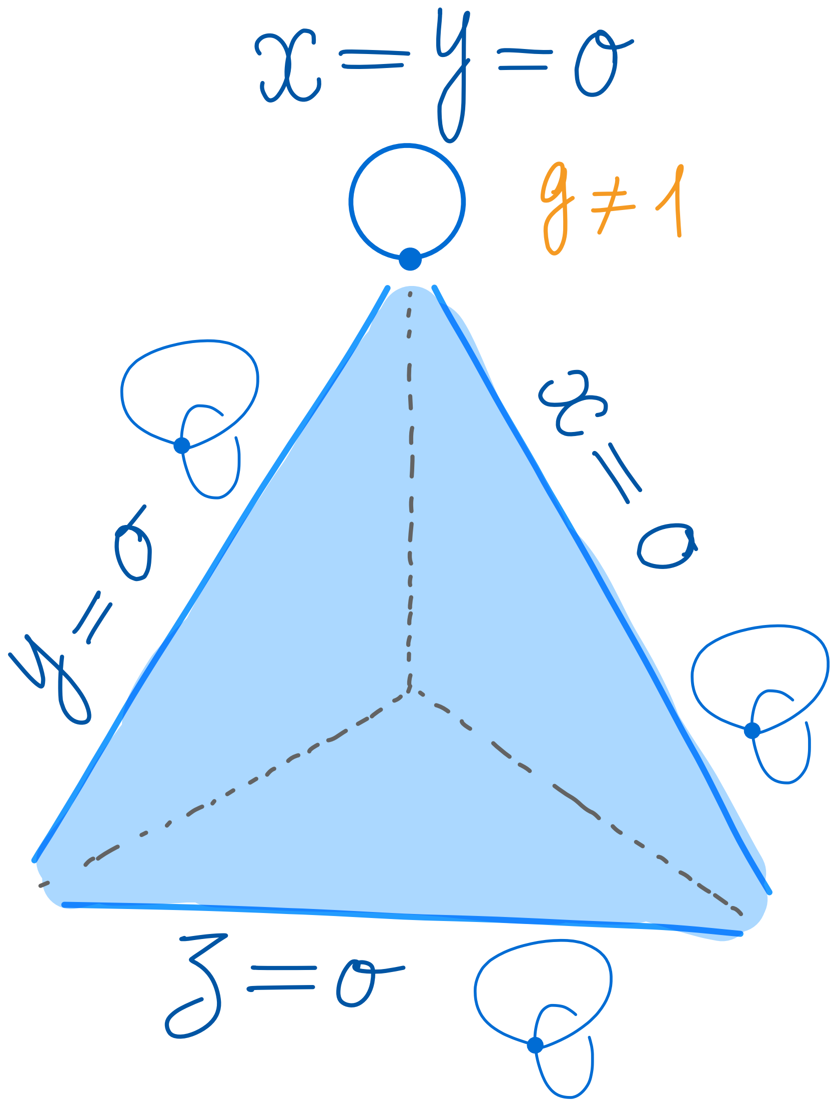
Why Teichmuller theory? $\mathcal{M}^{comb}_{1,1}(1)$

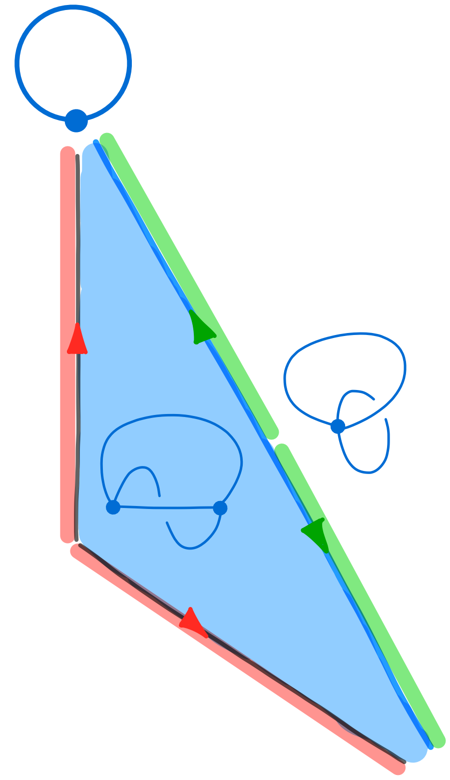




Why Teichmuller theory?

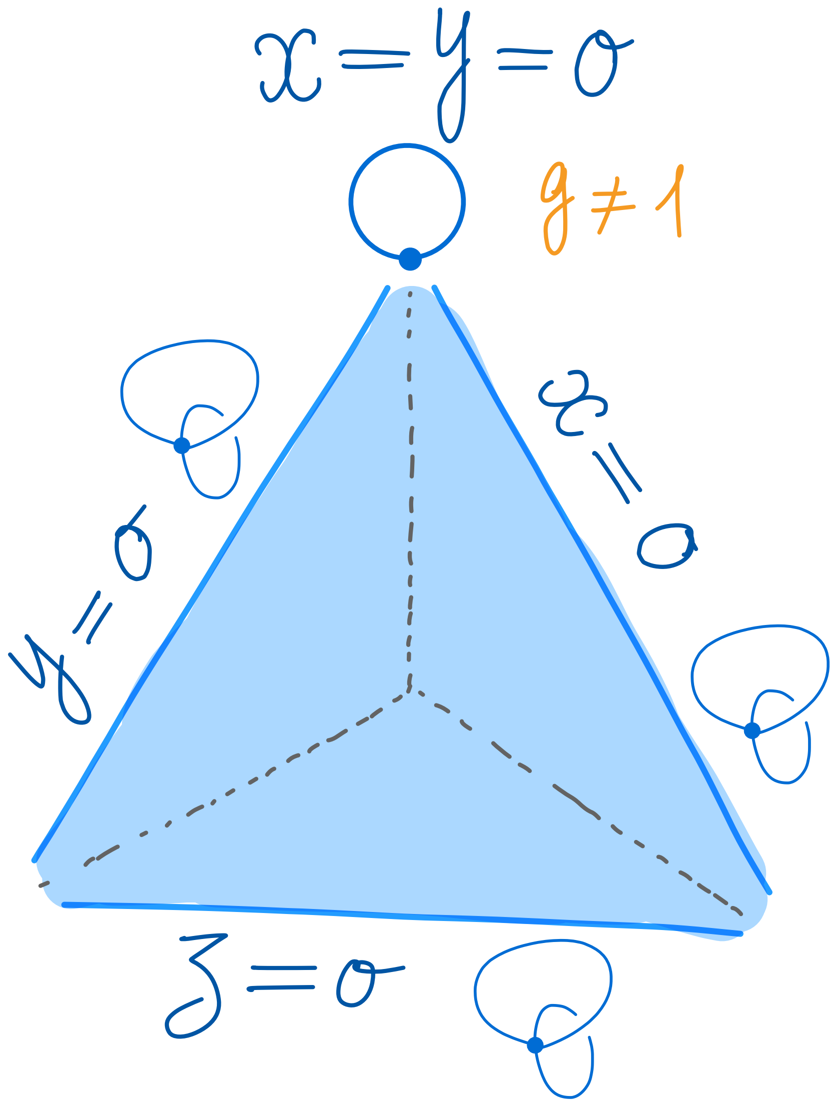
$\mathcal{M}_{1,1}^{comb}(1)$

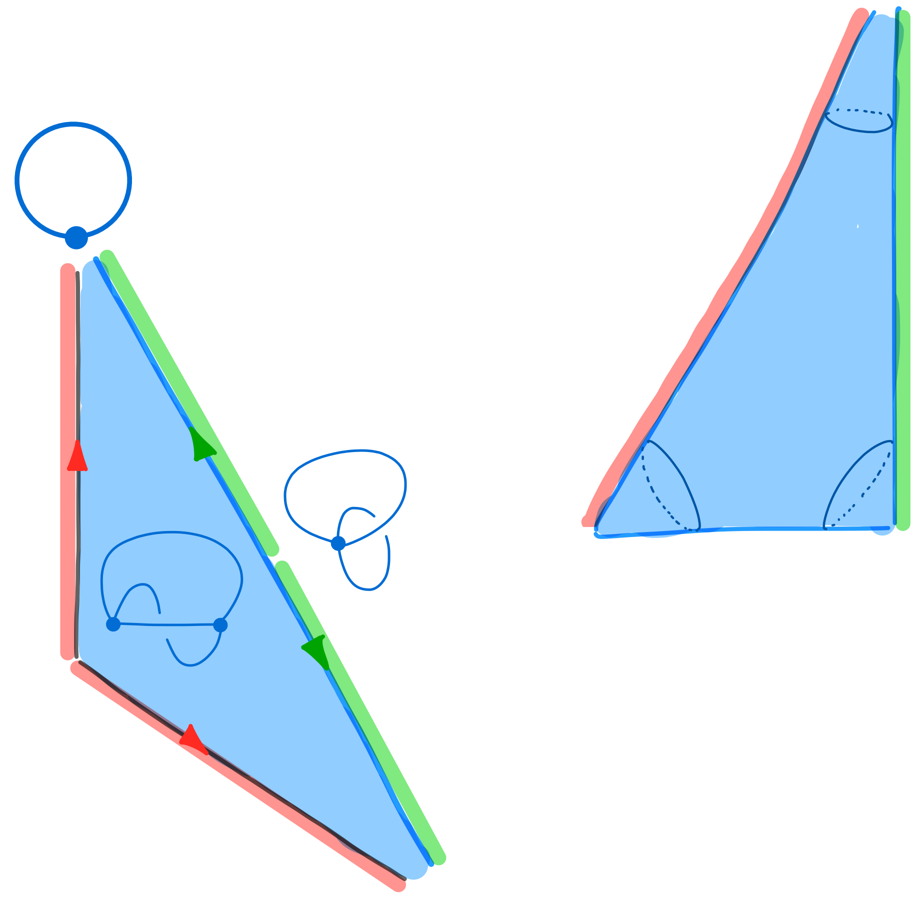




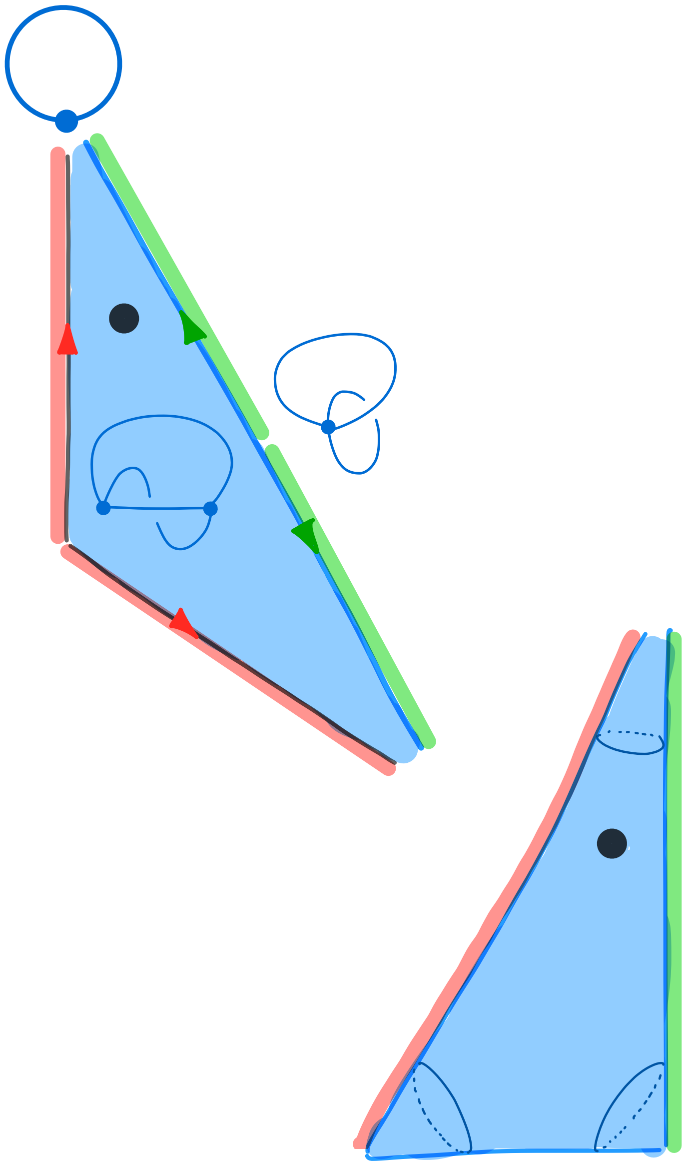
Why Teichmuller theory?

$\mathcal{M}_{1,1}^{comb}(1)$



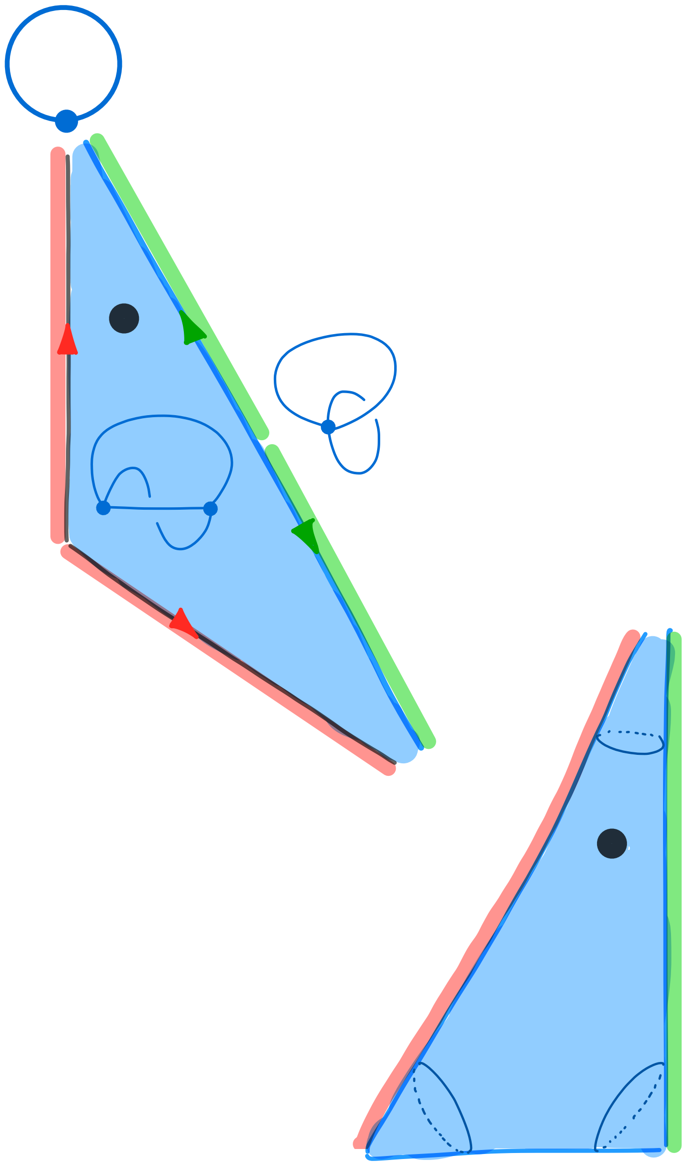


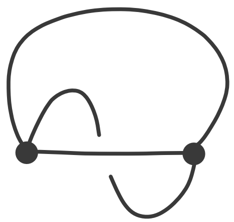
Why Teichmuller theory? $\mathcal{M}^{comb}_{1,1}(1)$



Why Teichmuller theory?

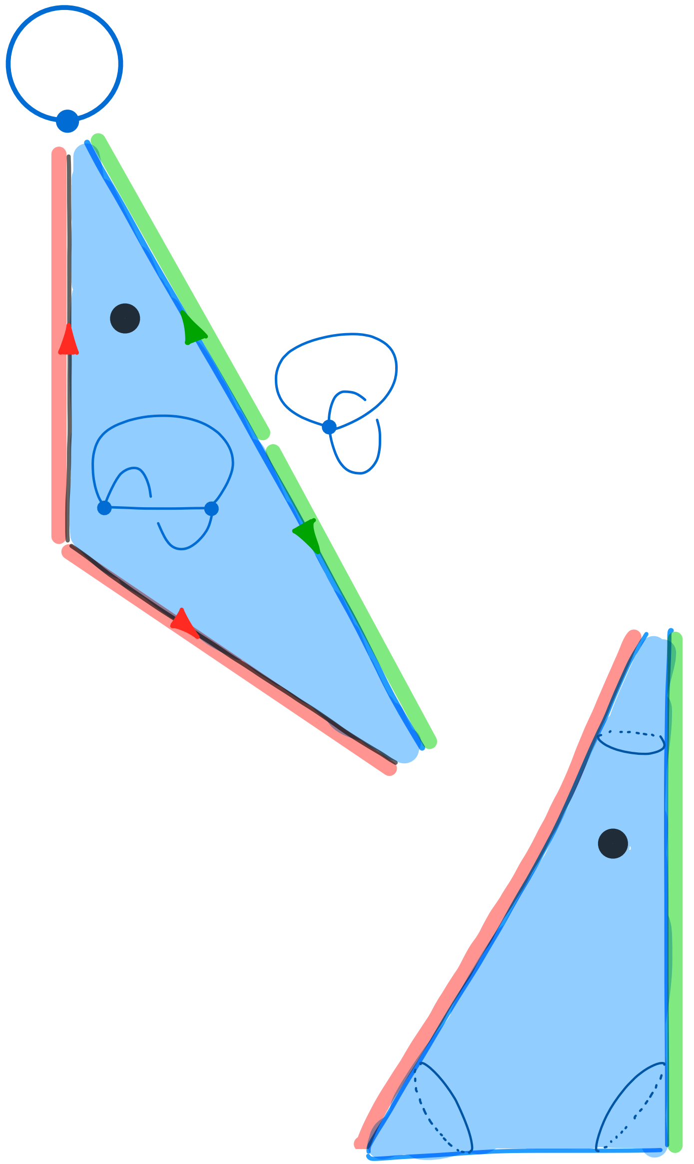
$\mathcal{M}^{comb}_{1,1}(1)$

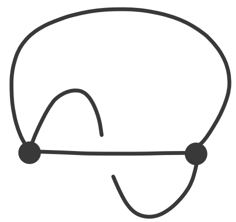




Why Teichmuller theory?

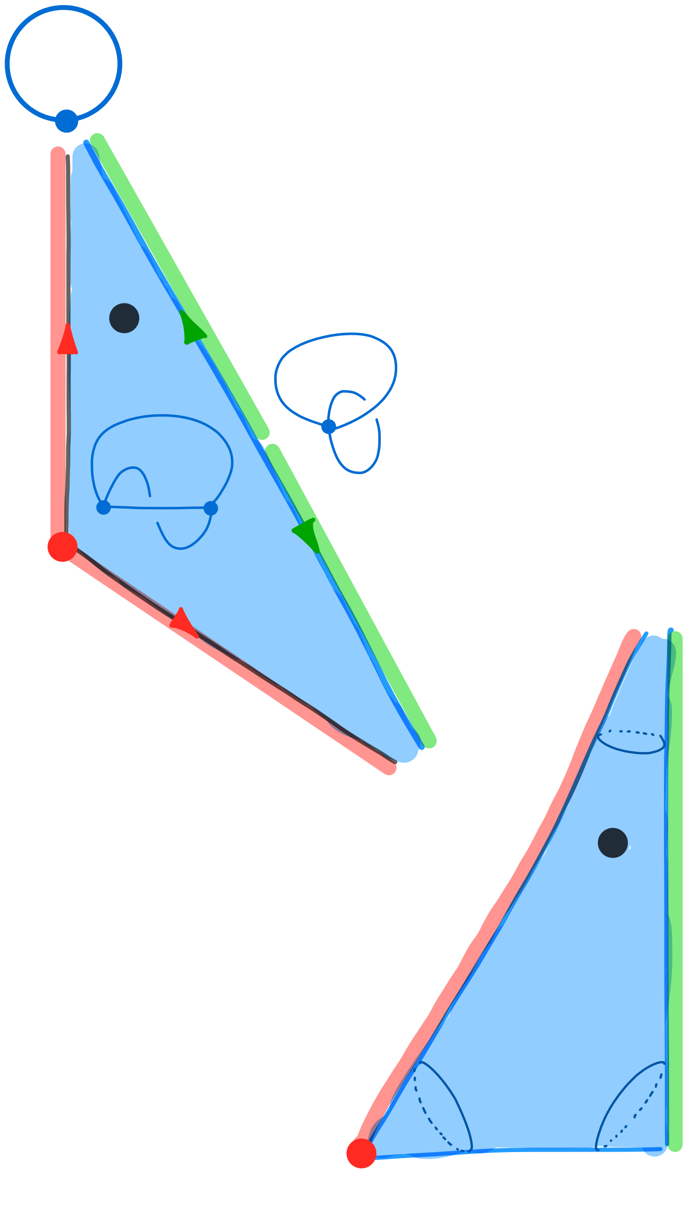
$\mathcal{M}_{1,1}^{comb}(1)$

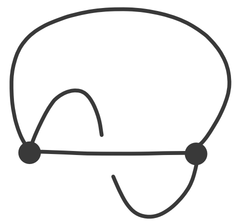


$\mathrm{Aut}\Big($  $\Big) \simeq \mathbb{Z}/2\mathbb{Z}$

Why Teichmuller theory?

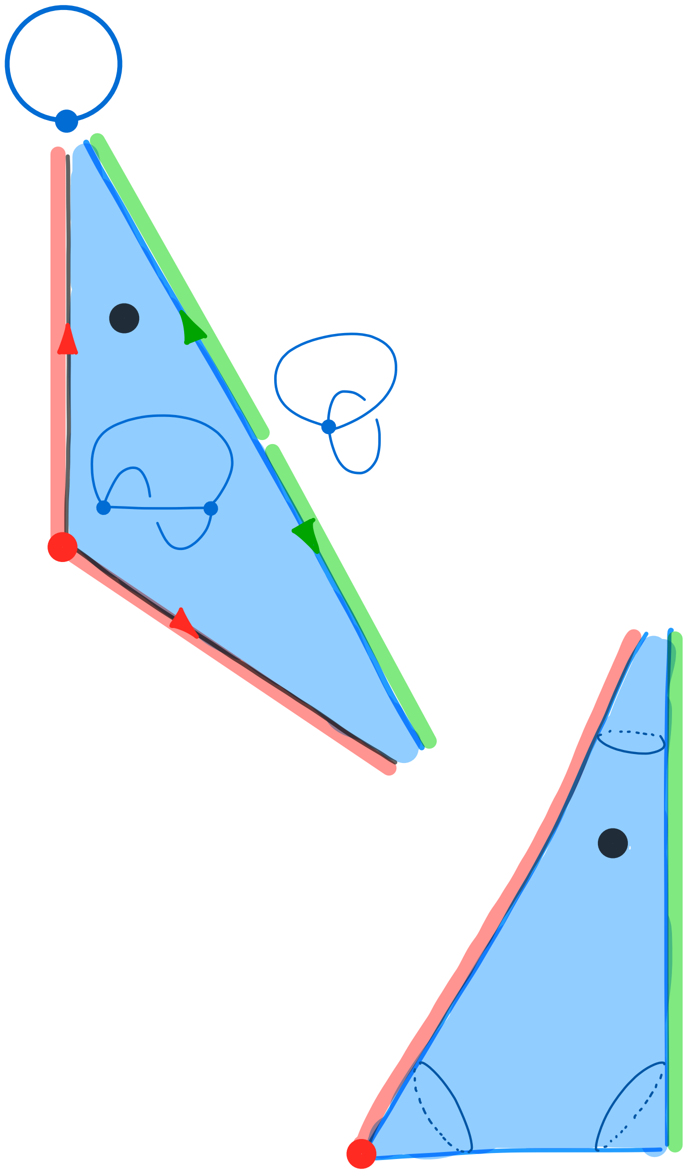
$\mathcal{M}_{1,1}^{comb}(1)$

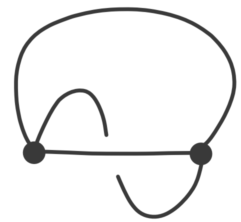


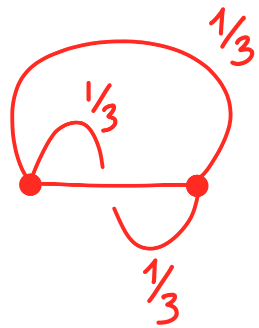
$\mathrm{Aut}\Big($  $\Big) \simeq \mathbb{Z}/2\mathbb{Z}$

Why Teichmuller theory?

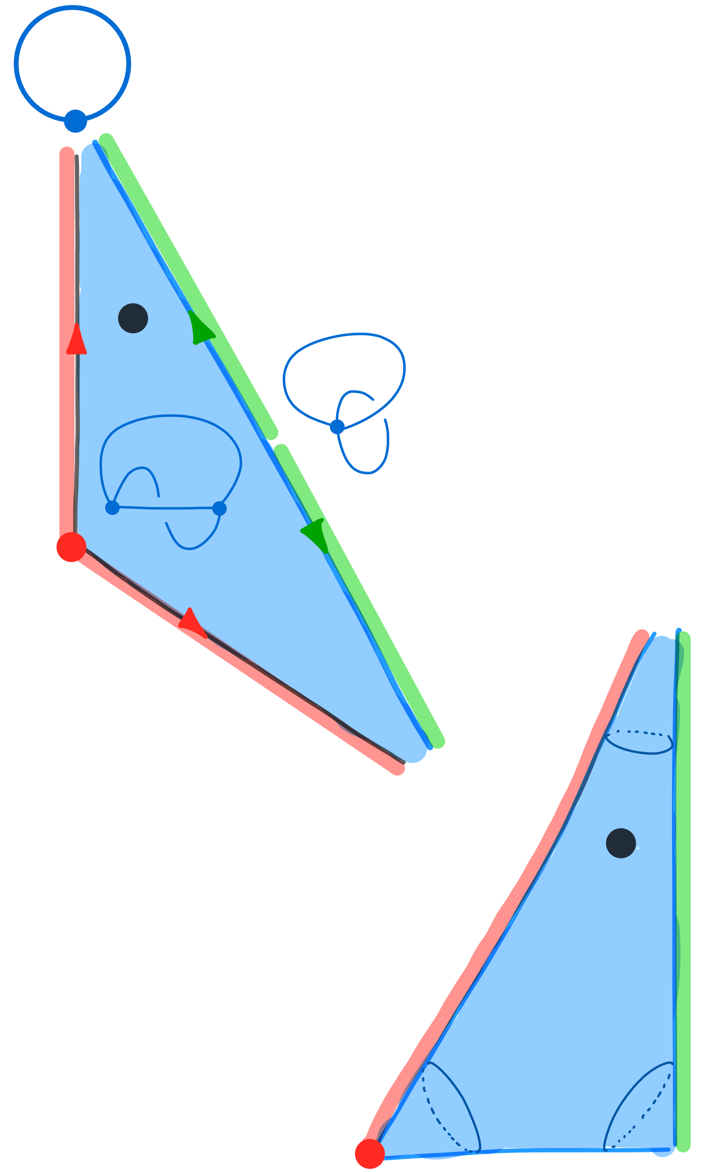
$\mathcal{M}^{comb}_{1,1}(1)$

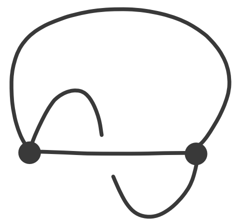


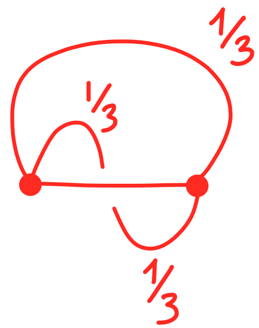
$\mathrm{Aut}\Big($  $\Big) \simeq \mathbb{Z}/2\mathbb{Z}$



Why Teichmuller theory? $\mathcal{M}^{\mathrm{comb}}_{1,1}(1)$

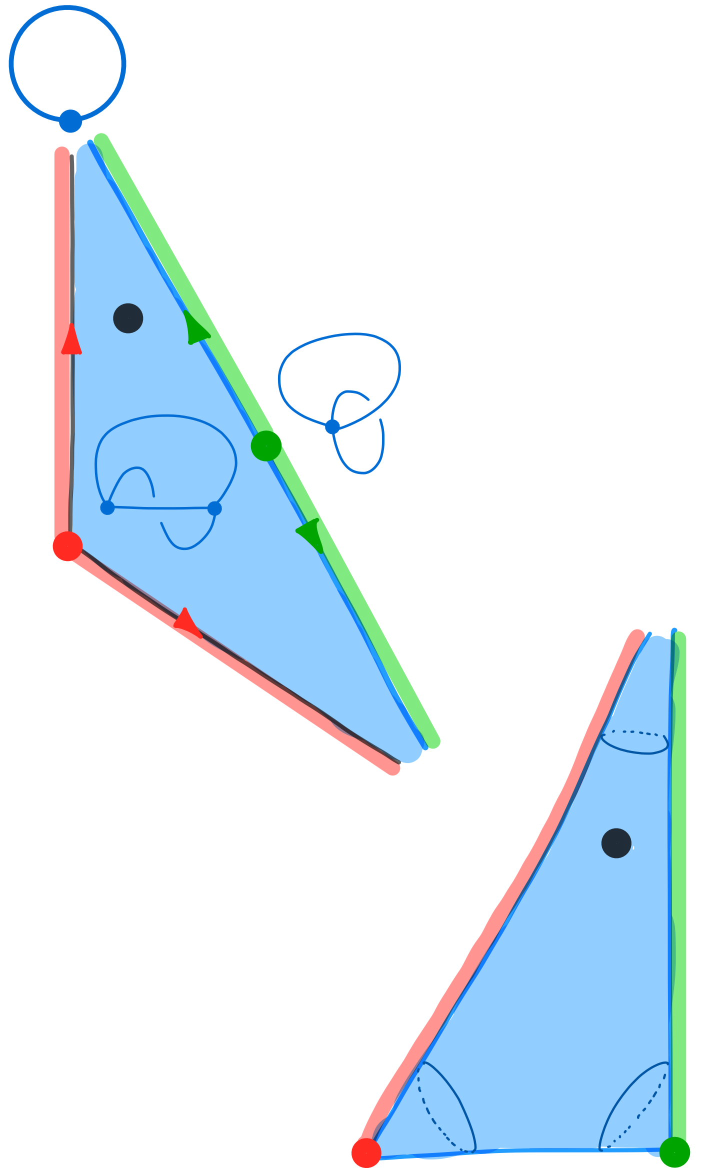


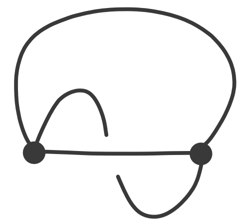
$\mathrm{Aut}\Big($  $\Big) \simeq \mathbb{Z}/2\mathbb{Z}$

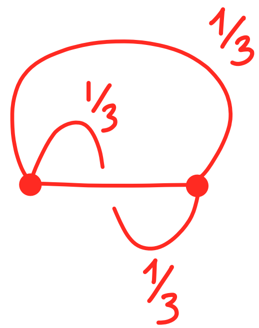
$\mathrm{Aut}\Big($ 

$\Big) \simeq \mathbb{Z}/6\mathbb{Z}$

Why Teichmuller theory? $\mathcal{M}^{comb}_{1,1}(1)$

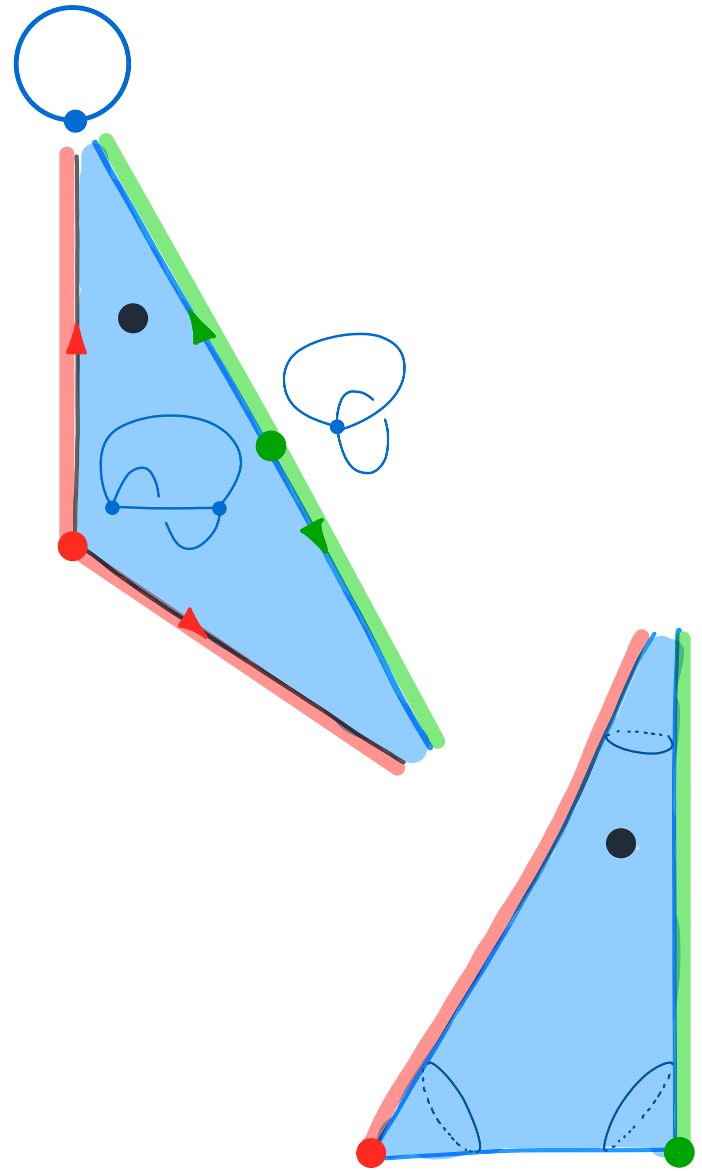


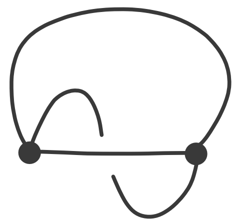
$\mathrm{Aut}\Big($  $\Big) \simeq \mathbb{Z}/2\mathbb{Z}$

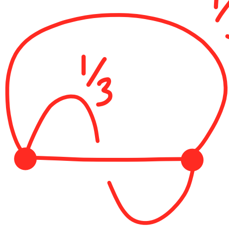
$\mathrm{Aut}\Big($  $\Big) \simeq \mathbb{Z}/6\mathbb{Z}$

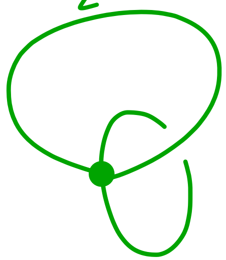
Why Teichmuller theory?

$\mathcal{M}^{\mathrm{comb}}_{1,1}(1)$



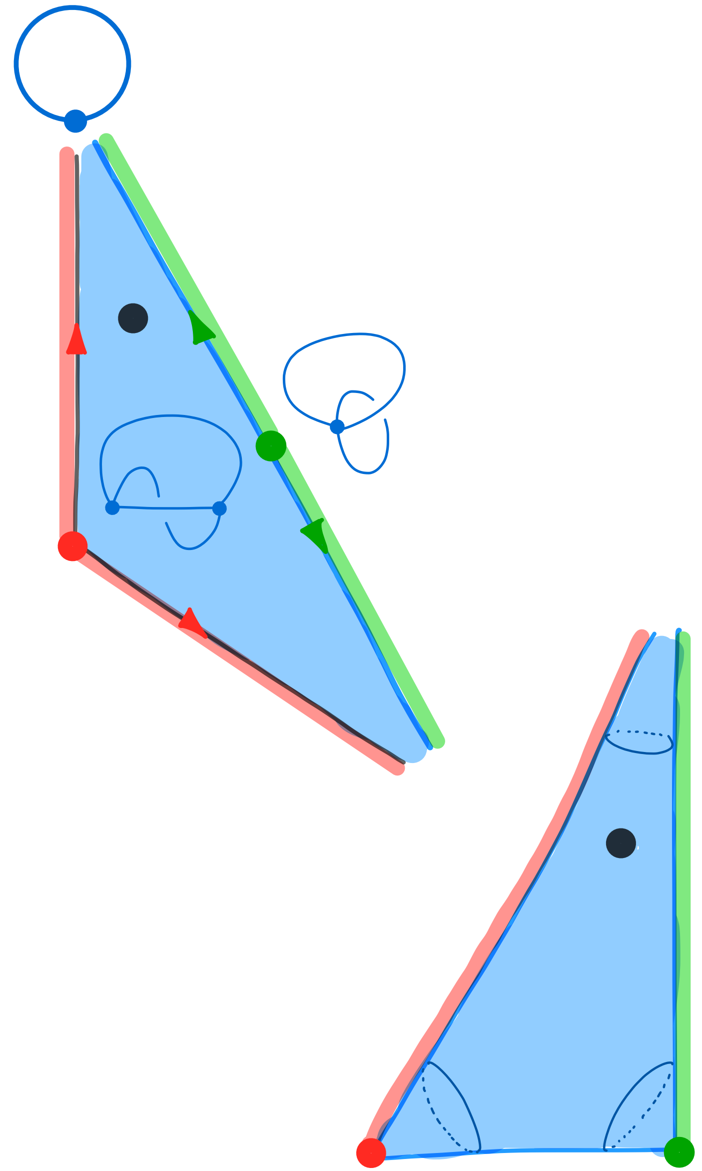
$\mathrm{Aut}\Big($  $\Big) \simeq \mathbb{Z}/2\mathbb{Z}$

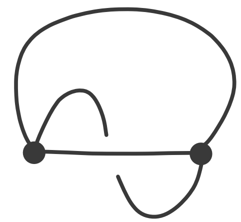
$\mathrm{Aut}\Big($  $\Big) \simeq \mathbb{Z}/6\mathbb{Z}$

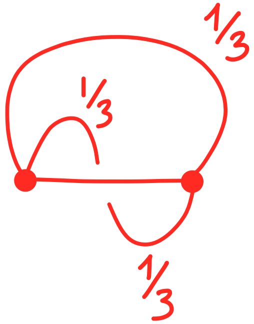


Why Teichmuller theory?

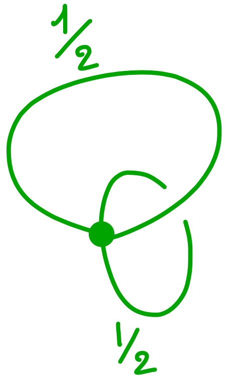
$$\mathcal{M}^{comb}_{1,1}(1)$$



$\mathrm{Aut}\Big($  $\Big) \simeq \mathbb{Z}/2\mathbb{Z}$

$\mathrm{Aut}\Big($ 

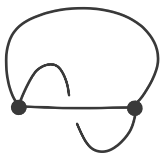
$\Big) \simeq \mathbb{Z}/6\mathbb{Z}$

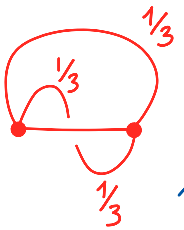
$\mathrm{Aut}\Big($ 

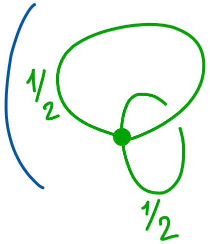
$\Big) \simeq \mathbb{Z}/4\mathbb{Z}$

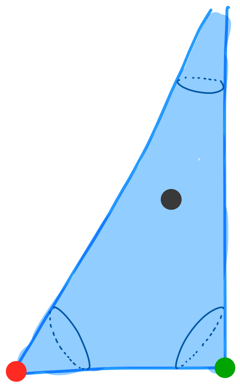
Why Teichmüller theory?

$$\mathcal{M}^{\mathrm{comb}}_{1,1}(1)$$

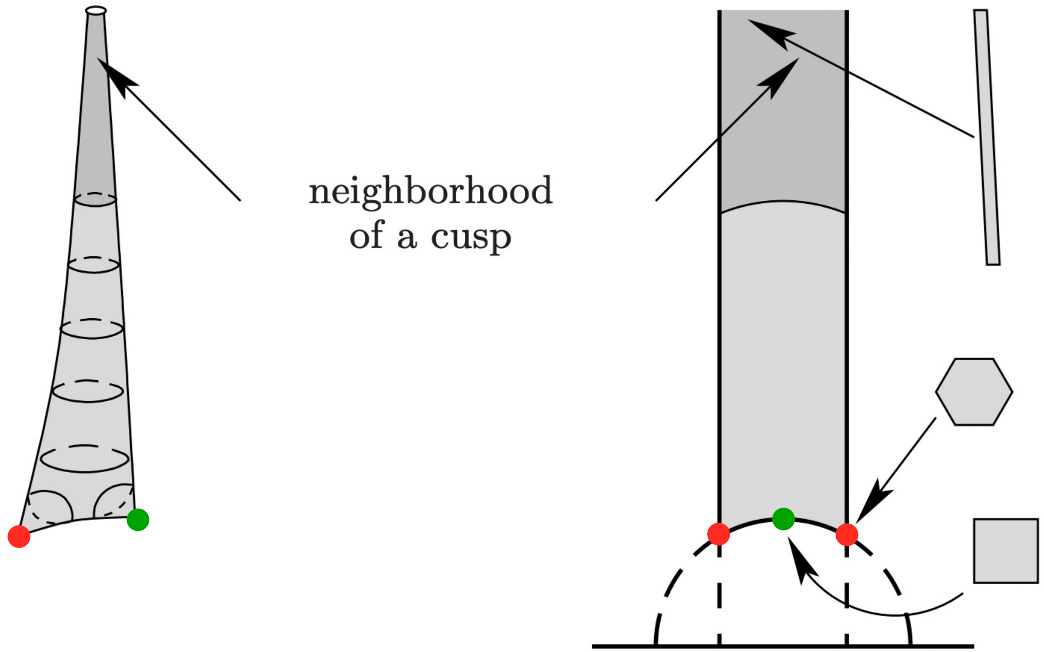
Aut() $\simeq \mathbb{Z}/2\mathbb{Z}$

Aut() $\simeq \mathbb{Z}/6\mathbb{Z}$

Aut() $\simeq \mathbb{Z}/4\mathbb{Z}$



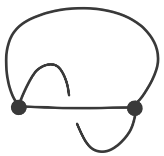
Anton Zorich

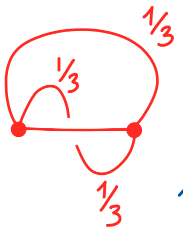


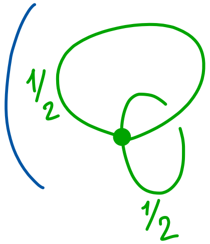
Why Teichmüller theory?

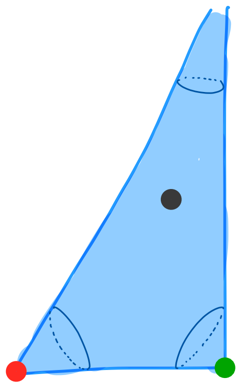
$$\mathcal{M}^{comb}_{1,1}(1)$$

Modular curve $\mathcal{H}/_{SL(2,\mathbb{Z})}$

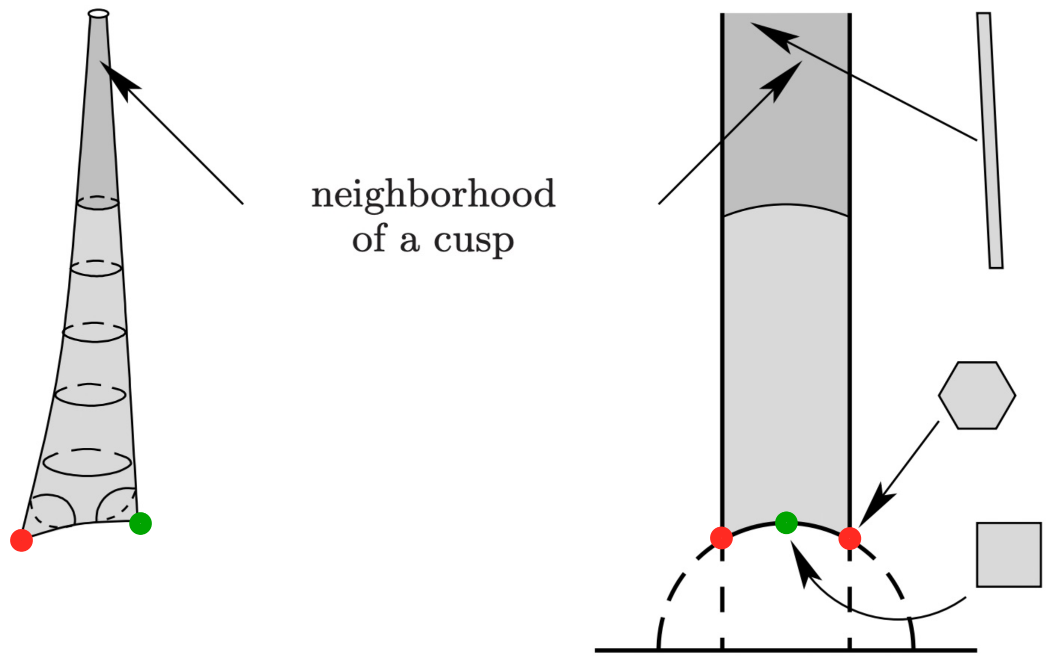
Aut() $\simeq \mathbb{Z}/_{2\mathbb{Z}}$

Aut() $\simeq \mathbb{Z}/_{6\mathbb{Z}}$

Aut() $\simeq \mathbb{Z}/_{4\mathbb{Z}}$



Anton Zorich

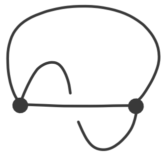


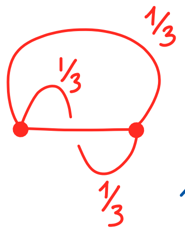
Why Teichmüller theory?

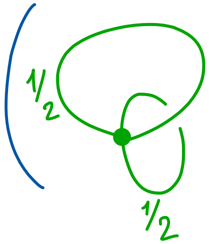
moduli space of Riemann surfaces genus 1 with 1 marked point

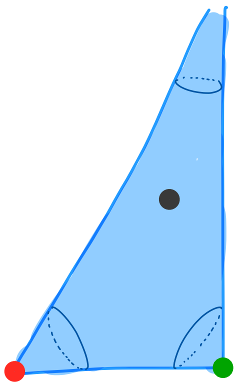
$$\mathcal{M}^{comb}_{1,1}(1) \qquad \mathcal{M}_{1,1}$$

Modular curve $\mathcal{H}/SL(2,\mathbb{Z})$

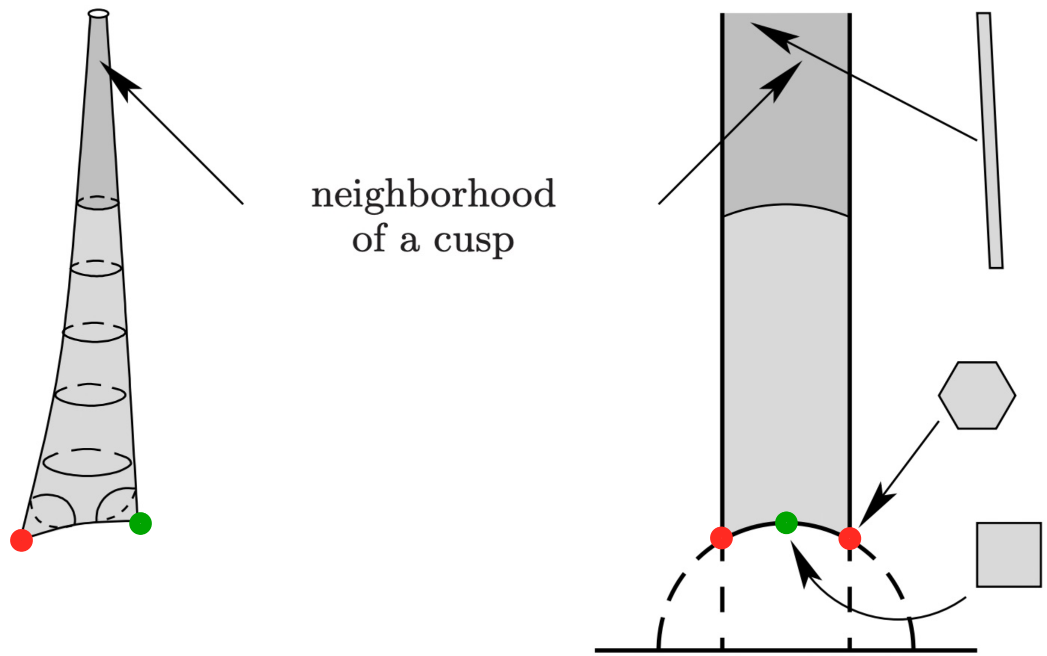
Aut() $\simeq \mathbb{Z}/2\mathbb{Z}$

Aut() $\simeq \mathbb{Z}/6\mathbb{Z}$

Aut() $\simeq \mathbb{Z}/4\mathbb{Z}$



Anton Zorich

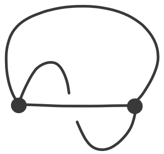


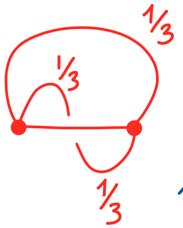
Why Teichmüller theory?

moduli space of Riemann surfaces genus 1 with 1 marked point

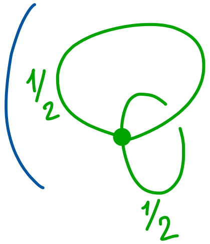
$$\mathcal{M}^{comb}_{1,1}(1) \underset{\text{homeo}}{\simeq} \mathcal{M}_{1,1}$$

Modular curve $\mathcal{H}/SL(2,\mathbb{Z})$

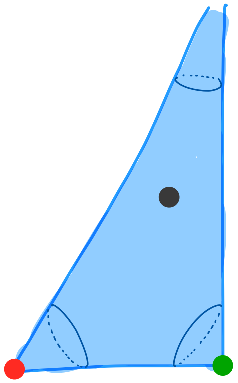
Aut() $\simeq \mathbb{Z}/2\mathbb{Z}$

Aut(

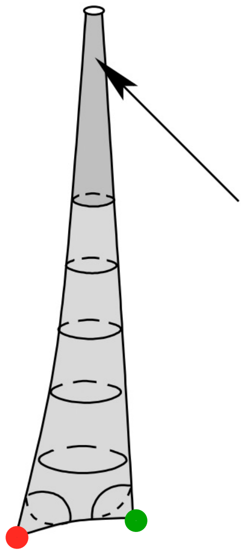
) $\simeq \mathbb{Z}/6\mathbb{Z}$

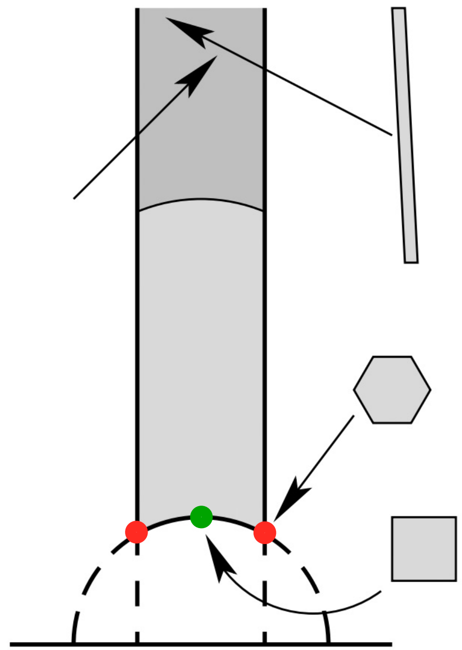
Aut(

) $\simeq \mathbb{Z}/4\mathbb{Z}$



Anton Zorich





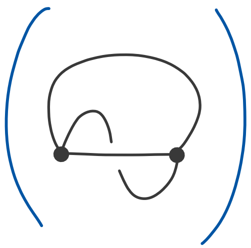
Why Teichmüller theory?

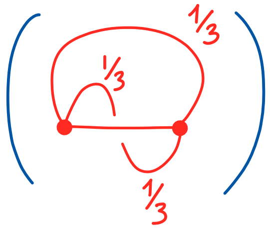
moduli space of Riemann surfaces

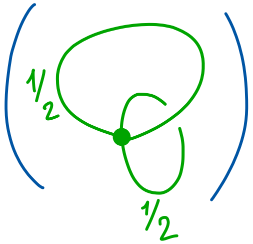
genus g with n marked points

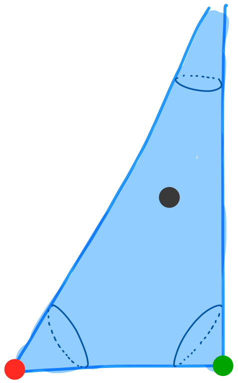
$$\mathcal{M}^{comb}_{g,n}(\vec{L}) \underset{\text{homeo}}{\simeq} \mathcal{M}_{g,n}$$

Modular curve $\mathcal{H}/SL(2,\mathbb{Z})$

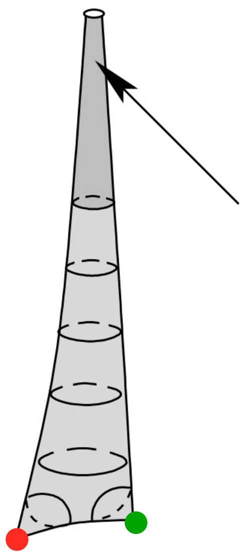
Aut() $\simeq \mathbb{Z}/2\mathbb{Z}$

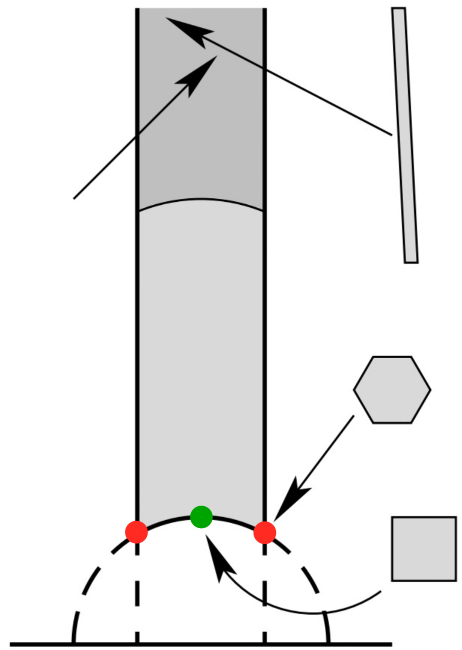
Aut() $\simeq \mathbb{Z}/6\mathbb{Z}$

Aut() $\simeq \mathbb{Z}/4\mathbb{Z}$



Anton Zorich





Mathematics > Differential Geometry

[Submitted on 22 Oct 2020 (v1), last revised 29 May 2021 (this version, v3)]

On the Kontsevich geometry of the combinatorial Teichmüller space

Jørgen Ellegaard Andersen, Gaëtan Borot, Séverin Charbonnier, Alessandro Giacchetto, Danilo Lewański, Campbell Wheeler

For bordered surfaces S, we develop a complete parallel between the geometry of the combinatorial Teichmüller space $\mathcal{T}_S^{\mathrm{comb}}$ equipped with Kontsevich symplectic form ω_K, and then the usual Weil-Petersson geometry of Teichmüller space $\mathcal{T}_S$. The basis for this is an identification of $\mathcal{T}_S^{\mathrm{comb}}$ with a space of measured foliations with transverse boundary conditions. We equip $\mathcal{T}_S^{\mathrm{comb}}$ with an analog of the Fenchel-Nielsen coordinates (defined similarly as Dehn-Thurston coordinates) and show they are Darboux for ω_K (analog of Wolpert formula). We then set up the geometric recursion of Andersen-Borot-Orantin to produce mapping class group invariants functions on $\mathcal{T}_S^{\mathrm{comb}}$ whose integration with respect to Kontsevich volume form satisfy topological recursion. Further we establish an analog of Mirzakhani-McShane identities, and provide applications to the study of the enumeration of multicurves with respect to combinatorial lengths and Masur-Veech volumes. The formalism allows us to provide uniform and completely geometric proofs of Witten's conjecture/Kontsevich theorem and Norbury's topological recursion for lattice point count in the combinatorial moduli space, parallel to Mirzakhani's proof of her recursion for Weil-Petersson volumes. We strengthen results of Mondello and Do on the convergence of hyperbolic geometry to combinatorial geometry along the rescaling flow, allowing us to flow systematically natural constructions on the usual Teichmüller space to their combinatorial analogue, such as a new derivation of the piecewise linear structure of $\mathcal{T}_S^{\mathrm{comb}}$ originally obtained in the work of Penner, as the limit under the flow of the smooth structure of $\mathcal{T}_S$.

Comments: 107 pages. v2: Section 1 explains better relations to previous works, in particular how Dehn-Thurston coordinates compare to Fenchel-Nielsen coordinates. The PL statement (Section 5) follows fro Penner's 1982 PhD thesis, this article provides a different proof via the rescaling flow on Teichmüller (we added Remark 5.9 in that proof to take into account twisting numbers at the boundaries)

Subjects: **Differential Geometry (math.DG)**; Mathematical Physics (math-ph); Geometric Topology (math.GT); Symplectic Geometry (math.SG)

MSC classes: 14H10, 14N10, 53C12, 57K20, 57M15

Cite as: arXiv:2010.11806 **[math.DG]**
(or arXiv:2010.11806v3 **[math.DG]** for this version)
https://doi.org/10.48550/arXiv.2010.11806

member

.0.11806

Search...

Help | Advar

al Geometry

ast revised 29 May 2021 (this version, v3)]

C vich geometry of the combinatorial Teichmüller space

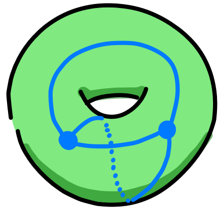
Jørgen E ndersen, Gaëtan Borot, Séverin Charbonnier, Alessandro Giacchetto, Danilo Lewański, Ca

For bordered surfaces S, we develop a complete parallel between the geometry of the combinatorial Teichmüller space T_S^{com} equipped with Kontsevich symplectic form ω_K, and then the usual Weil-Petersson geometry of Teichmüller space T_S. The basis for this is an identification of T_S^{comb} with a space of measured foliations with transverse boundary conditions. We equip T_S^{comb} with an analog of the Fenchel-Nielsen coordinates (defined similarly as Dehn-Thurston coordinates) and show they are Darboux for ω_K (analog of Wolpert formula). We then set up the geometric recursion of Andersen-Borot-Orantin to produce mapping class group invariants functions on T_S^{comb} whose integration with respect to Kontsevich volume form satisfy topological recursion. Further we establish an analog of Mirzakhani-McShane identities, and provide applications to the study of the enumeration of multicurves with respect to combinatorial lengths and Masur-Veech volumes. The formalism allows us to provide uniform and completely geometric proofs of Witten's conjecture/Kontsevich theorem and Norbury's topological recursion for lattice point count in the combinatorial moduli space, parallel to Mirzakhani's proof of her recursion for Weil-Petersson volumes. We strengthen results of Mondello and Do on the convergence of hyperbolic geometry to combinatorial geometry along the rescaling flow, allowing us to flow systematically natural constructions on the usual Teichmüller space to their combinatorial analogue, such as a new derivation of the piecewise linear structure of T_S^{comb} originally obtained in the work of Penner, as the limit under the flow of the smooth structure of T_S.

Comments:	107 pages. v2: Section 1 explains better relations to previous works, in particular how Dehn-Thurston coordinates compare to Fenchel-Nielsen coordinates. The PL statement (Section 5) follows fro Penner's 1982 PhD thesis, this article provides a different proof via the rescaling flow on Teichmüller (we added Remark 5.9 in that proof to take into account twisting numbers at the boundaries)
Subjects:	**Differential Geometry (math.DG)**; Mathematical Physics (math-ph); Geometric Topology (math.GT); Symplectic Geometry (math.SG)
MSC classes:	14H10, 14N10, 53C12, 57K20, 57M15
Cite as:	arXiv:2010.11806 **[math.DG]** (or arXiv:2010.11806v3 **[math.DG]** for this version) https://doi.org/10.48550/arXiv.2010.11806

Thank you!

Merci!

Thank yu!

erci!

つづく