Random geodesics on Hyperbolic Surfaces

Durham 26.2.2025

joint work with

Vincent Delecroix

Random Geodesics (Component Spectrum & Moduli Space) on Hyperbolic Surfaces

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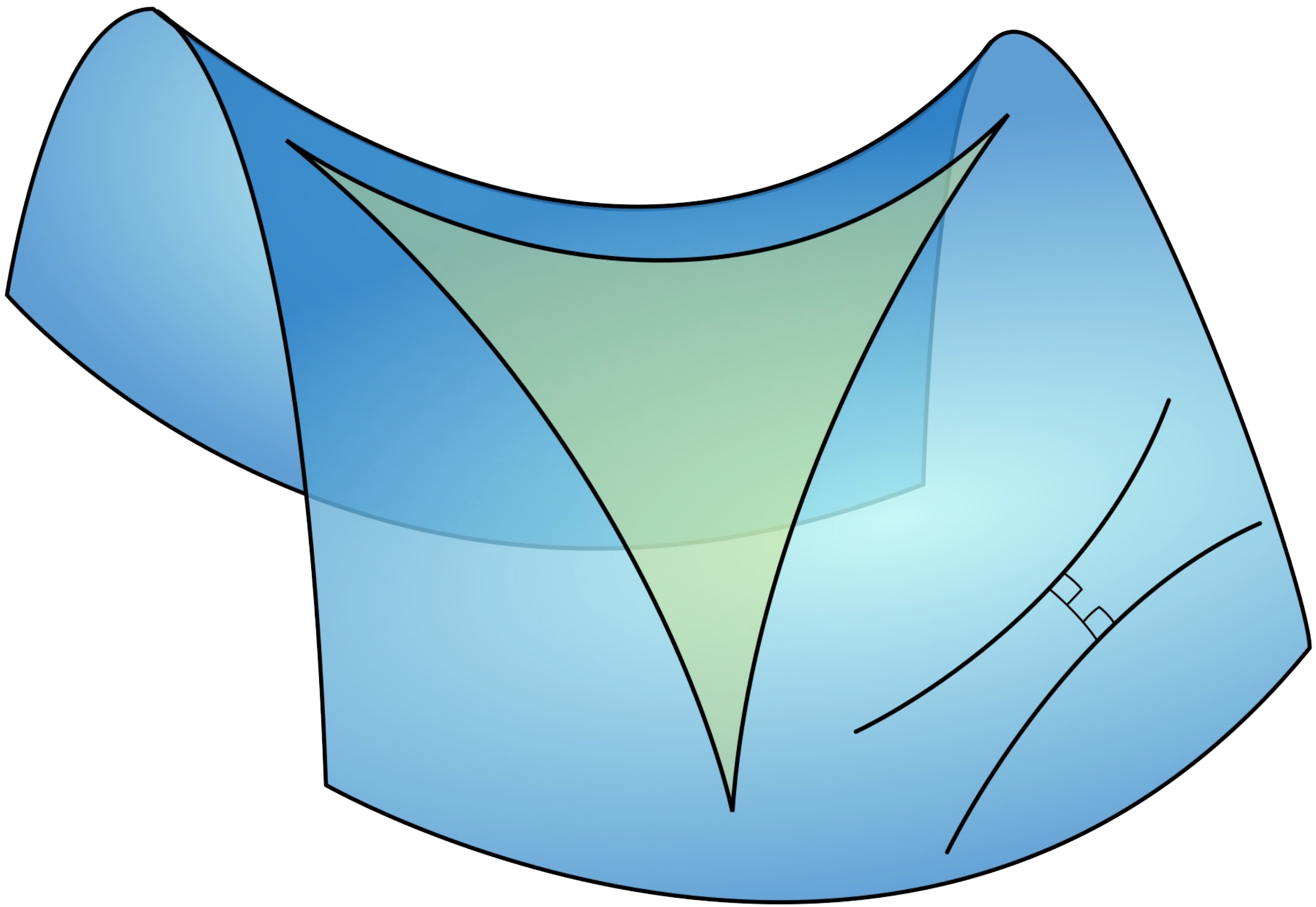
Definition a hyperbolic surface is a surface which locally looks like the hyperbolic plane



G. Meyer

X closed hyperbolic surface of genus $g \geq 2$

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γ closed geodesic on X

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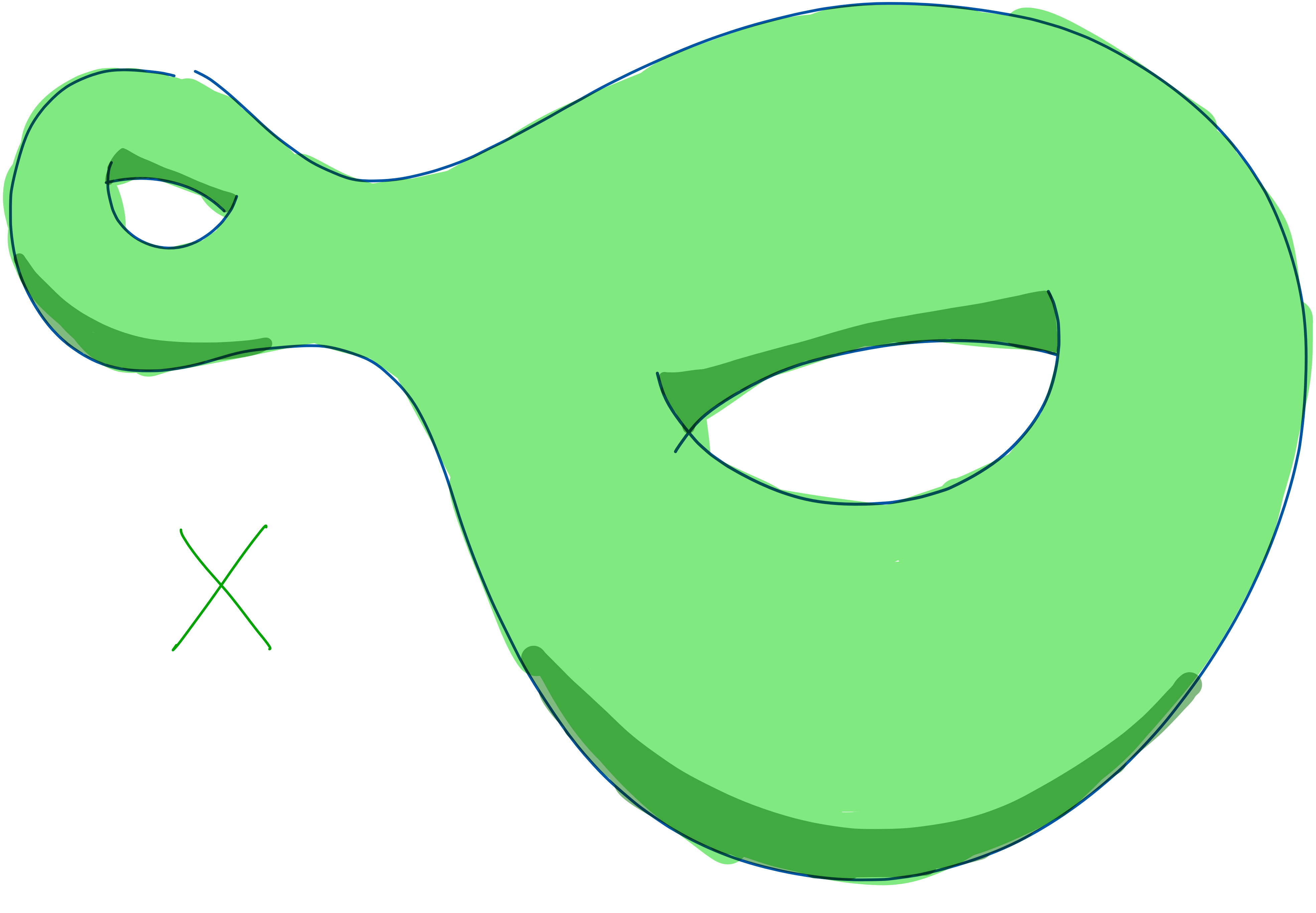
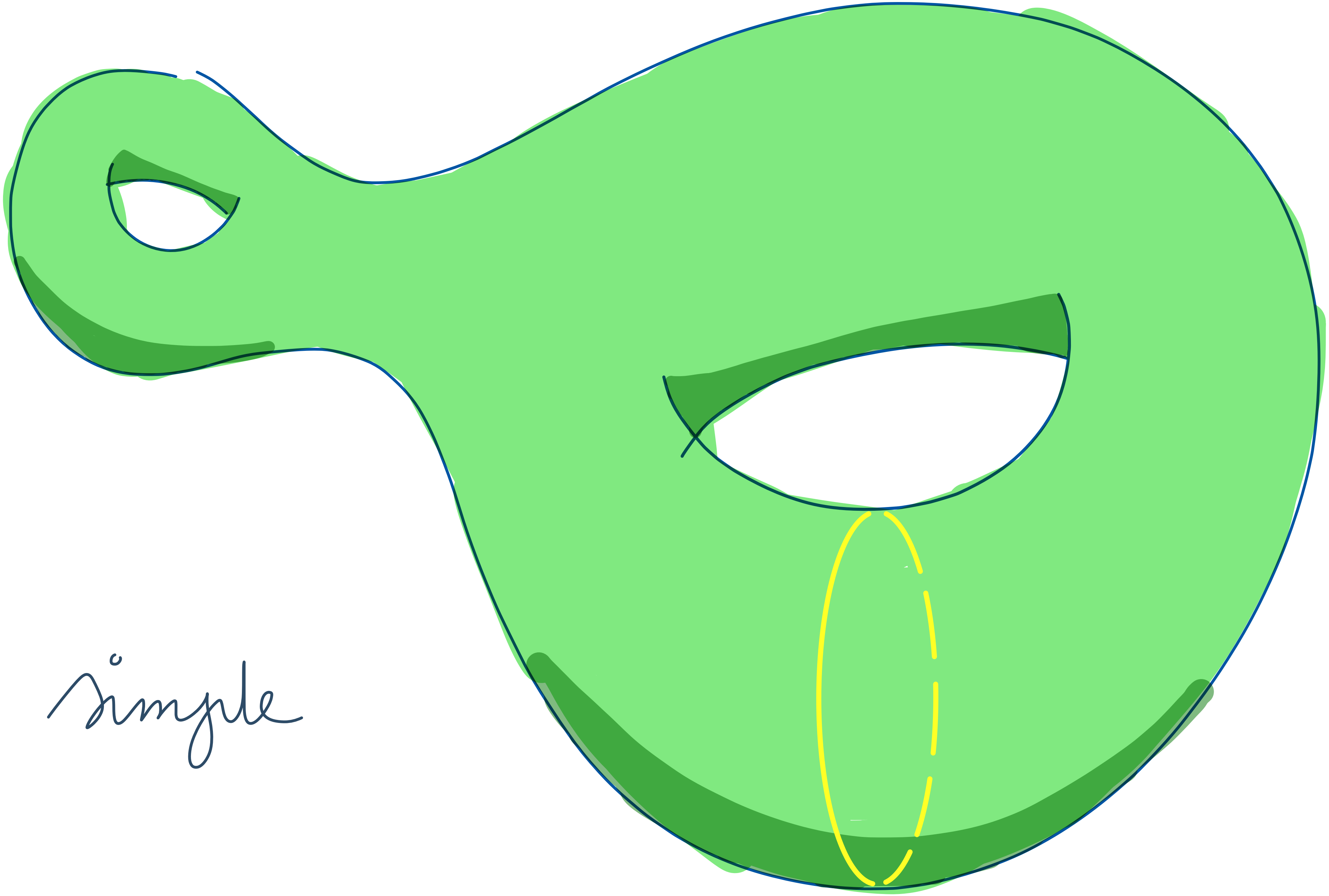
Definition

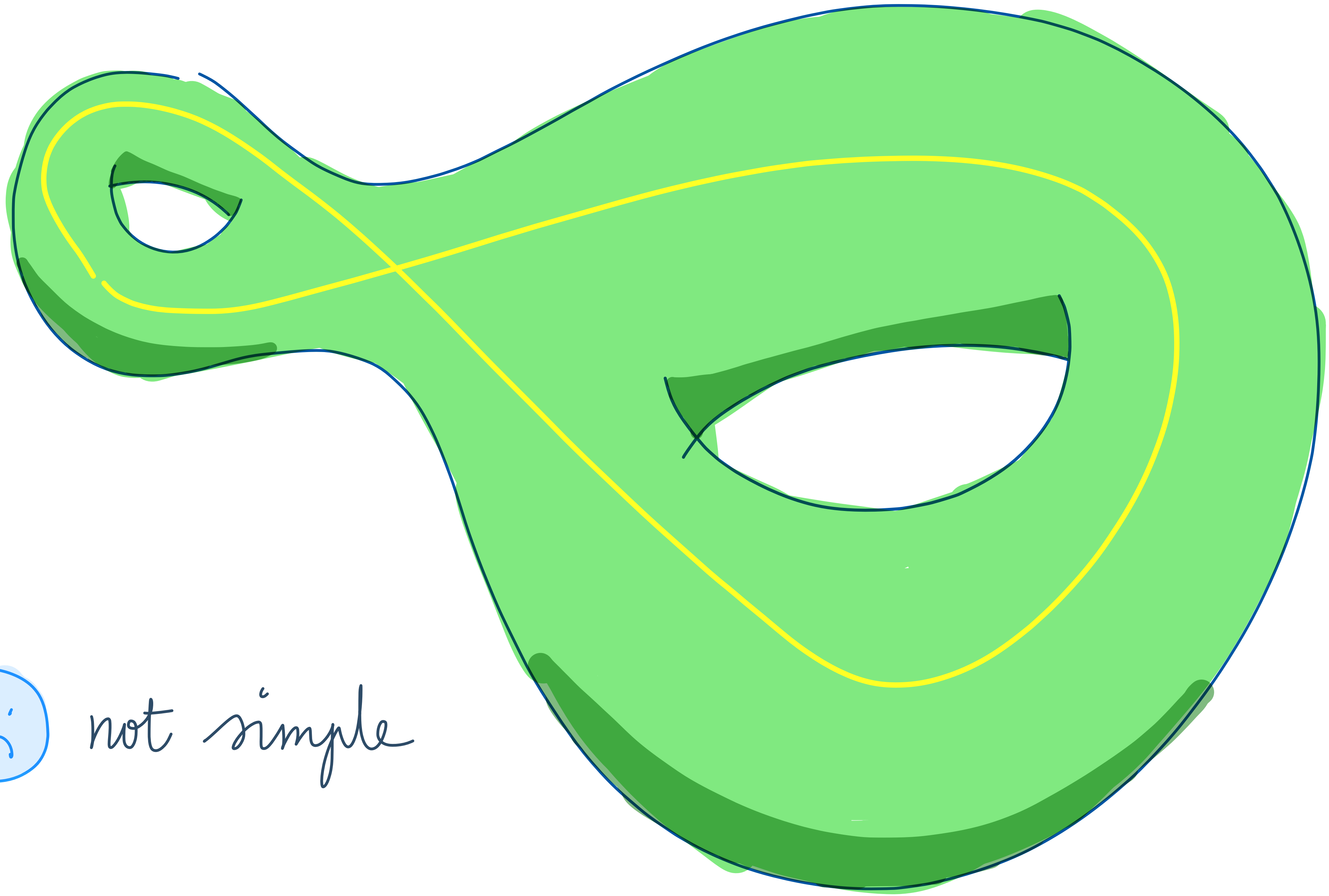
γ is **simple** if it doesn't intersect itself



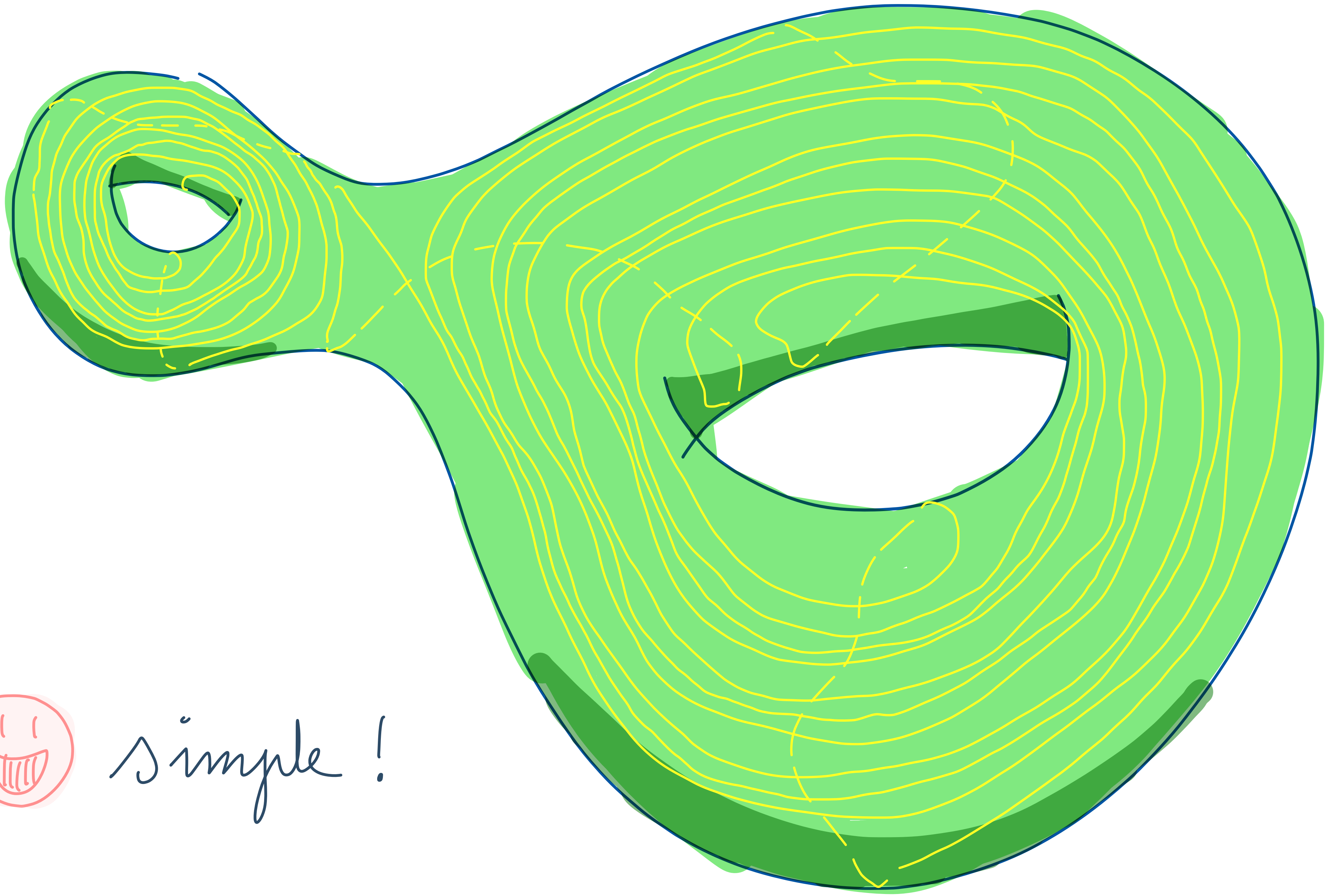
X



simple



not simple



simple !

multi geodesi

A (primitive) multigeodesic is a union of disjoint simple closed geodesics

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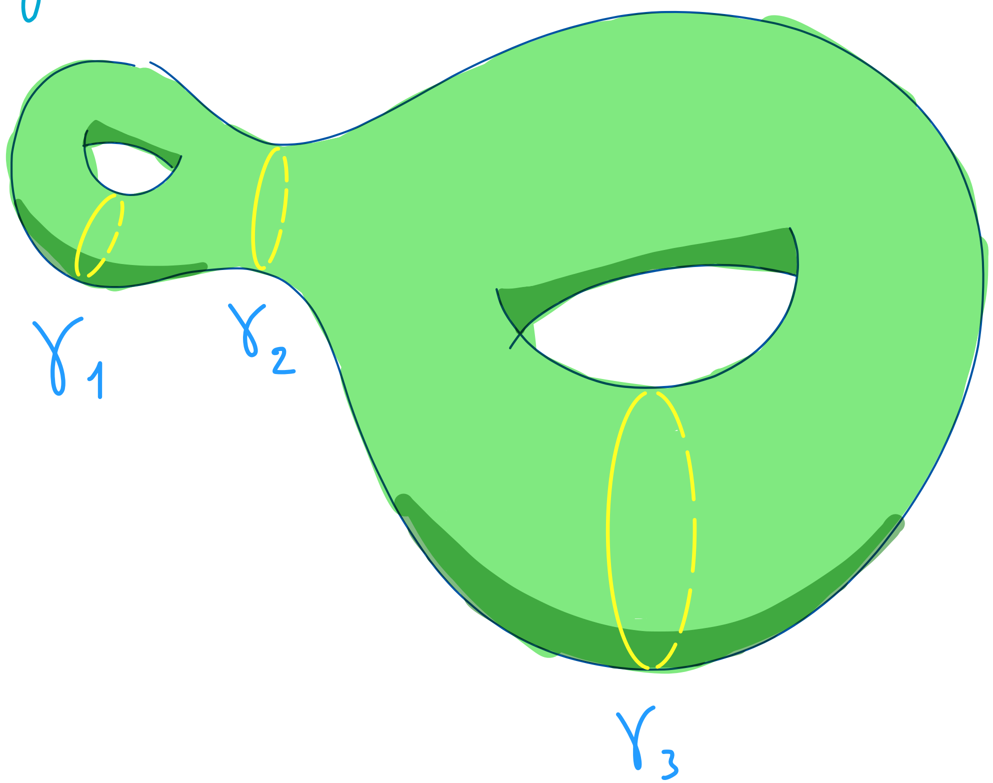

A (primitive) multigeodesic is a union of disjoint simple closed geodesics

$$\gamma = \gamma_1 + \gamma_2 + \dots + \gamma_k$$

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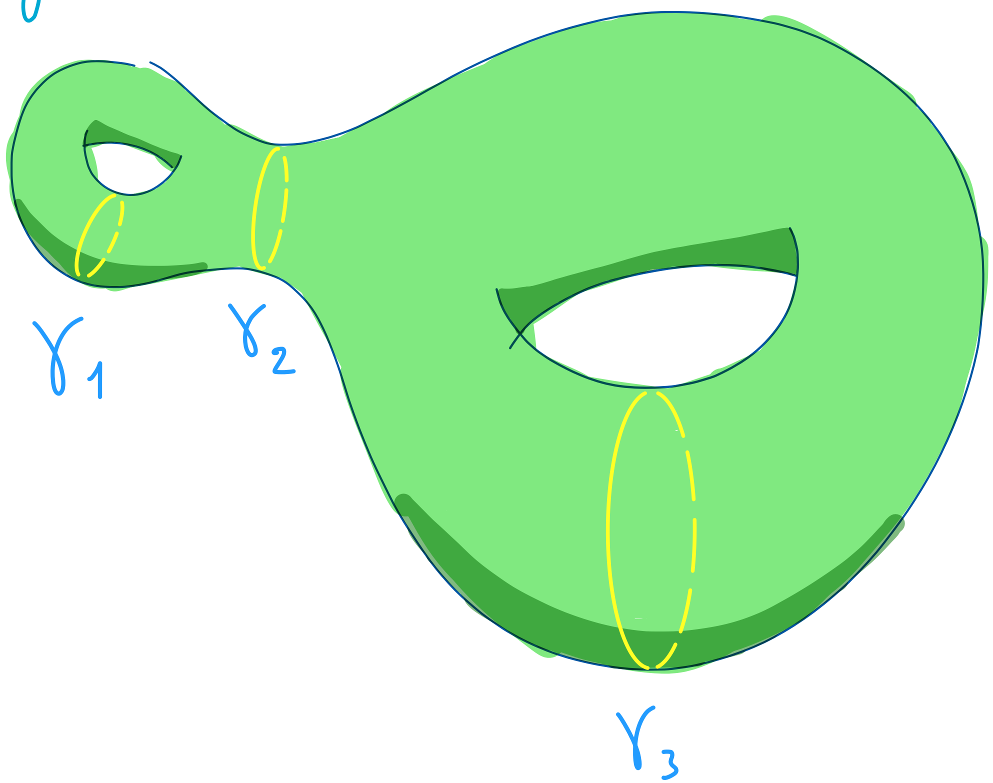

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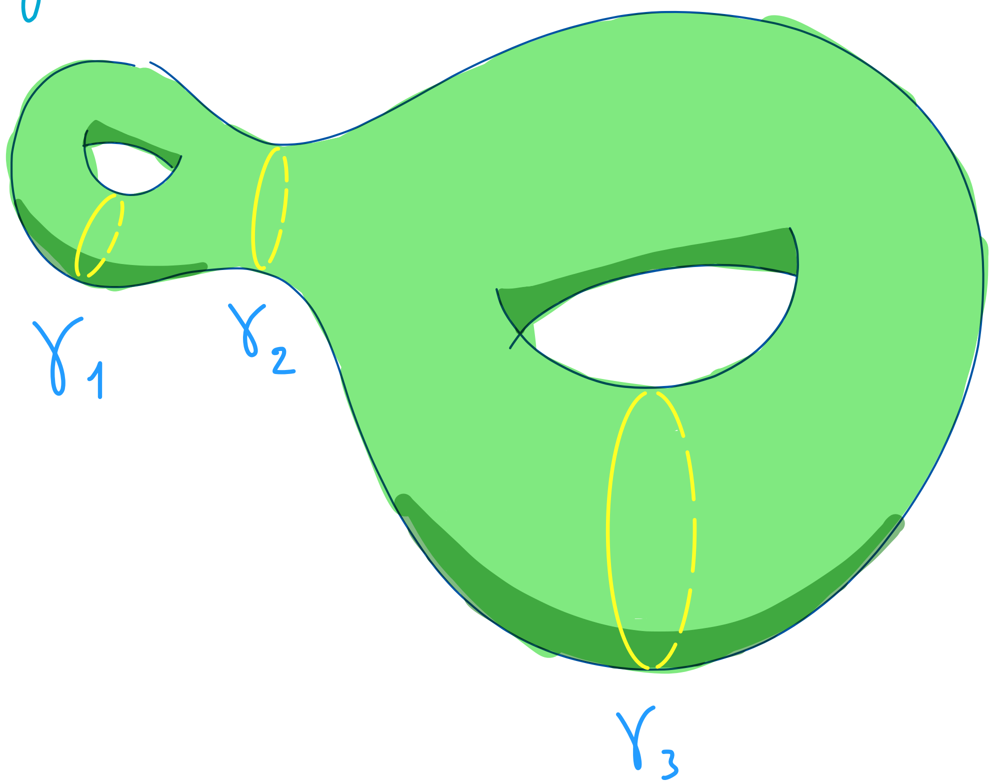

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$$k \leq 3g - 3$$

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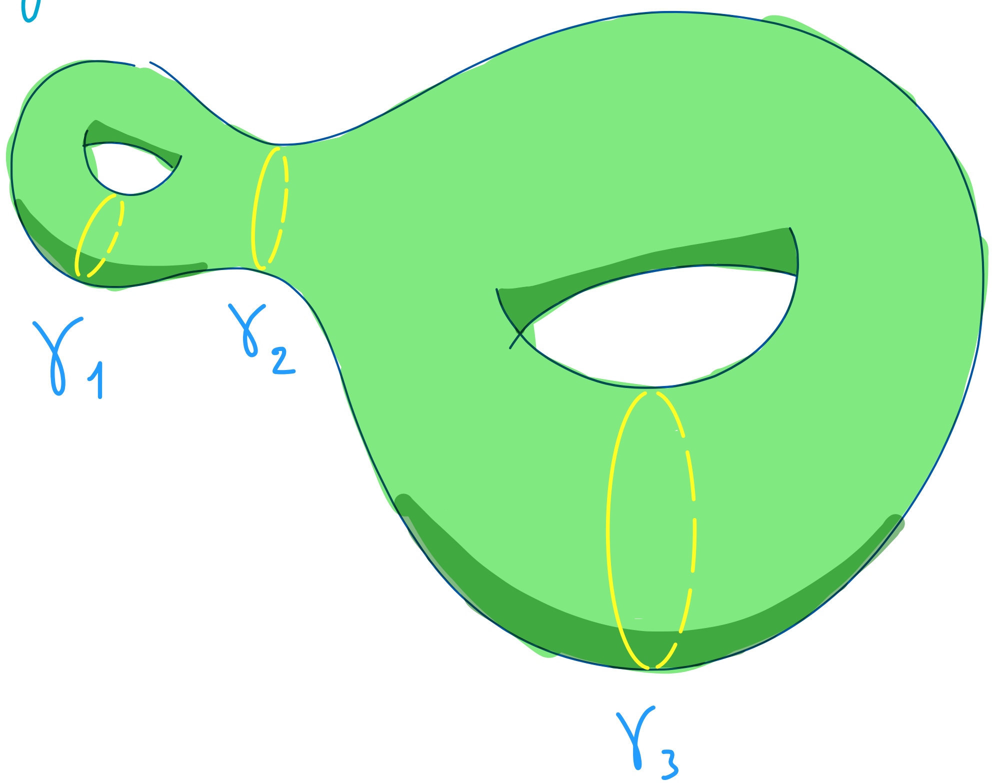

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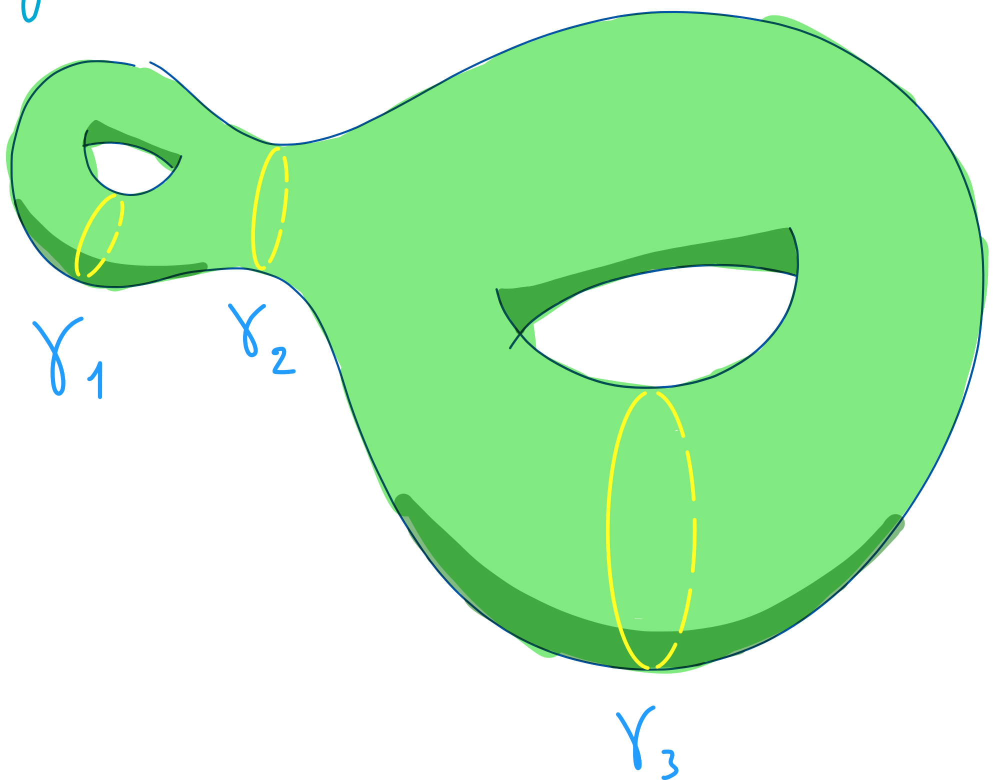

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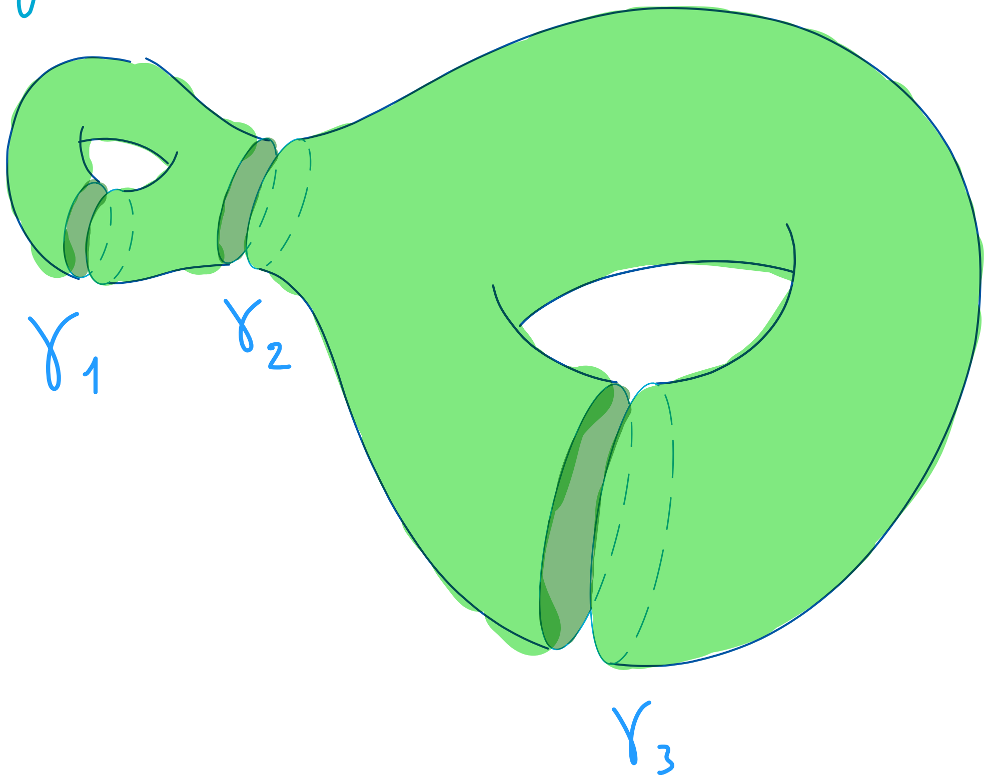

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What happens when $R \to \infty$?

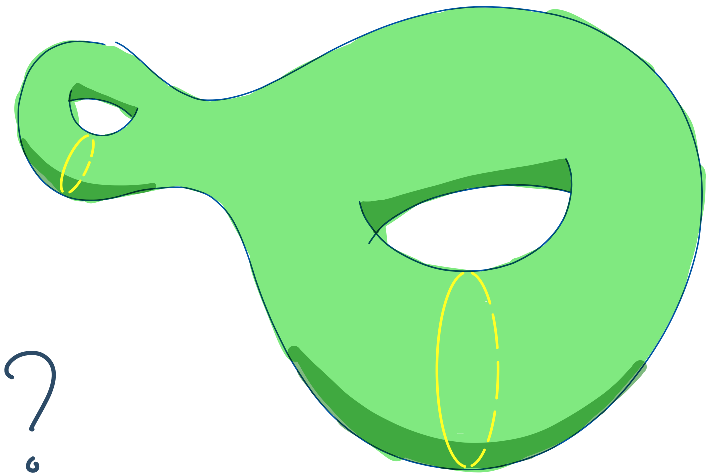
Topology

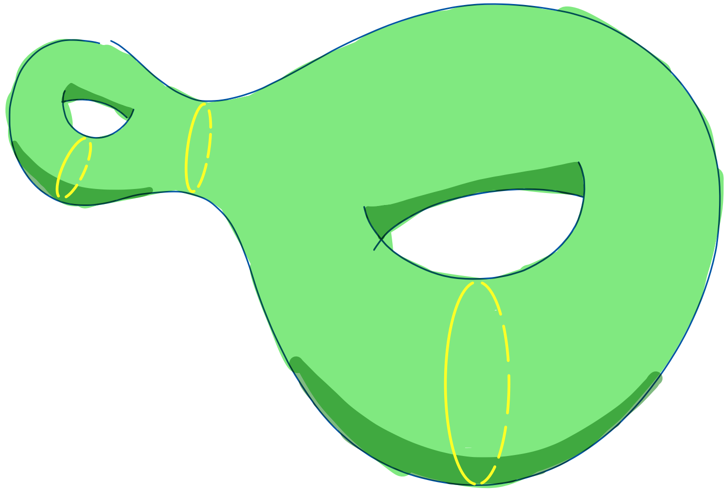
Topology

- separating?

Topology

◦ separating?





Topology

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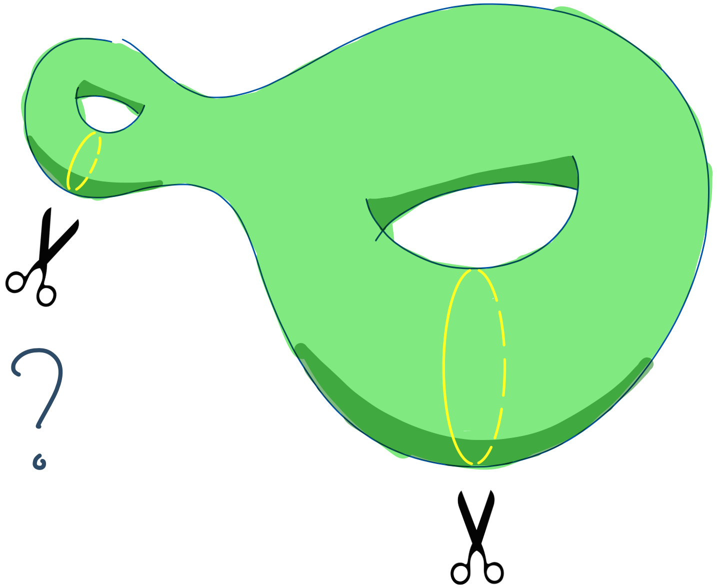

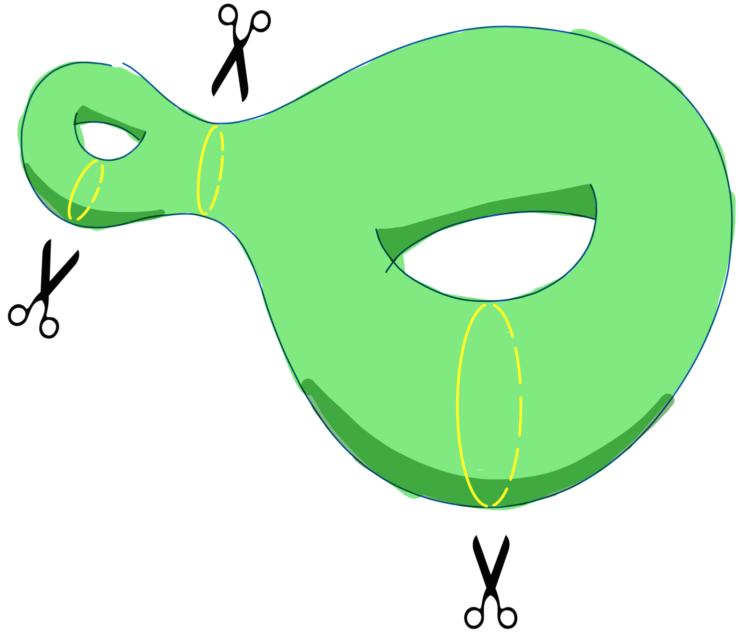

Topology

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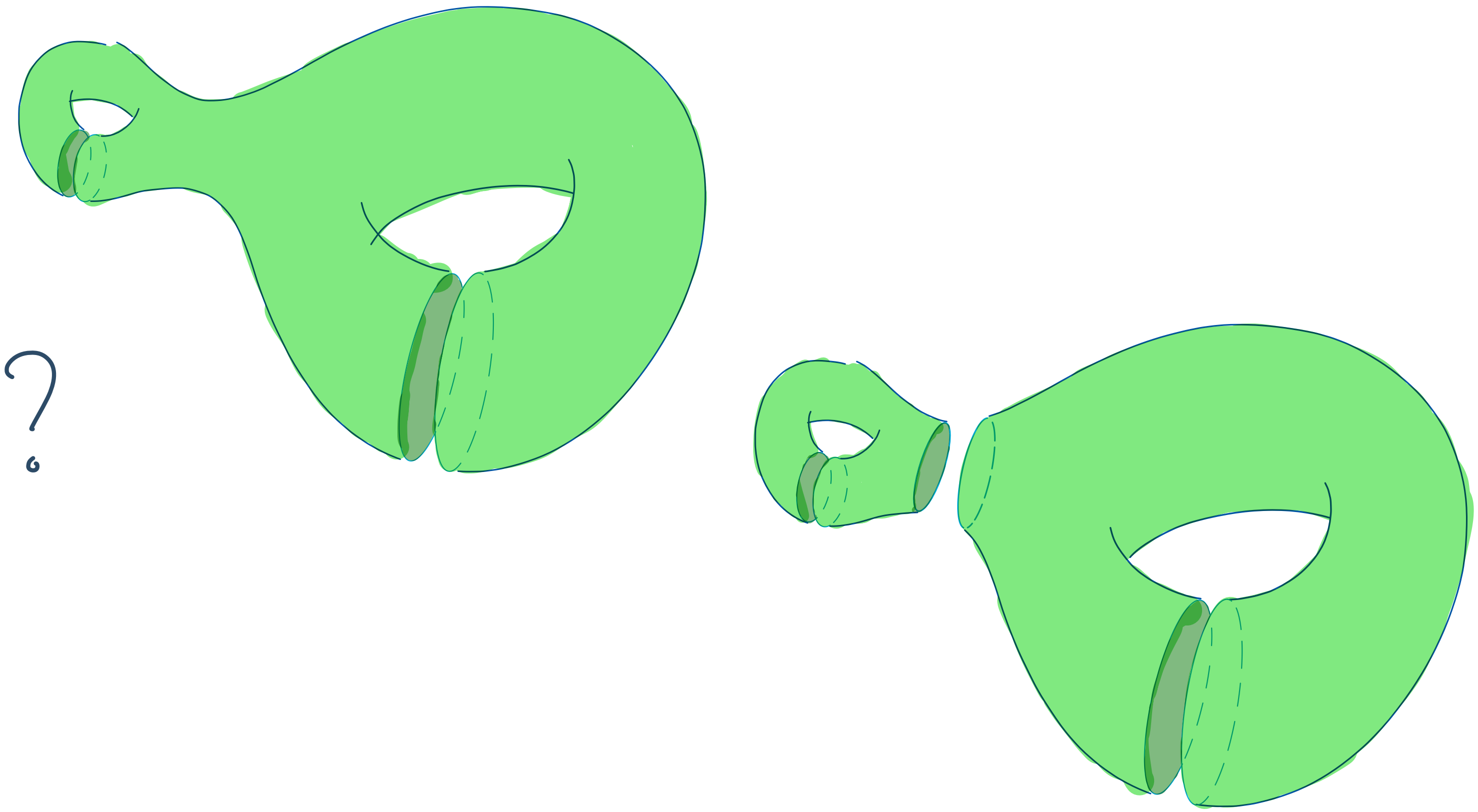

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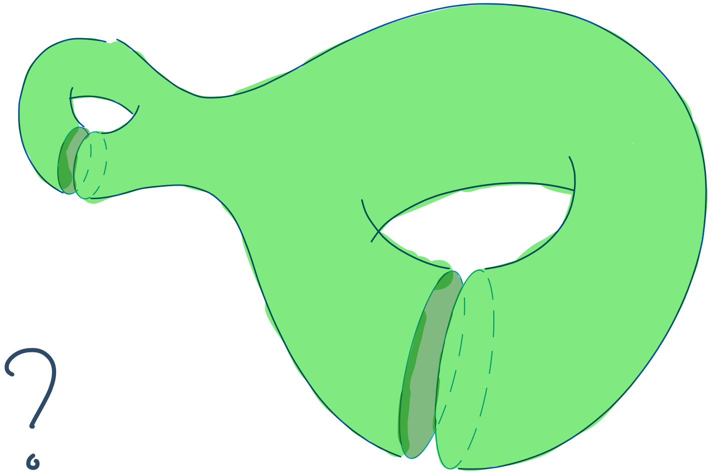
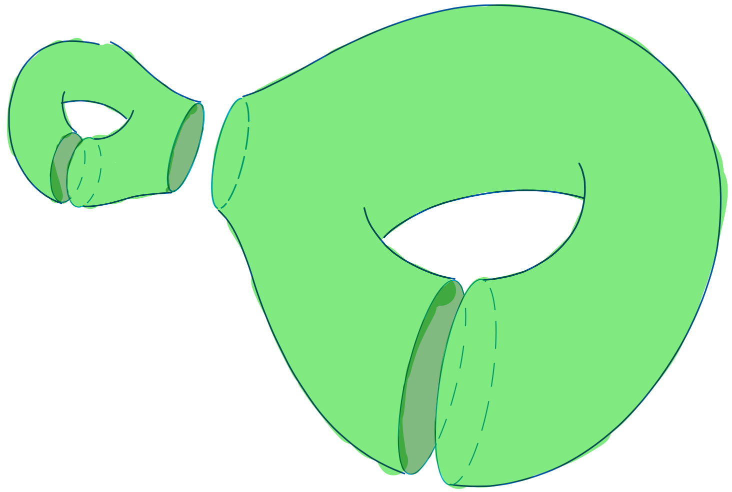


non-separating

separating



Topology

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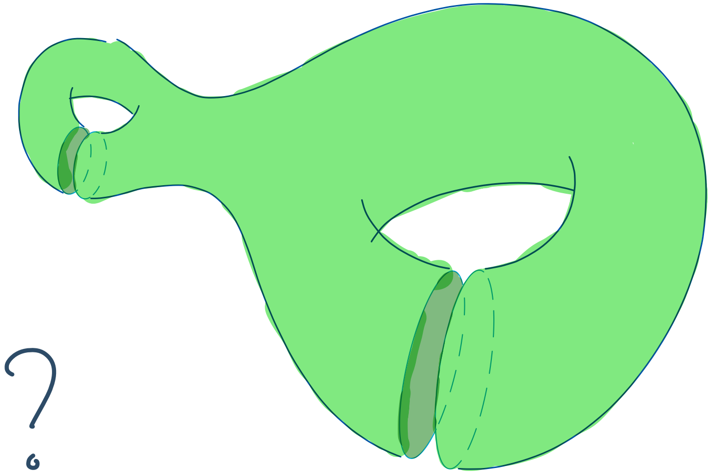
- number of components?



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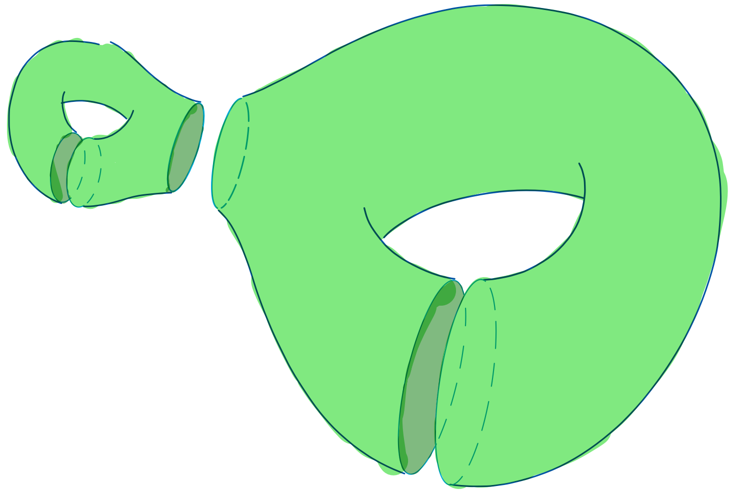

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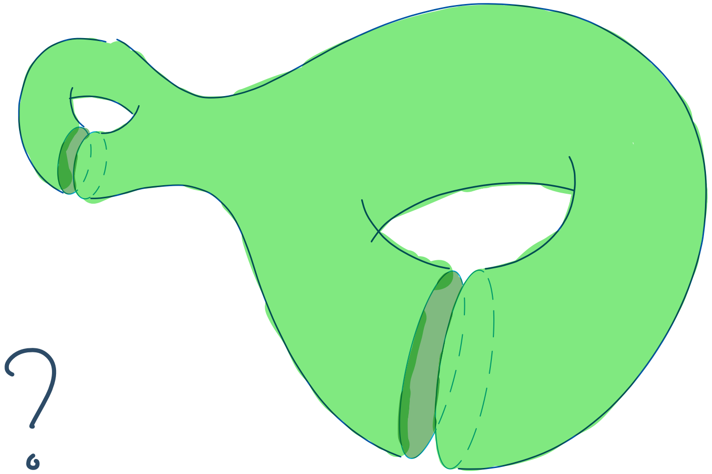

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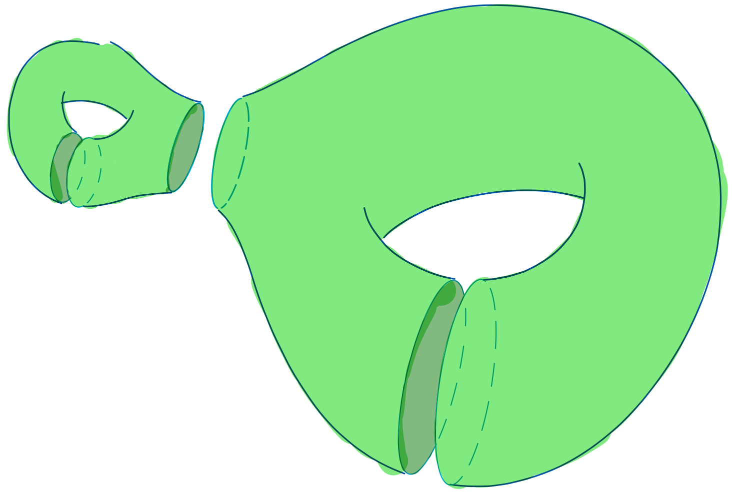

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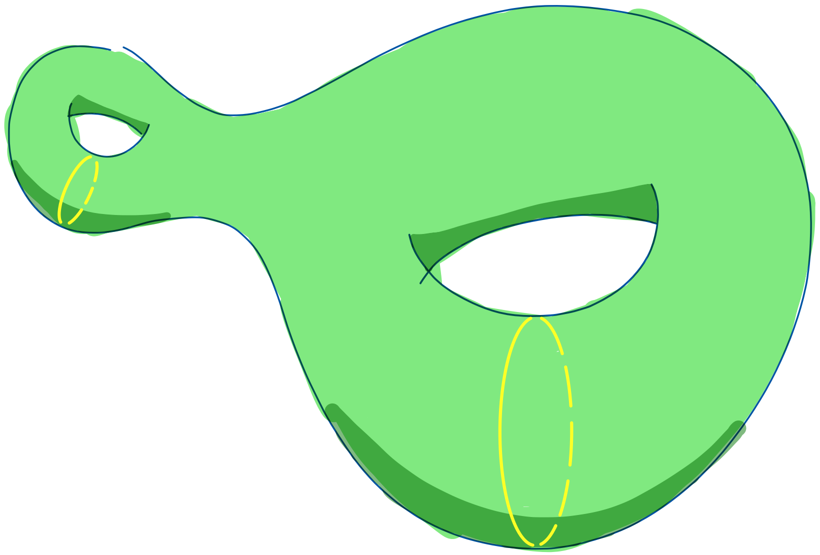

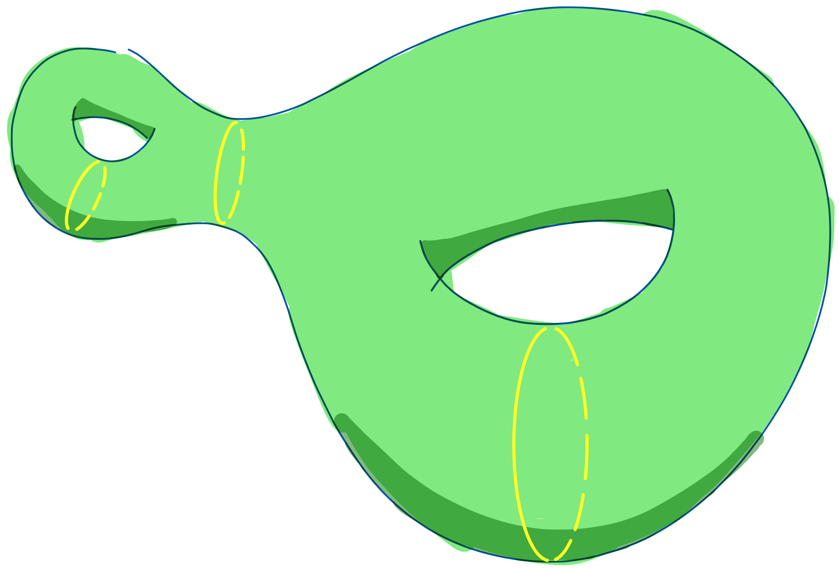

Topology

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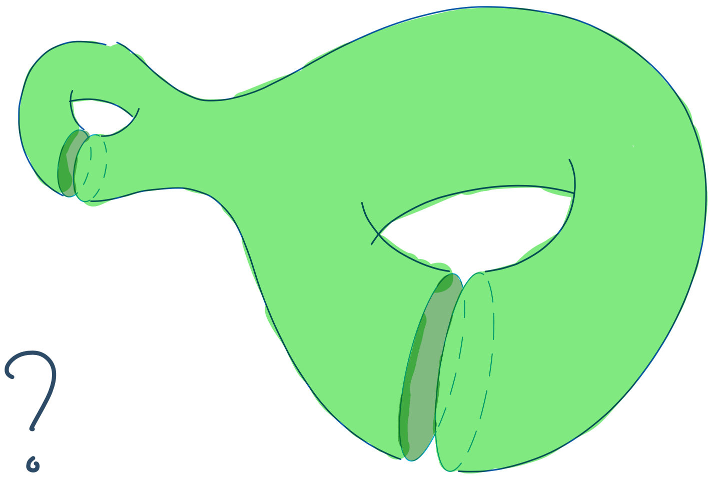

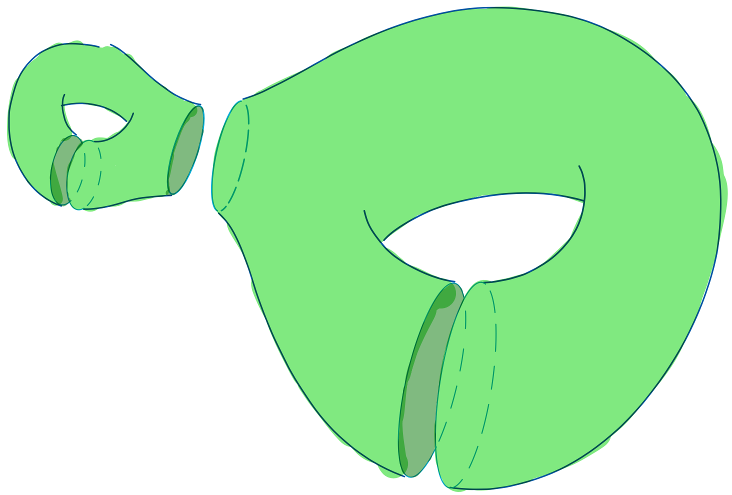

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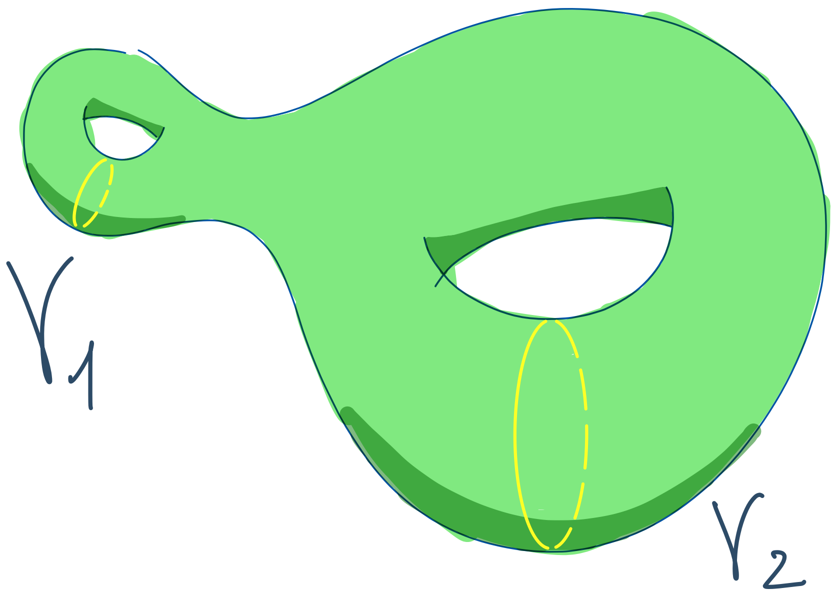

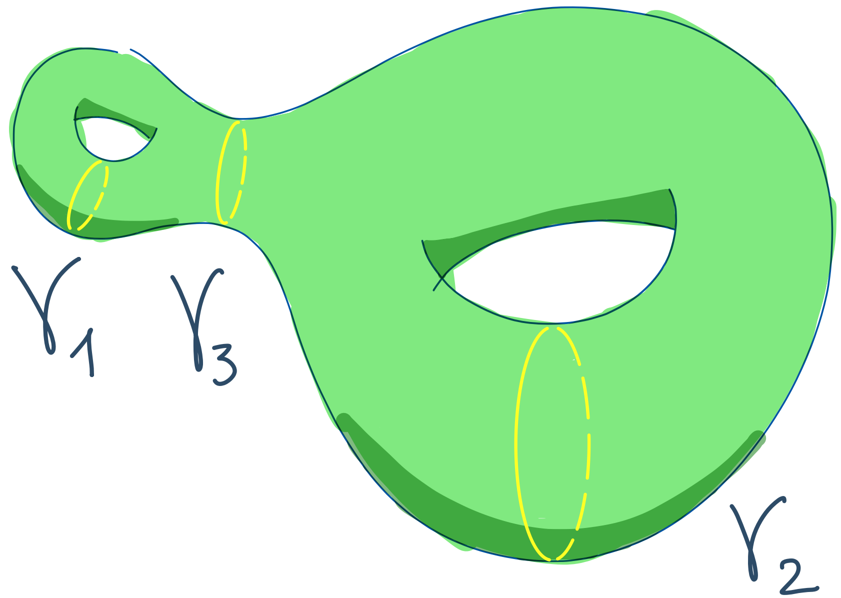

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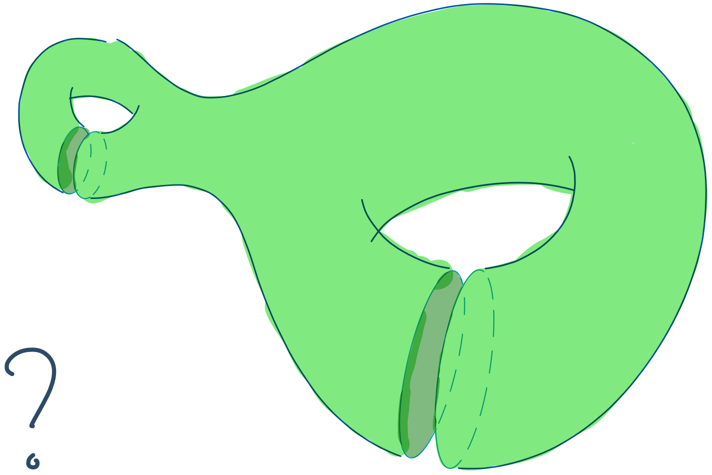

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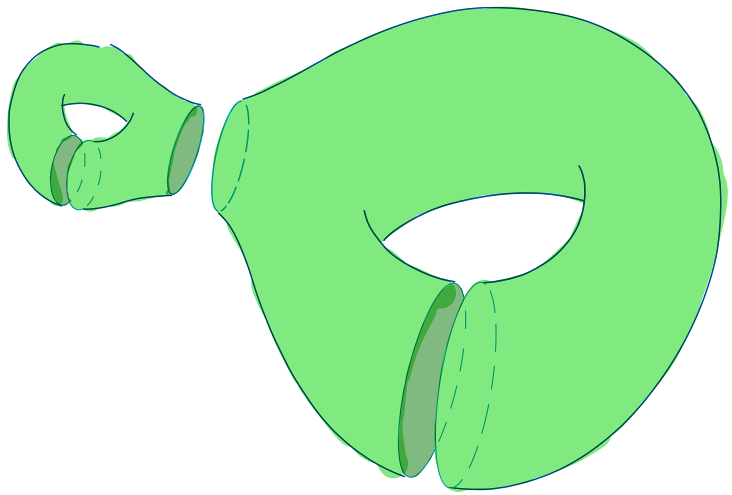

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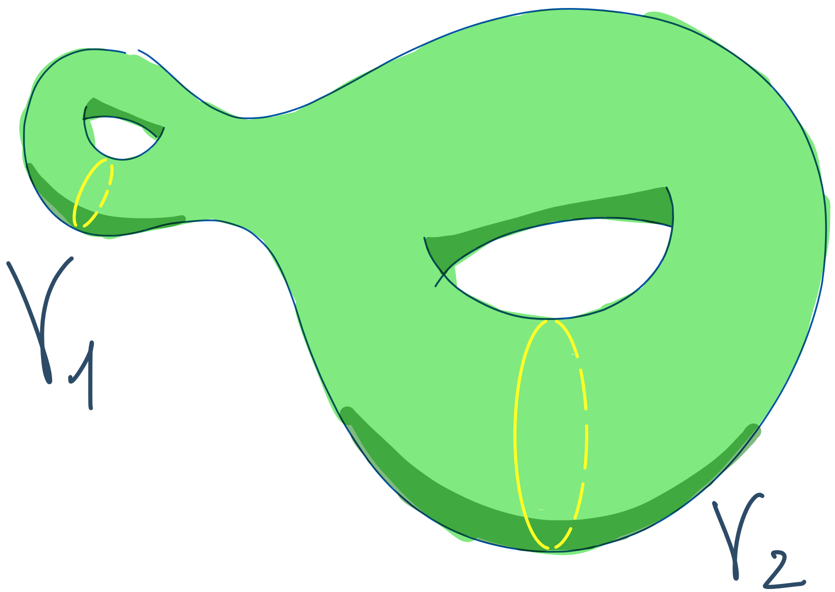
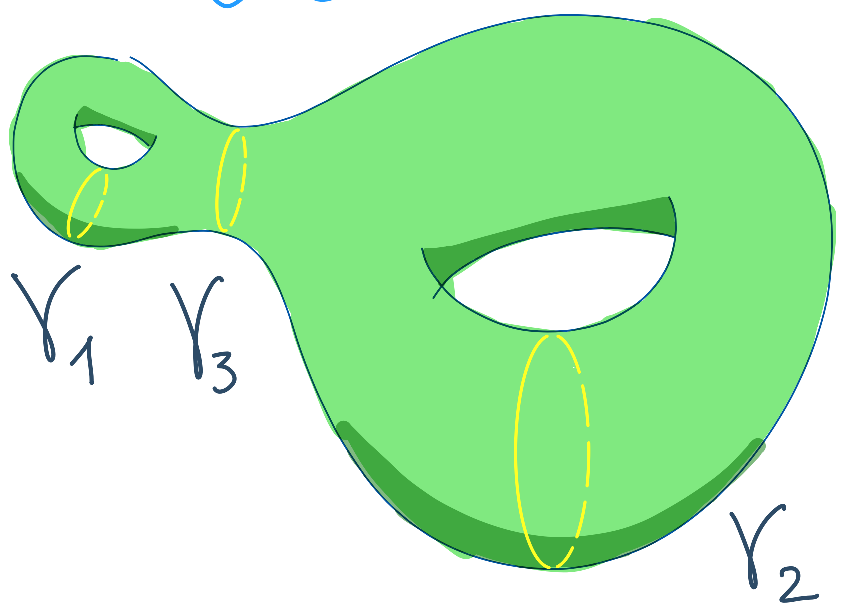
k=3



k=2



Theorem (Mirzakhani '08)

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$$\lim_{R\to\infty} \frac{\#\{\gamma \text{ simple closed geod on } X \mid \gamma \text{ sep}, \ \ell_X(\gamma) \leq R\}}{\#\{\gamma \text{ simple closed geod on } X \mid \ell_X(\gamma) \leq R\}}$$

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Geometry ?

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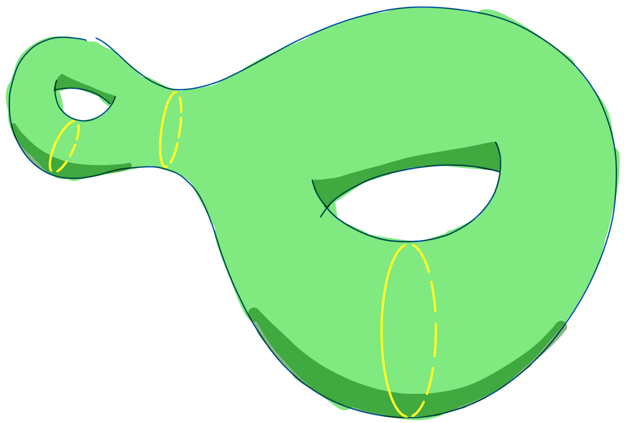

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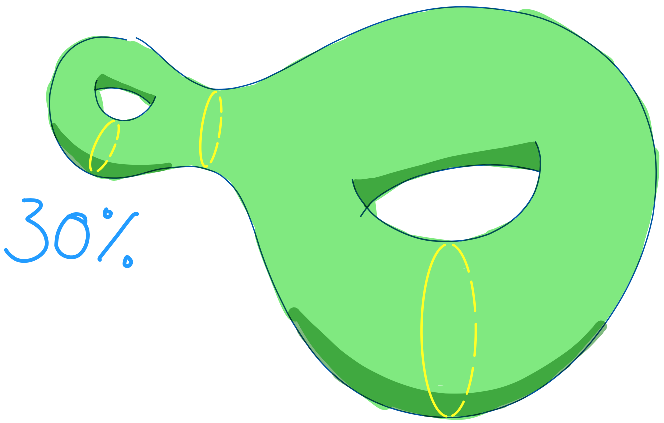

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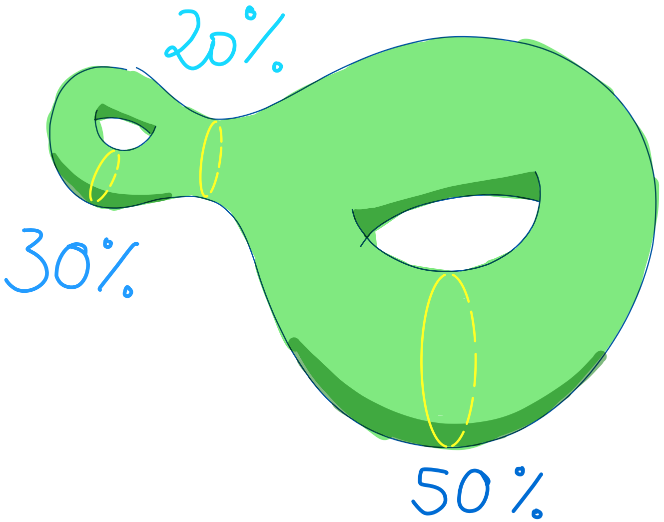

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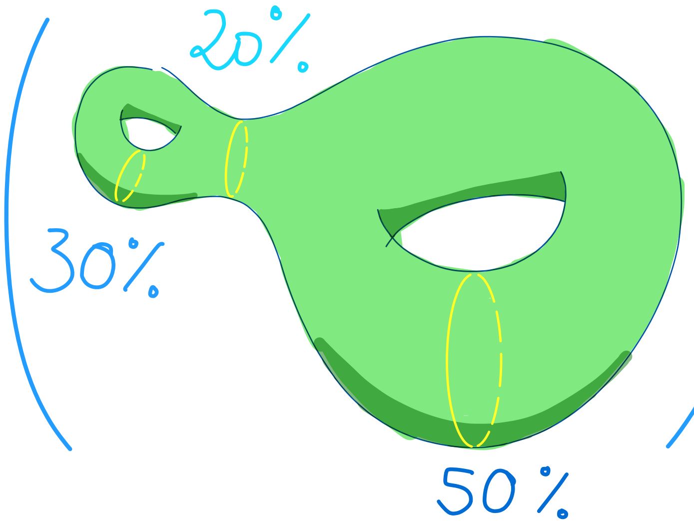
eg $\hat{\ell}^{\downarrow}\Bigg($  $\Bigg) = (50\%, 30\%, 20\%)$

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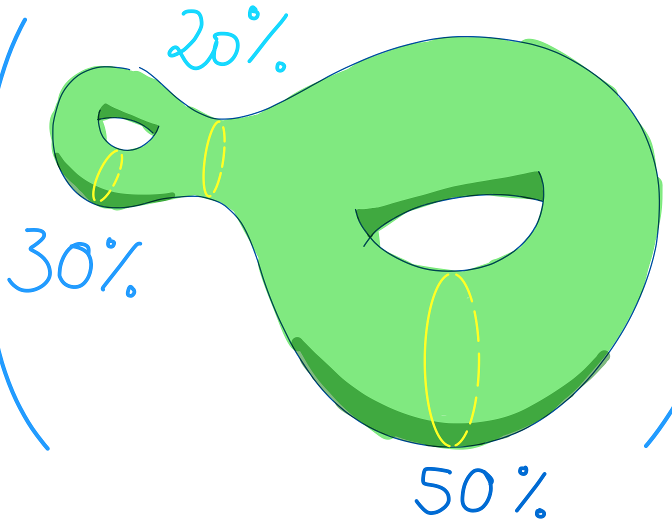
eg $\hat{\ell}^{\downarrow}\Big($ 

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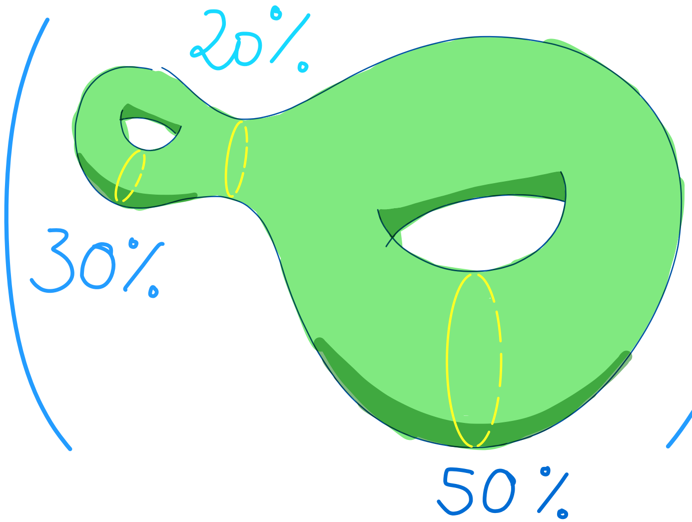
eg $\hat{\ell}^{\downarrow}\Bigg($ 

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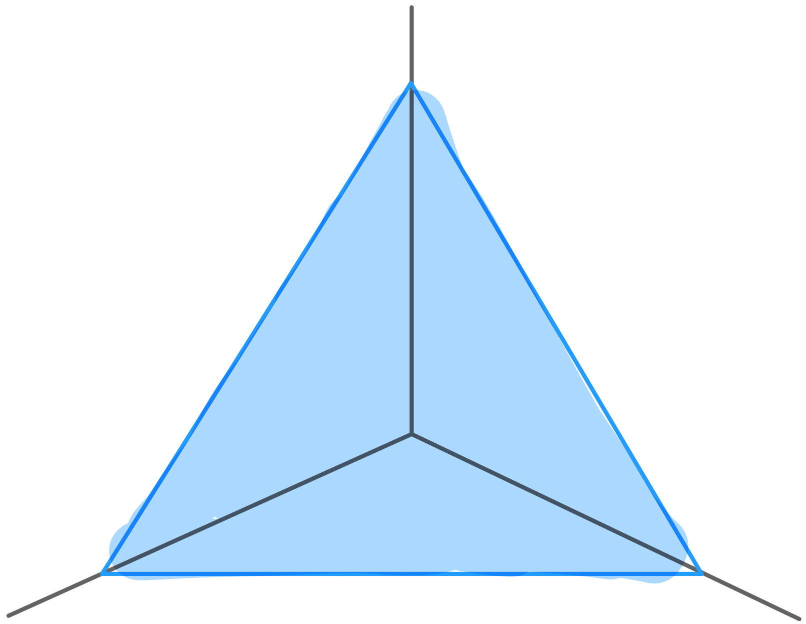
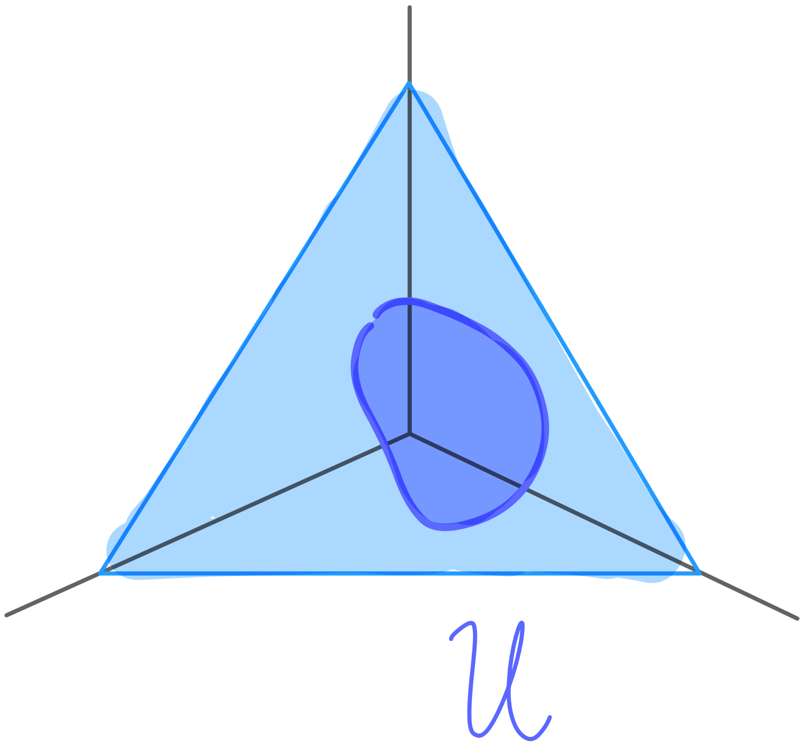
$$\Delta^{\infty} = \left\{ (x_1, x_2, \dots) \in \mathbb{R}^{\infty}_{\geq 0} \;\middle|\; \sum_{i=1}^{\infty} x_i = 1 \right\}$$

Δ^{∞}



$\mathcal{U} \subset \Delta^\infty$ open



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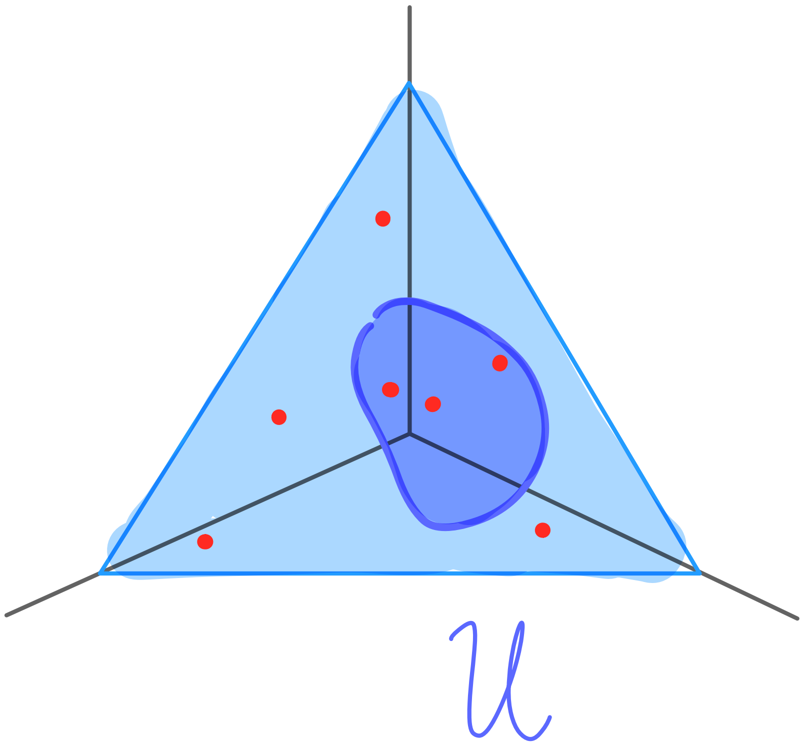
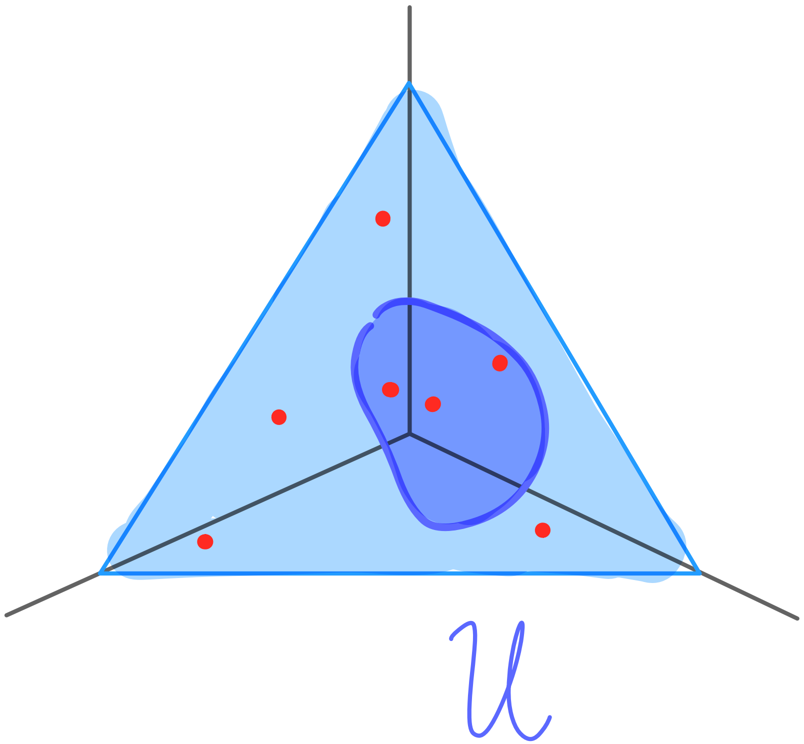


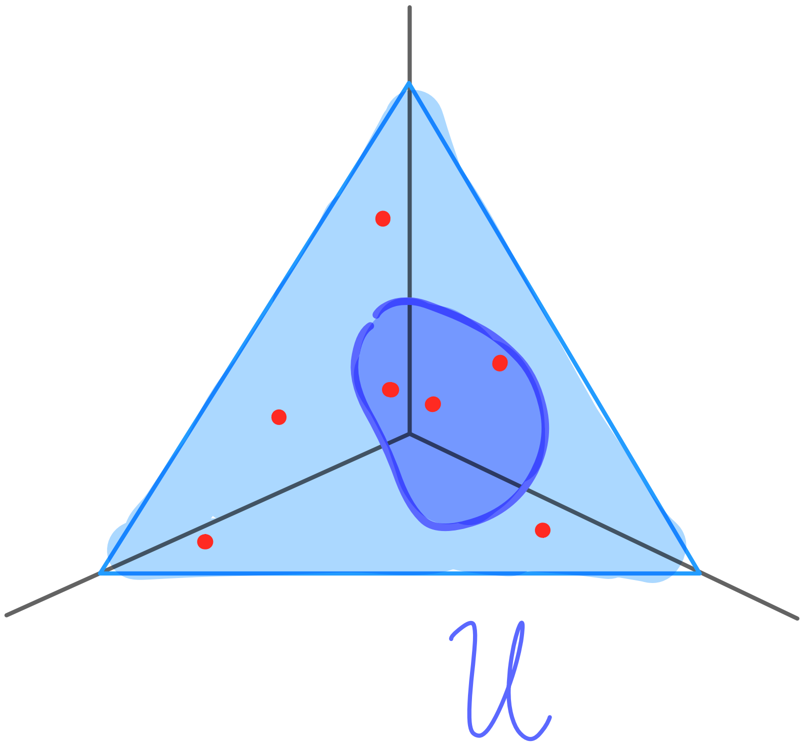
$\mathcal{U} \subset \Delta^{\infty}$ open

$$\mathbb{P}\left(\hat{\ell}^{\downarrow}(\gamma_{x,R}) \in \mathcal{U}\right)$$



$\mathcal{U} \subset \Delta^\infty$ open

$$\mathbb{P}\left(\hat{\ell}^{\downarrow}(\gamma_{x,R}) \in \mathcal{U}\right) = \frac{\#\{\bullet \mid \bullet \in \mathcal{U}\}}{\#\{\bullet\}}$$



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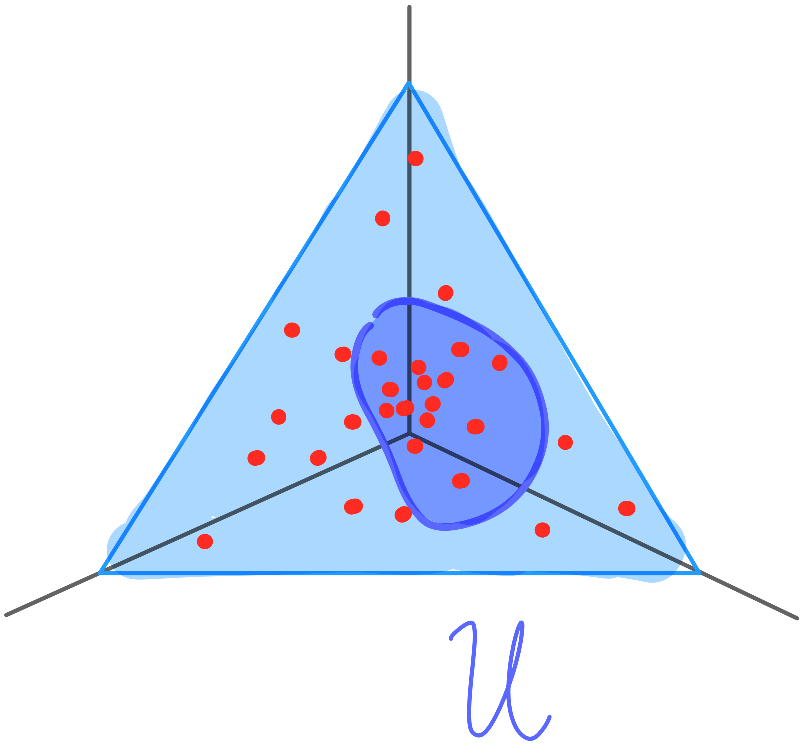

$\mathcal{U} \subset \Delta^{\infty}$ open

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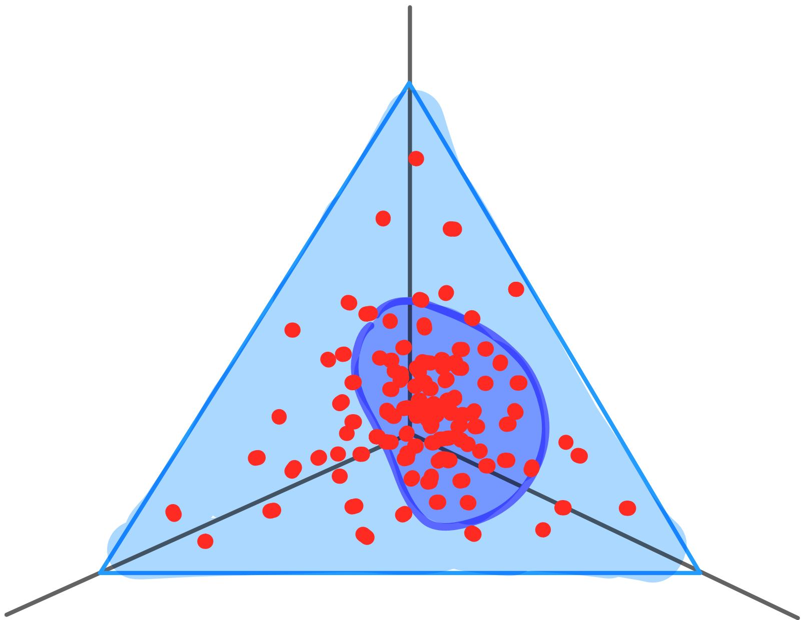

$\mathcal{U}$

$\mathcal{U} \subset \Delta^{\infty}$ open

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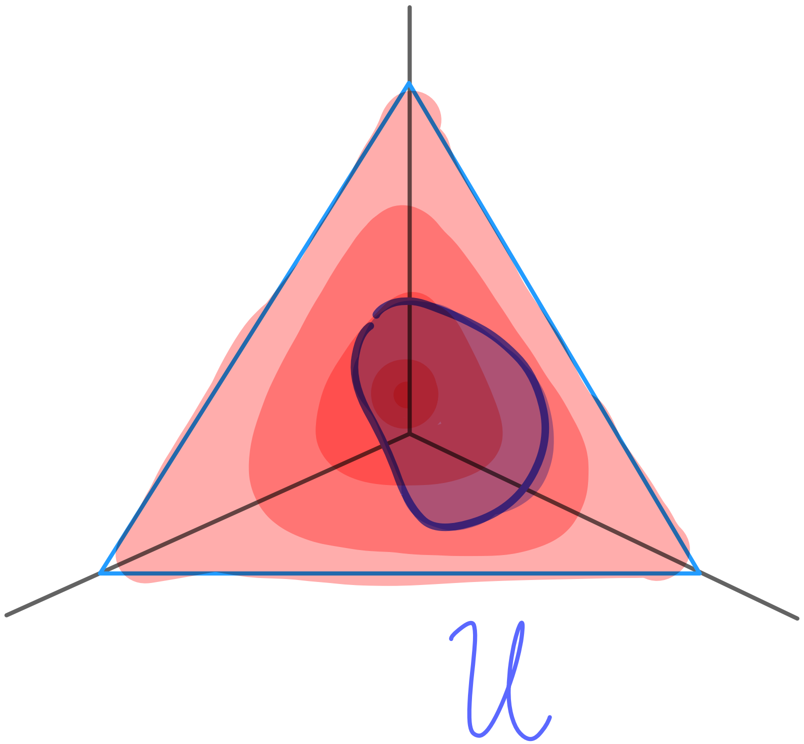


Theorem (Delecroix - L '22)

Theorem (Delecroix - L '22)

$$\hat{\ell}^{\downarrow}(\gamma_{X,R})$$

Theorem (Delecroix - L '22)

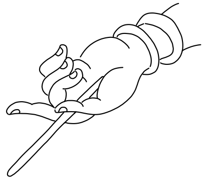
$$\hat{\ell}^{\downarrow}(\gamma_{x,R}) \xrightarrow[R \to \infty]{}$$

Theorem (Delecroix - L '22)

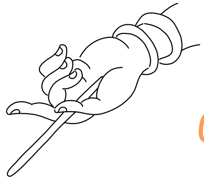
- $\hat{\ell}^{\downarrow}(\gamma_{X,R}) \xrightarrow[R\to\infty]{(d)} \mathcal{L}_g$

Theorem (Delecroix – L '22) ∘ does not depend on X

∘ $\hat{\ell}^{\downarrow}(\gamma_{X,R}) \xrightarrow[R\to\infty]{(d)} \mathcal{L}_g$



Theorem (Delecroix - L '22)

- $\hat{\ell}^{\downarrow}(\gamma_{X,R}) \xrightarrow[R\to\infty]{(d)} \mathcal{L}_g$ 

- does not depend on X
- explicit

Theorem (Delecroix – L '22)

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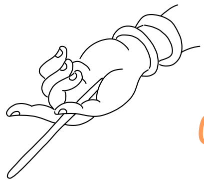

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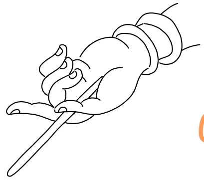
$$\mathcal{L}_g \xrightarrow[g\to\infty]{}$$

Theorem (Delecroix – L '22)

- $\hat{\ell}^{\downarrow}(\gamma_{X,R}) \xrightarrow[R \to \infty]{(d)} \mathcal{L}_g$ 

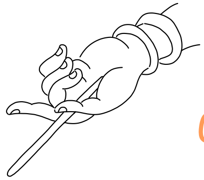
- $\mathcal{L}_g \xrightarrow[g \to \infty]{(d)}$ Poisson–Dirichlet distribution of parameter $1/2$

∘ does not depend on X

∘ explicit

Theorem (Delecroix – L '22)

- $\hat{\ell}^{\downarrow}(\gamma_{X,R}) \xrightarrow[R\to\infty]{(d)} \mathcal{L}_g$



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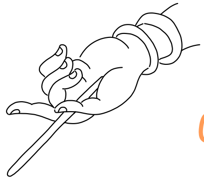
Corollary As $g \to \infty$, on average

Theorem (Delecroix – L '22)

- does ~~not~~ not depend on X
- explicit



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Corollary As $g \to \infty$, on average longest component

Theorem (Delecroix – L '22)

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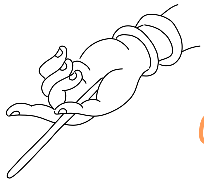
Corollary As $g \to \infty$, on average longest component $\to 75.8\%$

Theorem (Delecroix – L '22)

- $\hat{\ell}^{\downarrow}(\gamma_{X,R}) \xrightarrow[R\to\infty]{(d)} \mathcal{L}_g$



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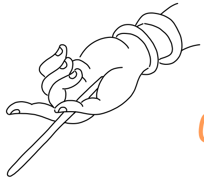
Corollary As $g \to \infty$, on average

longest component $\longrightarrow 75.8\%$

$2^{\text{nd}} \longrightarrow 17.1\%$

Theorem (Delecroix – L '22)

- $\hat{\ell}^{\downarrow}(\gamma_{X,R}) \xrightarrow[R\to\infty]{(d)} \mathcal{L}_g$ 
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Corollary As $g \to \infty$, on average

longest component $\longrightarrow$ 75.8%

2^{nd} $\longrightarrow$ 17.1%

3^{rd} $\longrightarrow$ 4.9%

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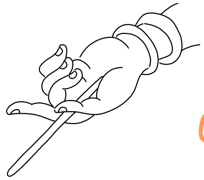
- $\hat{\ell}^{\downarrow}(\gamma_{X,R}) \xrightarrow[R\to\infty]{(d)} \mathcal{L}_g$ 
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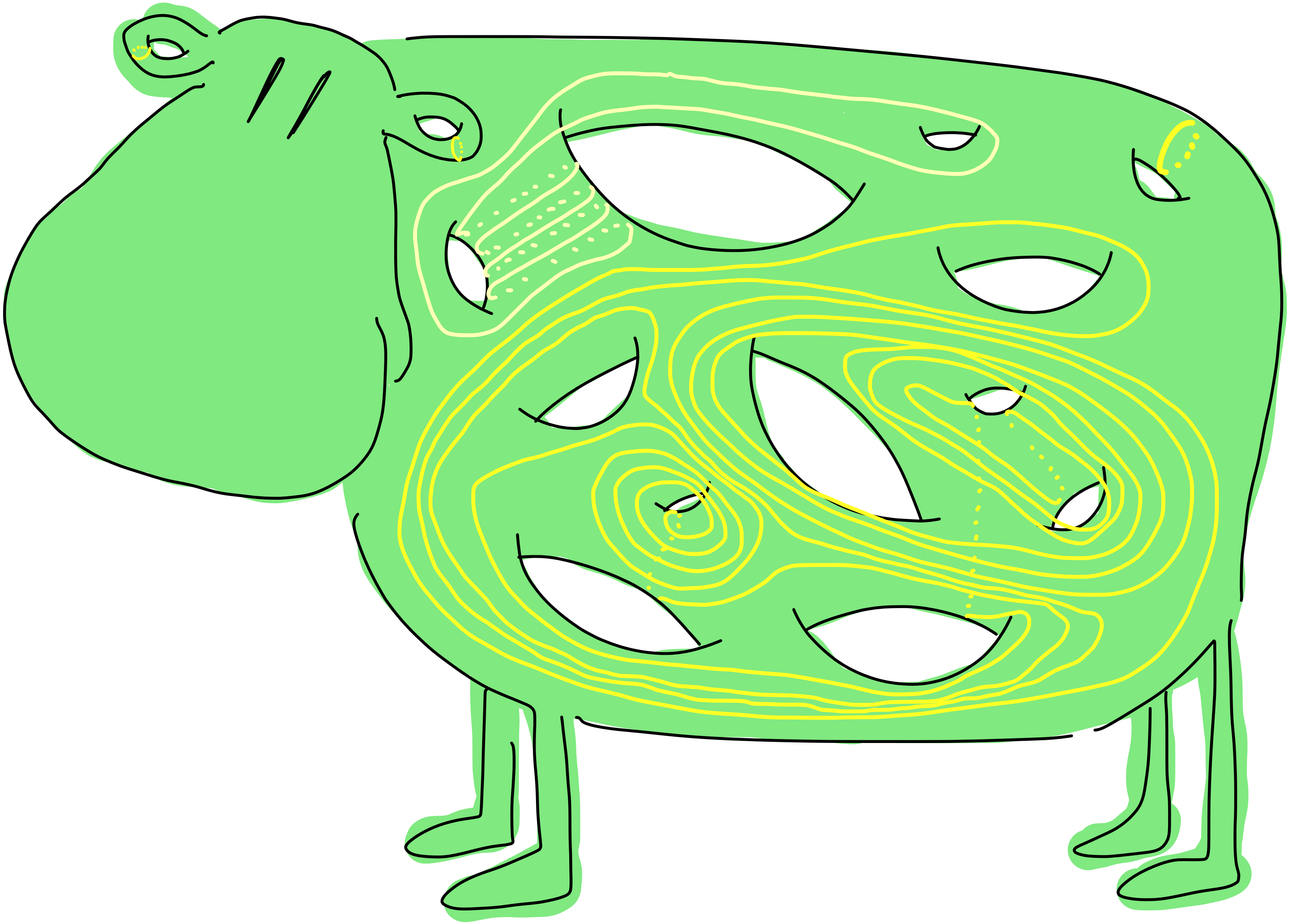
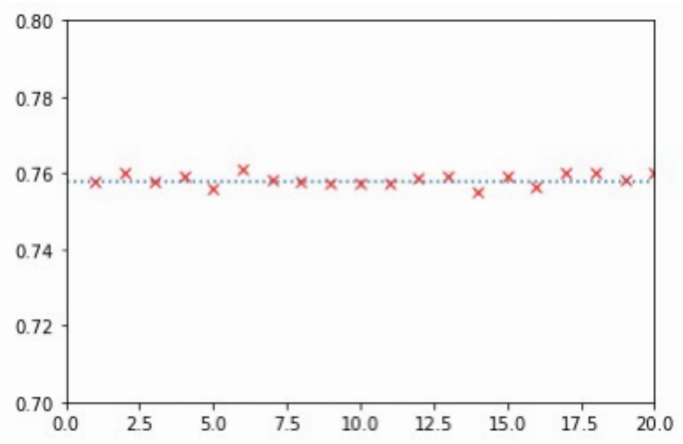
Corollary As $g \to \infty$, on average

longest component $\longrightarrow$ 75.8%
2^{nd} $\longrightarrow$ 17.1%
3^{rd} $\longrightarrow$ 4.9%

$\Big\}$ 97.8%

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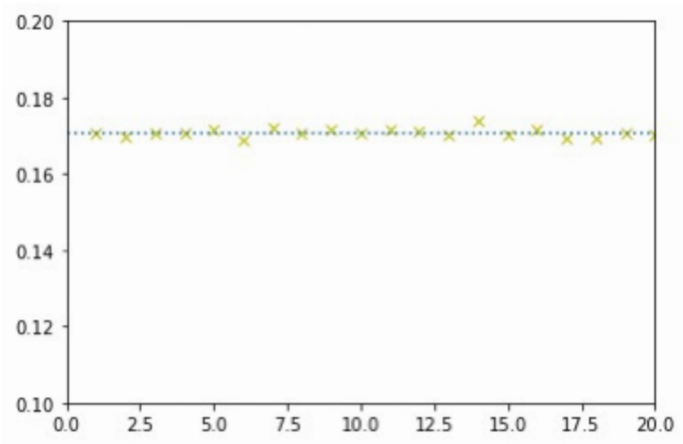
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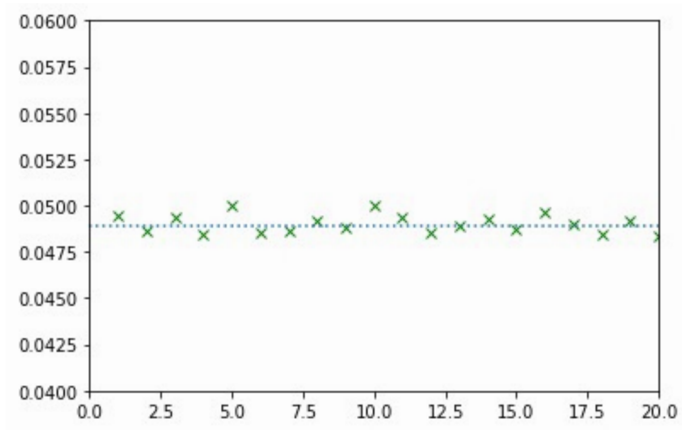
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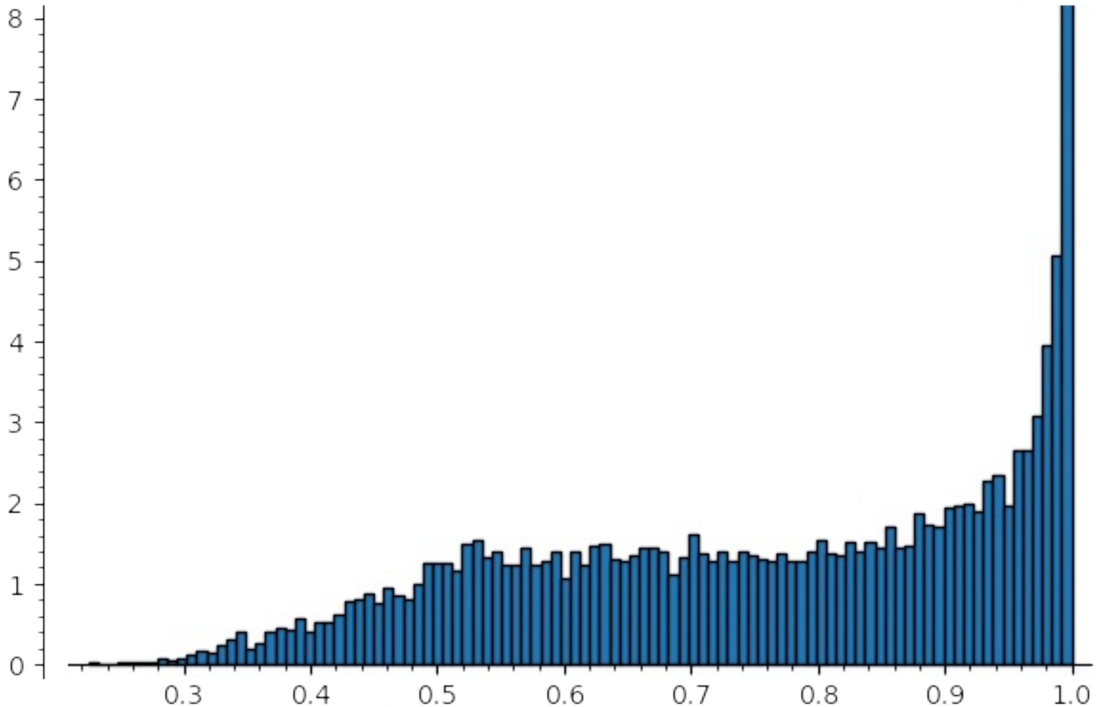
1st 2nd 3rd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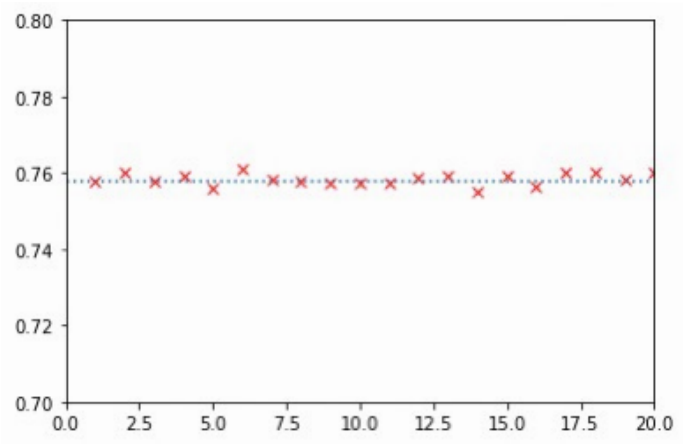
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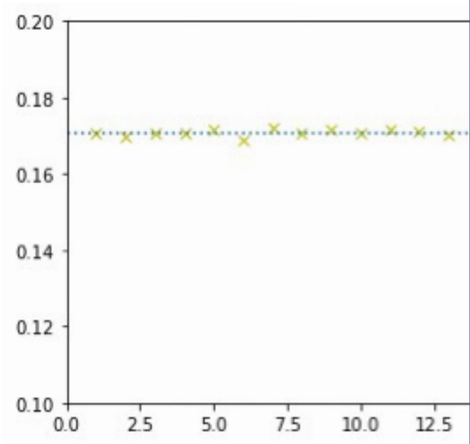
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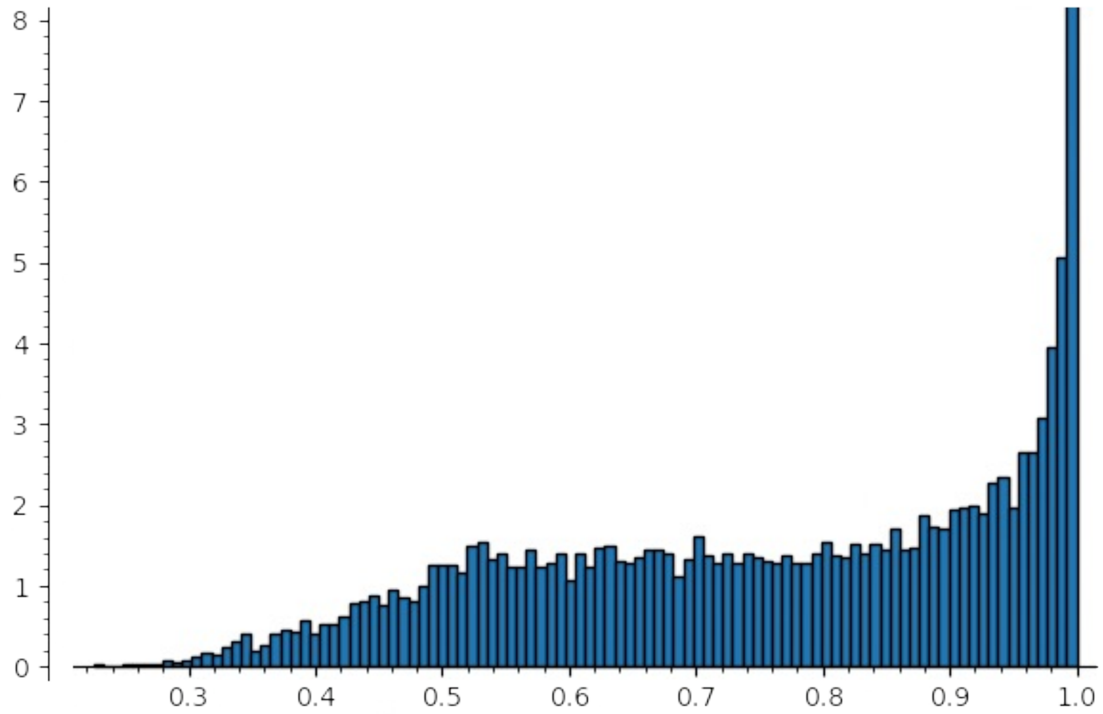
1st

2nd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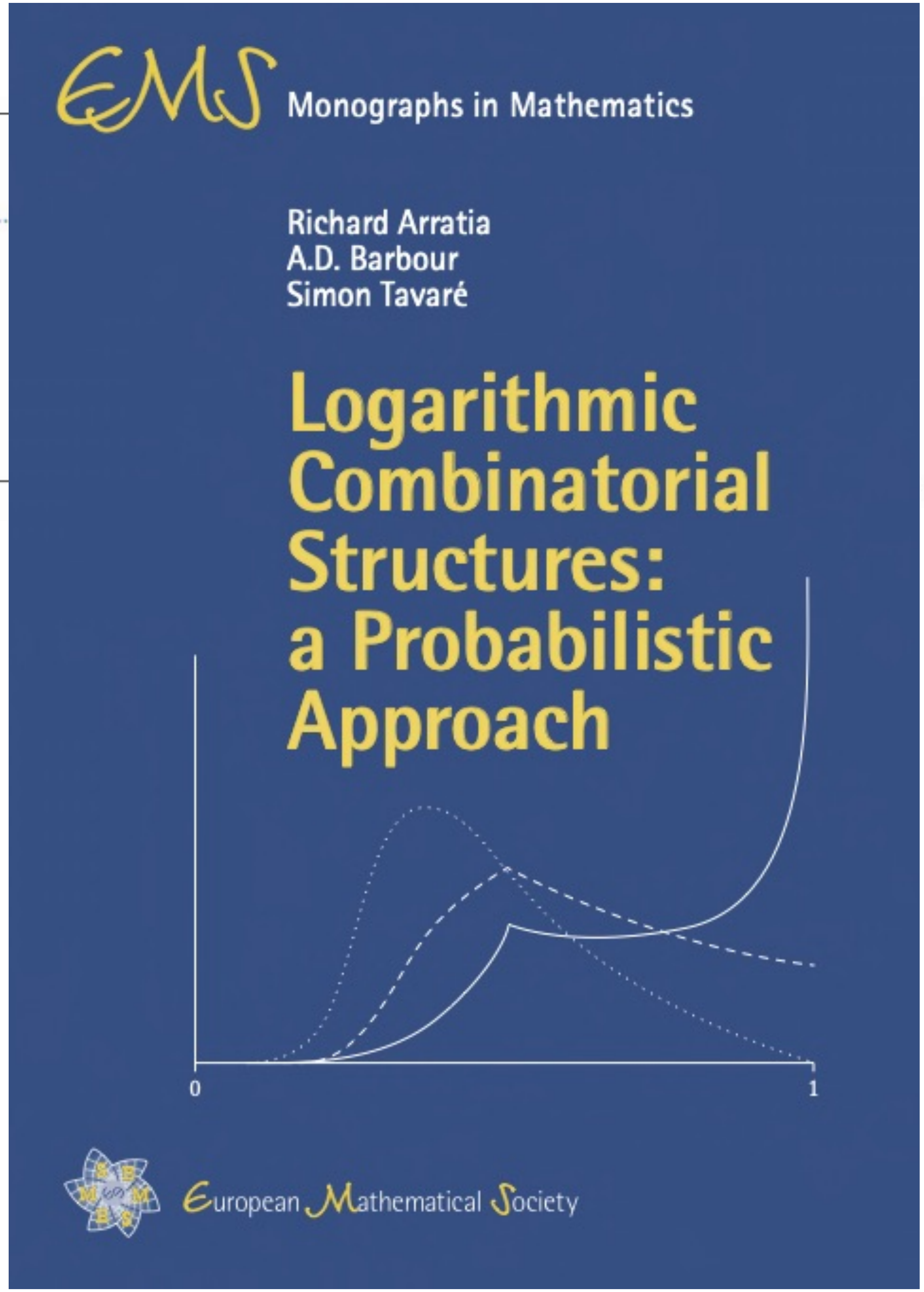
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Richard Arratia
A.D. Barbour
Simon Tavaré
Logarithmic Combinatorial Structures: a Probabilistic Approach
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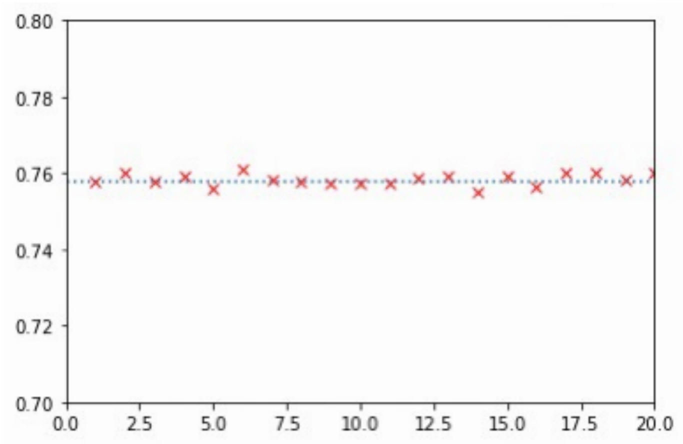
European Mathematical Society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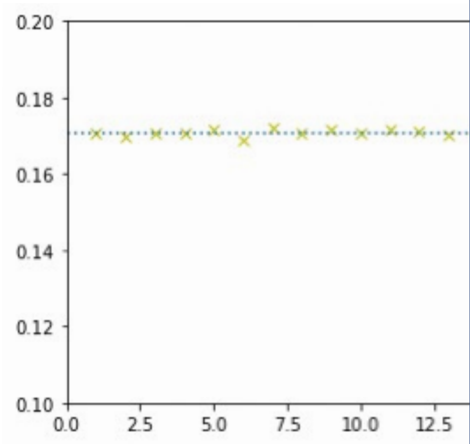
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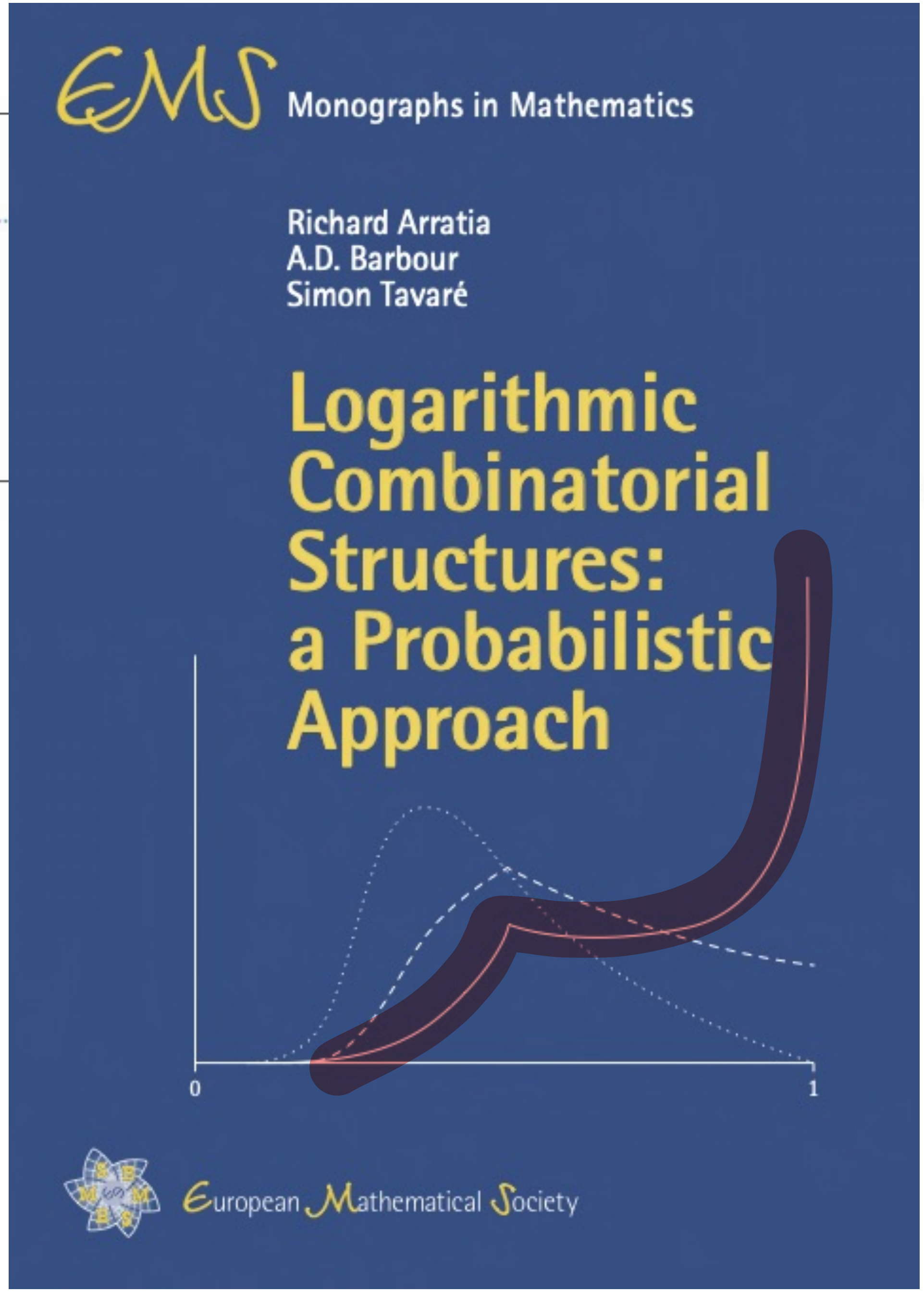
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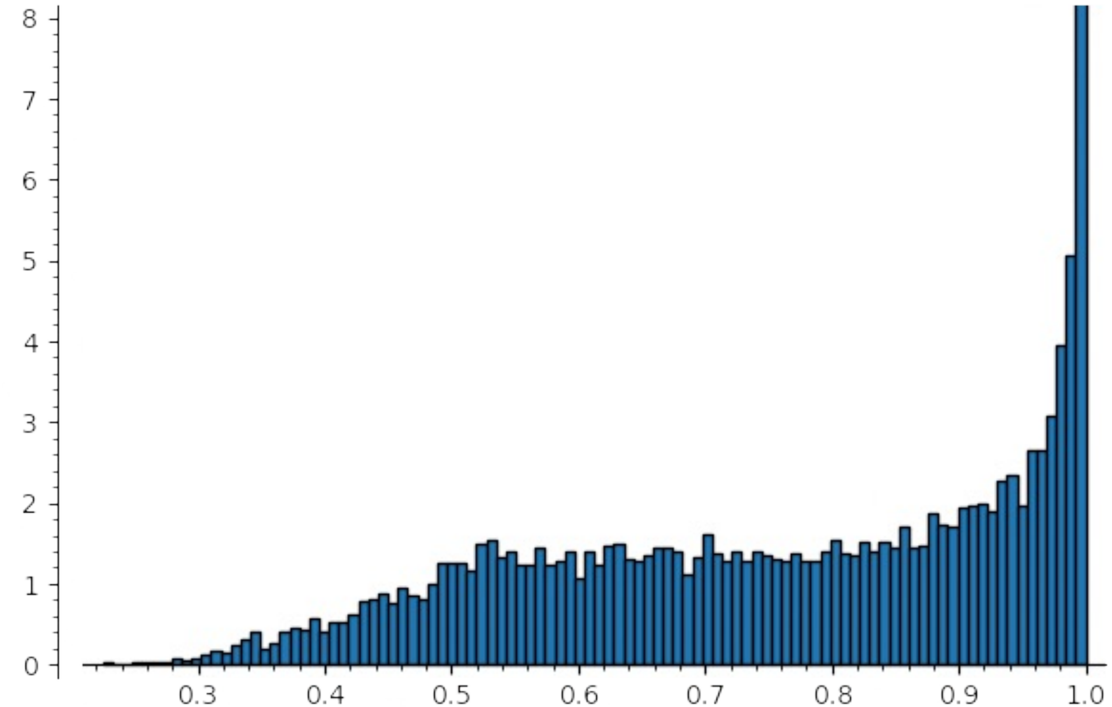
1st

2nd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Poisson–Dirichlet distribution

What is a

Poisson–Dirichlet distribution?

What is a Poisson–Dirichlet distribution?

many equivalent definitions

What is a

Poisson–Dirichlet distribution?

many equivalent definitions

- normalization of a Poisson point process

What is a Poisson–Dirichlet distribution?

many equivalent definitions

- normalization of a Poisson point process
- limit Dirichlet distributions

What is a

Poisson – Dirichlet distribution ?

many equivalent definitions

- normalization of a Poisson point process
- limit Dirichlet distributions
- stick – breaking process

What is a
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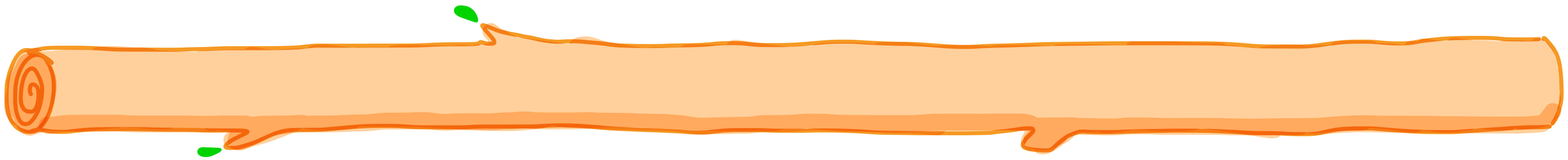
Poisson–Dirichlet distribution

stick-breaking process

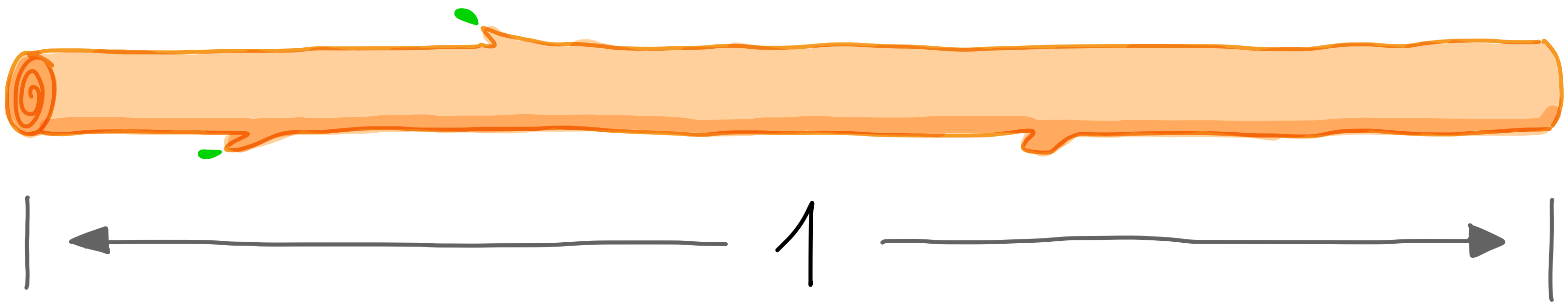
Poisson–Dirichlet distribution

stick-breaking process



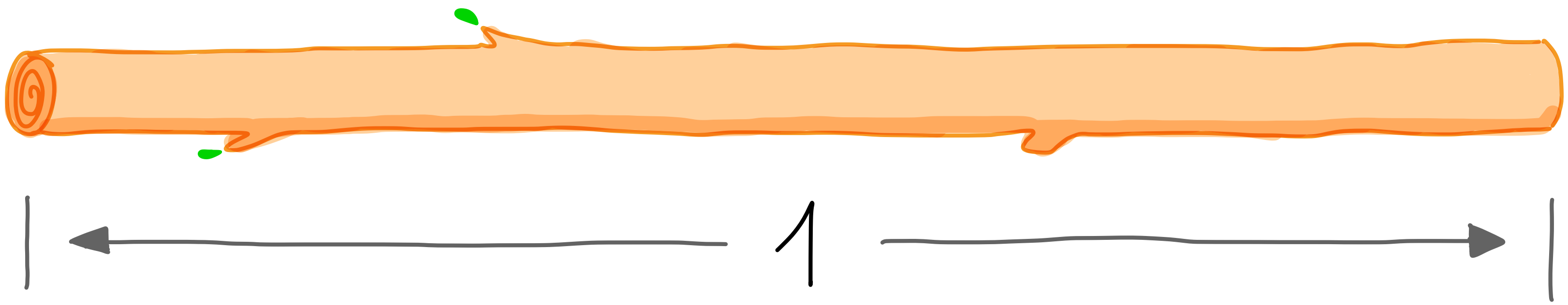
Poisson–Dirichlet distribution

stick-breaking process



Poisson–Dirichlet distribution

$U_1, U_2, \ldots$ a sequence of iid Uniform$([0,1])$ random variables



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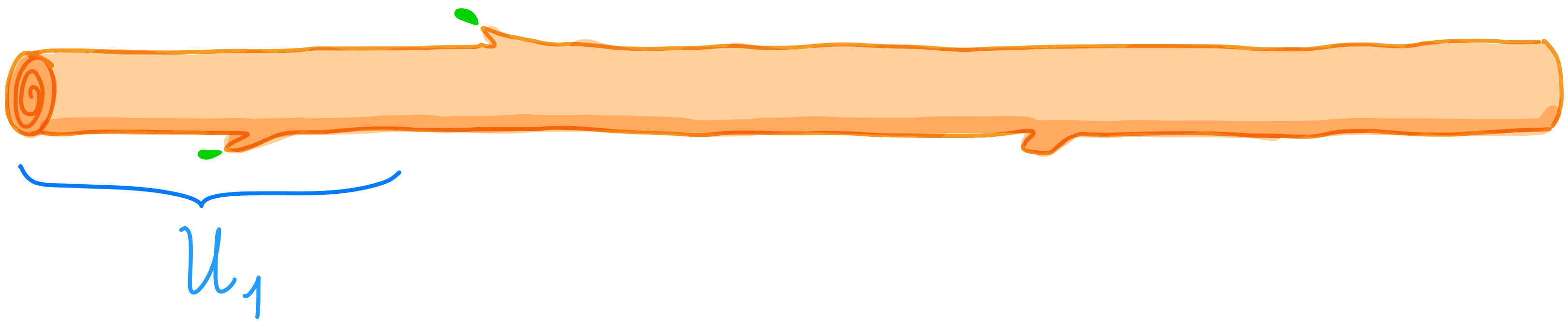

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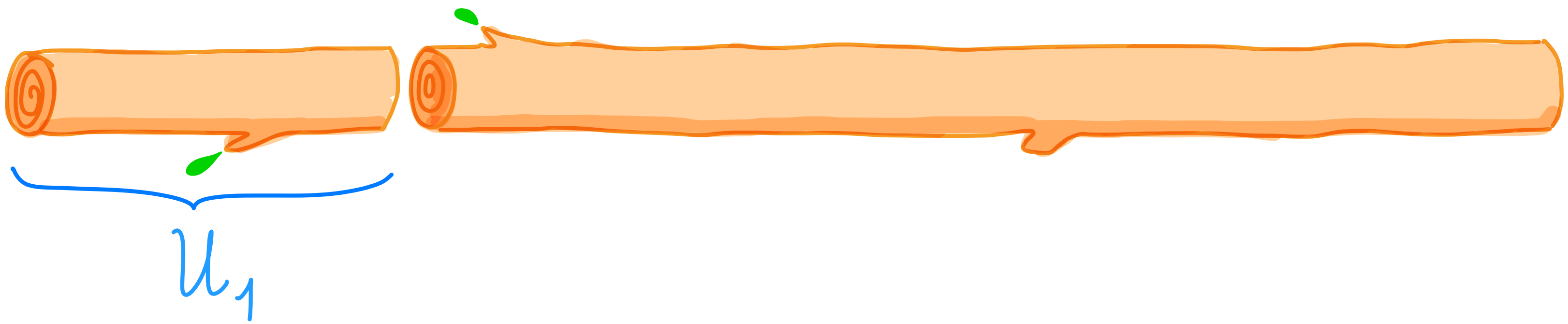
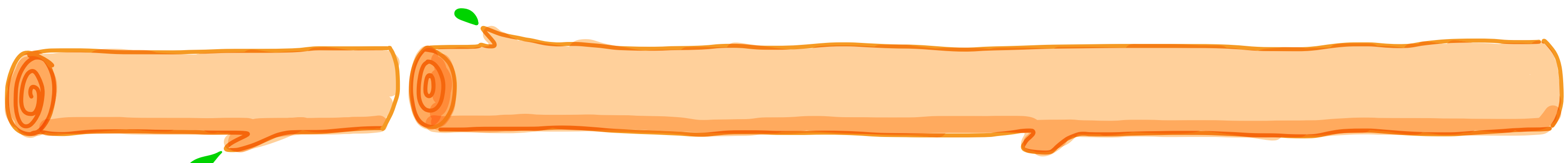
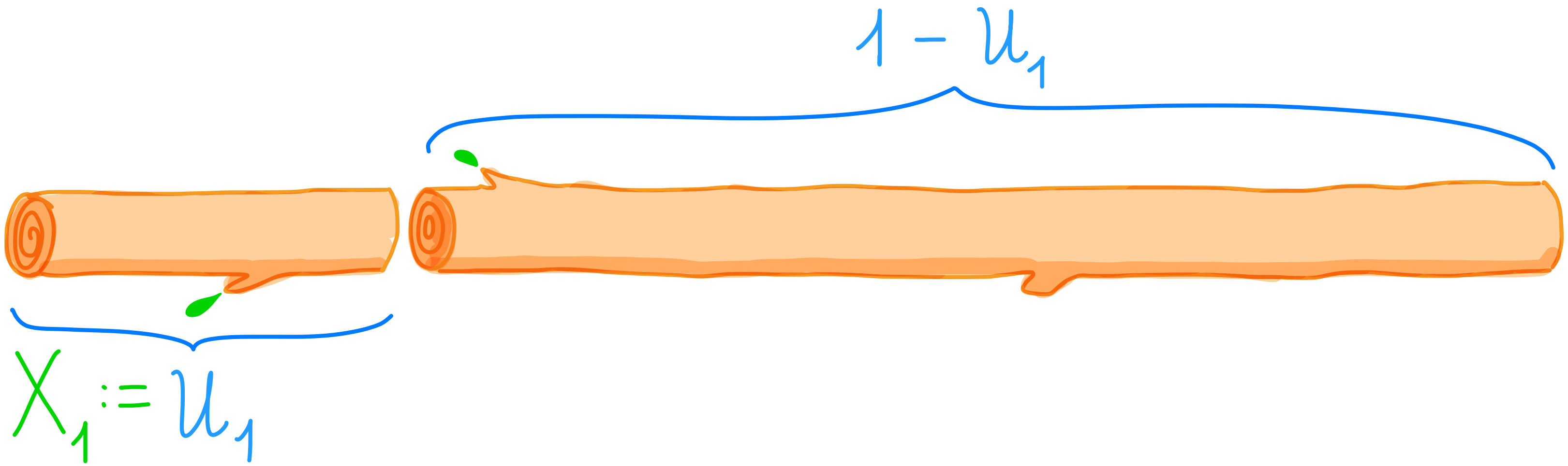


Poisson–Dirichlet distribution

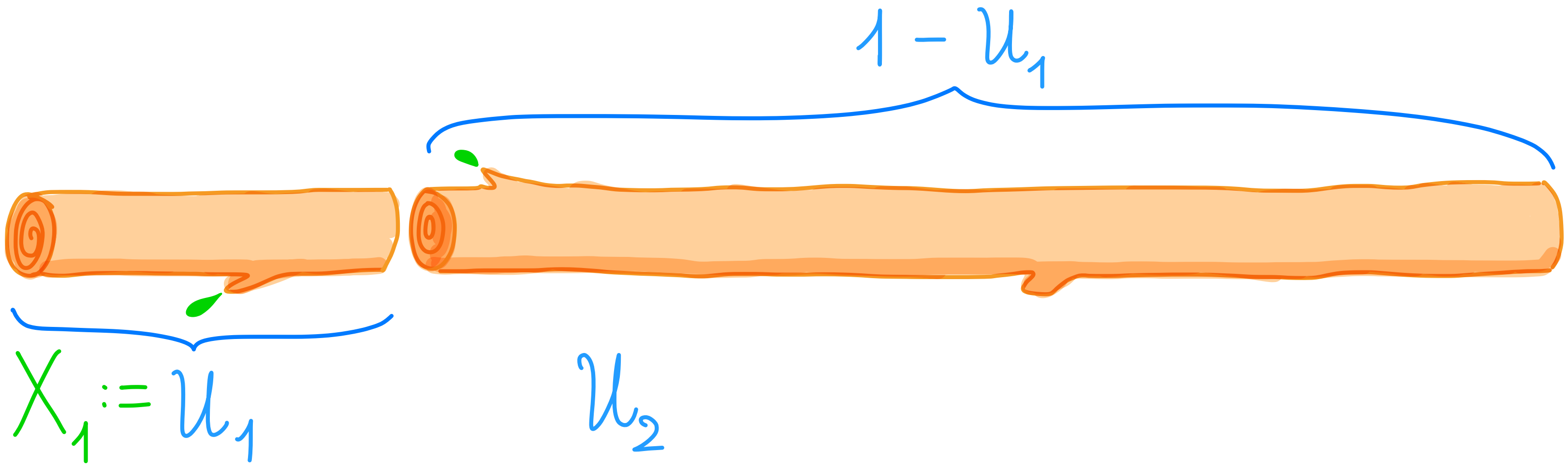


$$\underbrace{}_{X_1 := U_1}$$

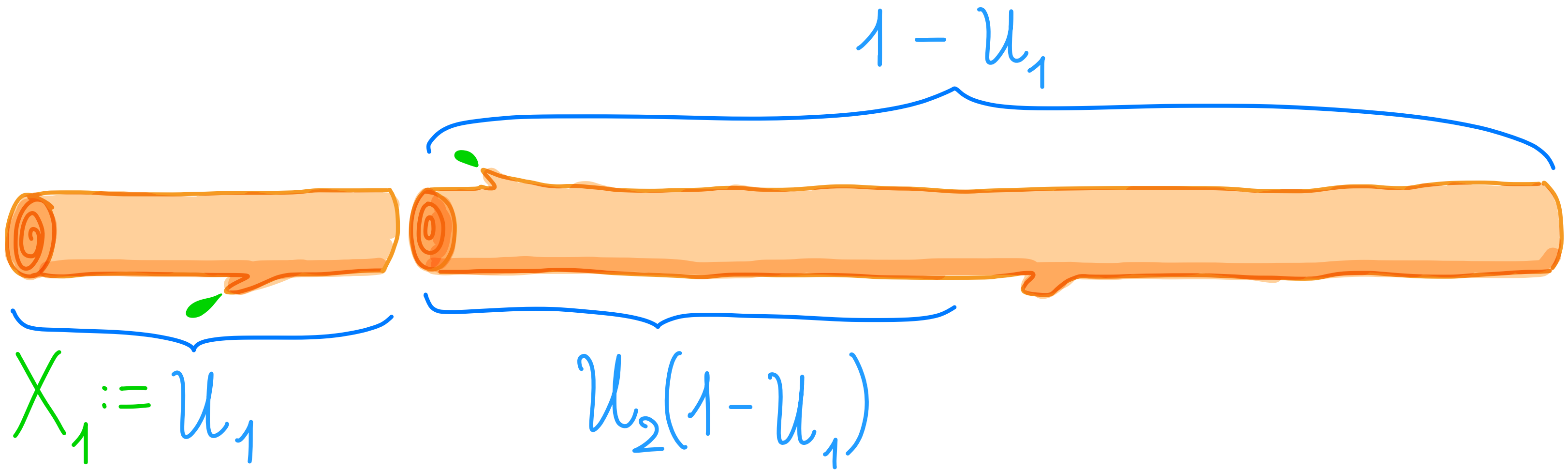
Poisson–Dirichlet distribution



Poisson – Dirichlet distribution



Poisson–Dirichlet distribution



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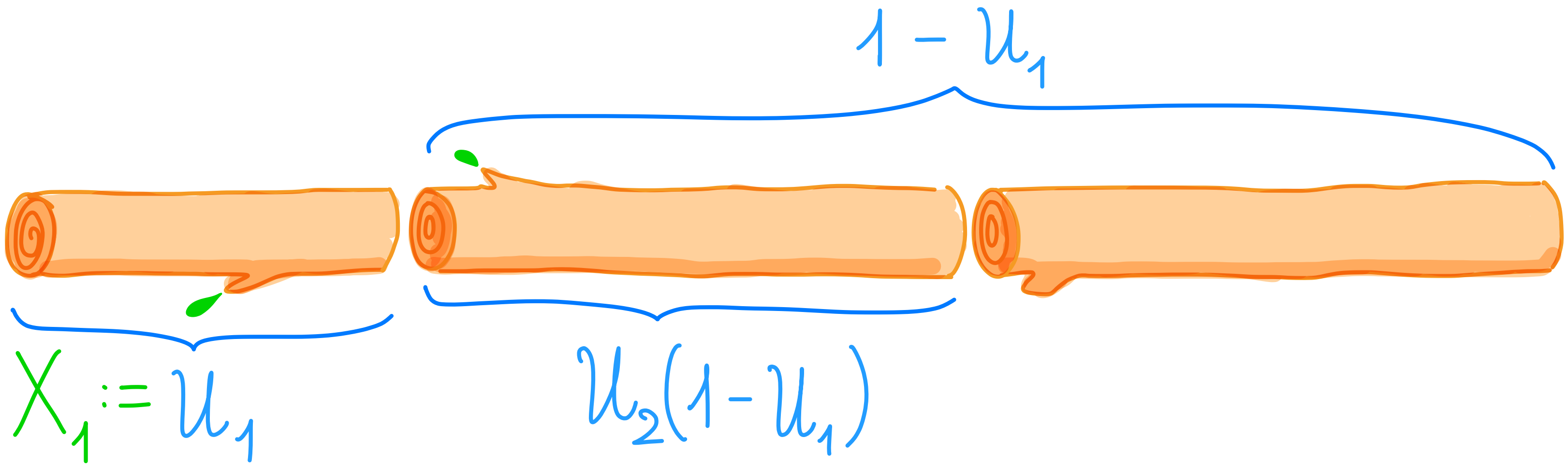
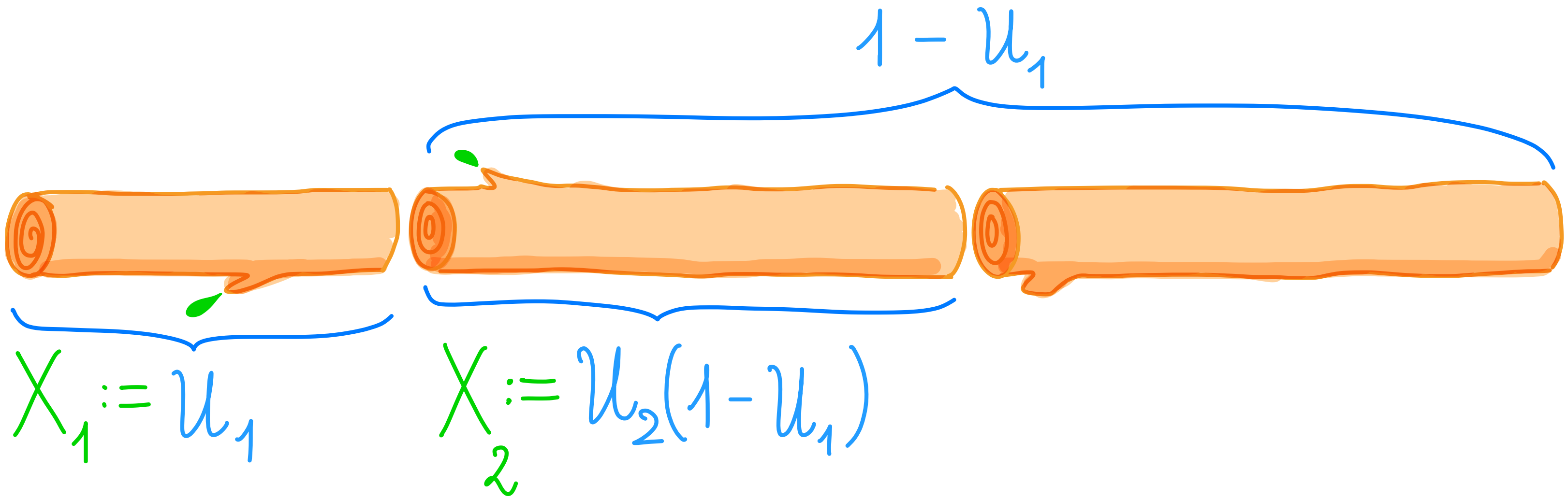
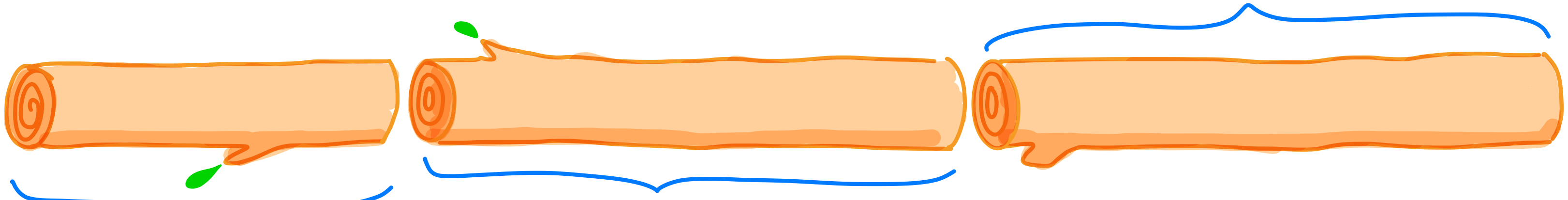


Poisson–Dirichlet distribution



Poisson–Dirichlet distribution

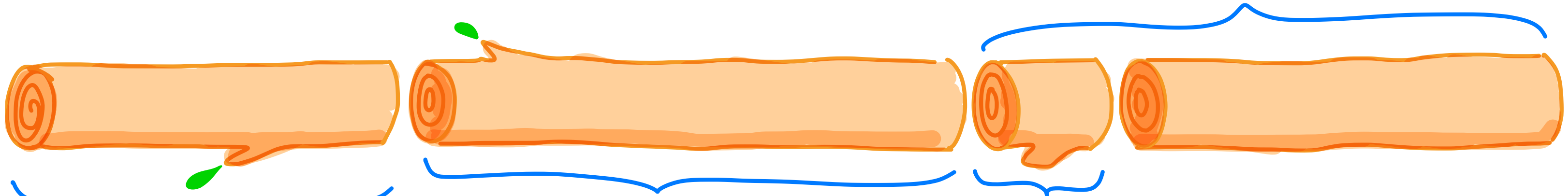
$$(1-U_2)(1-U_1)$$



$$X_1 := U_1 \qquad X_2 := U_2(1-U_1)$$

Poisson–Dirichlet distribution

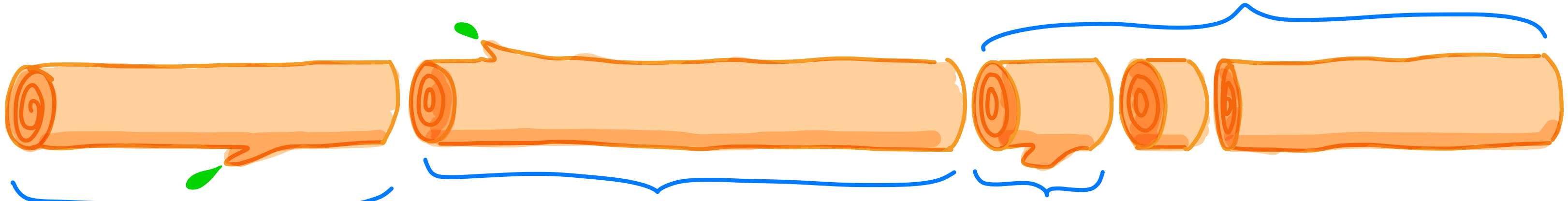
$(1-U_2)(1-U_1)$



$X_1 := U_1 \qquad X_2 := U_2(1-U_1) \qquad X_3 := U_3(1-U_2)(1-U_1)$

Poisson–Dirichlet distribution

$(1-U_2)(1-U_1)$



$X_1 := U_1$ $\qquad$ $X_2 := U_2(1-U_1)$ $\qquad$ $X_3 := U_3(1-U_2)(1-U_1)$

Poisson–Dirichlet distribution

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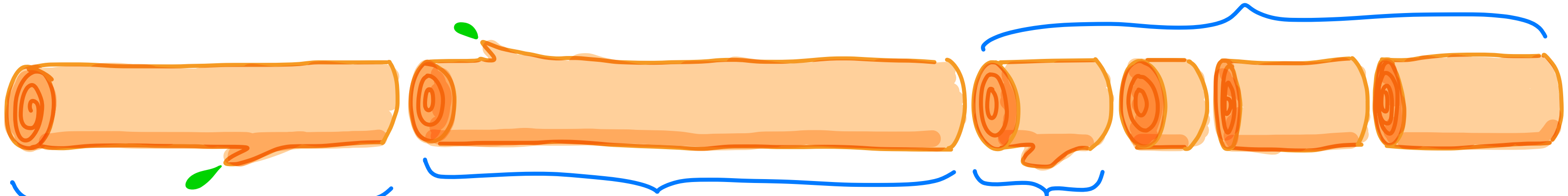

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Poisson–Dirichlet distribution

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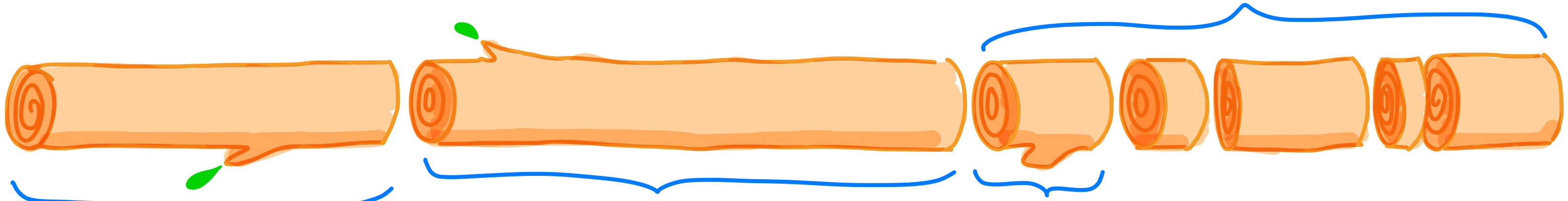
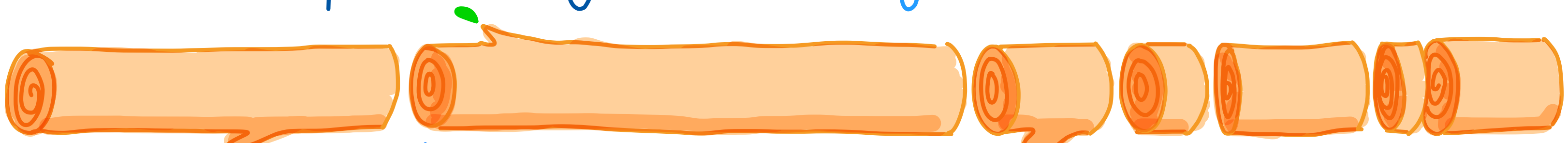


$X_1 := U_1 \qquad X_2 := U_2(1-U_1) \qquad X_3 := U_3(1-U_2)(1-U_1)$

Poisson–Dirichlet distribution

$\mathcal{U}_1, \mathcal{U}_2, \ldots$ a sequence of iid Uniform$([0,1])$ random variables

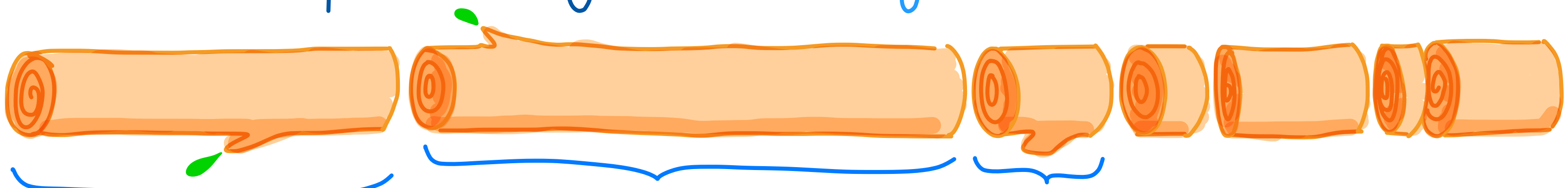


$X_1 := \mathcal{U}_1$ $\quad$ $X_2 := \mathcal{U}_2(1-\mathcal{U}_1)$ $\quad$ $X_3 := \mathcal{U}_3(1-\mathcal{U}_2)(1-\mathcal{U}_1)$

$$X_1 = \mathcal{U}_1$$
$$X_2 = \mathcal{U}_2(1-\mathcal{U}_1)$$
$$X_3 = \mathcal{U}_3(1-\mathcal{U}_2)(1-\mathcal{U}_1)$$

Poisson–Dirichlet distribution

$U_1, U_2, \ldots$ a sequence of iid Uniform([0,1]) random variables



$X_1 := U_1$ $\quad X_2 := U_2(1-U_1)$ $\quad X_3 := U_3(1-U_2)(1-U_1)$

$$X_1 = U_1$$

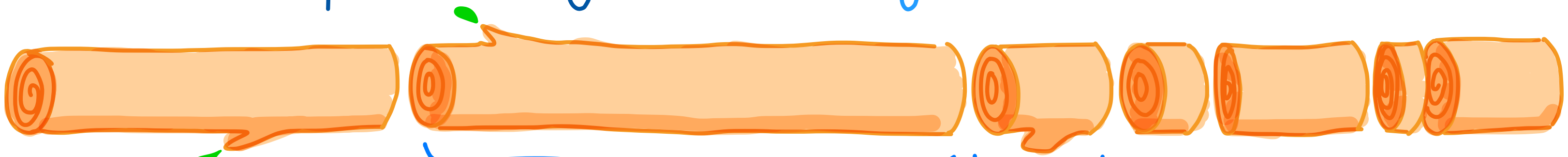
$$X_2 = U_2(1-U_1)$$

$$X_3 = U_3(1-U_2)(1-U_1)$$

almost surely, $X_1 + X_2 + \cdots = 1$

Poisson–Dirichlet distribution

$U_1, U_2, \ldots$ a sequence of iid Uniform$([0,1])$ random variables



$X_1 := U_1$ $\quad$ $X_2 := U_2(1-U_1)$ $\quad$ $X_3 := U_3(1-U_2)(1-U_1)$

$$X_1 = U_1$$
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almost surely, $X_1 + X_2 + \cdots = 1$

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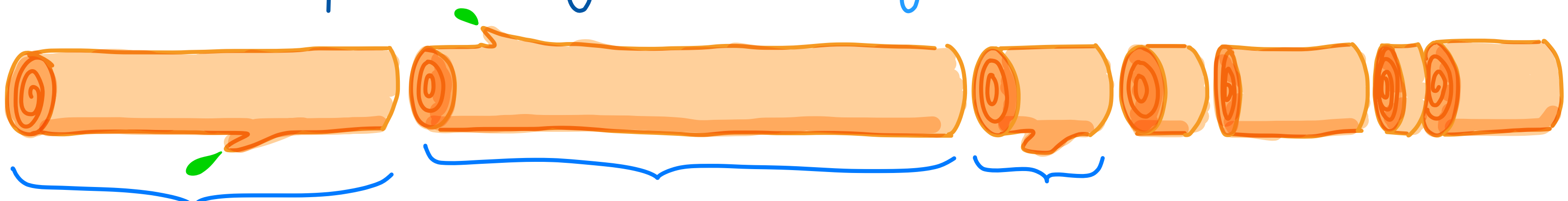
Poisson–Dirichlet distribution

$U_1, U_2, \dots$ a sequence of iid Uniform$([0,1])$ random variables



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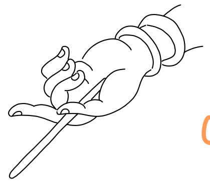
Definition $\quad X^\downarrow \sim PD(1)$

Theorem (Delecroix – L '22)

- $\hat{\ell}^{\downarrow}(\gamma_{X,R}) \xrightarrow[R \to \infty]{(d)} \mathcal{L}_g$ 
- $\mathcal{L}_g \xrightarrow[g \to \infty]{(d)}$ Poisson–Dirichlet distribution of parameter $1/2$

- does not depend on X
- explicit

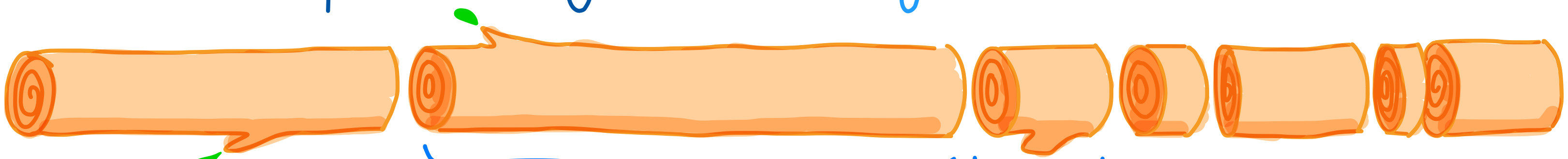
Corollary As $g \to \infty$, on average

longest component $\longrightarrow$ 75.8%
2nd $\longrightarrow$ 17.1%
3rd $\longrightarrow$ 4.9%

$\Bigg\}$ 97.8%

Poisson–Dirichlet distribution

$U_1, U_2, \ldots$ a sequence of iid Uniform([0,1]) random variables



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1/2 ?

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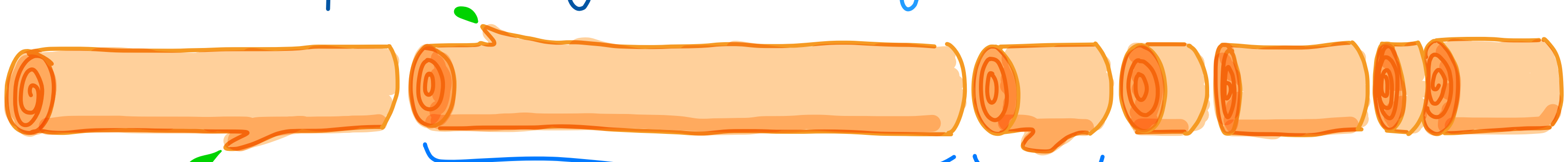

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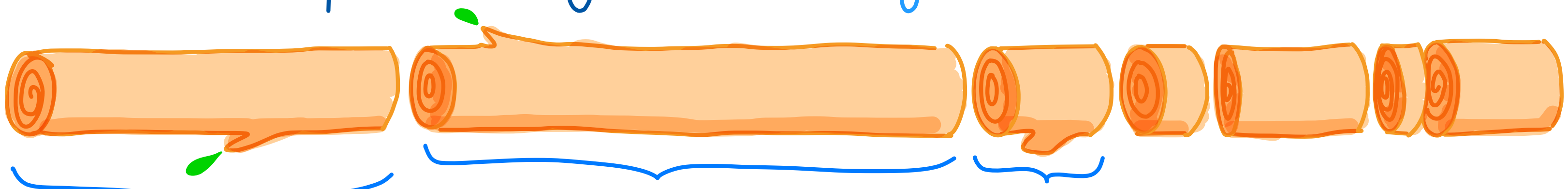

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Poisson–Dirichlet distribution

density

$\mathbb{1}_{(0,1)}(x) \cdot \theta \cdot (1-x)^{\theta-1}$

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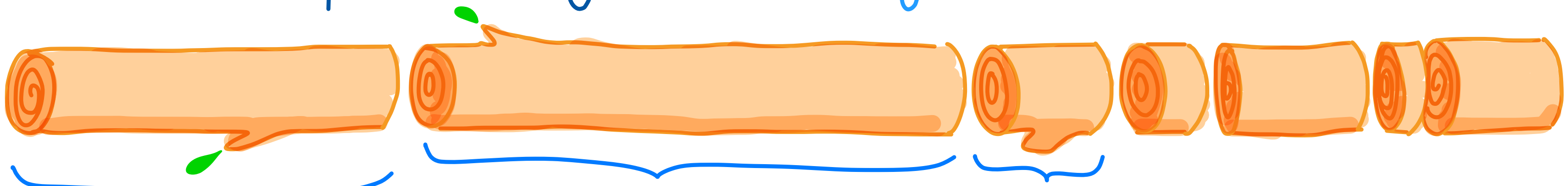

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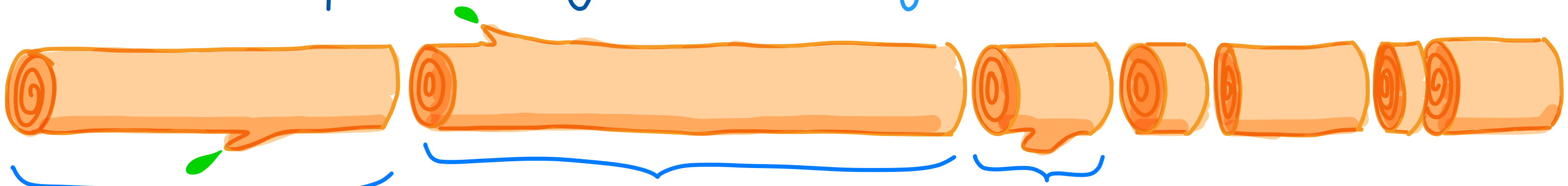

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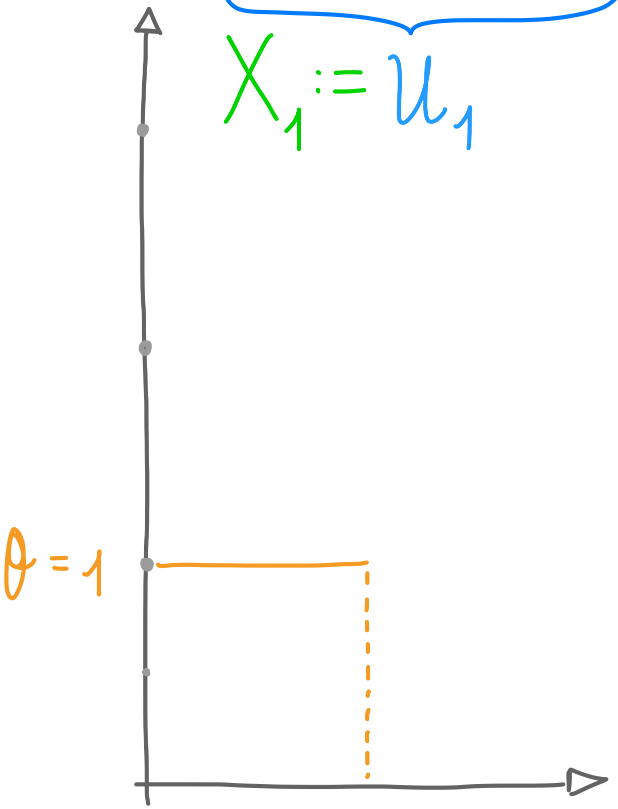

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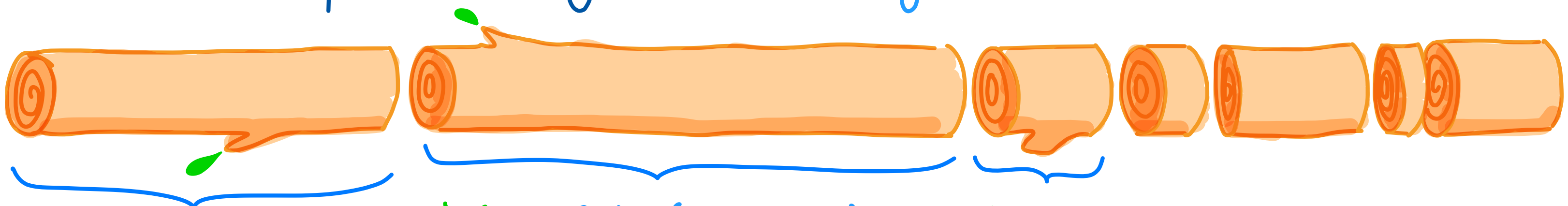

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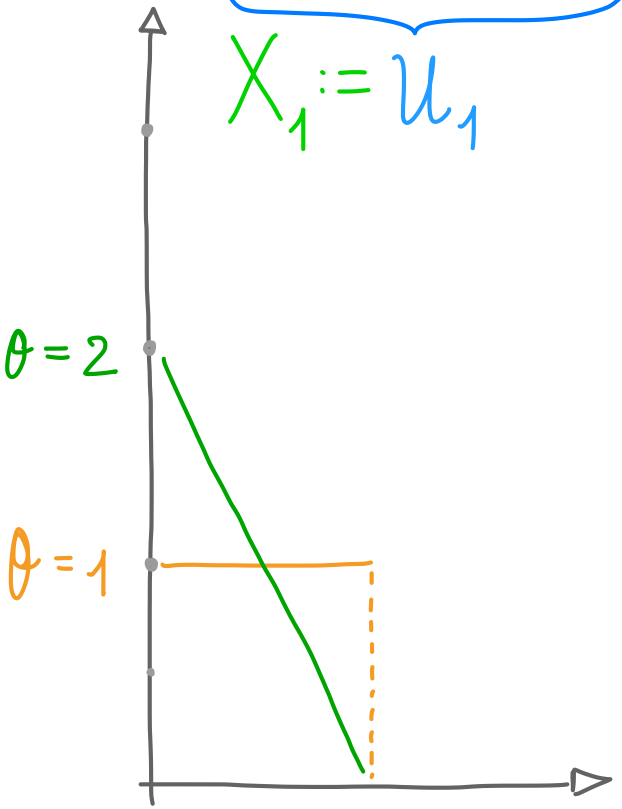

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Poisson–Dirichlet distribution

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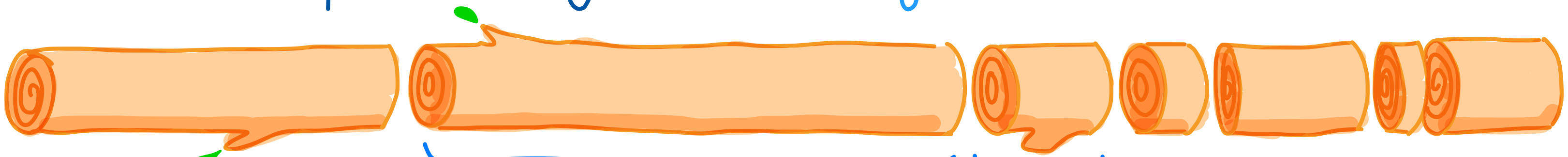

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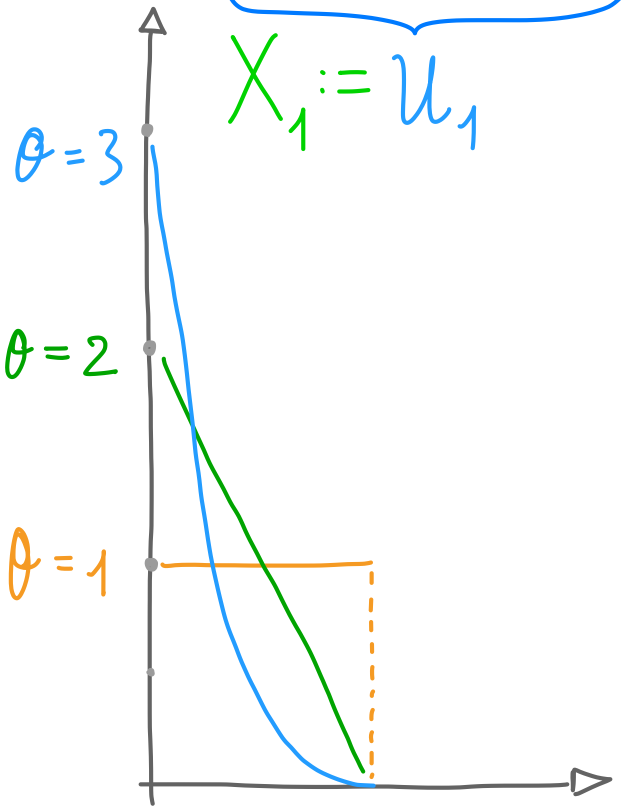

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Definition $X^\downarrow \sim PD(1)$ $\theta > 0$

Poisson–Dirichlet distribution

density

$\mathbb{1}_{(0,1)}(x)\cdot\theta\cdot(1-x)^{\theta-1}$

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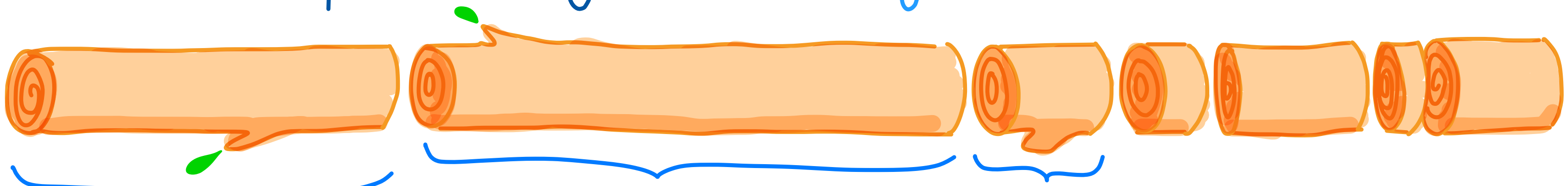

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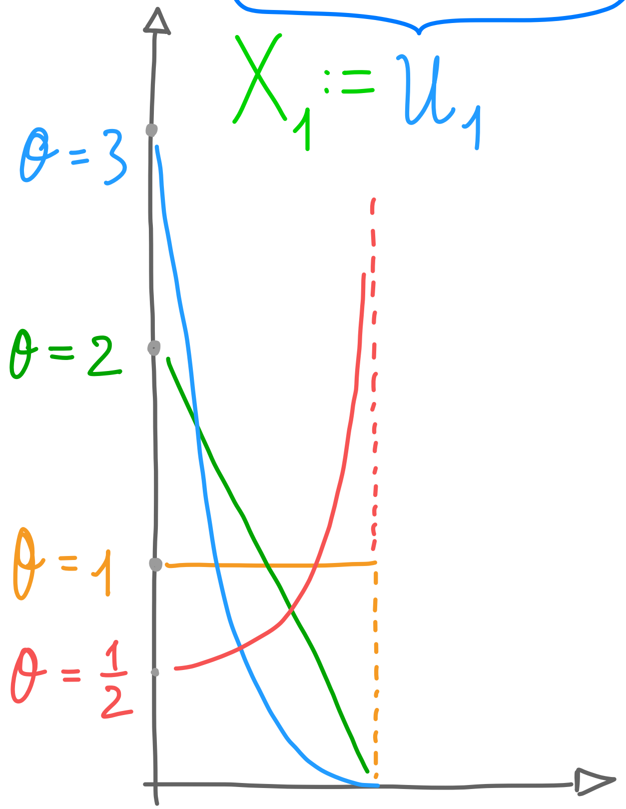

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Definition $X^{\downarrow} \sim PD(\cancel{1}\ \theta)$, $\theta > 0$

Poisson–Dirichlet distribution

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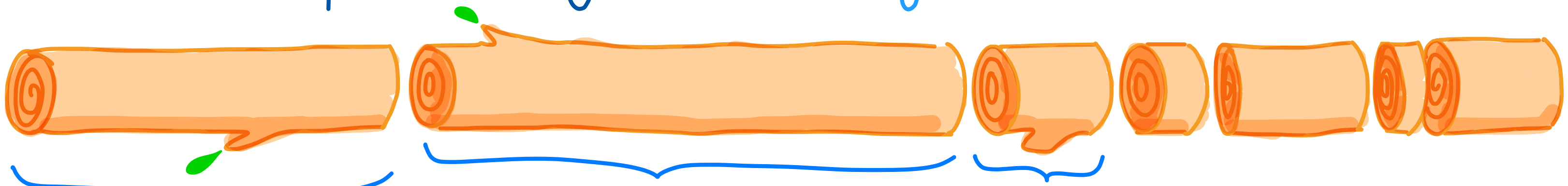

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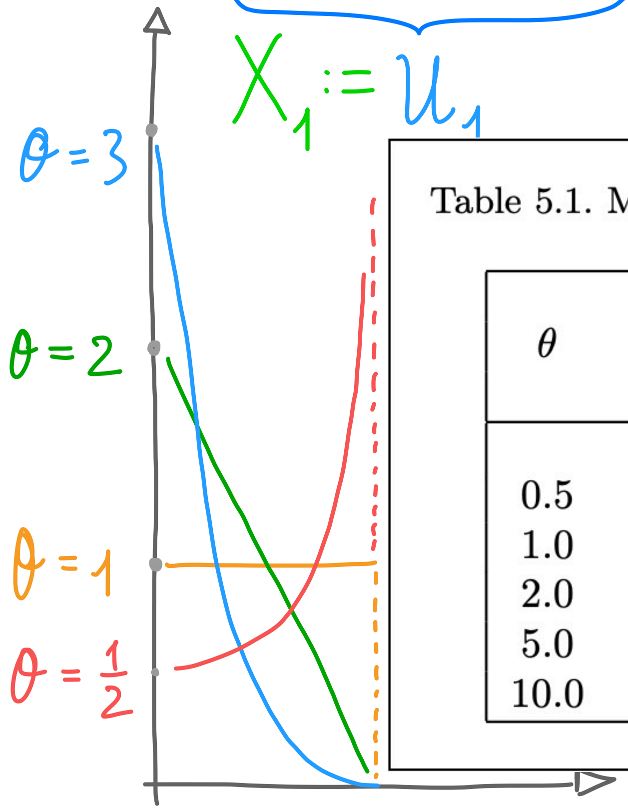

Table 5.1. Mean and variance of L_1

θ	$\mathbb{E}L_1$	$\mathrm{Var}(L_1)$
0.5	0.7578	0.0370
1.0	0.6243	0.0369
2.0	0.4756	0.0271
5.0	0.2973	0.0116
10.0	0.1949	0.0047

…ost surely, $X_1 + X_2 + \dots = 1$

$\Rightarrow X = (X_1, X_2, \dots) \in \Delta^\infty$

Definition $\quad X^{\downarrow} \sim \mathrm{PD}(\cancel{1})$ $\quad \theta > 0$

Example: Integer factorization

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$k \geq 2$ an integer

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$\parallel$

9450

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$k \geq 2$ an integer

$$\begin{array}{l} k \\ \| \\ 9450 = 7 \cdot 5 \cdot 5 \cdot 3 \cdot 3 \cdot 3 \cdot 2 \end{array}$$

Example: Integer factorization

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$$k = 9450 = \underbrace{7}_{= P_1(k)} \cdot 5 \cdot 5 \cdot 3 \cdot 3 \cdot 3 \cdot 2$$

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$$\underset{k}{\overset{\|}{9450}} = \underset{P_1(k)}{\overset{}{\underbrace{7}}} \cdot \underset{P_2(k)}{\underbrace{5}} \cdot 5 \cdot 3 \cdot 3 \cdot 3 \cdot 2$$

Example: Integer factorization

$k \geq 2$ an integer

$$\underset{\displaystyle 9450}{\overset{\displaystyle k}{\|}} = \underbrace{7}_{P_1(k)} \cdot \underbrace{5}_{P_2(k)} \cdot \overbrace{5}^{P_3(k)} \cdot 3 \cdot 3 \cdot 3 \cdot 2$$

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$$k = 9450 = \underbrace{7}_{P_1(k)} \cdot \underbrace{5}_{P_2(k)} \cdot \underbrace{5}_{P_3(k)} \cdot 3 \cdot 3 \cdot 3 \cdot 2$$

$\hat{\ell}^{\downarrow} : \mathbb{Z}_{\geq 2}$

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$$\underset{\|}{k}$$

$$9450 = \underbrace{7}_{P_1(k)} \cdot \underbrace{5}_{P_2(k)} \cdot \overbrace{5}^{P_3(k)} \cdot 3 \cdot 3 \cdot 3 \cdot 2$$

$$\hat{\ell}^{\downarrow} : \mathbb{Z}_{\geq 2}$$

$$k \mapsto \Big(P_1(k), \; P_2(k), \; P_3(k), \ldots$$

Example: Integer factorization

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$$k \mapsto \frac{1}{\log k}\left(\log P_1(k), \log P_2(k), \log P_3(k), \dots\right.$$

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Example : Integer factorization

$\mathbf{k_n}$ a uniform random integer in $\{2, 3, \ldots, n\}$

$k \geq 2$ an integer

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$\mathbf{k_n}$ a uniform random integer in $\{2, 3, \ldots, n\}$

$k \geq 2$ an integer $\quad P_3(k)$

$\| $

945 $5 \cdot 3 \cdot 3 \cdot 3 \cdot 2$

$k)$

$\hat{l} \downarrow$

$\rightarrow \Delta^\infty$

$\left(\log P_1(k), \log P_2(k), \log P_3(k), \ldots\right.$

Theorem (Billingsley '72)

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Another example: Cycle decomposition for permutations

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$\sigma \in S_n$

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$$\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 8 & 6 & 4 & 3 & 2 & 5 & 7 & 1 & 9 \end{array}$$

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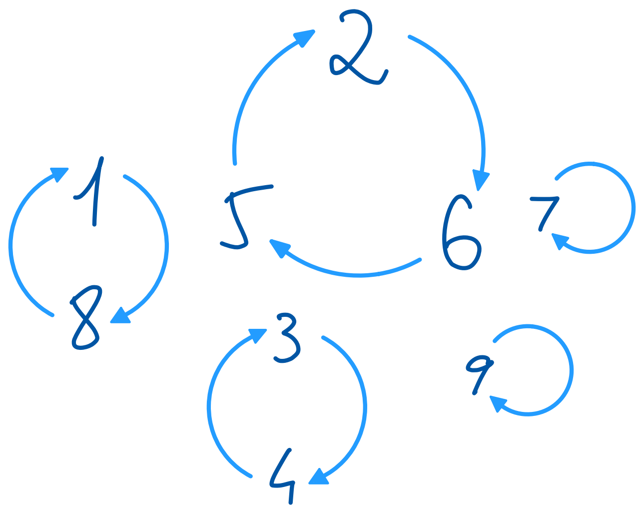

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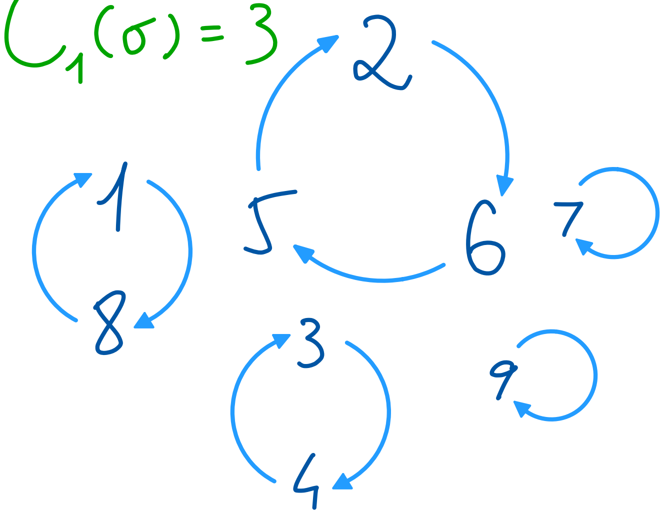
$C_1(\sigma) = 3$



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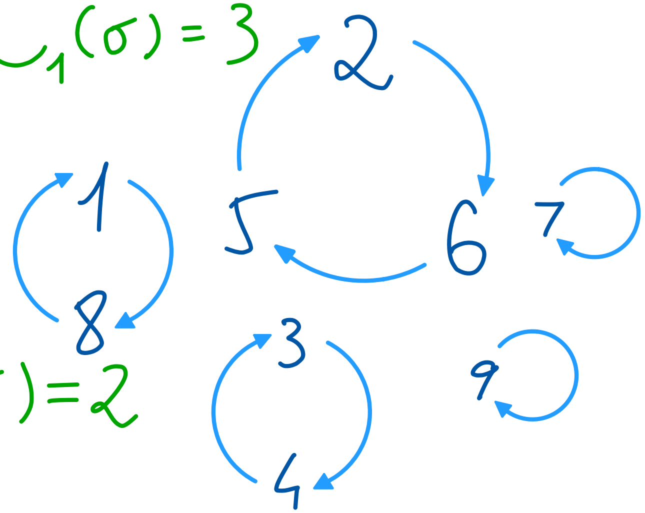

$$C_1(\sigma) = 3$$

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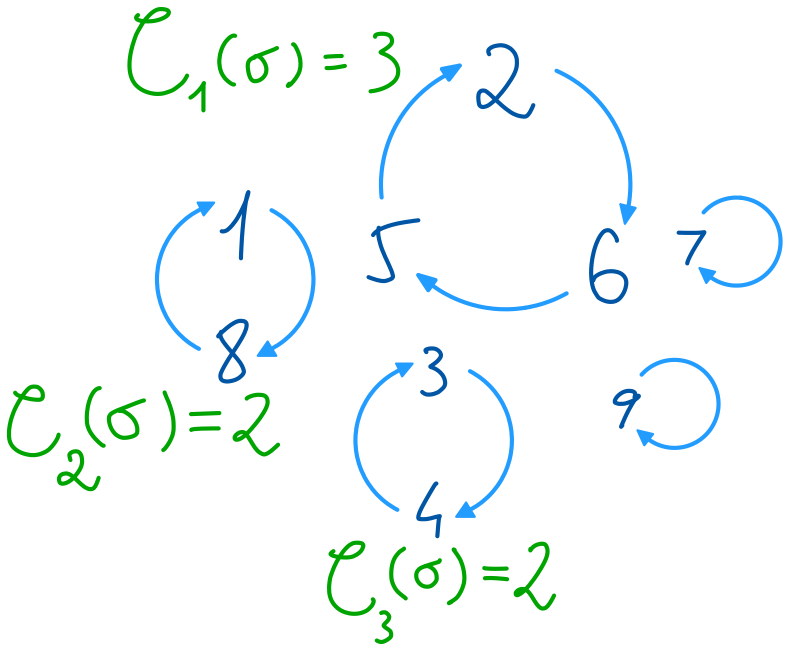

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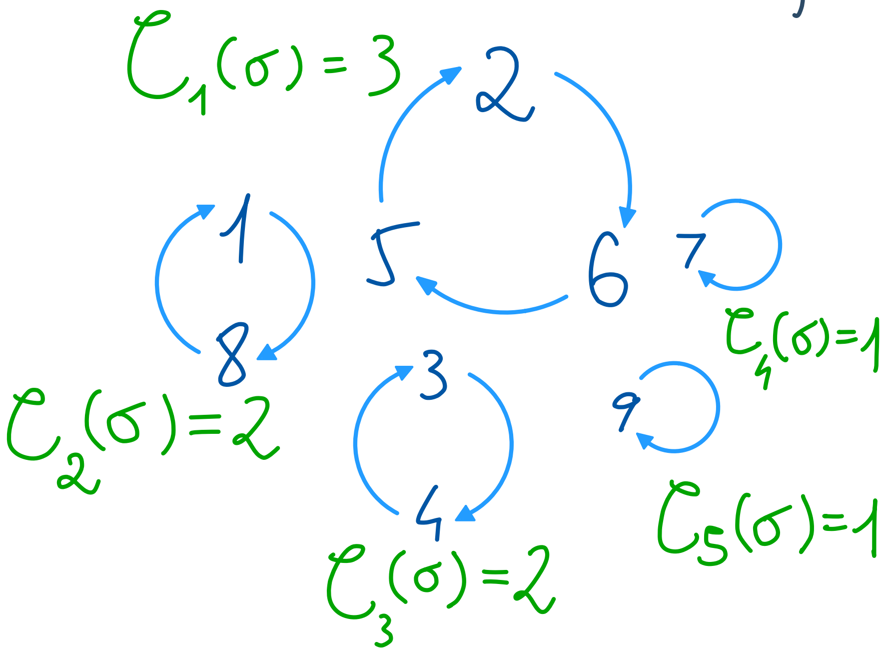

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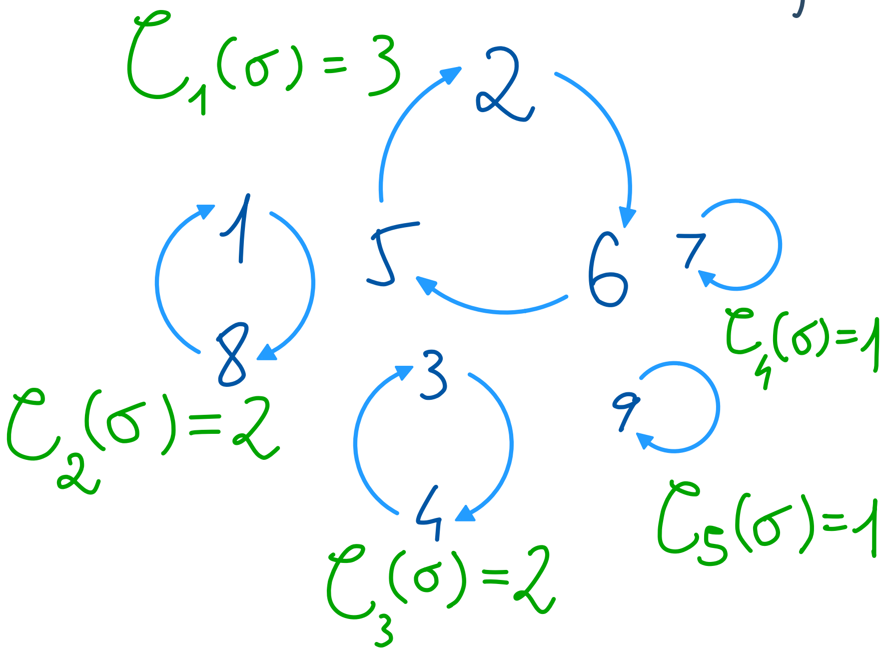

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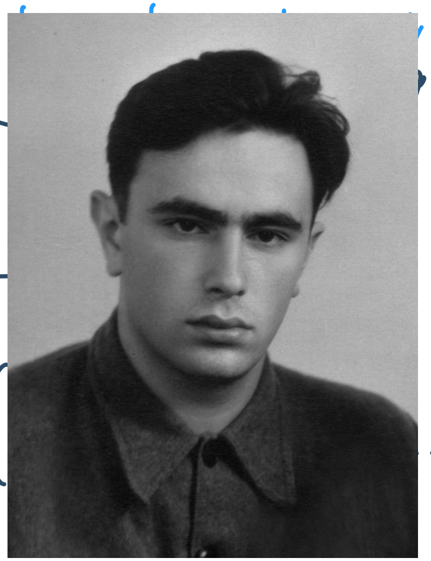

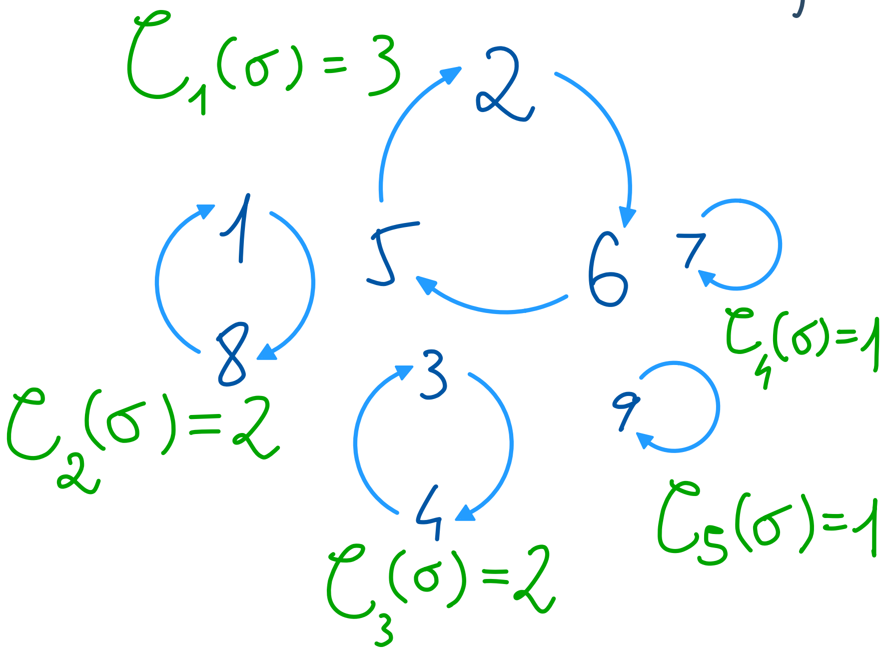

$\hat{\ell}^{\downarrow}$:

Theorem (Kingman, Vershik–Shmidt '77)

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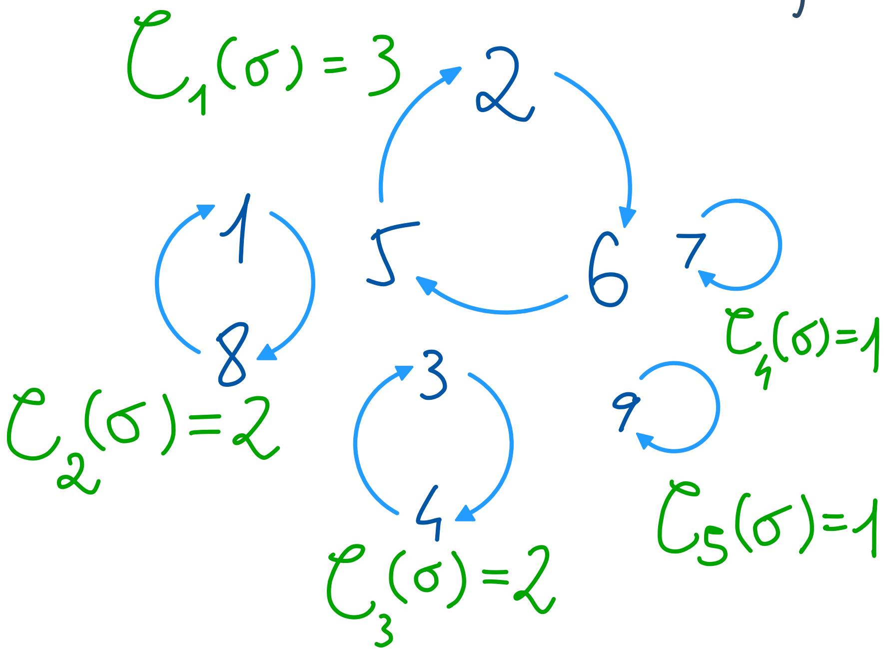

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Theorem (Kingman, Vershik–Shmidt '77) $\sigma_n \in S_n$ uniform random

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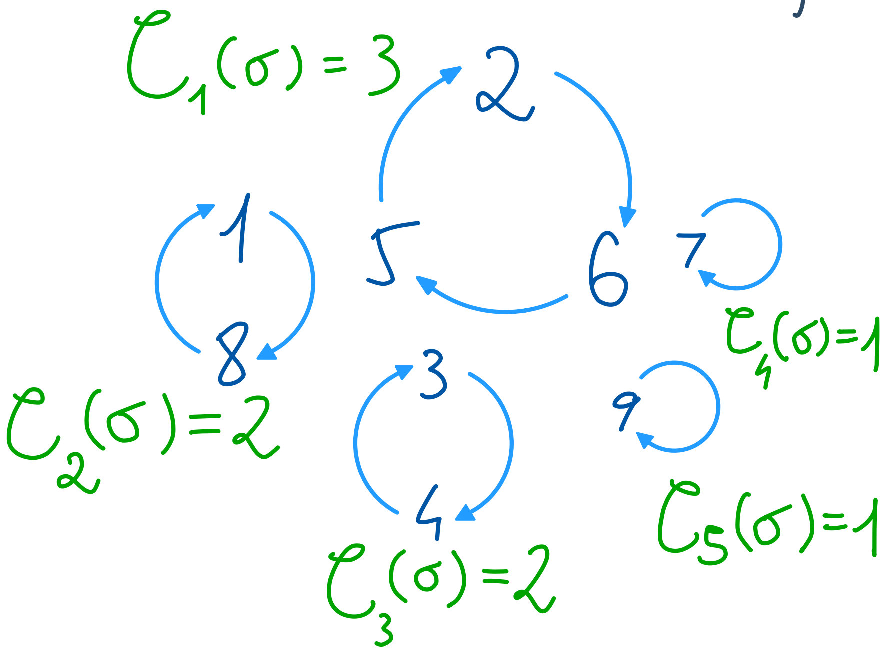

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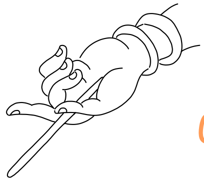
Theorem (Delecroix – L '22)

- $\hat{\ell}^{\downarrow}(\gamma_{X,R}) \xrightarrow[R\to\infty]{(d)} \mathcal{L}_g$ 
  - does ~~not~~ not depend on X
  - explicit
- $\mathcal{L}_g \xrightarrow[g\to\infty]{(d)}$ Poisson–Dirichlet distribution of parameter $1/2$

Corollary As $g \to \infty$, on average

$$\left.\begin{array}{lcl} \text{longest component} & \longrightarrow & 75.8\% \\ 2^{\text{nd}} & \longrightarrow & 17.1\% \\ 3^{\text{rd}} & \longrightarrow & 4.9\% \end{array}\right\} 97.8\%$$

$\underline{PD(\theta)?}$

PD(θ)?

S_n

PD(θ)? $\sigma_n \in S_n$

PD(θ)? $\sigma_{n,\theta} \in S_n$

PD(θ)? $\sigma_{n,\theta} \in S_n$

weight $w_\theta(\sigma)$

$PD(\theta)$? $\sigma_{n,\theta} \in S_n$

weight $w_\theta(\sigma) = \theta^{K(\sigma)}$

PD(θ)? $\sigma_{n,\theta} \in S_n$

weight $w_\theta(\sigma) = \theta^{K(\sigma)}$ n° of cycles in σ

$PD(\theta)$?

$\sigma_{n,\theta} \in S_n$

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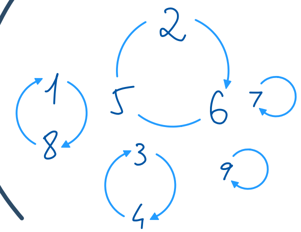
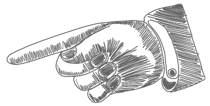
$K(\sigma)$ n° of cycles in σ

eg. $K\Big($ 

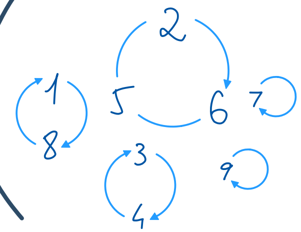
$\Big) = 5$

PD(θ)? $\quad \sigma_{n,\theta} \in S_n$

weight $\quad w_\theta(\sigma) = \theta^{K(\sigma)}$



n° of cycles in σ

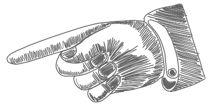
eg. $K\Big($ 

$\Big) = 5$

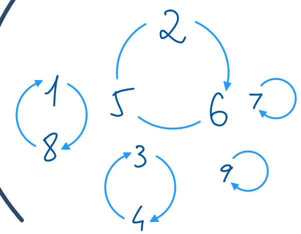
$\mathbb{P}(\sigma_{n,\theta} = \sigma)$

$PD(\theta)$? $\quad \sigma_{n,\theta} \in S_n$

weight $\quad w_\theta(\sigma) = \theta^{K(\sigma)}$

$K(\sigma)$  n° of cycles in σ

$$\mathbb{P}(\sigma_{n,\theta} = \sigma) = \frac{w_\theta(\sigma)}{\sum_{\tau \in S_n} w_\theta(\tau)}$$

e.g. $K\Big($ 

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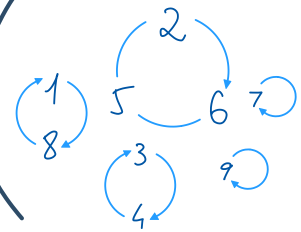
eg. $K\Bigg($ 

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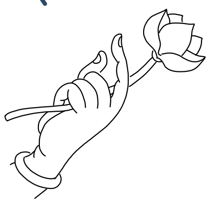
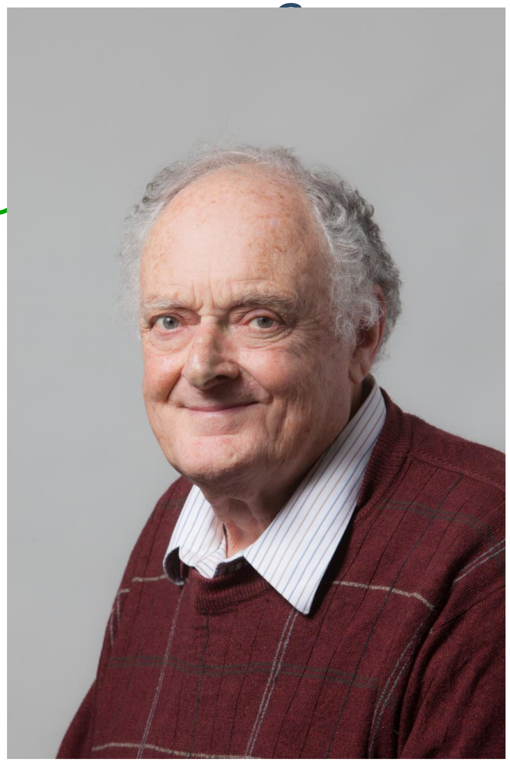


Ewens measur



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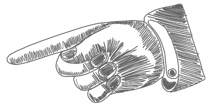
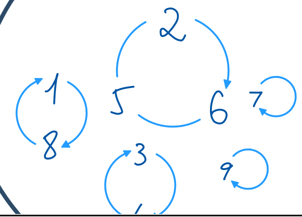
weight $w_\theta(\sigma) = \theta^{K(\sigma)}$    n° of cycles in σ

eg. $K\Big($  $\Big) = 5$

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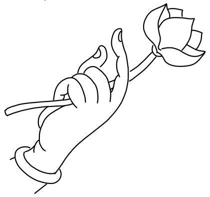

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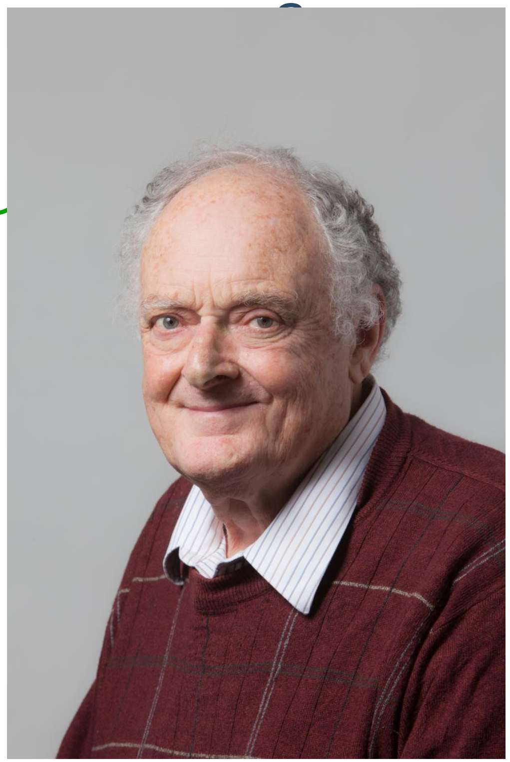

THEORETICAL POPULATION BIOLOGY 3, 87–112 (1972)

The Sampling Theory of Selectively Neutral Alleles*

W. J. Ewens†

Department of Zoology, University of Texas at Austin, Austin, Texas, 78712

Received August 17, 1971

DEDICATED TO THE MEMORY OF KEN KOJIMA

In this paper a beginning is made on the sampling theory of neutral alleles. That is, we consider deductive and subsequently inductive questions relating to a sample of genes from a selectively neutral locus. The inductions concern estimation, confidence intervals and hypothesis testing. In particular the test of the hypothesis that the alleles being sampled are indeed selectively neutral will be considered. In view of the large amount of data currently being obtained by electrophoretic methods on allele frequencies and numbers, and the current interest in the possibility of extensive "non-Darwinian" evolution, such a sampling theory seems necessary. However, a large number of unsolved problems in this area remain, a partial listing being given towards the end of this paper.

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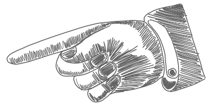
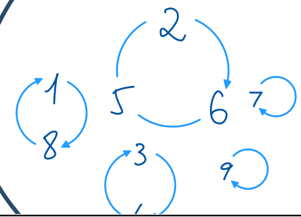
weight $\quad w_\theta(\sigma) = \theta^{\boxed{K(\sigma)}}$



n° of cycles in σ

eg. $K\Big($  $\Big) = 5$

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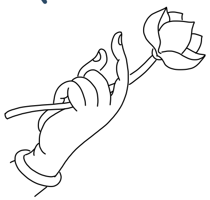

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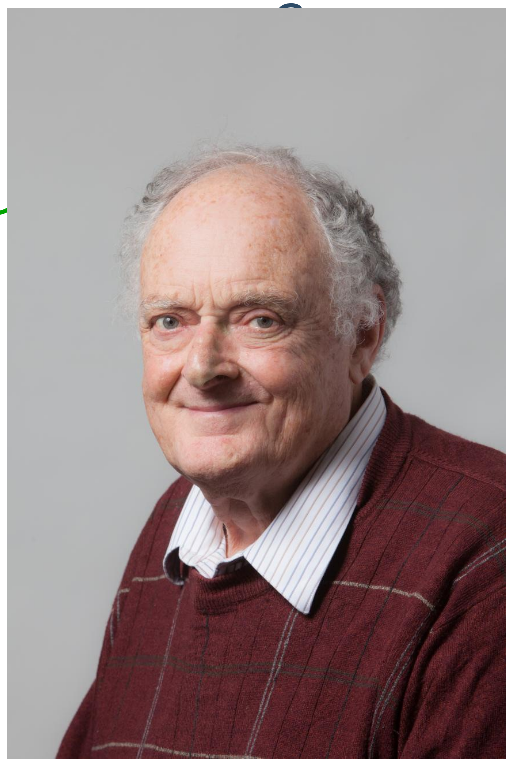

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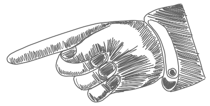
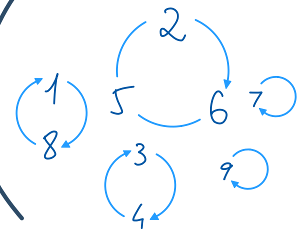
PD(θ)?

$\sigma_{n,\theta} \in S_n$

weight $\quad w_\theta(\sigma) = \theta^{K(\sigma)}$  n° of cycles in σ

eg. $\quad K\Big($  $\Big) = 5$

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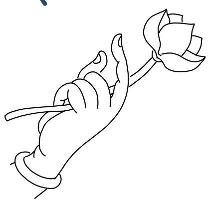


Ewens measure

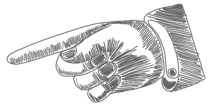
Remark

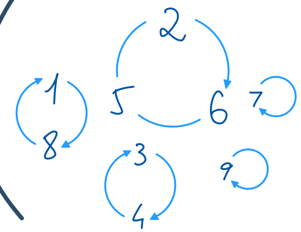
$\theta = 1 \Rightarrow \sigma_{n,1} = \sigma_n$ uniform

Theorem

$$\hat{\ell}^{\downarrow}(\sigma_{n,\theta}) \xrightarrow[n\to\infty]{(d)} PD(\theta)$$

$PD(\theta)$? $\sigma_{n,\theta} \in S_n$

weight $w_\theta(\sigma) = \theta^{K(\sigma)}$  n° of cycles in σ

e.g. $K\Big($  $\Big) = 5$

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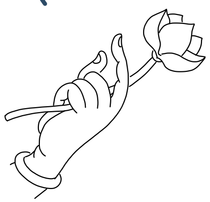

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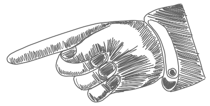
Ewens measure

Theorem • $\hat{\ell}^{\downarrow}(\sigma_{n,\theta}) \xrightarrow[n\to\infty]{(d)} PD(\theta)$

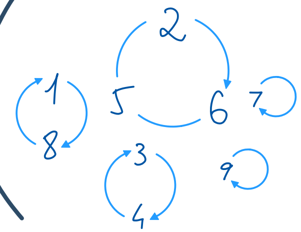
$K(\sigma_{n,\theta})$

PD(θ)? $\quad \sigma_{n,\theta} \in S_n$



$K(\sigma)$ n° of cycles in σ

weight $\quad w_\theta(\sigma) = \theta^{K(\sigma)}$

eg. $K\left(\right.$ 

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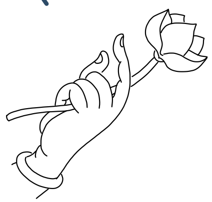

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Theorem

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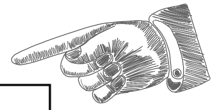
- $\mathbb{E}(K(\sigma_{n,\theta})) \underset{n\to\infty}{\sim} \theta \cdot \log n$

$PD(\theta)$? $\sigma_{n,\theta} \in S_n$

$K(\sigma)$



n° of cycles in σ

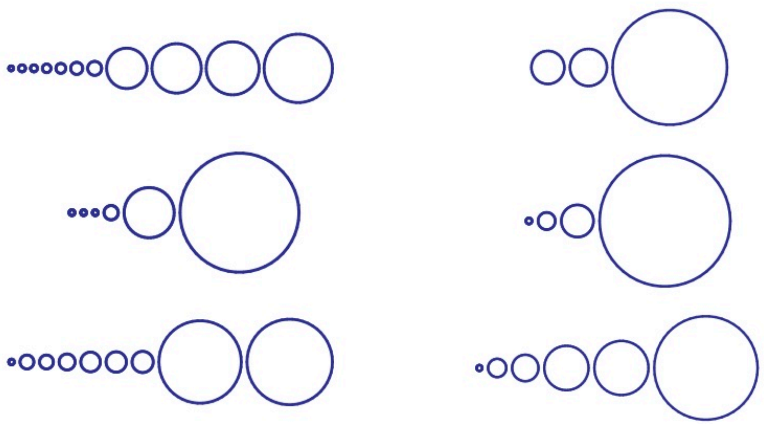
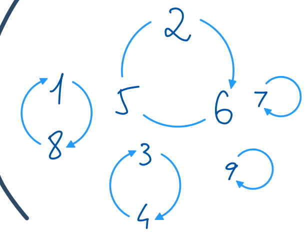


Figure III.11. The profile of permutations: a rendering of the cycle structure of six random permutations of size 500, where circle areas are drawn in proportion to cycle lengths. Permutations tend to have a few small cycles (of size $O(1)$), a few large ones (of size $\Theta(n)$), and altogether have $H_n \sim \log n$ cycles on average.

$K\Big($  $\Big) = 5$

$\Rightarrow \sigma_{n,1} = \sigma_n$ uniform

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- $\hat{\ell}^{\downarrow}(\sigma_{n,\theta}) \xrightarrow[n\to\infty]{(d)} PD(\theta)$
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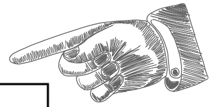
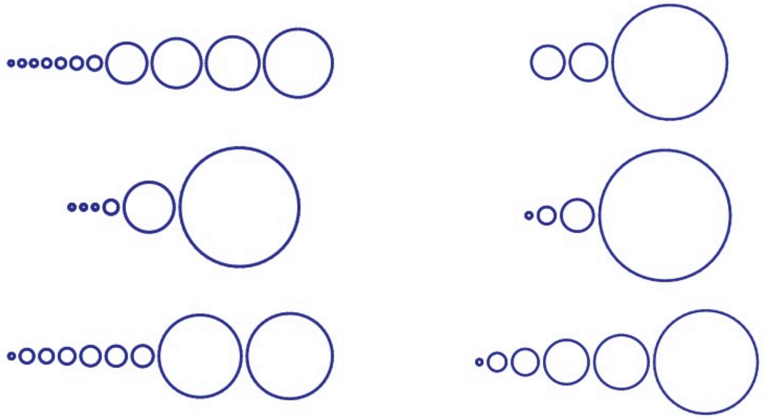
$PD(\theta)$? $\sigma_{n,\theta} \in S_n$

$K(\sigma)$  n° of cycles



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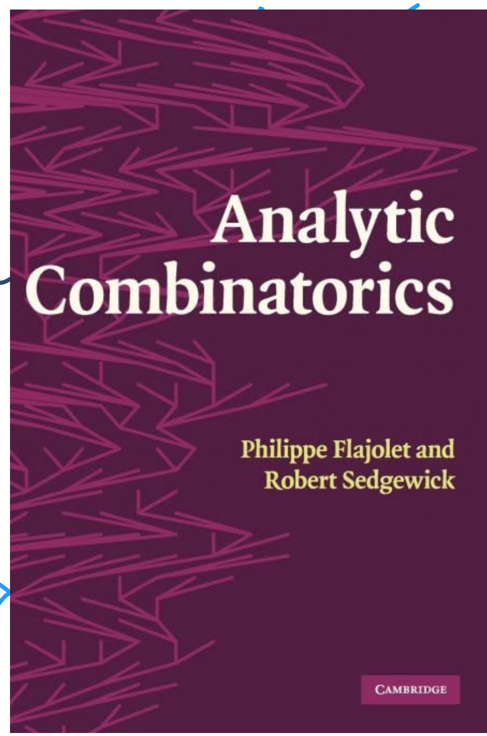

$= 5$

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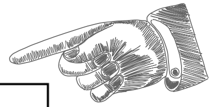
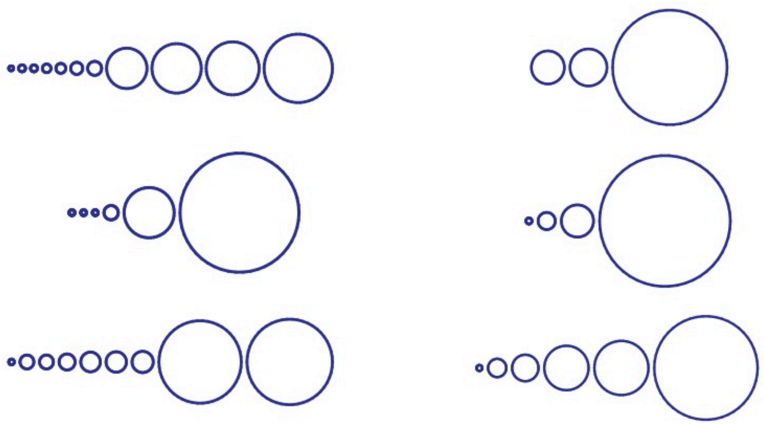
PD(θ)? $\sigma_{n,\theta} \in S_n$

$K(\sigma)$ n° of cycles



176 *III. PARAMETERS AND MULTIVARIATE GFS*



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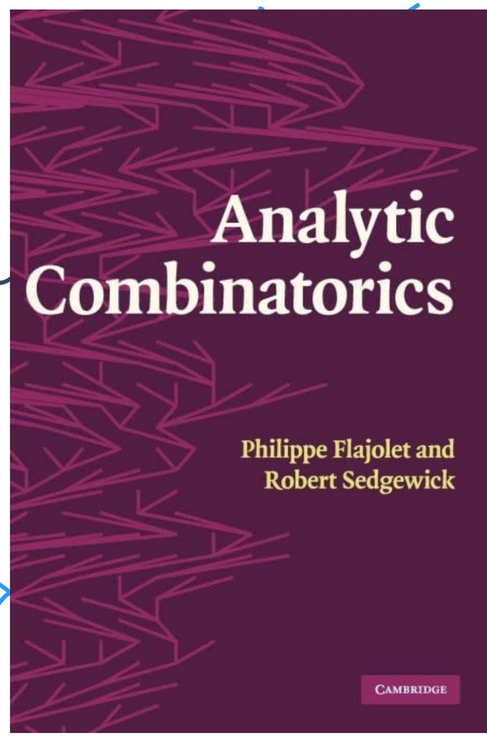

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[DGZZ]:

$\mathbb{E}(\text{n° of components of } \gamma) \underset{g\to\infty}{\sim} \frac{1}{2}\log g$

Small cycles in a random permutation

Small cycles in a random permutation

$K_1(\sigma)$ number of cycles of size 1 in $\sigma \in S_n$

Small cycles in a random permutation

uniform proba measure

$K_1(\sigma)$ number of cycles of size 1 in $\sigma \in S_n$

σ_n

Small cycles in a random permutation

uniform proba measure

$K_1(\sigma)$ number of cycles of size 1 in $\sigma \in S_n$

$K_1(\sigma_n)$

Small cycles in a random permutation

uniform proba measure

$K_1(\sigma)$ number of cycles of size 1 in $\sigma \in S_n$

- $\mathbb{E}(K_1(\sigma_n)) = 1$

Small cycles in a random permutation

uniform proba measure

$K_1(\sigma)$ number of cycles of size 1 in $\sigma \in S_n$

- $\mathbb{E}(K_1(\sigma_n)) = 1$
- $K_1(\sigma_n) \xrightarrow[n\to\infty]{(d)} \text{Poisson}(1)$

Small cycles in a random permutation

uniform proba measure

$K_i(\sigma)$ number of cycles of size i in $\sigma \in S_n$

- $\mathbb{E}(K_i(\sigma_n)) = 1/i$
- $K_i(\sigma_n) \xrightarrow[n\to\infty]{(d)} \text{Poisson}(1/i)$

Small cycles in a random permutation Ewens proba measure

$K_i(\sigma)$ number of cycles of size i in $\sigma \in S_n$

- $\mathbb{E}(K_i(\sigma_{n,\theta})) = \theta/i$
- $K_i(\sigma_{n,\theta}) \xrightarrow[n\to\infty]{(d)} \text{Poisson}(\theta/i)$

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<u>Conjecture</u> number of components of a random multigeod $K_{a,b}(\gamma_g)$ of (normalized) length in $\left[\frac{a}{6g}, \frac{b}{6g}\right]$

Small cycles in a random permutation

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Conjecture number of components of a random multigeod

$K_{a,b}(\gamma_g)$ of (normalized) length in $\left[\frac{a}{6g}, \frac{b}{6g}\right]$

$$K_{a,b}(\gamma_g) \xrightarrow[g\to\infty]{(d)} \text{Poisson}\left(\int_a^b e^{-x} \cdot \frac{\cosh(x)-1}{x}\, dx\right)$$

Small cycles in a random permutation Ewens proba measure

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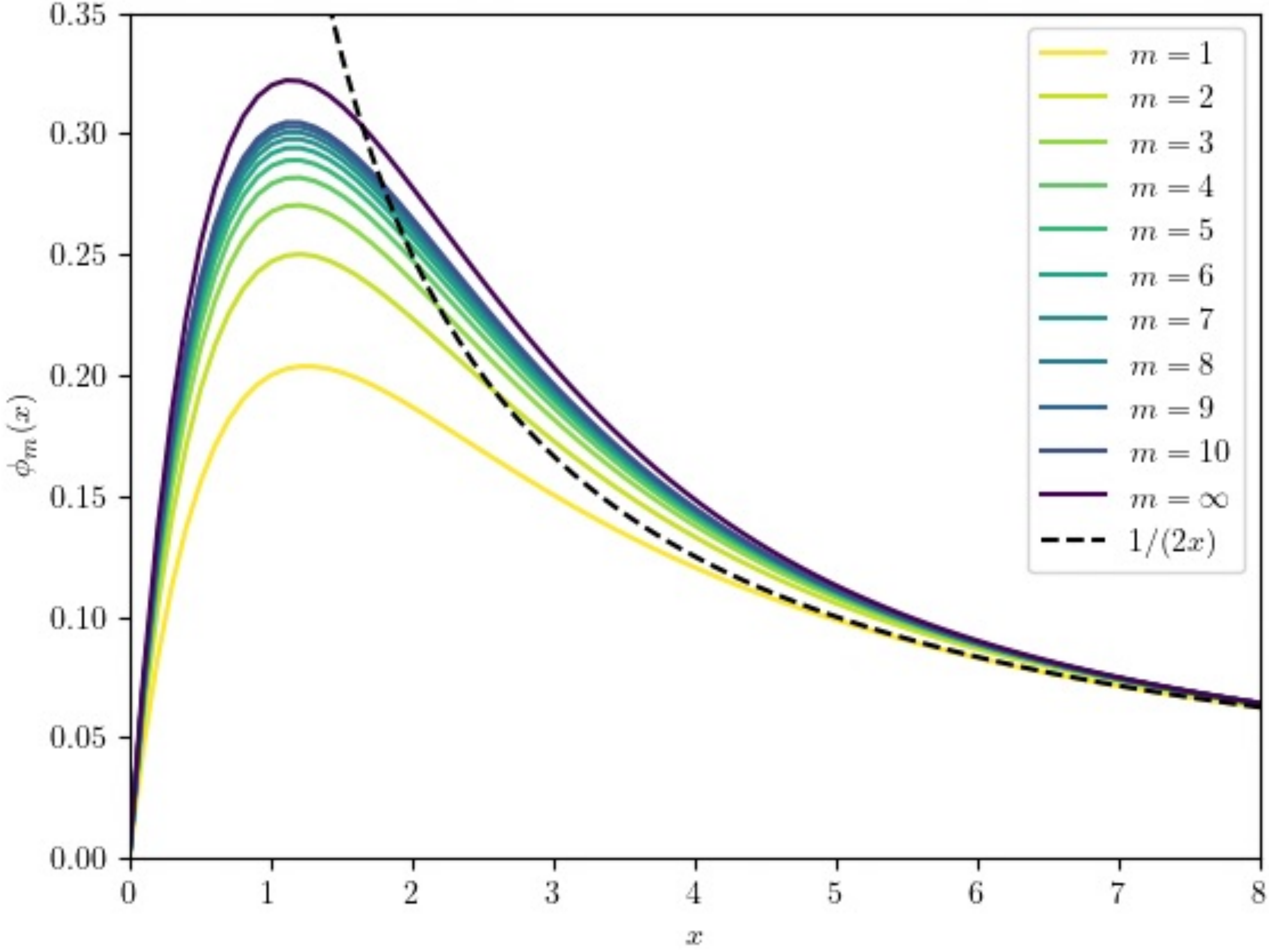


0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
$\phi_m(x)$
0
1
2
3
4
5
6
7
8
x
$m = 1$
$m = 2$
$m = 3$
$m = 4$
$m = 5$
$m = 6$
$m = 7$
$m = 8$
$m = 9$
$m = 10$
$m = \infty$
$1/(2x)$

Proof

Proof

- Divide and Conquer

Proof

- Divide and Conquer

α, β curves on X

Proof

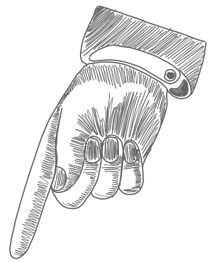
- Divide and Conquer

α, β curves on X

Definition $\alpha \overset{\text{topo}}{\underset{\text{type}}{\sim}} \beta$ if $\alpha \equiv \beta$ mod diffeo of X

Proof

$\exists\, h \in \text{Mapping Class Group}(X)$ s.t. $\beta = h \cdot \alpha$



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α, β curves on X

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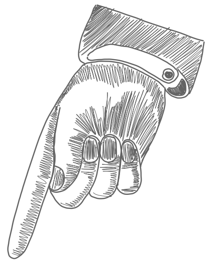

α, β curves on X

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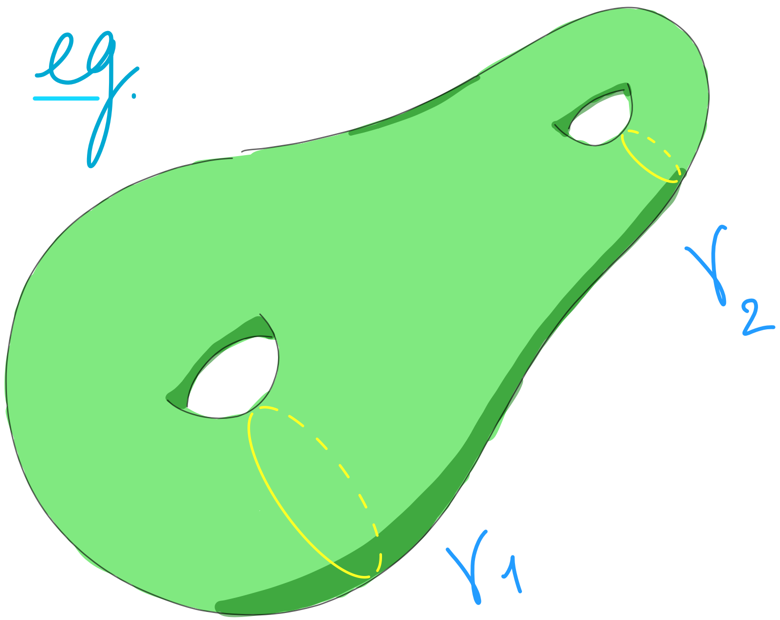
eg.



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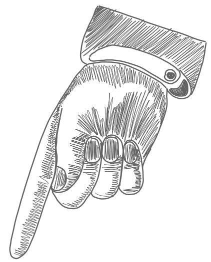

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α, β curves on X

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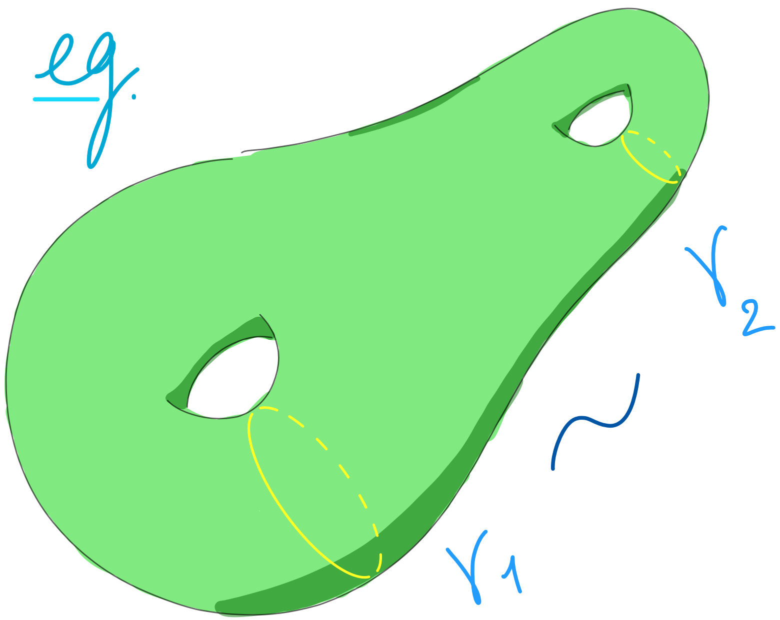

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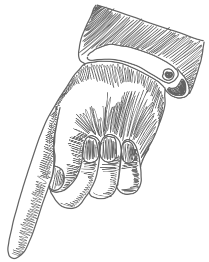

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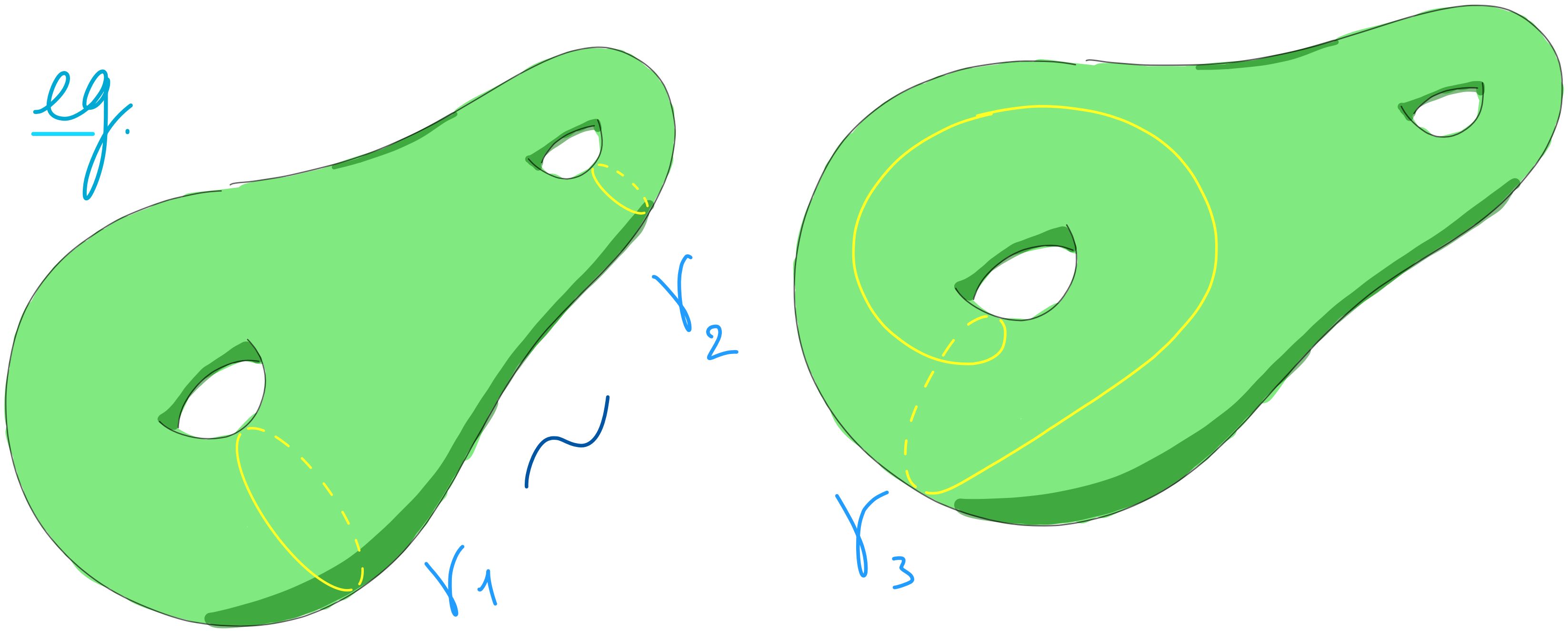

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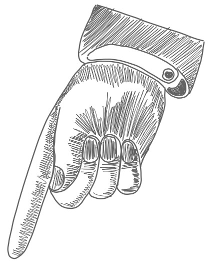

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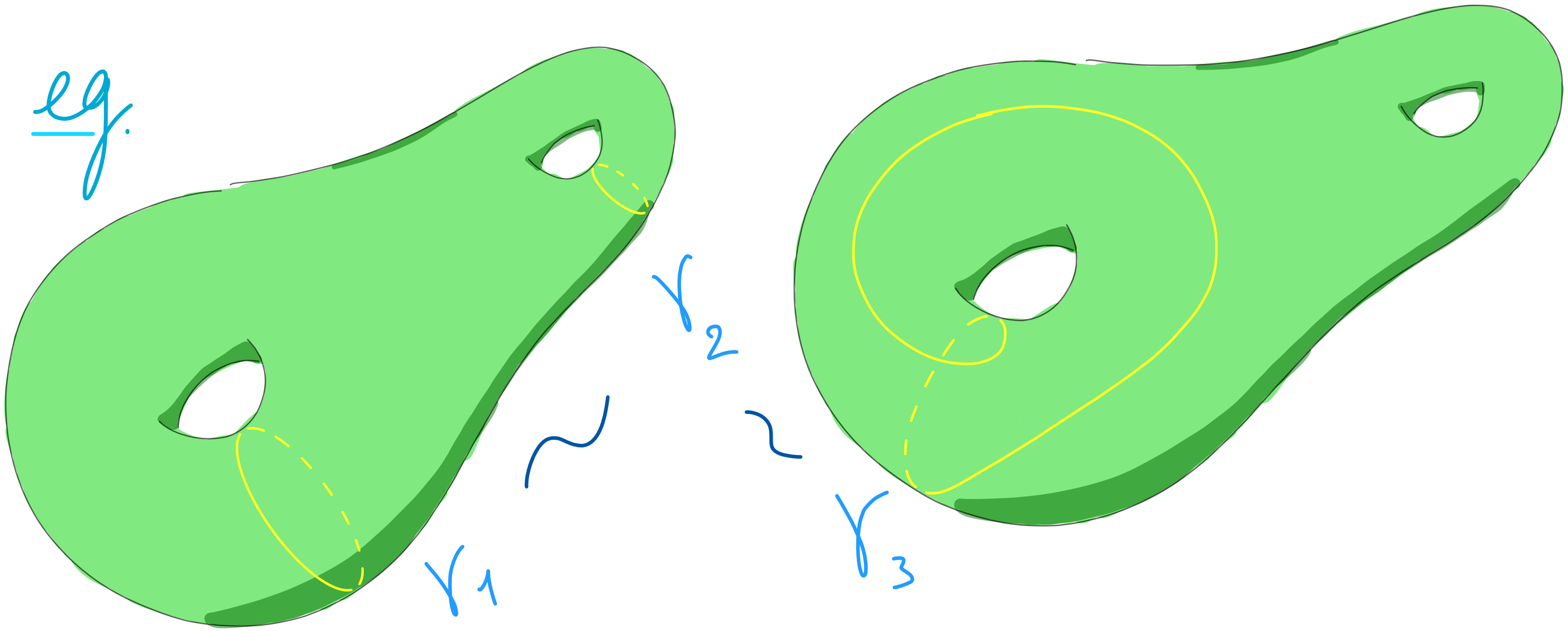

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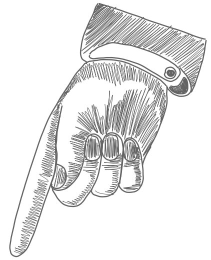

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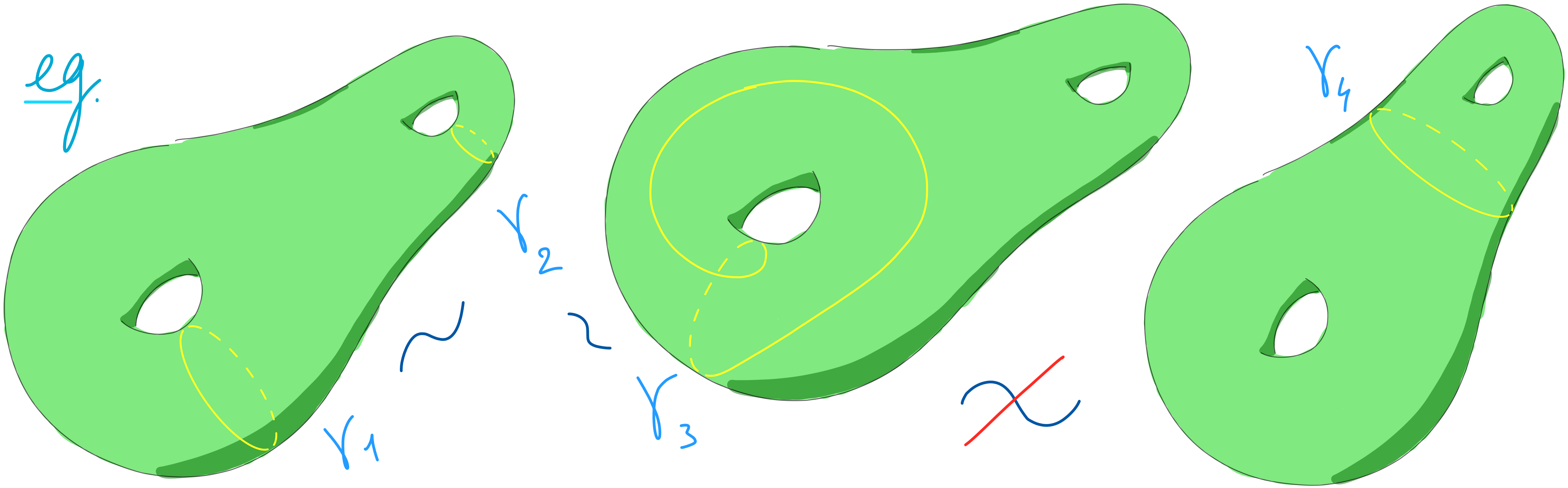

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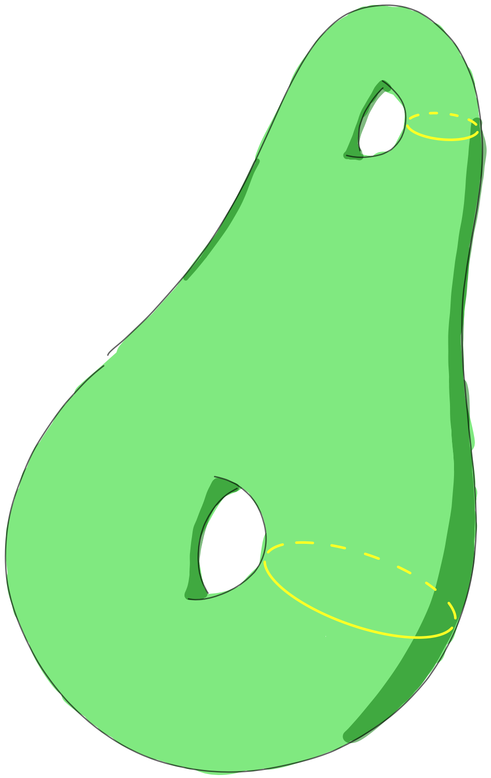

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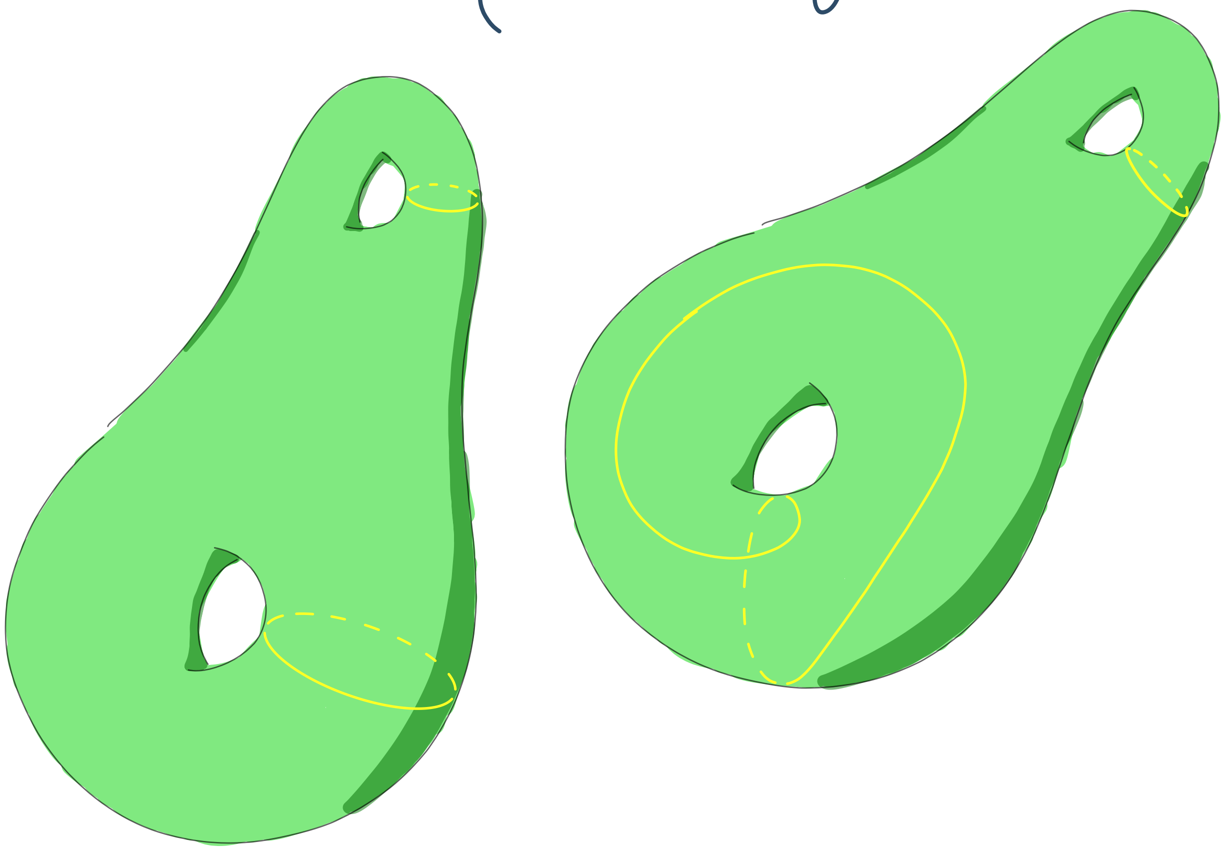

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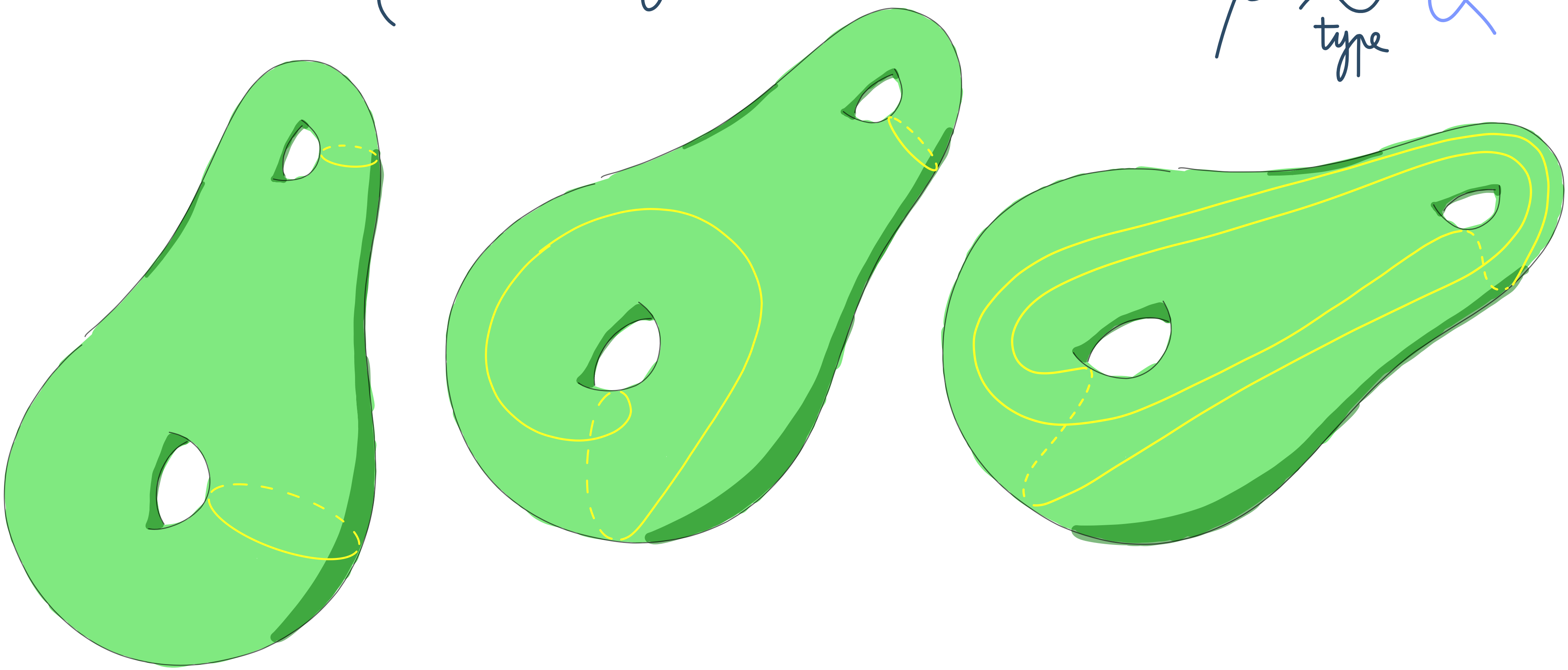

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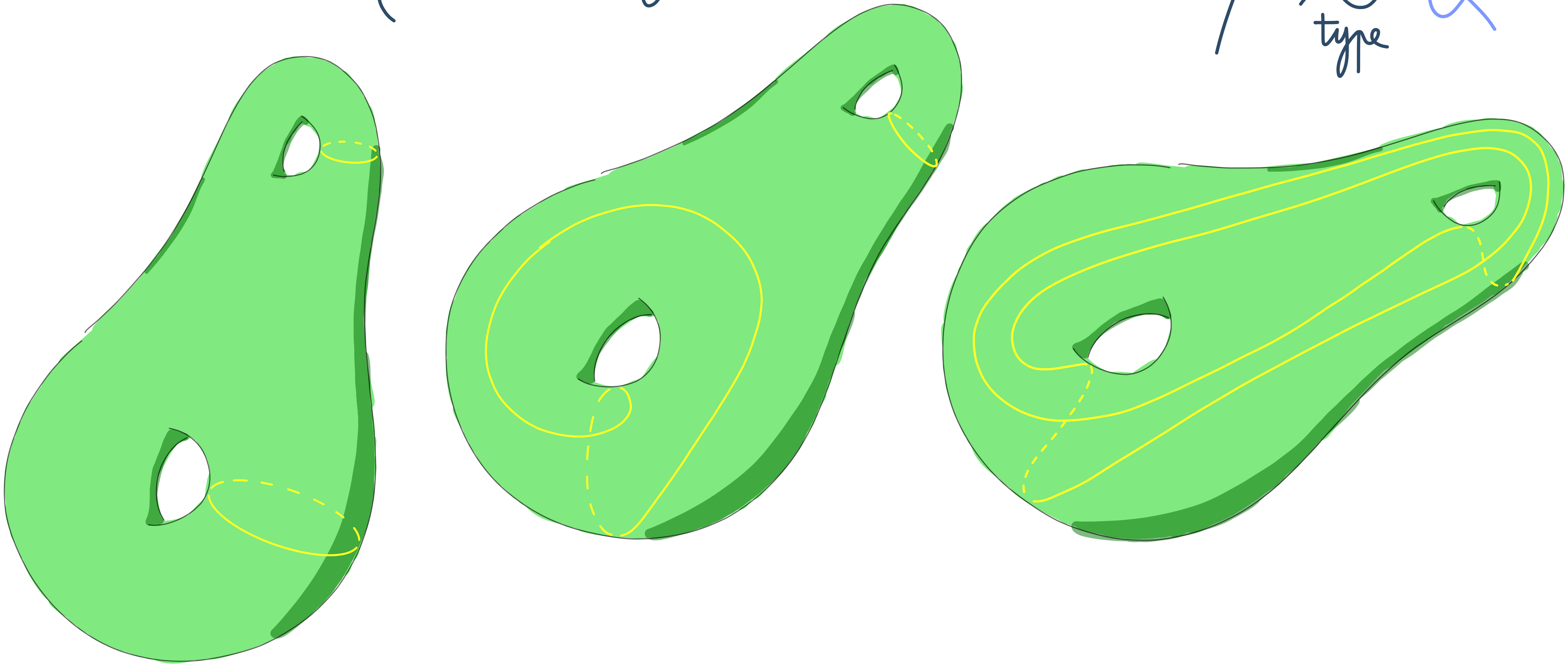

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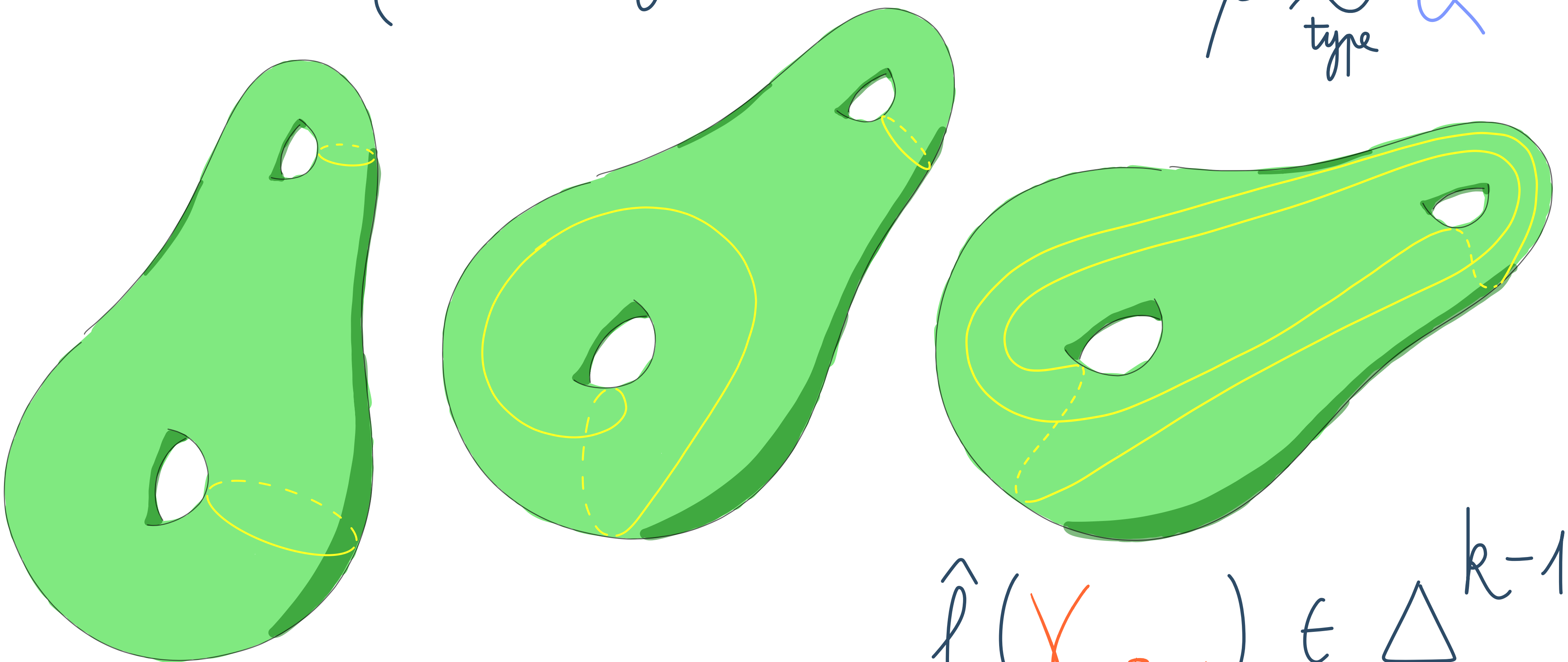

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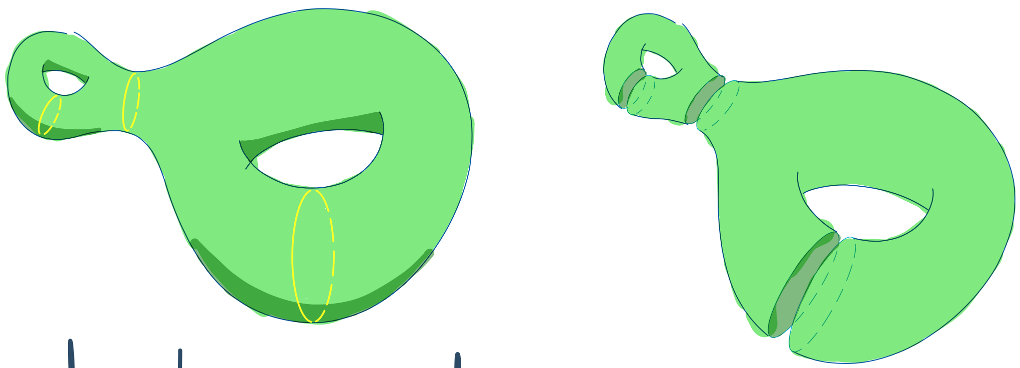

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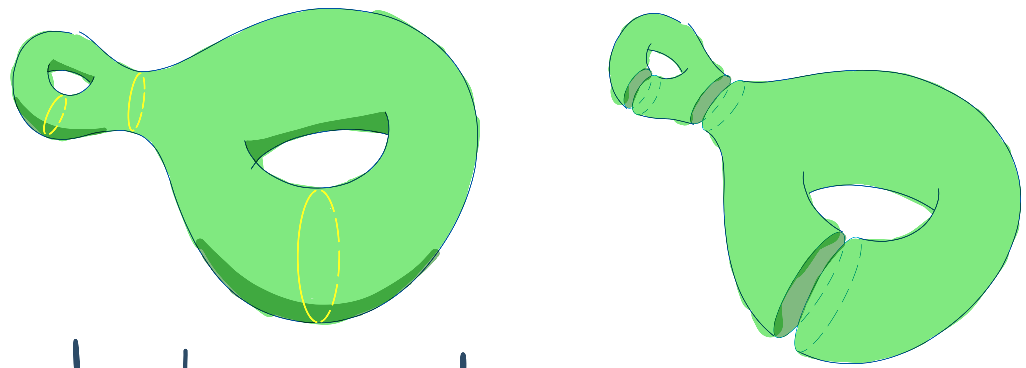

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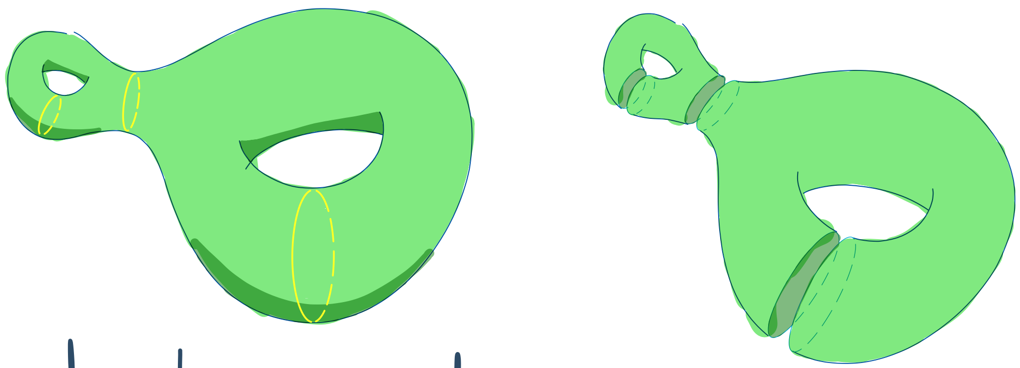

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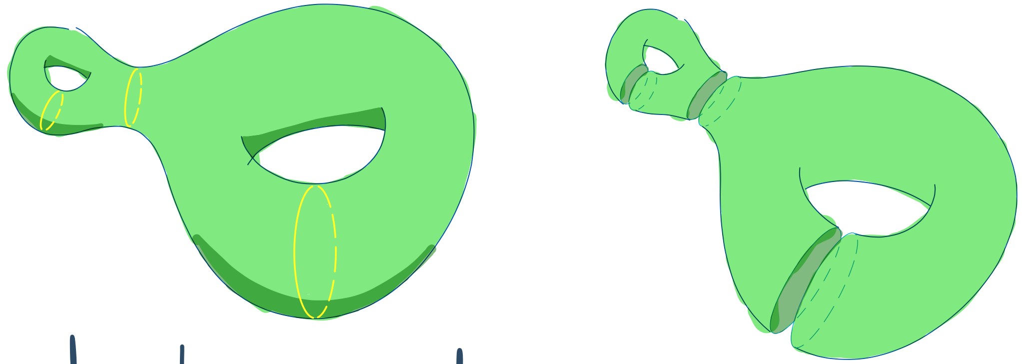

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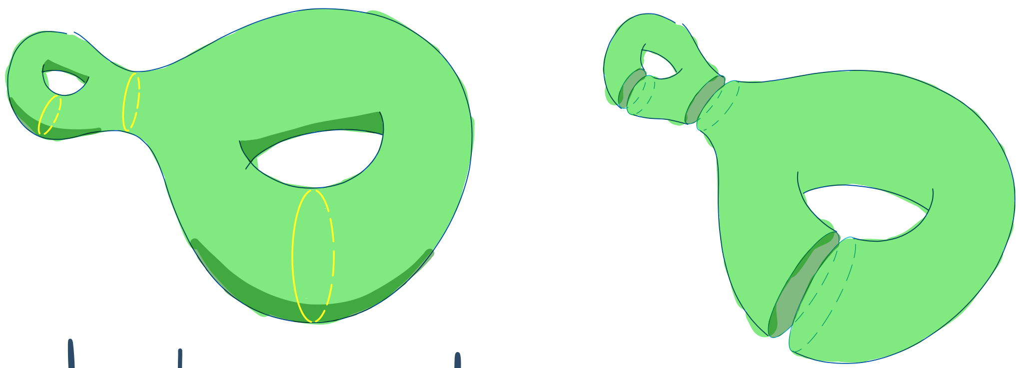

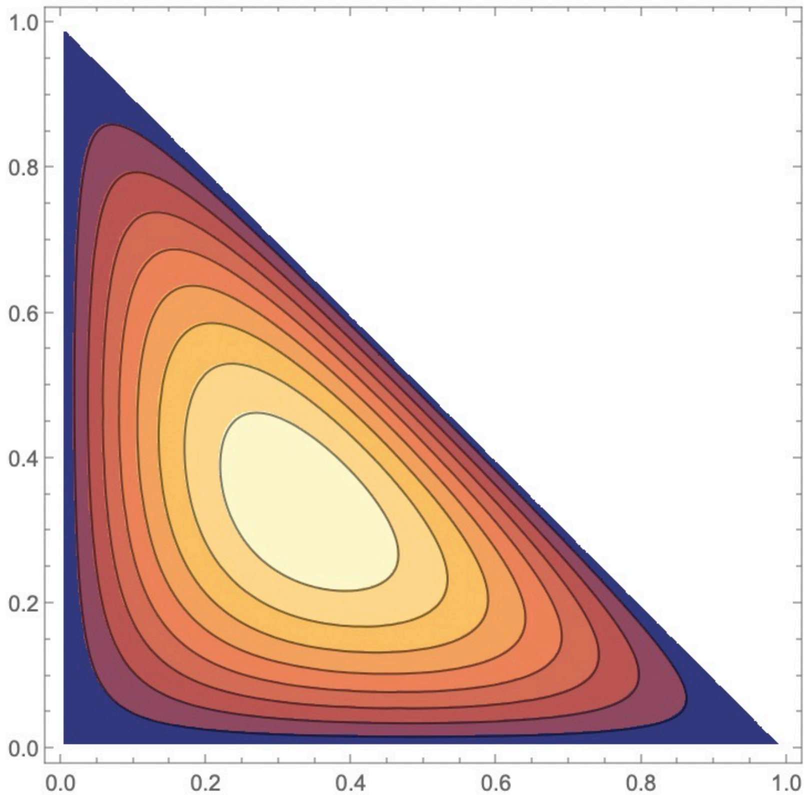

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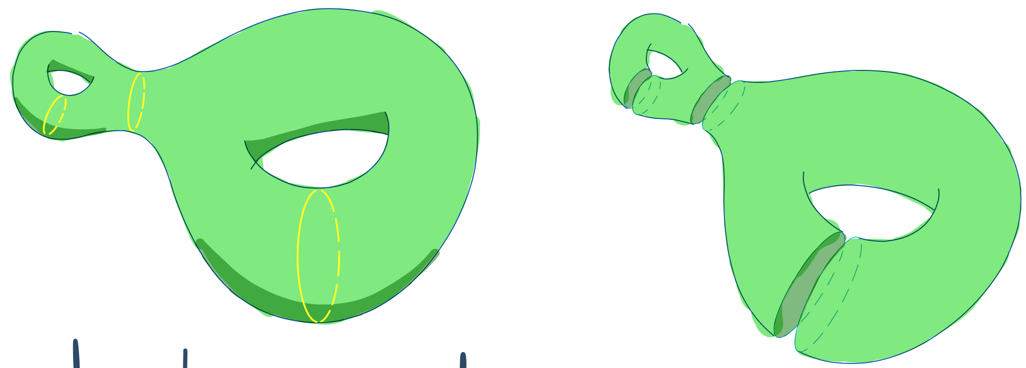

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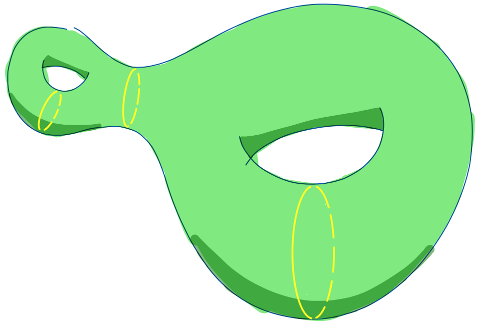

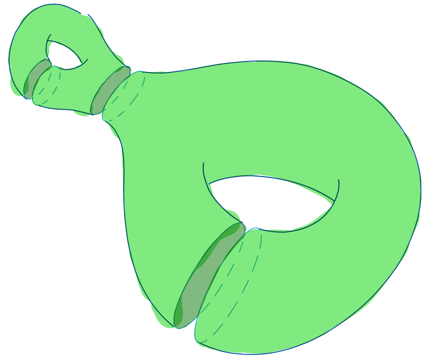

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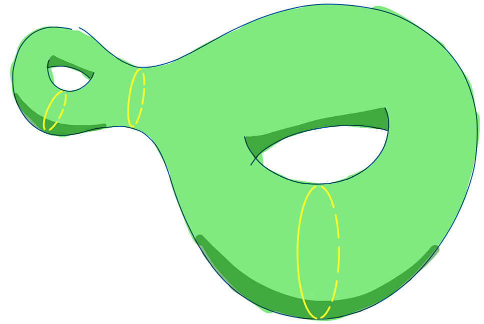

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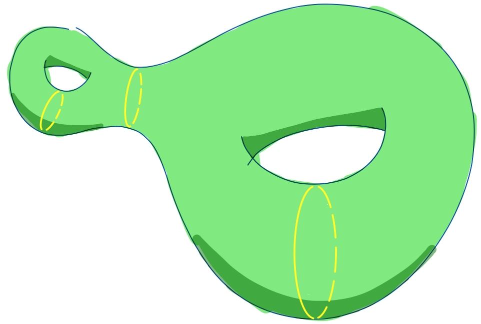

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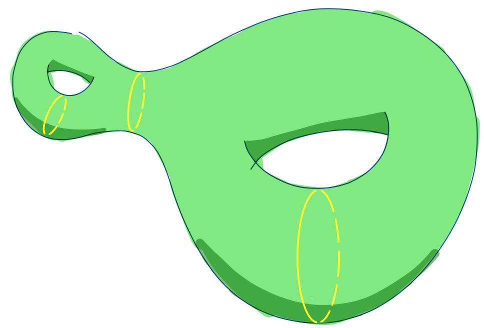

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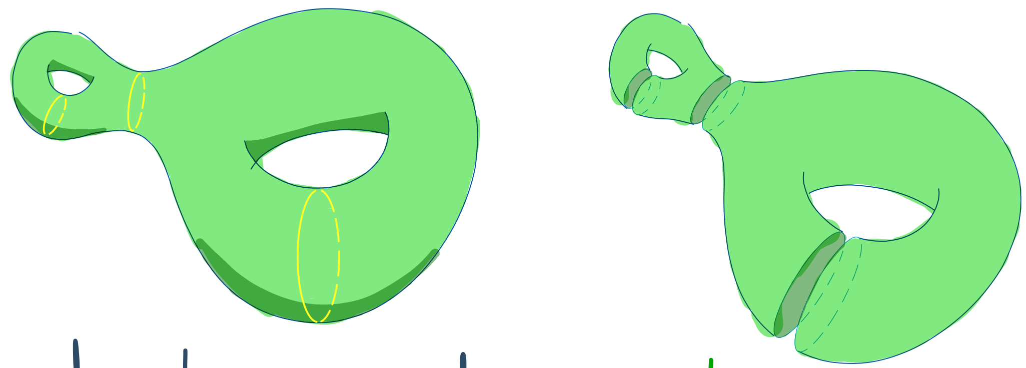

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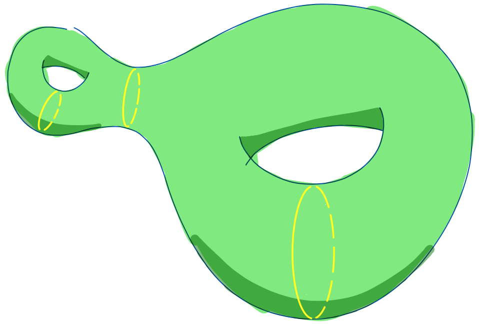 

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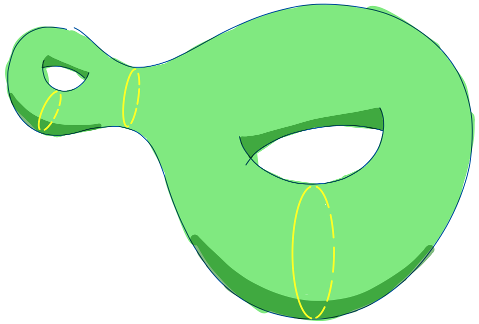 

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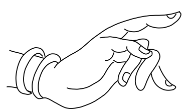
a polynomial  P_α

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P_α is a polynomial which can be expressed in terms of

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n}$$

intersection numbers of ψ-classes

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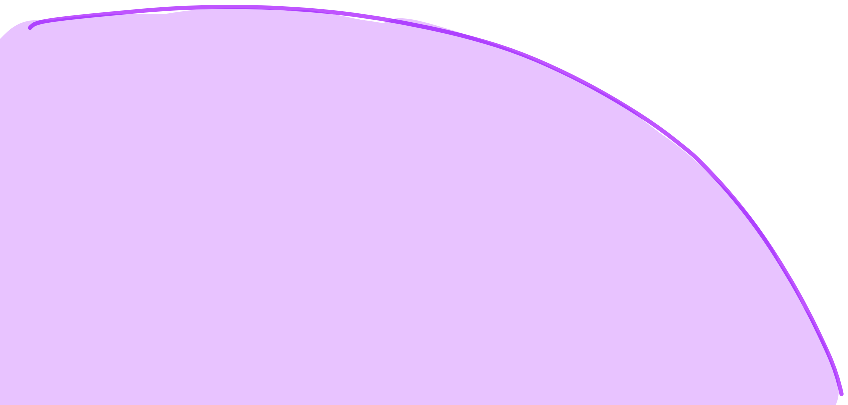

$\mathcal{M}_{g,n}$

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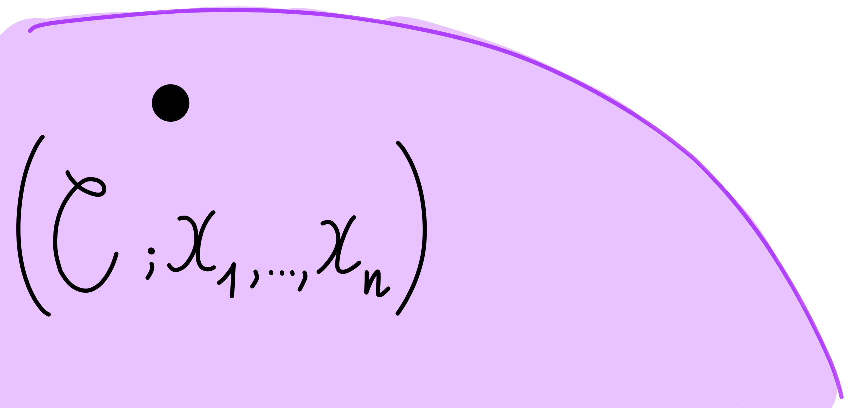

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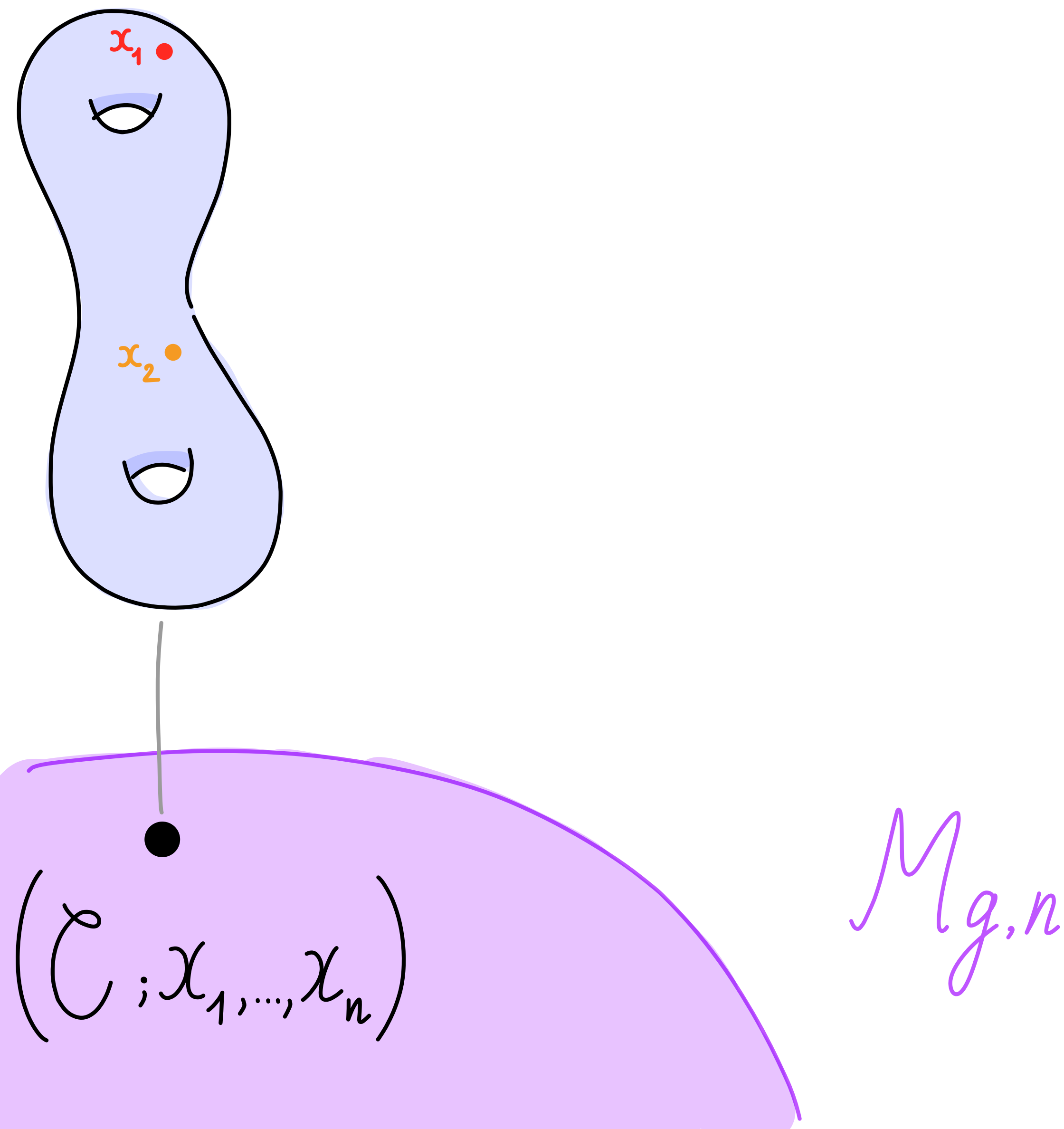

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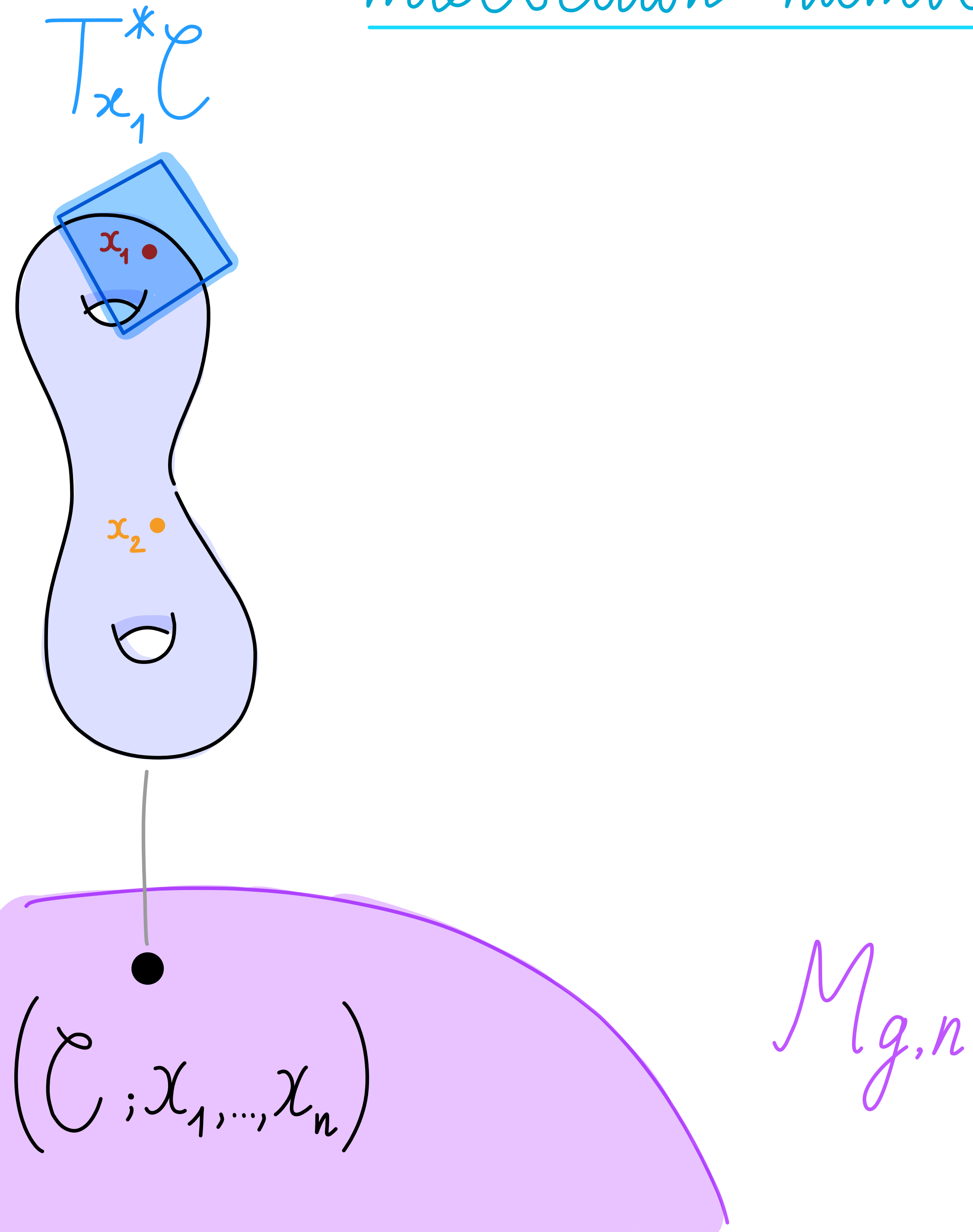

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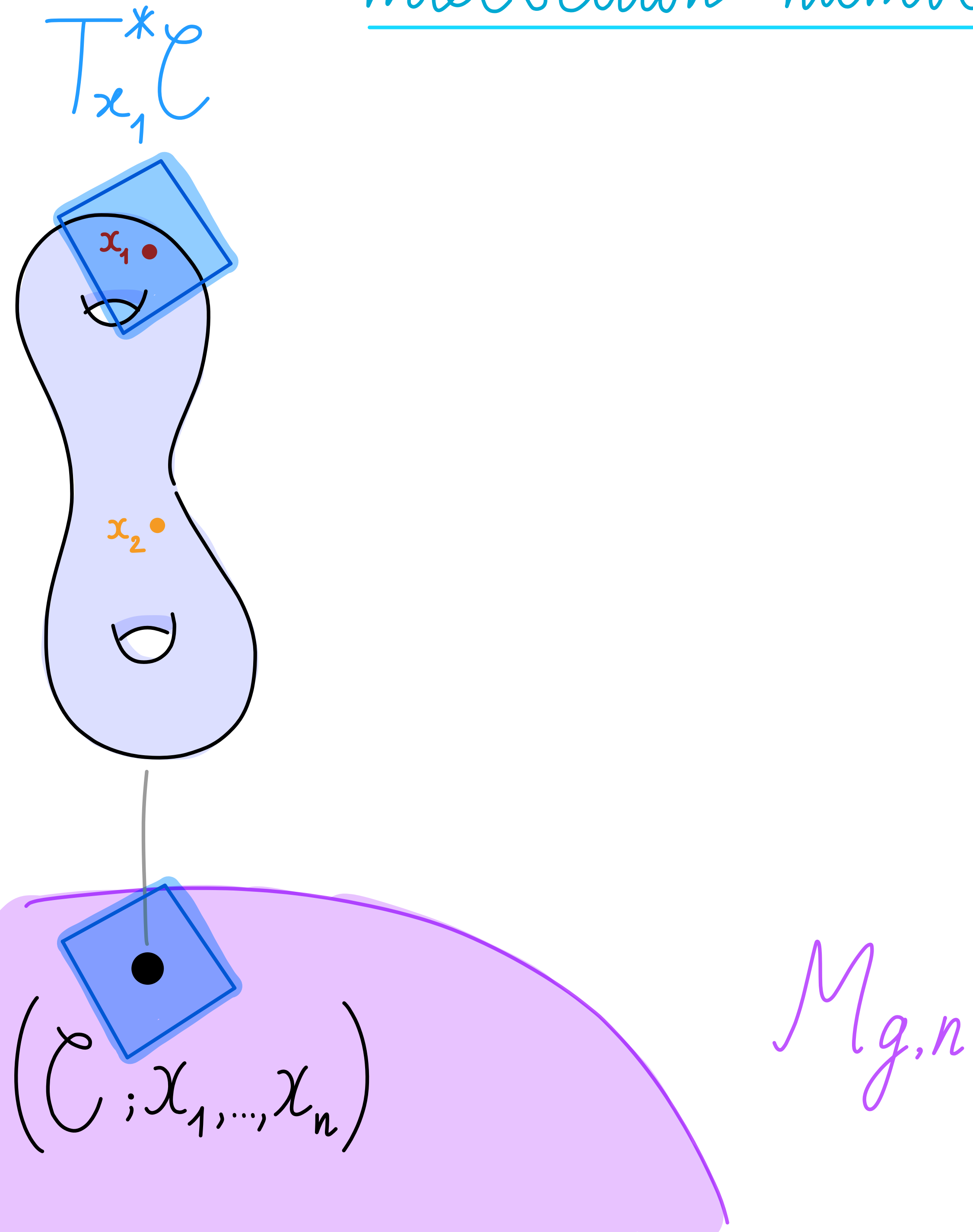

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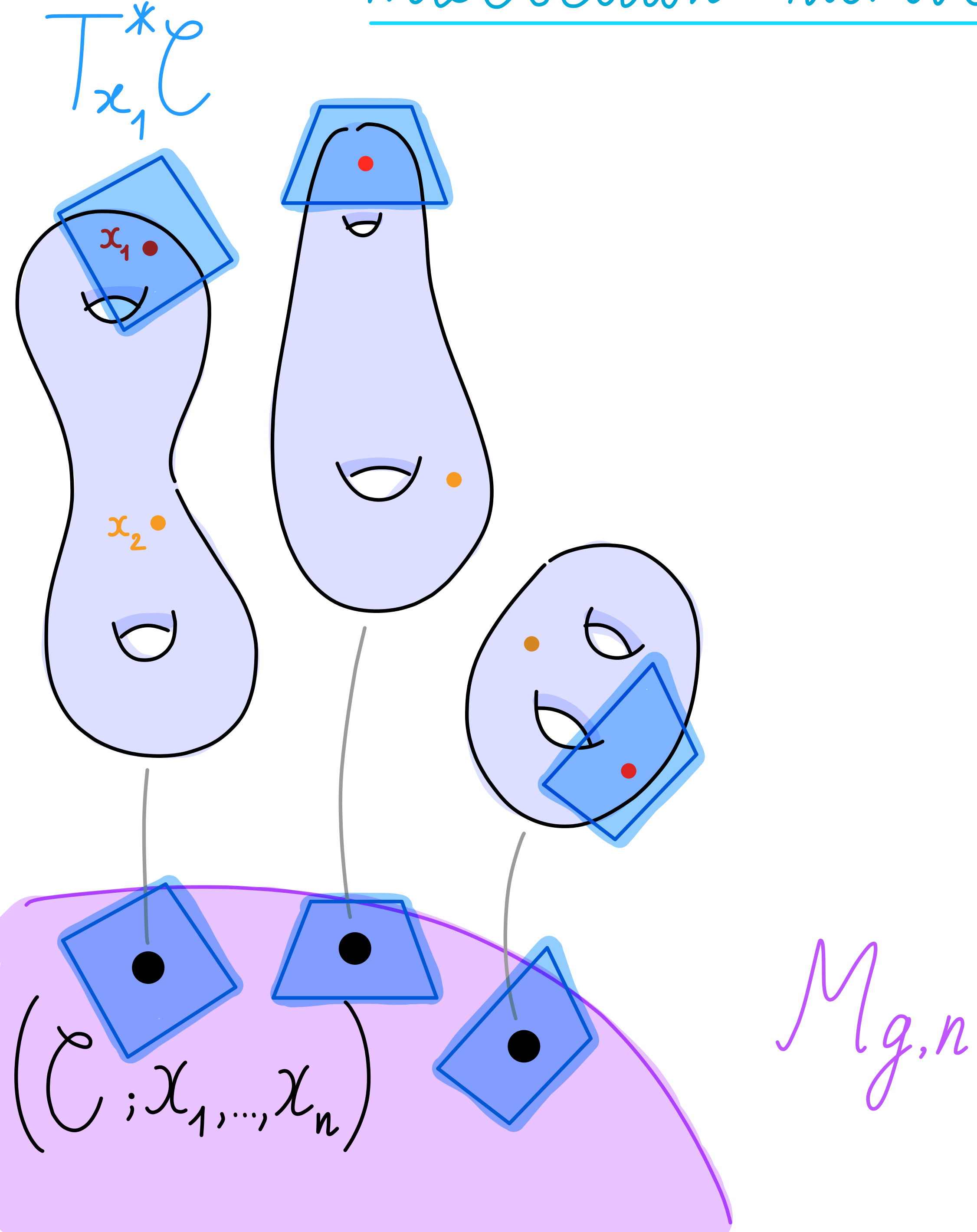

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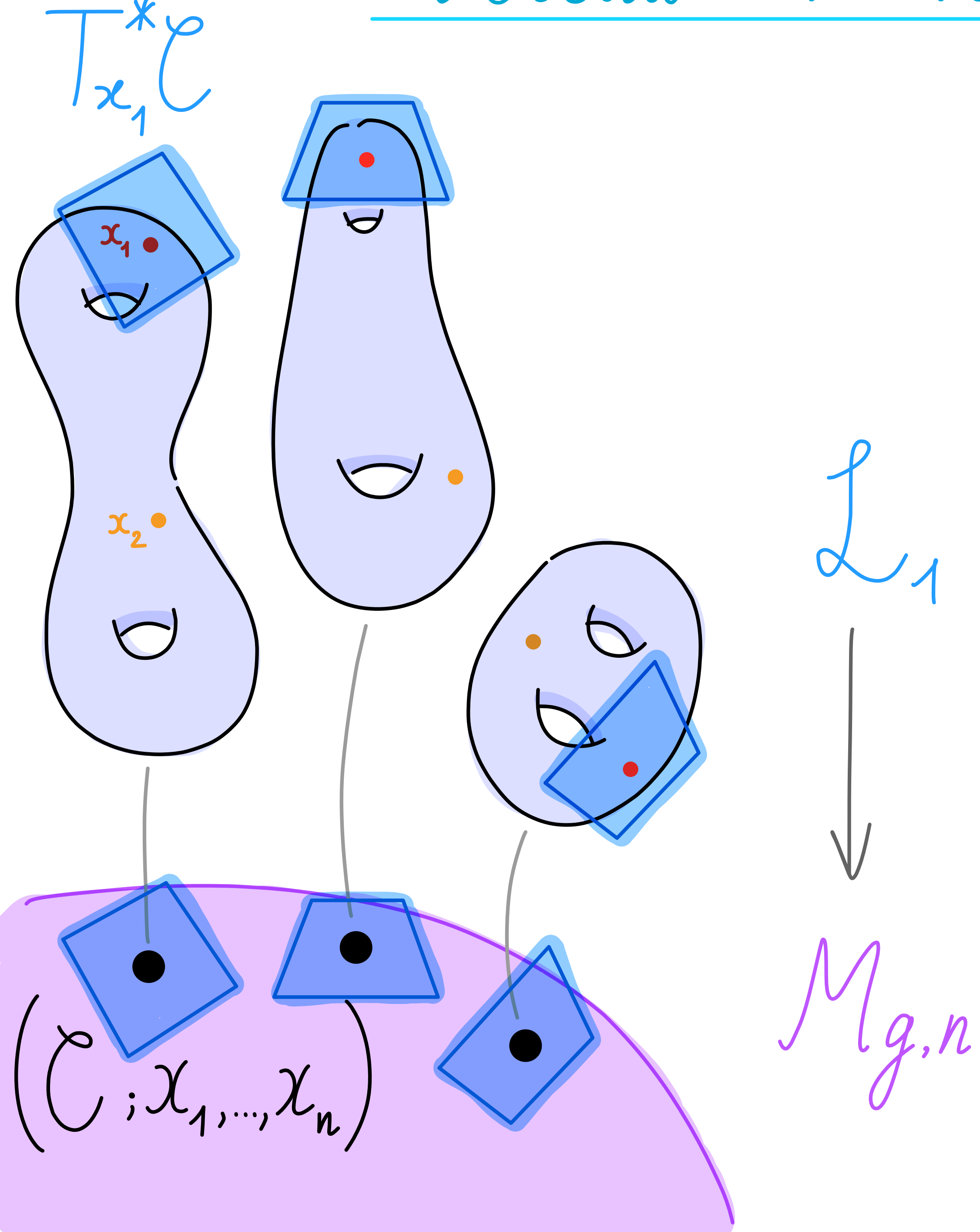

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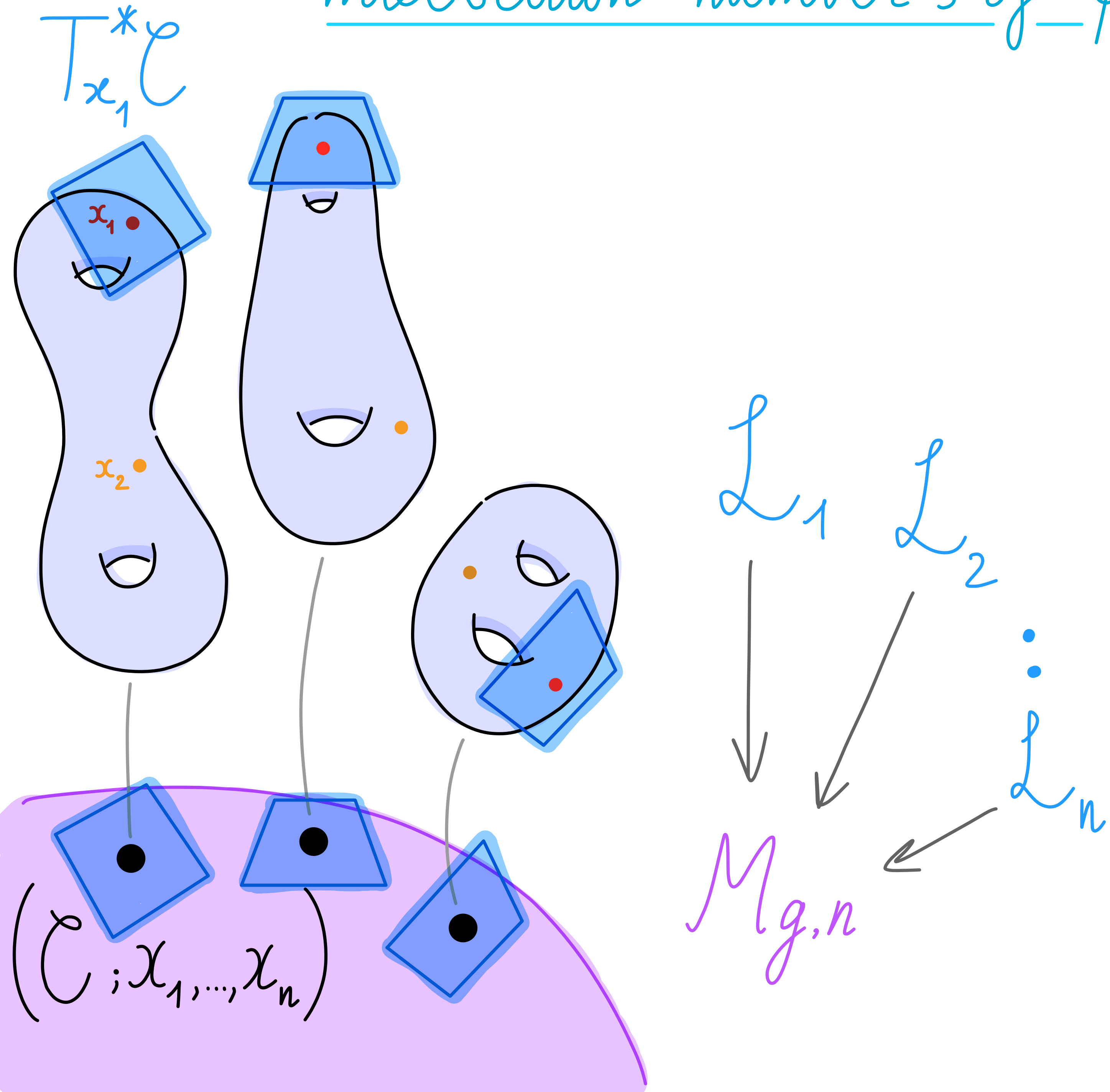

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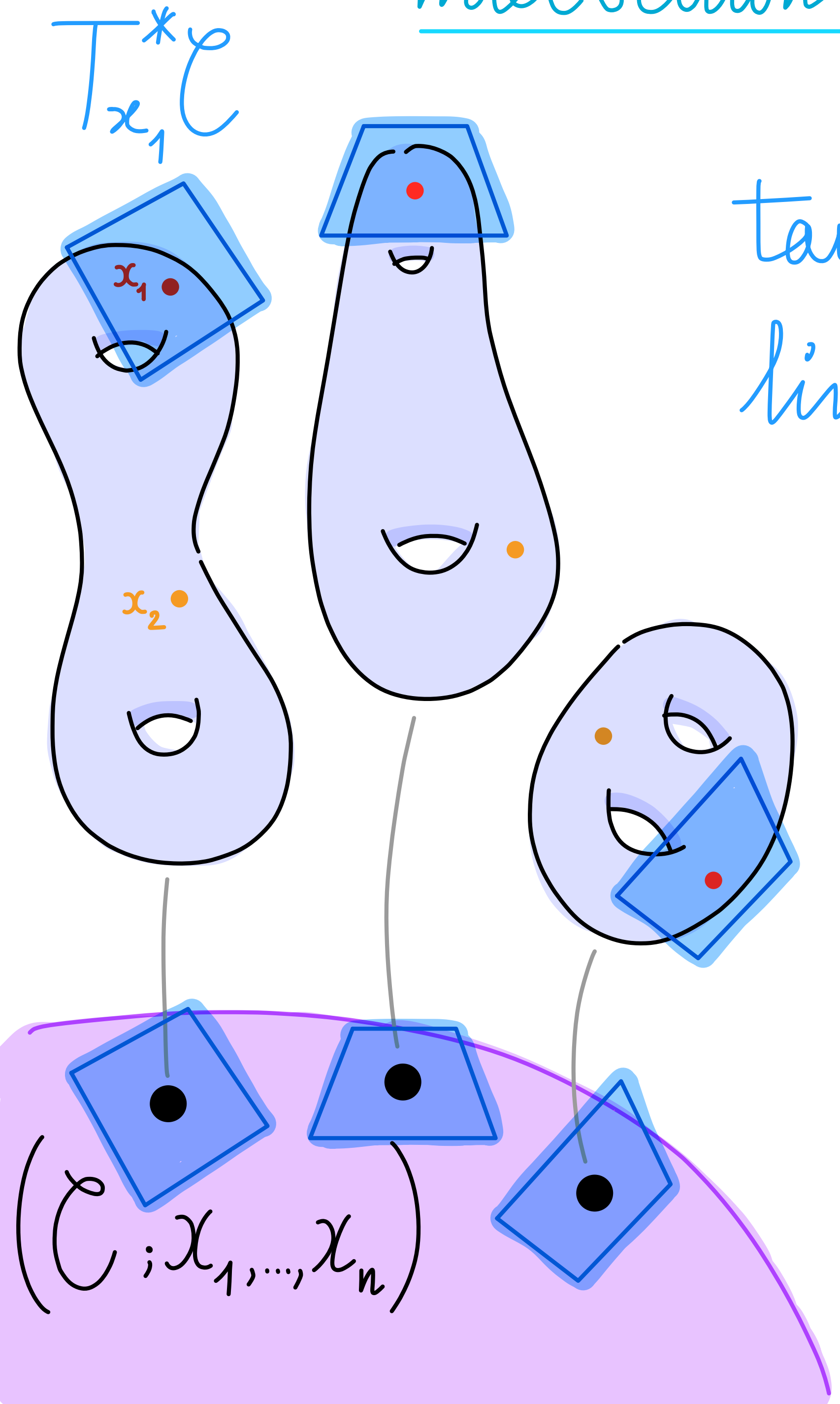
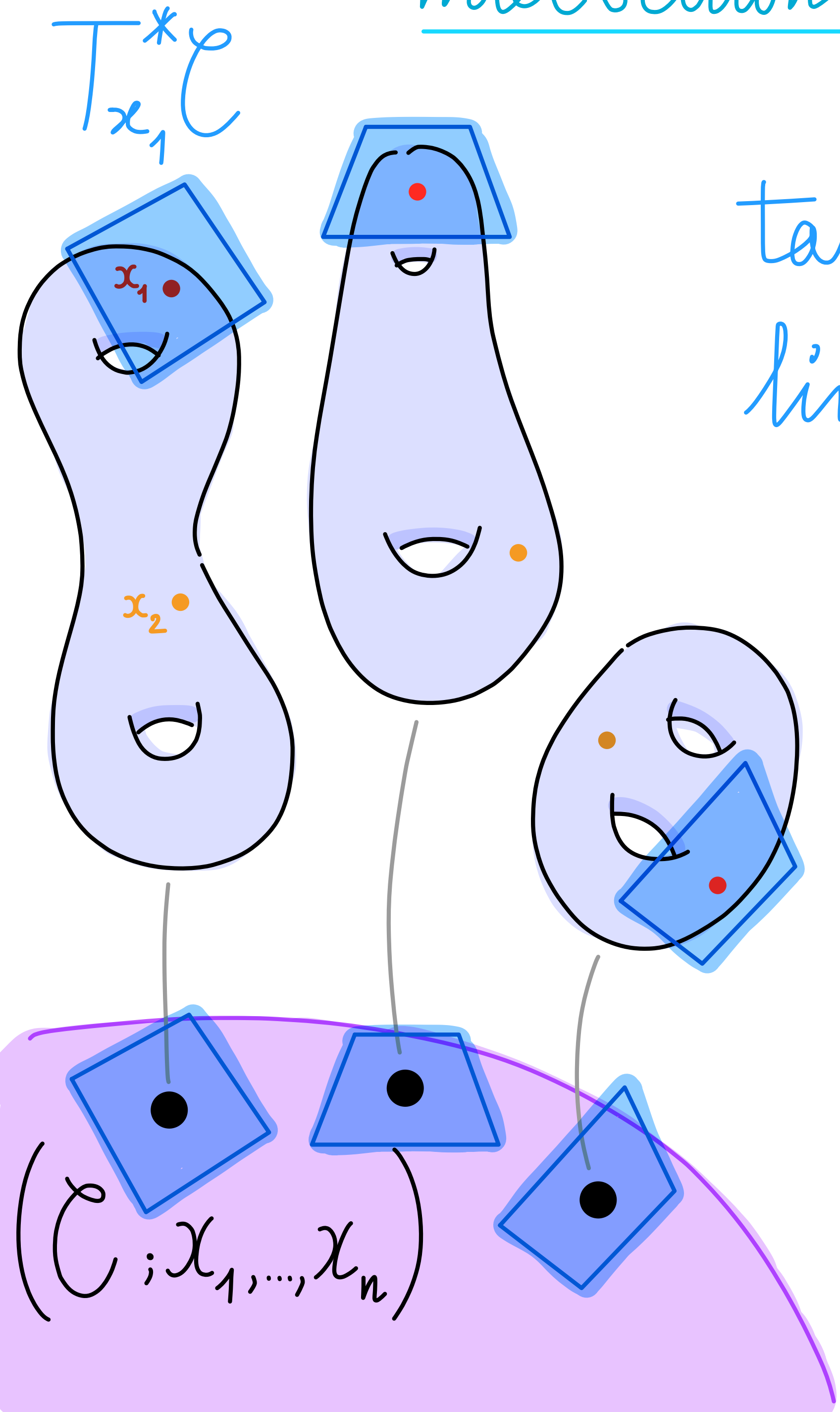


tautological line bundles

$$\mathcal{L}_1 \quad \mathcal{L}_2 \quad \vdots \quad \mathcal{L}_n \longrightarrow \mathcal{M}_{g,n}$$

intersection numbers of ψ-classes



tautological line bundles

$$\mathcal{L}_1,\ \mathcal{L}_2,\ \dots,\ \mathcal{L}_n \longrightarrow \mathcal{M}_{g,n}$$

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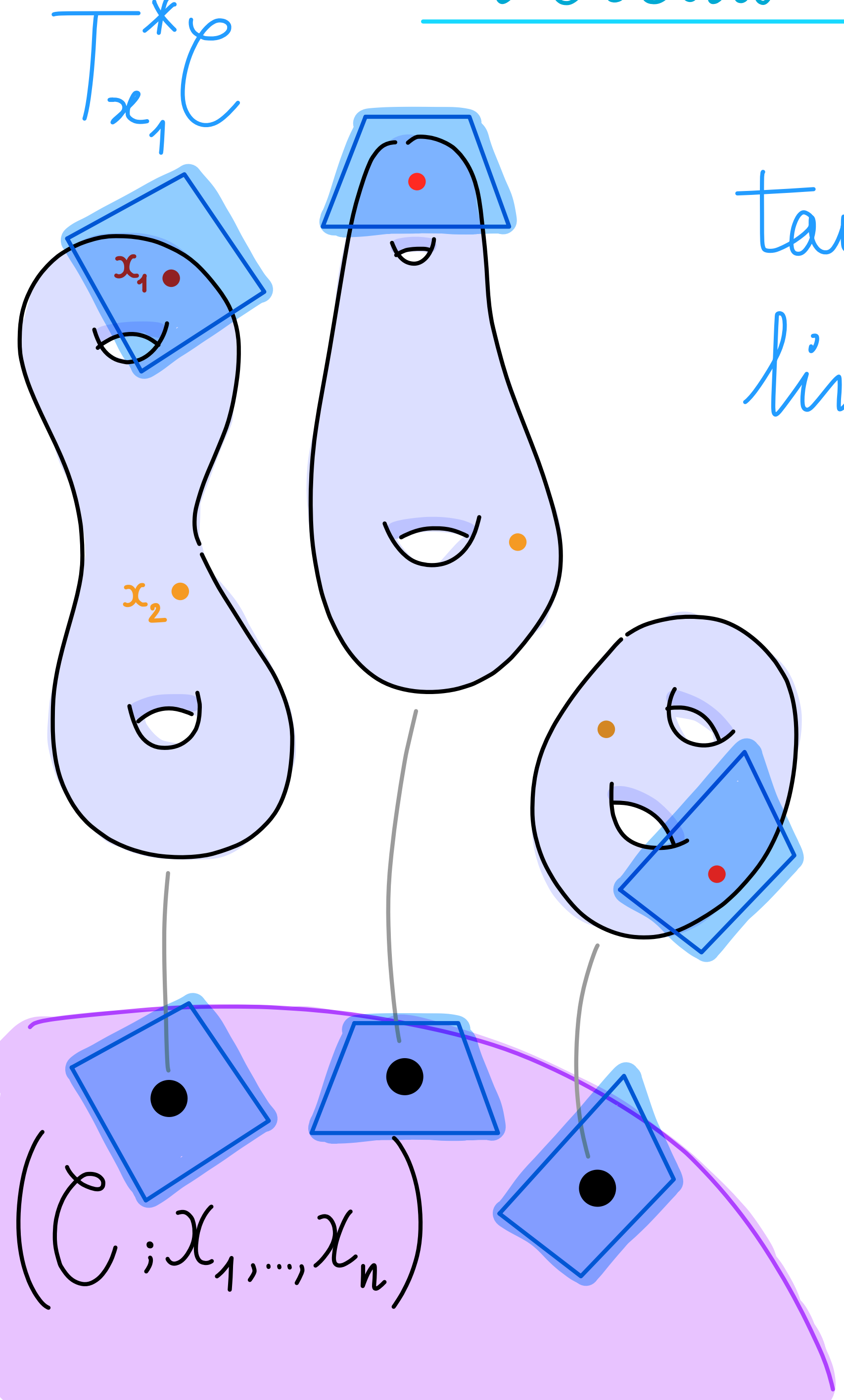

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$$\psi_i := c_1(\mathcal{L}_i)$$

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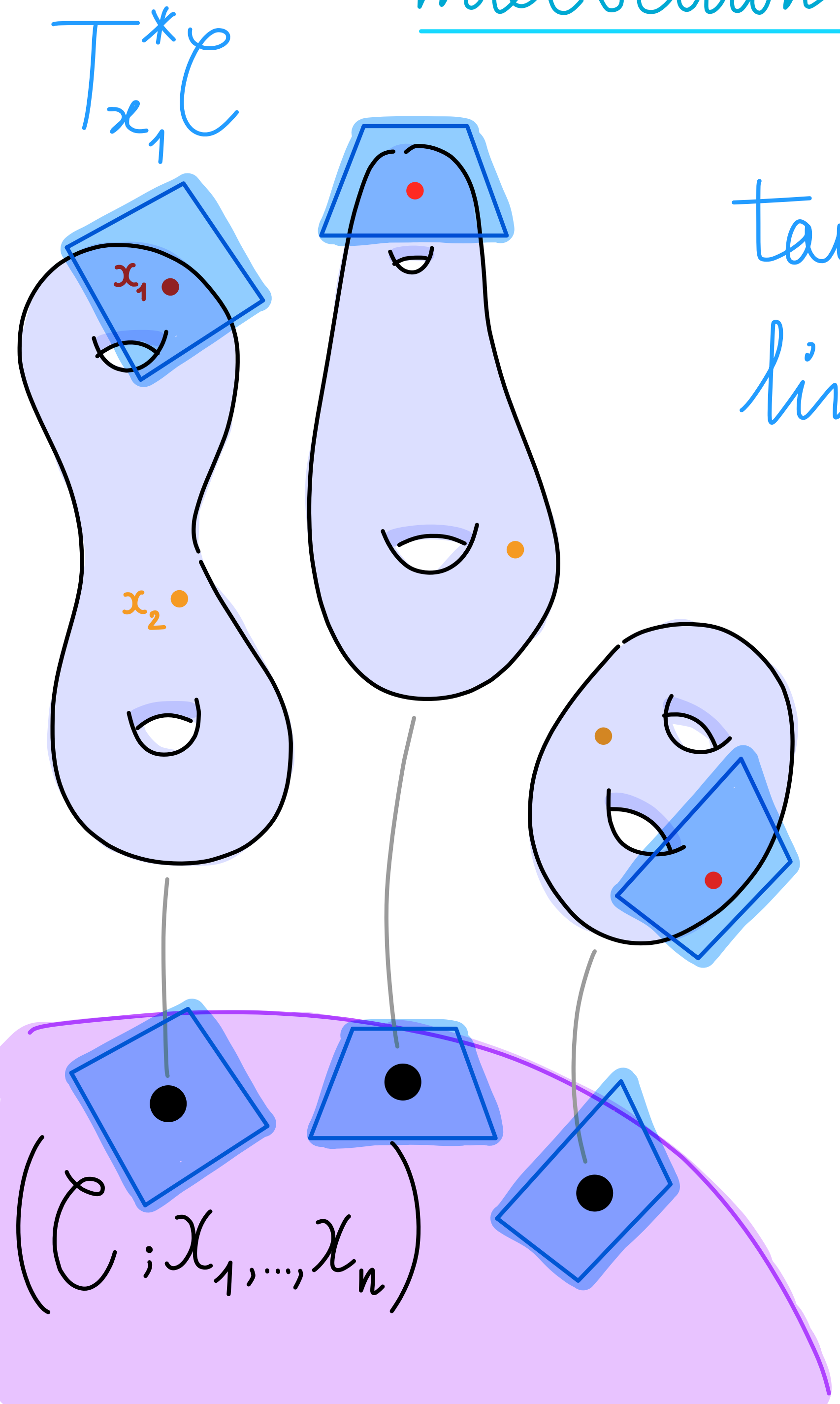

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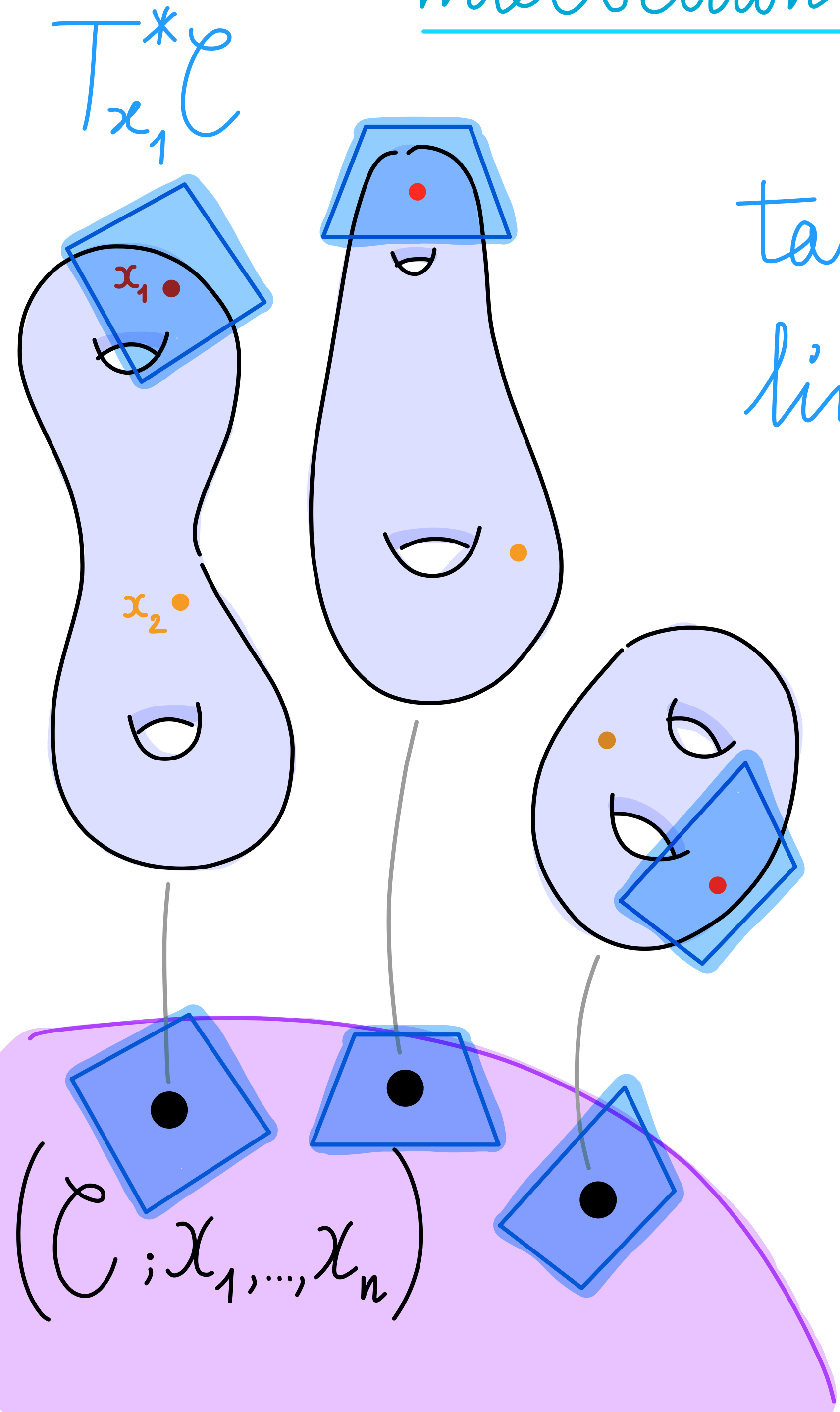

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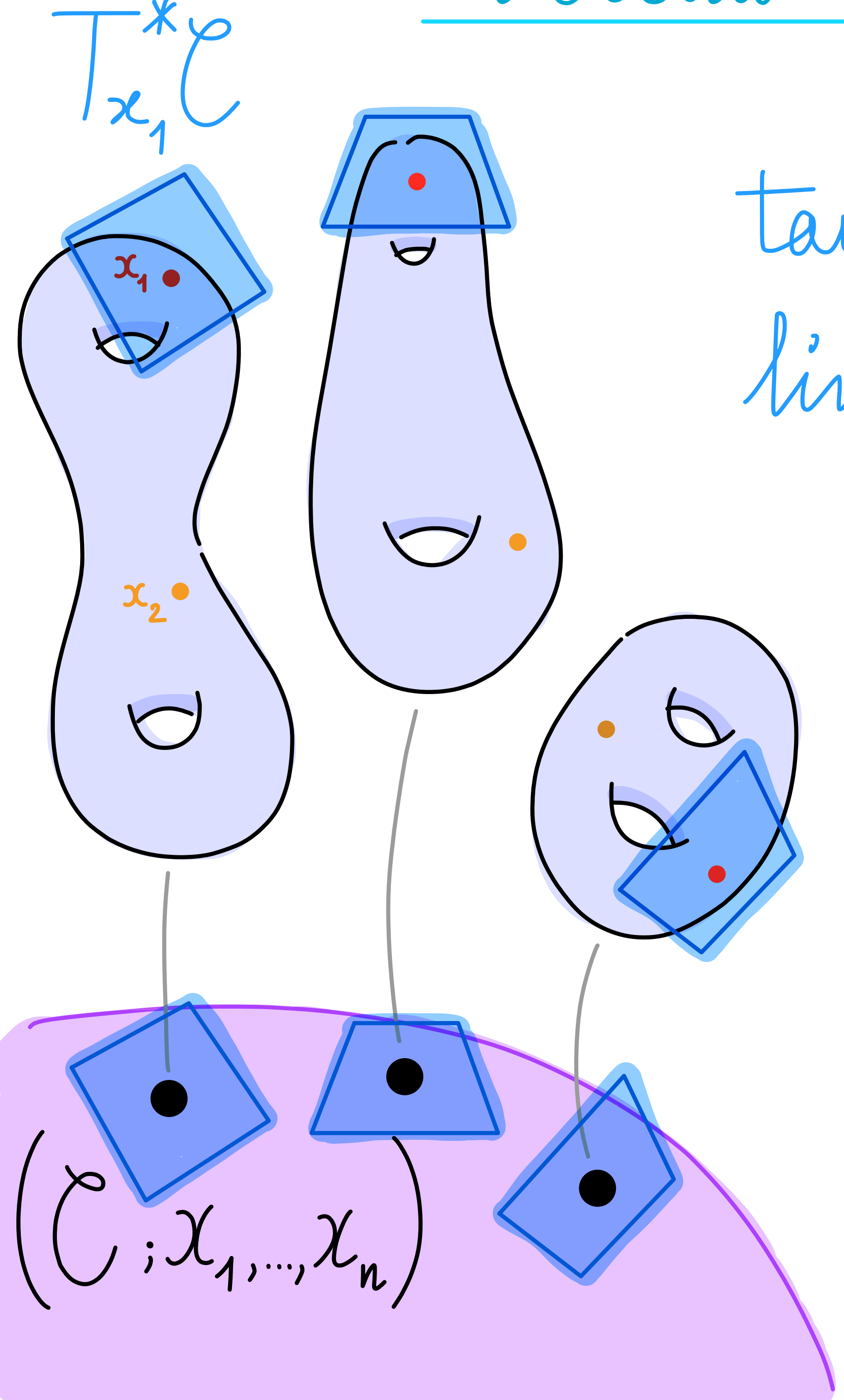

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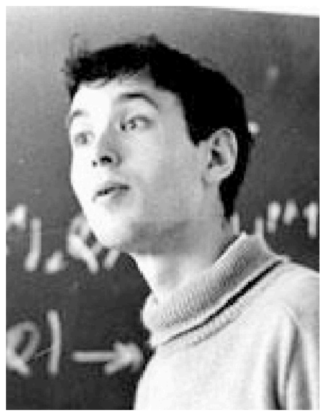

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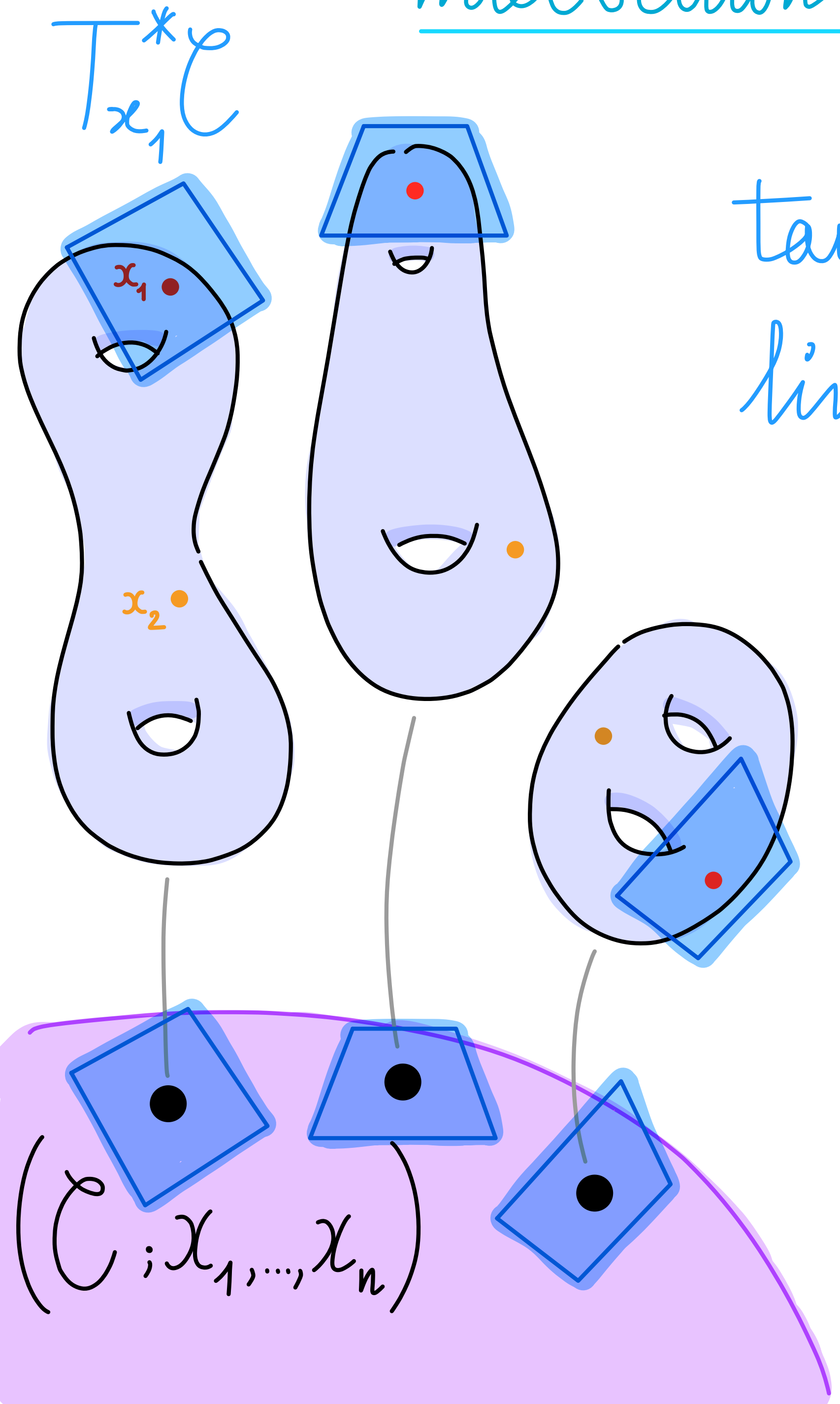

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$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \in \mathbb{Q}$$

Density polynomial $\alpha = \alpha_1 + \cdots + \alpha_k$, $\mathcal{U} \subset \Delta^{k-1}$ open

$$\mathbb{P}\left(\hat{\ell}(\gamma_{X,R,\alpha}) \in \mathcal{U}\right) \xrightarrow[R\to\infty]{} \int_{\mathcal{U}} P_\alpha(x_1, \ldots, x_k)$$

P_α is a polynomial which can be expressed in terms of

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where e_k is the kth elementary symmetric fct $\sum_{i_1 < \cdots < i_k} d_{i_1} \cdots d_{i_k}$

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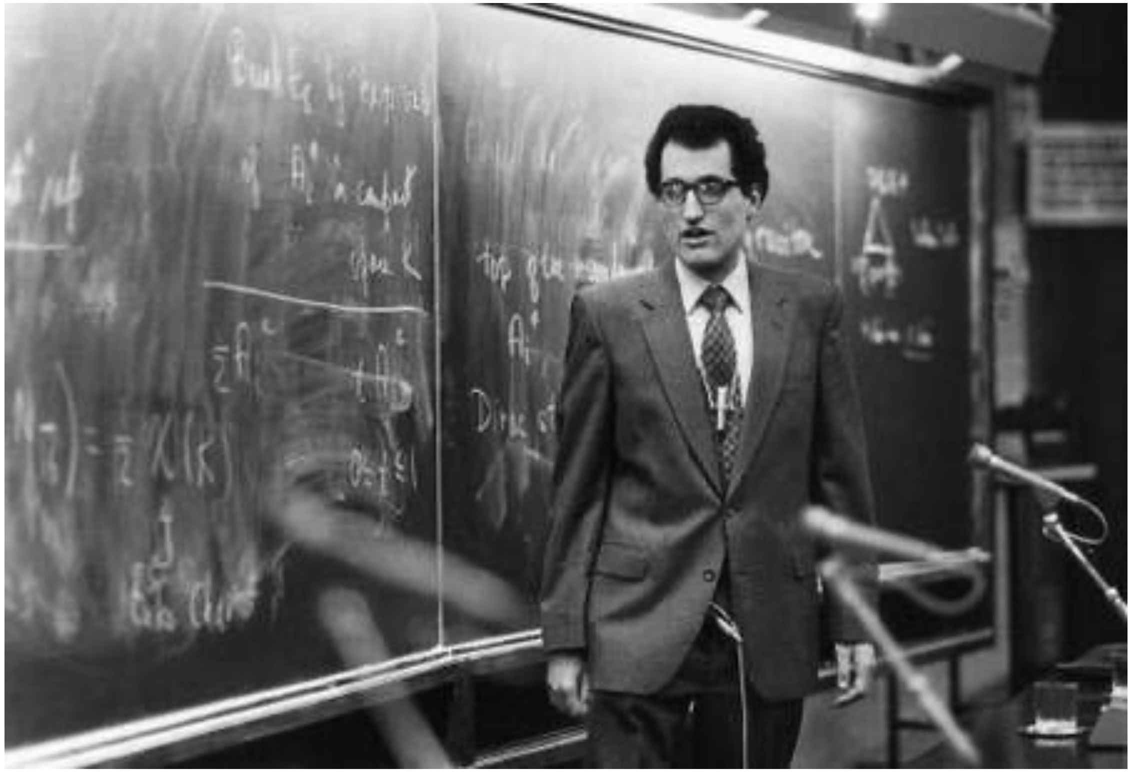

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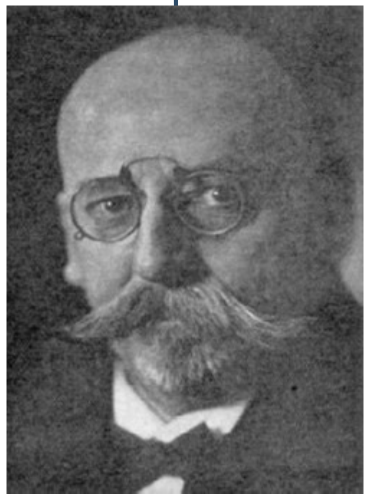
P_α is a polynomial which can be  d  of

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$\dots + \alpha_k$, $U \subset \Delta^{k-1}$ open

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$$\int_{\overline{\mathcal{M}}_{g,n+1}} \psi_1^{d_1} \cdots \psi_n^{d_n} \psi_{n+1}^{m+1}$$

$$(2m+3)\cdot\int_{\overline{M}_{g,n+1}} \psi_1^{d_1}\cdots\psi_n^{d_n}\,\psi_{n+1}^{m+1}$$

$$= \sum_{i=1}^{n} \frac{(2d_i+2m+1)!!}{(2d_i-1)!!}\int_{\overline{M}_{g,n}} \psi_1^{d_1}\cdots\psi_i^{d_i+m}\cdots\psi_n^{d_n}$$

$$+ \sum_{a+b=m-1} (2a+1)!!\,(2b+1)!!\int_{\overline{M}_{g-1,n+2}} \psi_1^{d_1}\cdots\psi_n^{d_n}\cdot\psi_{n+1}^{a}\cdot\psi_{n+2}^{b}$$

$$+ \sum_{\substack{a+b=m-1\\ I\sqcup J=\{1,\dots,n\}\\ g_1+g_2=g}} (2a+1)!!\,(2b+1)!!\cdot\int_{\overline{M}_{g_1,I\cup\{n+1\}}} \psi_{n+1}^{a}\prod_{i\in I}\psi_i^{d_i}\cdot\int_{\overline{M}_{g_2,J\cup\{n+2\}}} \psi_{n+2}^{b}\prod_{j\in J}\psi_j^{d_j}$$

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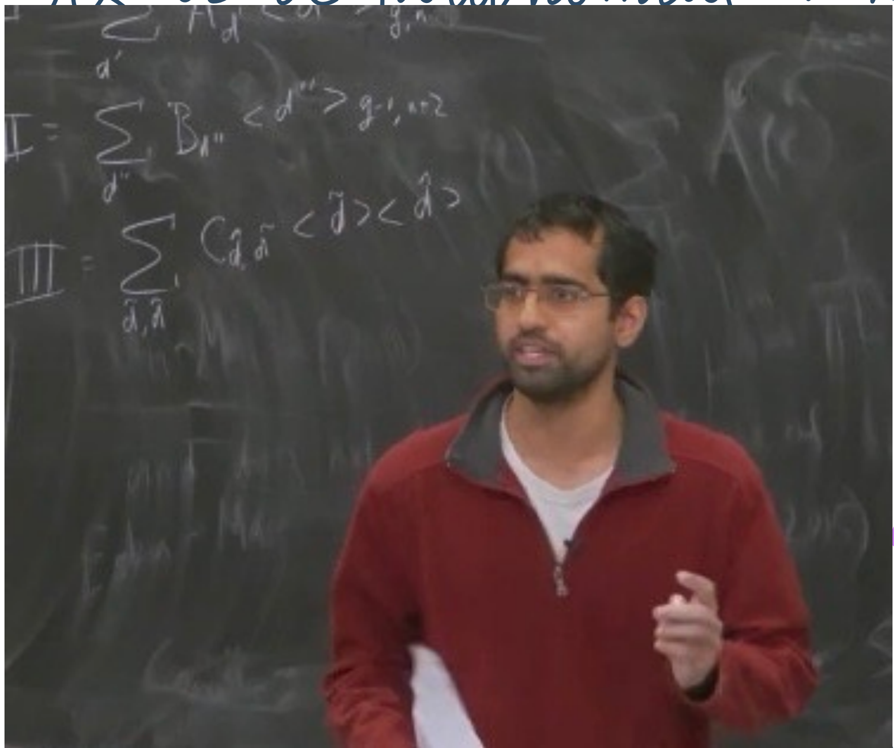

$$\underset{g\to\infty}{\sim} \frac{(6g-5+2n)!!}{(2d_1+1)!!\cdots(2d_n+1)!!} \cdot g.$$

91: recurrence relations KdV hierarchy

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Thank you!