(Random) Hyperbolic Surfaces with LARGE Systole

Palermo 26.7.2024

joint work with

Bram Petri

Systole

<u>Systole</u> X closed hyperbolic surface

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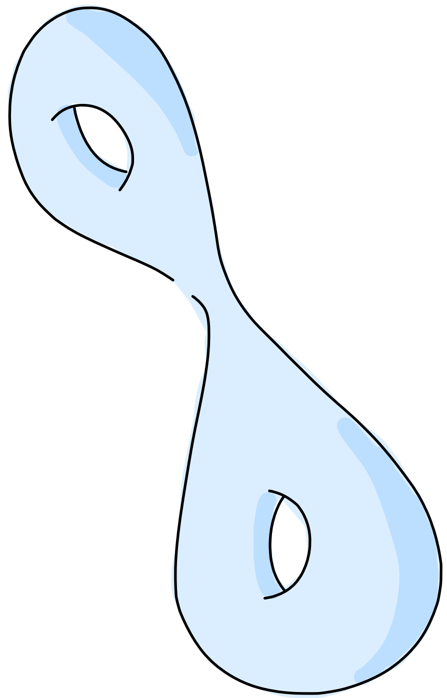
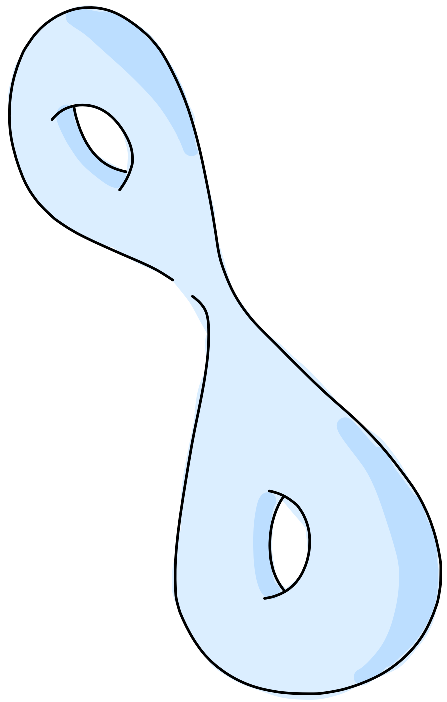


Systole X closed hyperbolic surface

$\mathrm{sys}(X) :=$ length of the shortest closed geodesics on X



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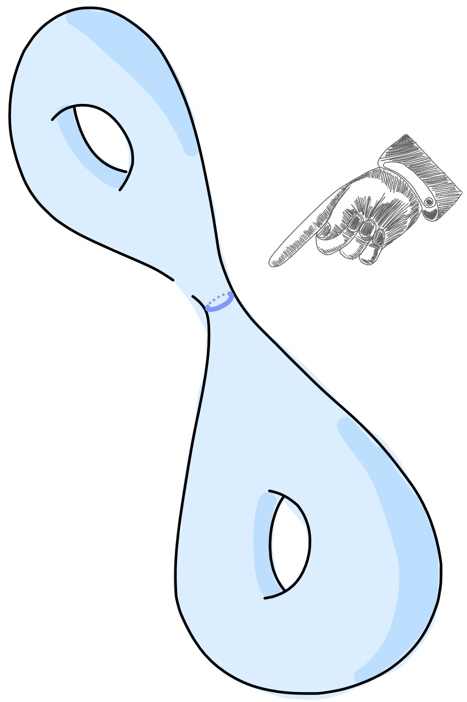

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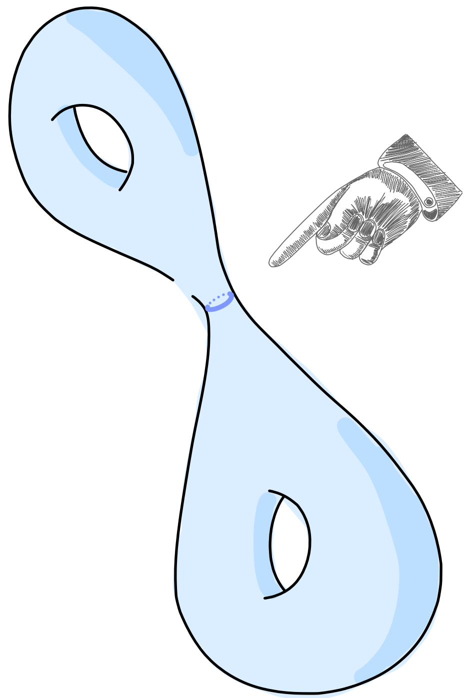

M_g

Systole

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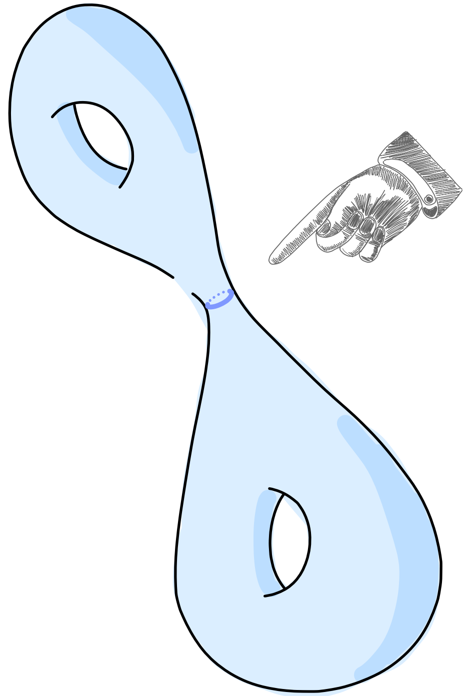

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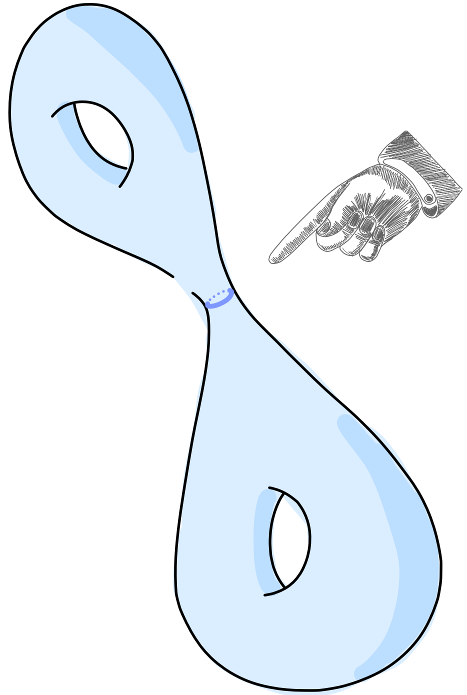

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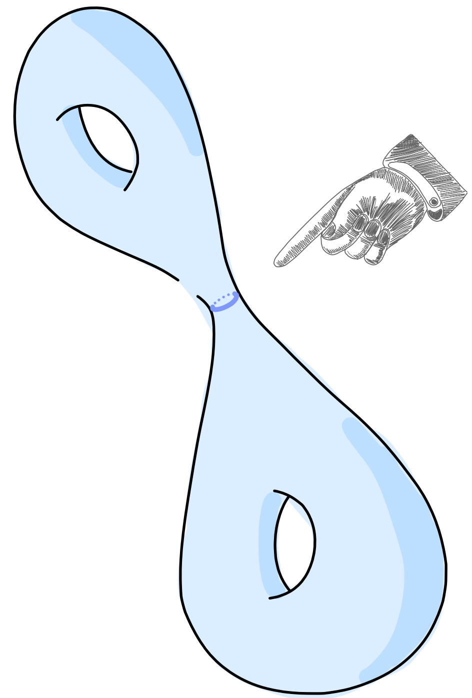

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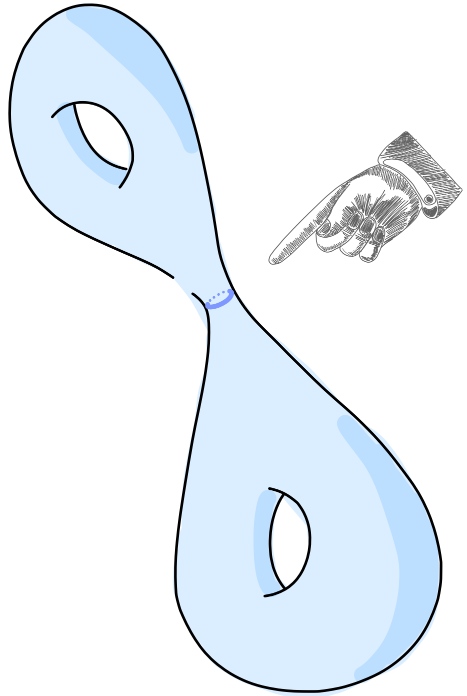

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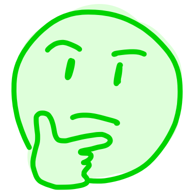

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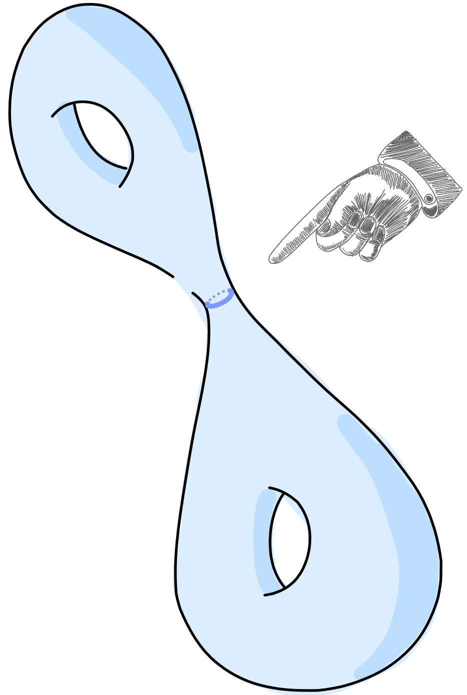

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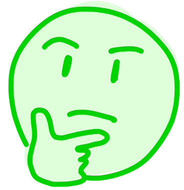

Lemma for any $X \in \mathcal{M}_g$, we have

$$\mathrm{sys}(X) \leqslant 4 \operatorname{arcsinh}\left(\sqrt{g-1}\right)$$

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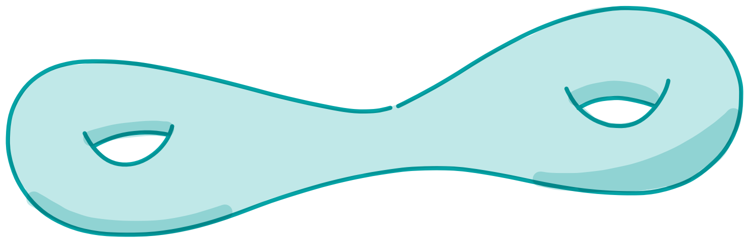

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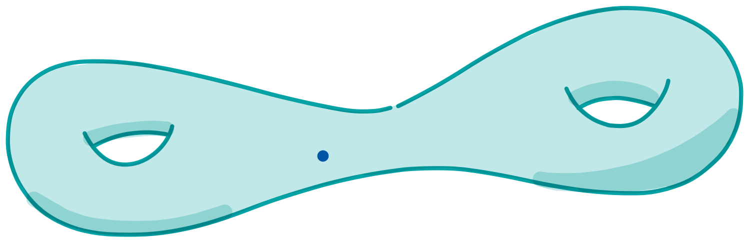

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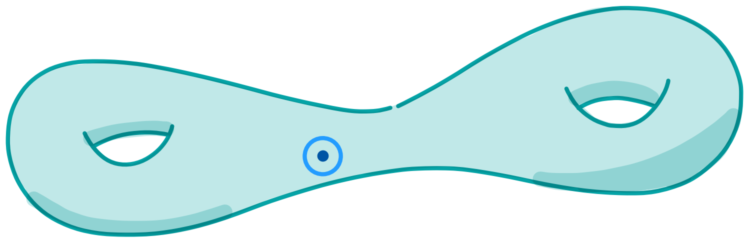

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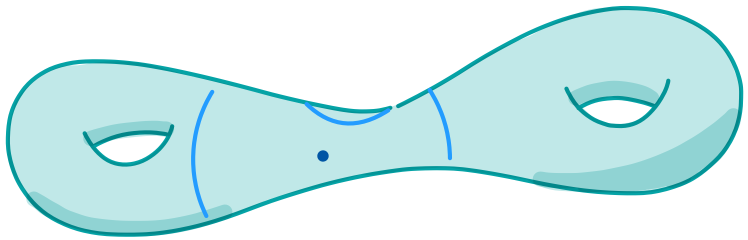

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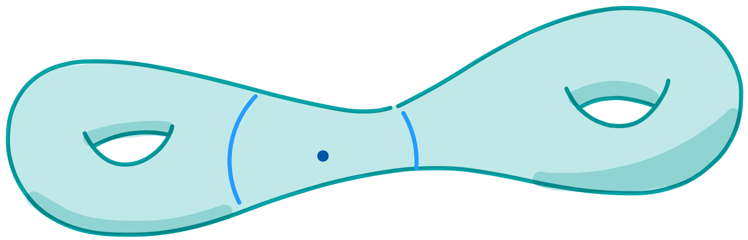

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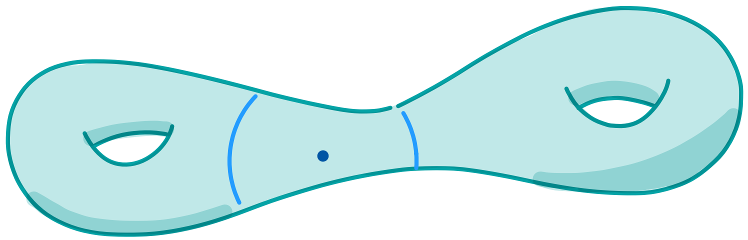

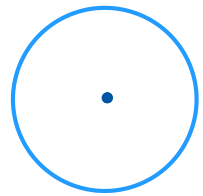

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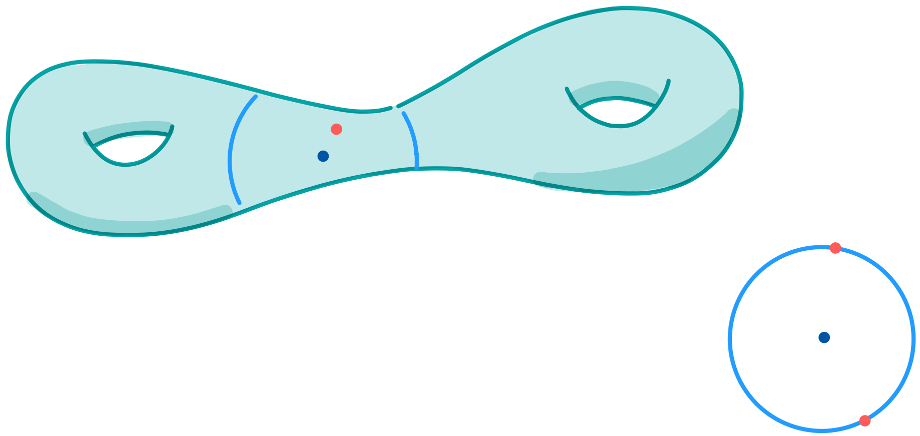

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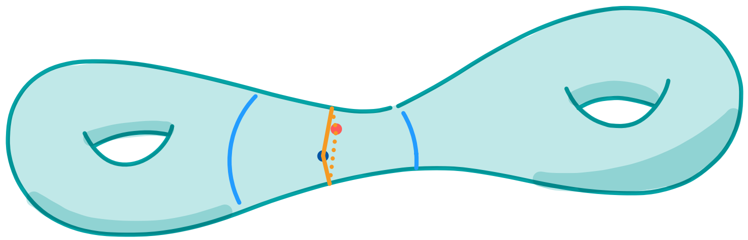

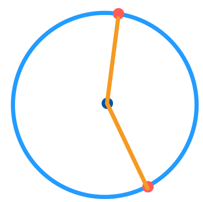

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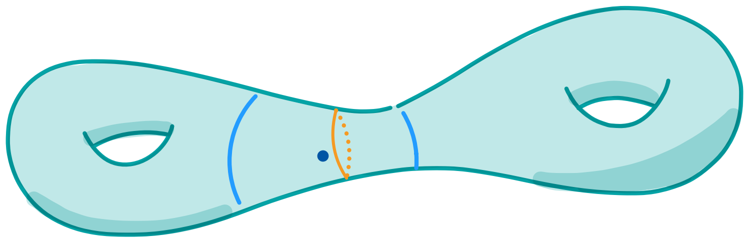

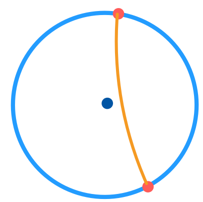

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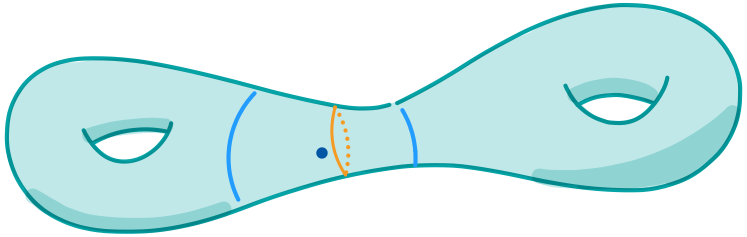

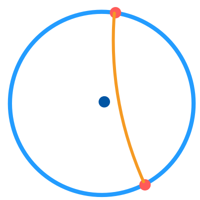

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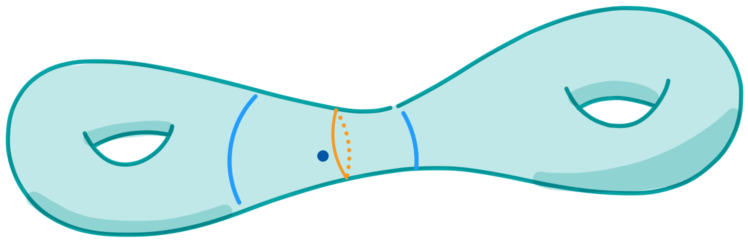

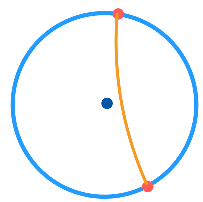

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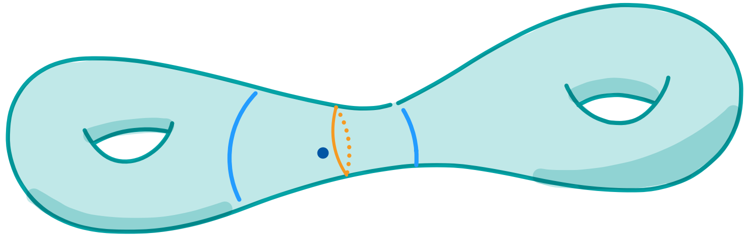

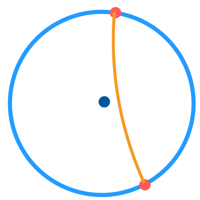

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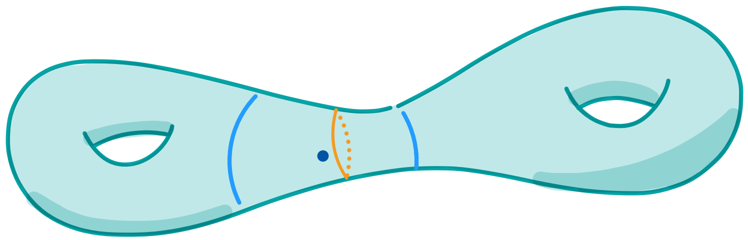

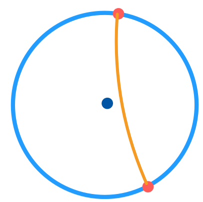

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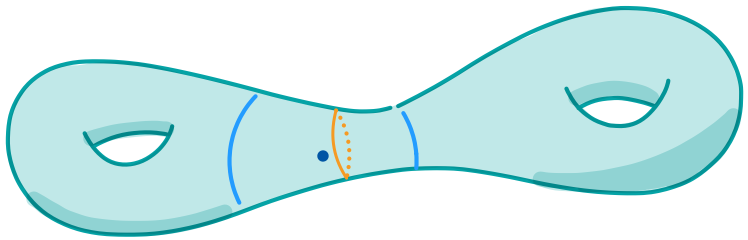

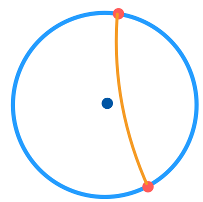

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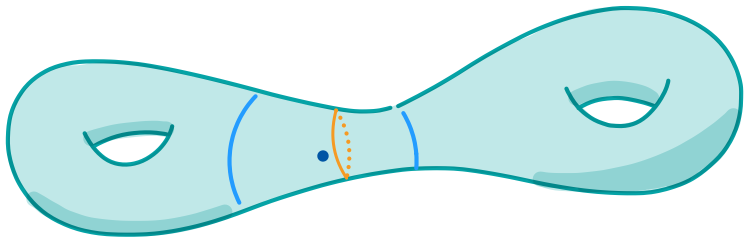

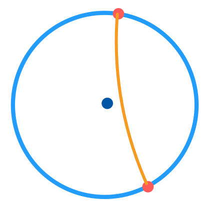

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$$\mathrm{area}(X) \overset{\text{Gauss-Bonnet}}{=} 2\pi(2g-2)$$

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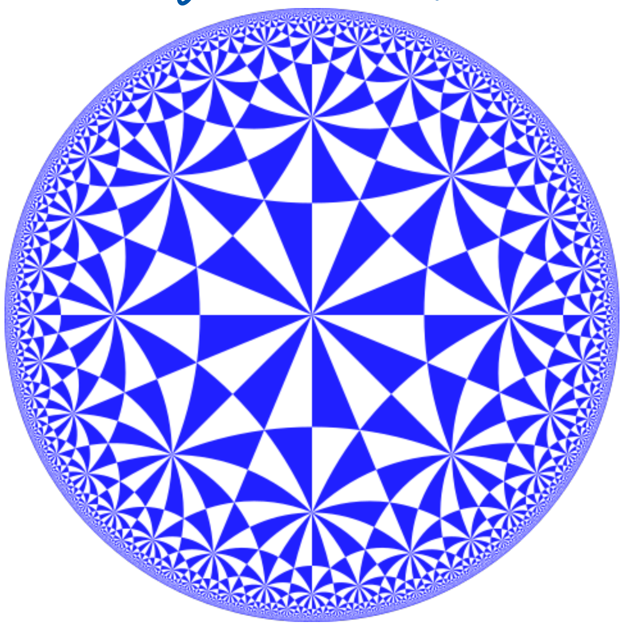

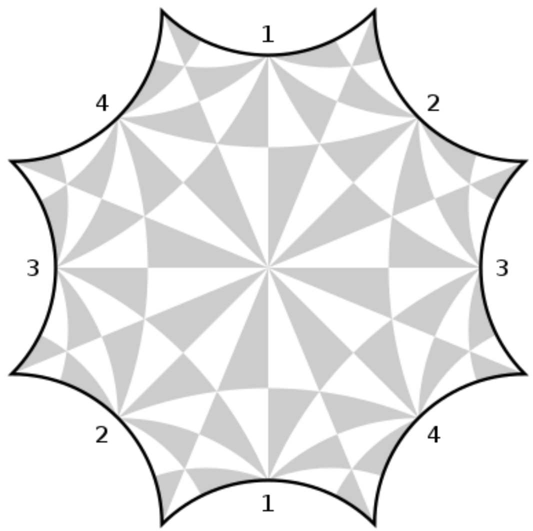

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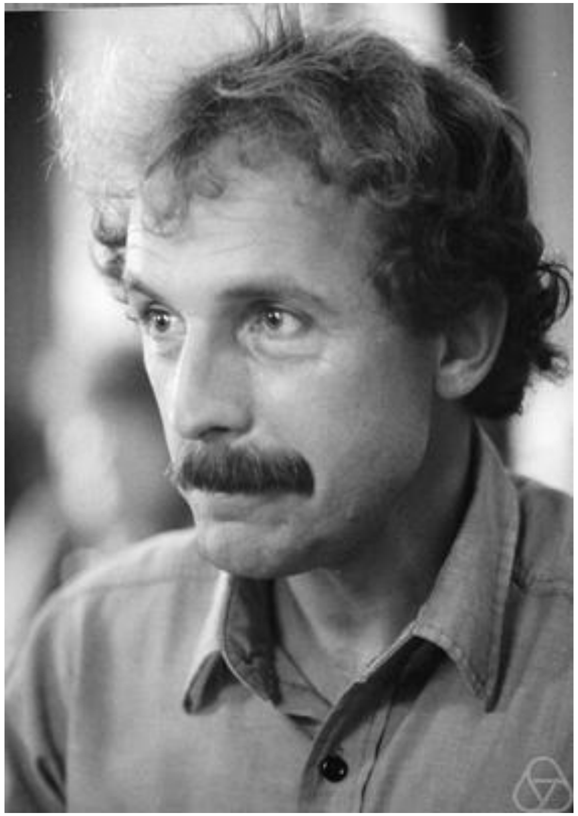
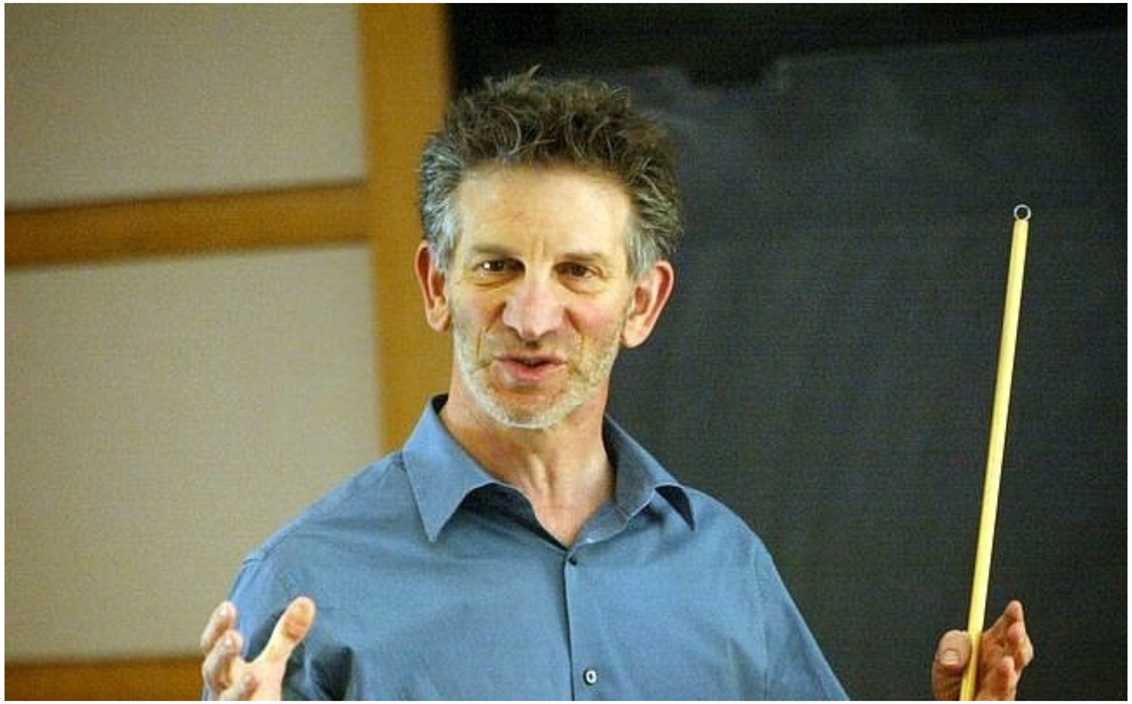
eg $\exists$? a sequence $(X_k)_k$ s.t. as $k \to \infty$, $\operatorname{sys}(X_k) \to \infty$

Theorem (Buser-Sarnak, 1994)





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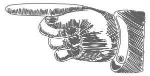

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some universal constant

X_k's are congruence covers of an arithmetic surface $/\mathbb{Q}$

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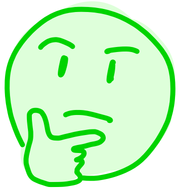

$$\lim_{g \to \infty} \frac{\max_{X \in \mathcal{M}_g} sys(X)}{\log g} = ?$$

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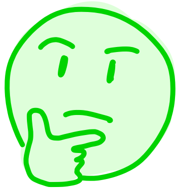

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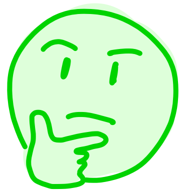
We don't know if the limit exists...

We know

$$\forall X, \quad \mathrm{sys}(X) \leqslant 2\log g + C_1$$

$$\exists (X_k)_k, \quad \mathrm{sys}(X_k) \geqslant \frac{4}{3}\log g_k - C_2$$



$$\lim_{g\to\infty} \frac{\max_{X\in\mathcal{M}_g} \mathrm{sys}(X)}{\log g} = \,?$$

We don't know if the limit exists...

We don't even know if max sys(g) is increasing...

Theorem (Buser–Sarnak, 1994)

$\exists$ a sequence $(X_k)_k$ such that $g_k = \text{genus}(X_k) \xrightarrow[k\to\infty]{} \infty$

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X_k's are congruence covers of an arithmetic surface $/\mathbb{Q}$

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Katz – Schaps – Vishne 2007

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a number field

Theorem (easy) $X \in \mathcal{M}_g$

$$\operatorname{sys}(X) \leq 4 \operatorname{arcsinh}\left(\sqrt{g-1}\right)$$

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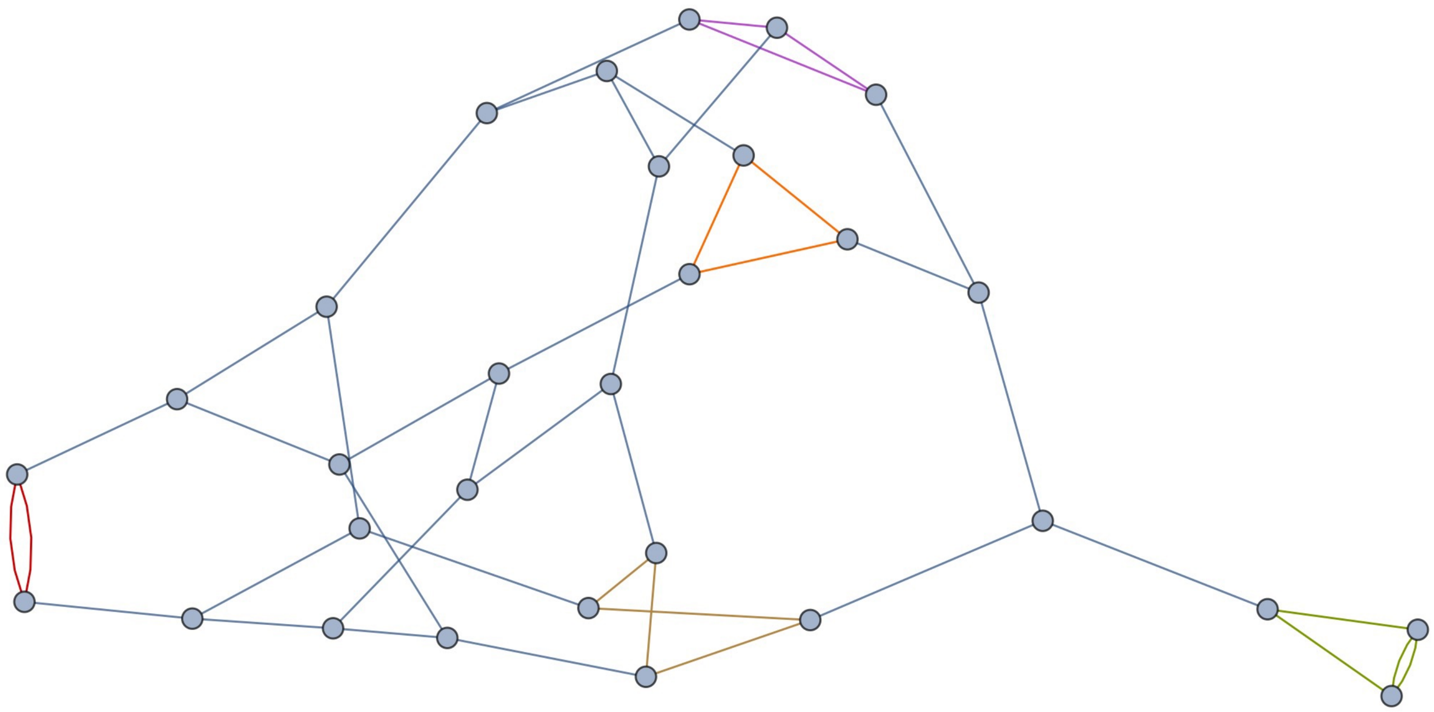
with cusps [Schmutz 1994, Fanoni-Parlier 2015]



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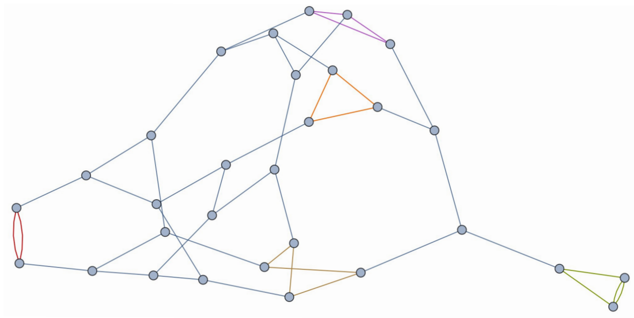

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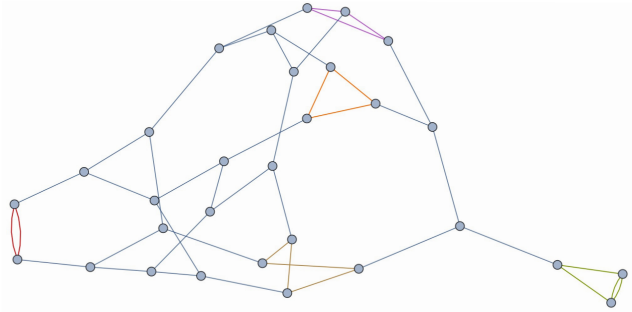

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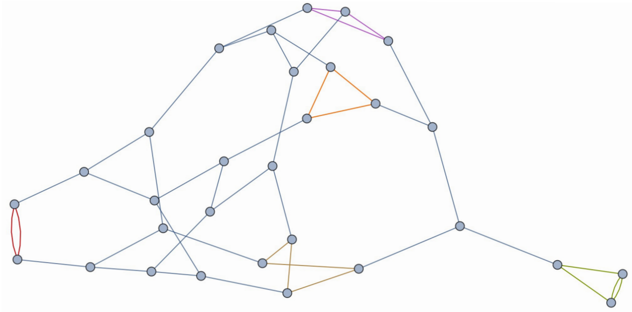

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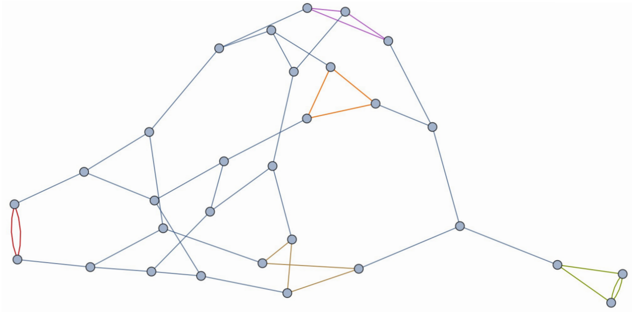

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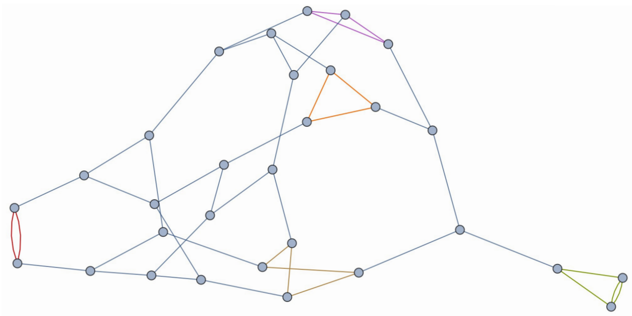

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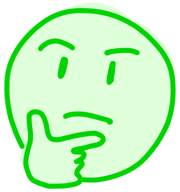

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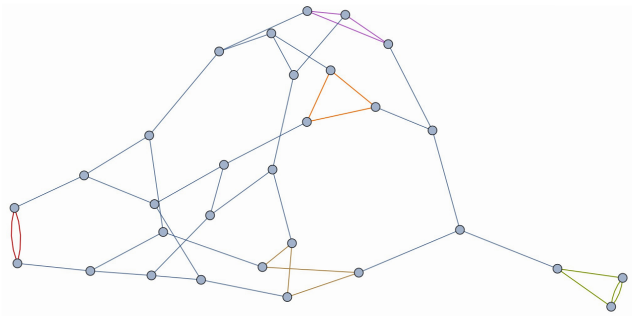

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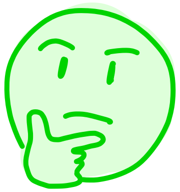

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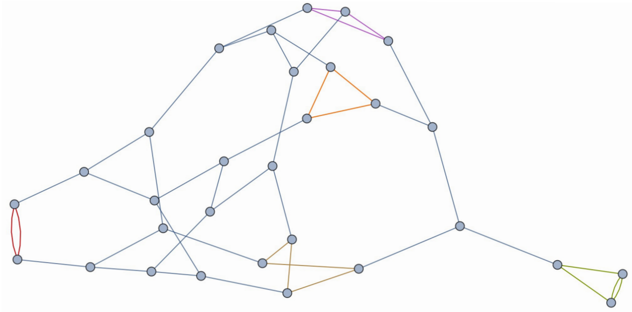

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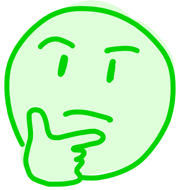

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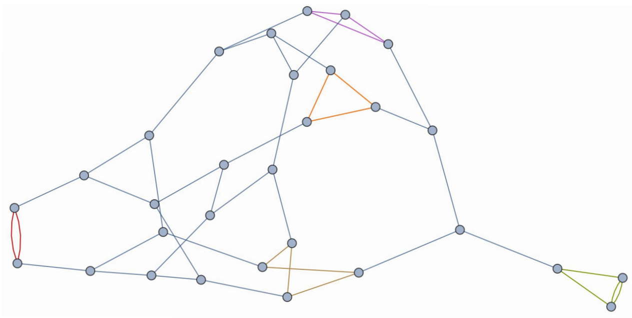

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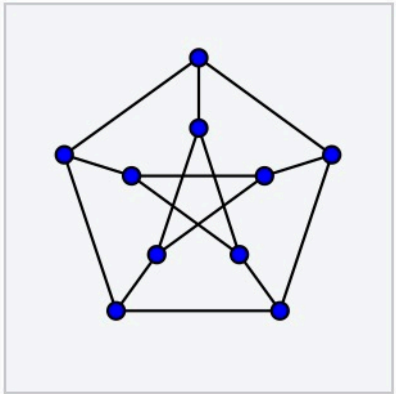
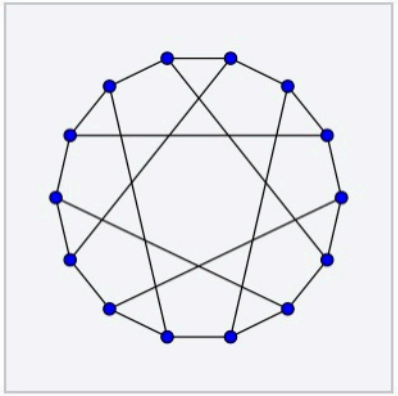
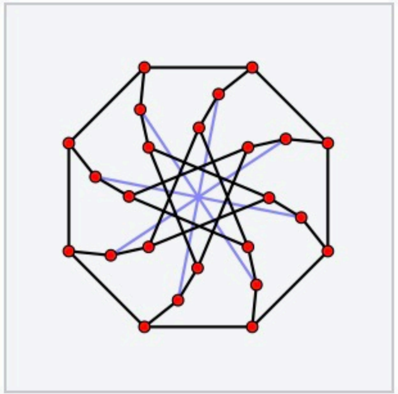
Γ_n ... es}



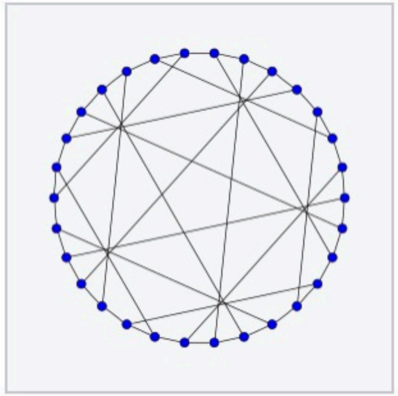
The Petersen graph has a girth of 5



The Heawood graph has a girth of 6



The McGee graph has a girth of 7

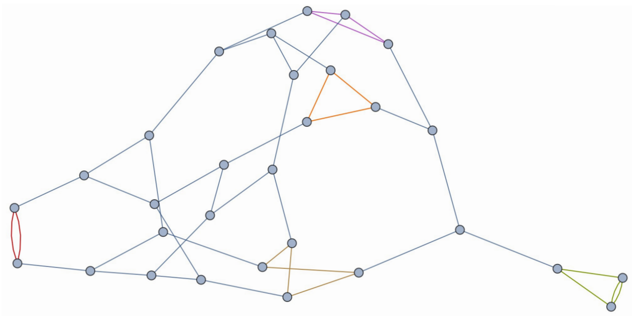


The Tutte–Coxeter graph (*Tutte eight cage*) has a girth of 8

Similarly, $\mathrm{gir}(G) \lesssim 2 \log_2 n$

$\exists (G_k)_k$ with $n_k \to \infty$ as $k \to \infty$ s.t. $\mathrm{gir}(G_k) \gtrsim \frac{4}{3} \log_2 n_k$

Girth



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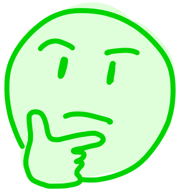

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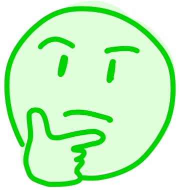

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Theorem (Buser–Sarnak)

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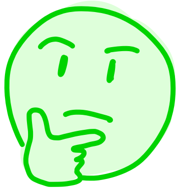

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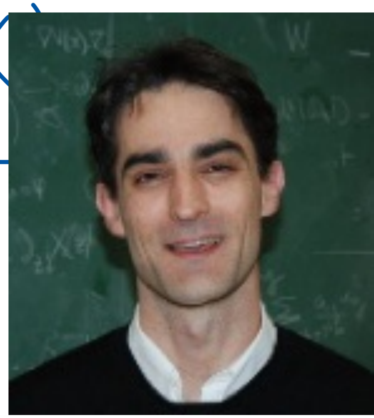

Theorem (Buser–Sarnak, Katz–Sabourau)

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How?

Random surfaces

Why should I care about random stuff?

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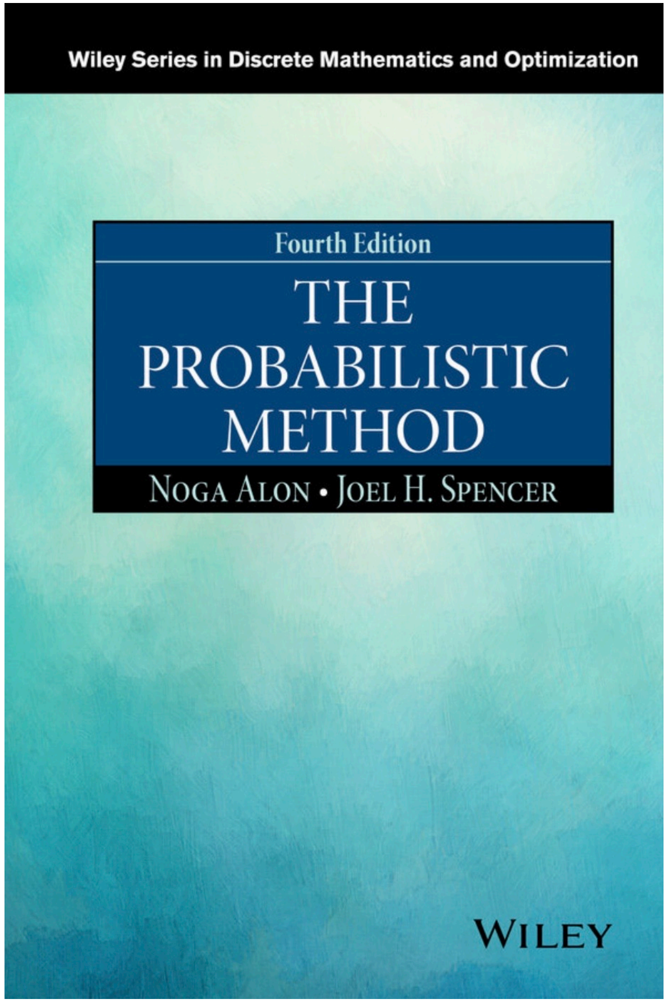

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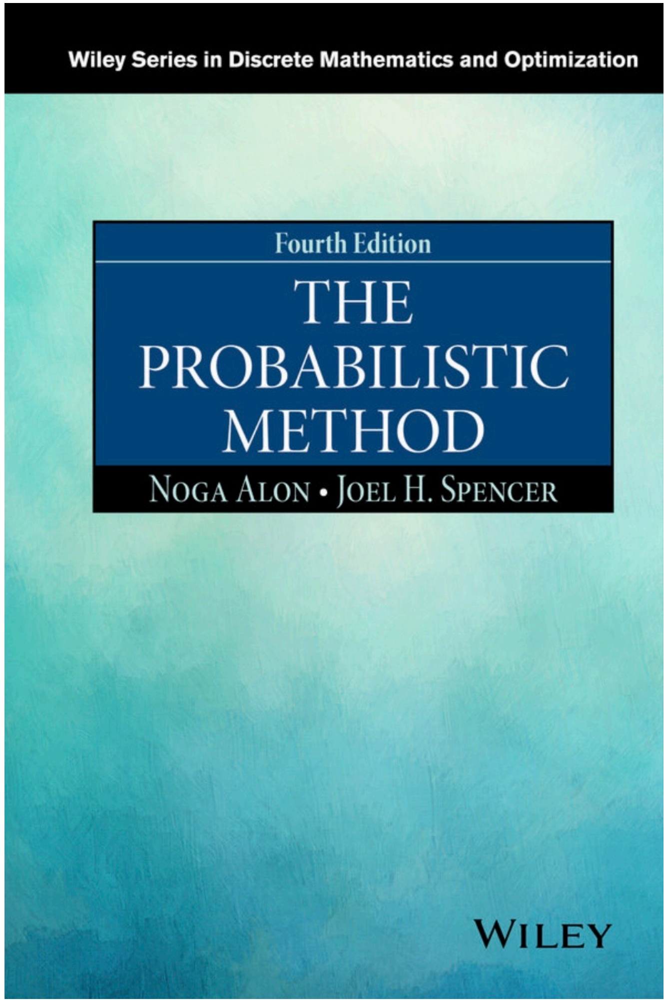

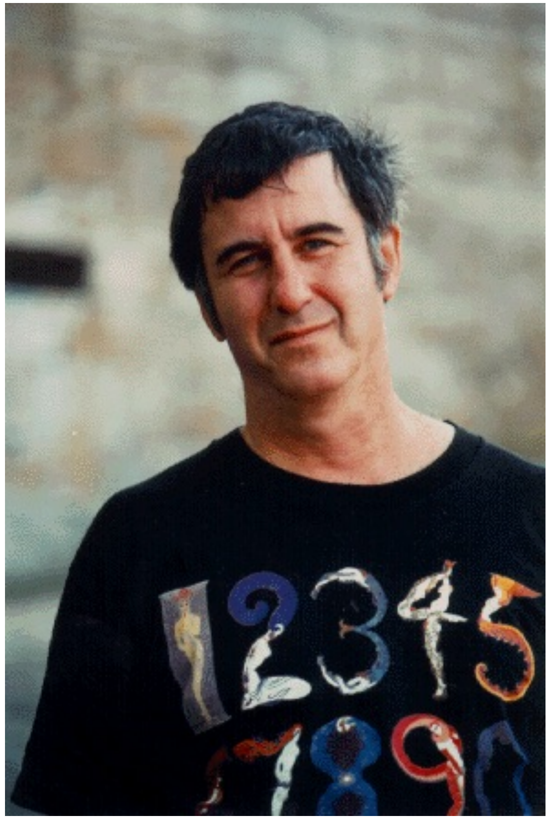

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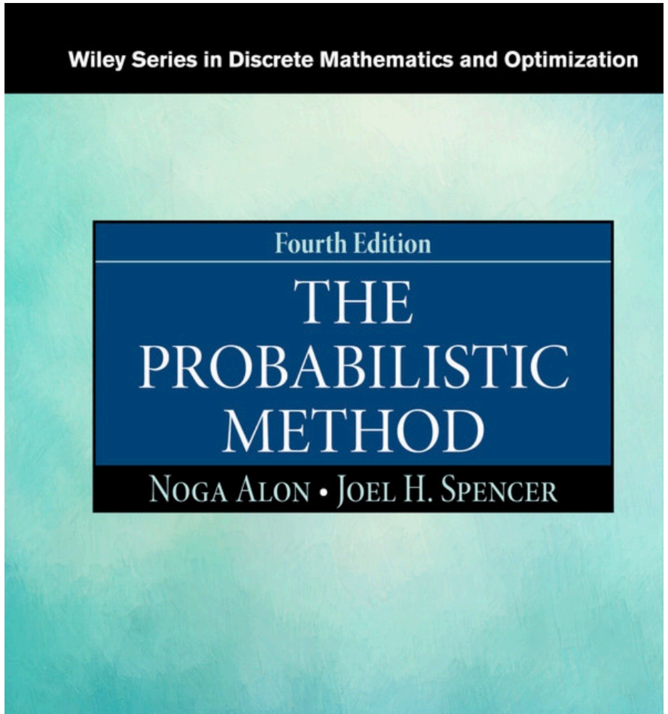

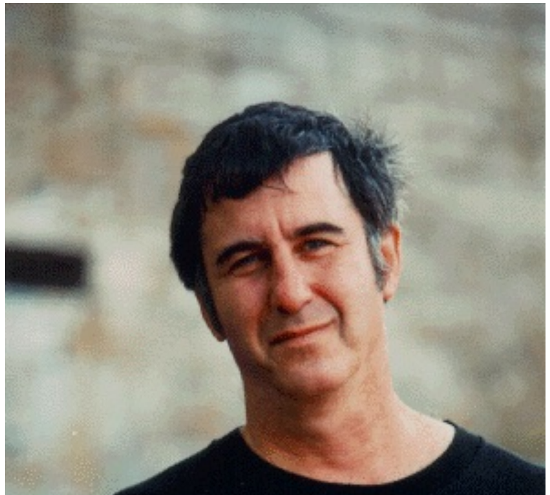

The basic probabilistic method can be described as follows: In order to prove the existence of a combinatorial structure with certain properties, we construct an appropriate probability space and show that a randomly chosen element in this space has the desired properties with positive probability. This method was initiated by Paul Erdős,

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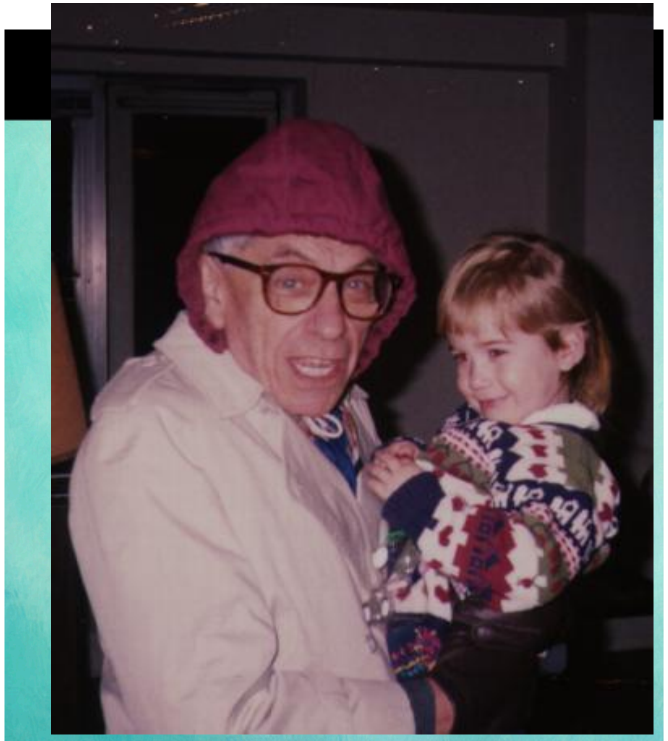

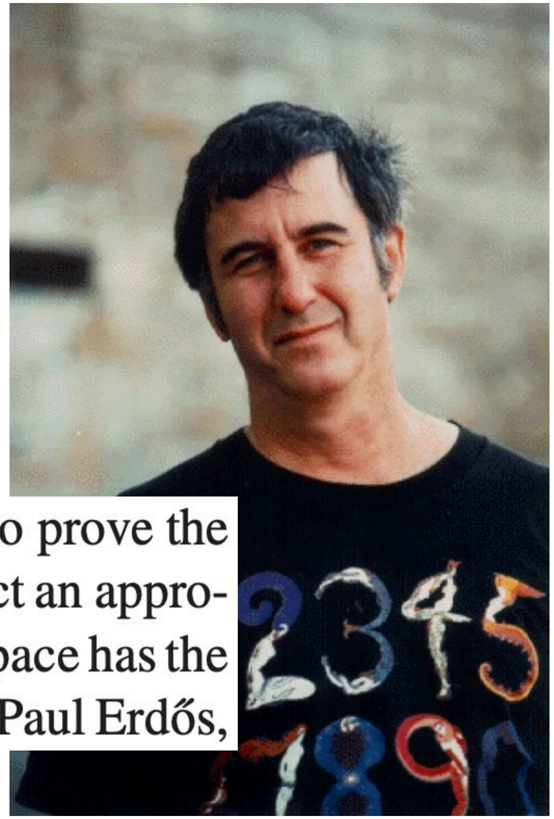

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two models

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both are inspired by graph theory

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1. a random greedy algorithm

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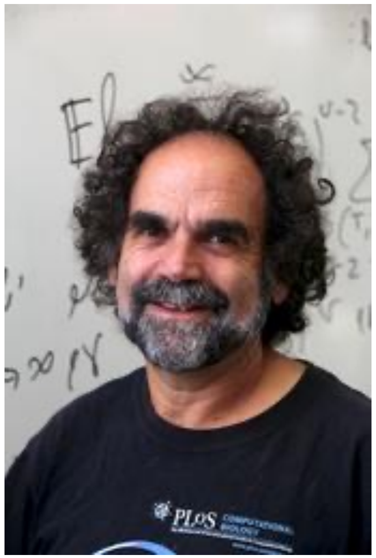

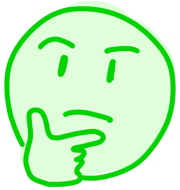

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Gamburd – Hoory – Shahshahani – Shalev – Virág

Random Galois covers

Random Galois covers

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Thank you!

Grazie!